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Essential Killing helices of order less than five on a non-flat complex space form

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Abstract. We study lengths of helices of orders 3 and 4 which are generated by some Killing vector fields on a complex projective plane and on a complex hyperbolic plane. We consider the moduli space of such helices under the congruence relation and give a lamination structure on this space which are closely related with the length spectrum. This shows that the moduli space does not form a canonical building structure with respect to the length spectrum.

1. Introduction.

When we study submanifolds in a non-flat complex space form, which is either a complex projective space or a complex hyperbolic space, we know that some extrinsic helical property of curves on submanifolds characterize these submanifolds. For example, S. Maeda [**11**] characterized Veronese embeddings by a circular property of extrinsic shapes of circles. It is also known that helices on a non-flat complex space form have many different properties compared with helices on a real space form, which is one of a standard sphere, a Euclidean space and a real hyperbolic space. Every helix on a real space form is generated by some Killing vector field, but not for all on a non-flat complex space form. In the preceding paper [**12**] Maeda and the author give a condition that helices on a non-flat complex space form to be generated by some Killing vector field, and in [**7**], we show that there are bounded helices of proper order 3 on a complex hyperbolic space. Needless to say that all helices of proper order 3 are unbounded on a Euclidean space and on a real hyperbolic space. We are hence interested in more on geometric properties of helices on a non-flat complex space form; their closedness, their lengths and so on.

In this paper we restrict ourselves to helices of proper order less than 5 on

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a non-flat complex space form which are generated by some Killing vector fields and each of which lies on some totally geodesic complex plane. We call such helices essential and Killing. As a sequel of the preceding papers [**7**], [**12**] we study bounded property and closedness of essential Killing helices, and also give their lengths. Our idea is based on combining a geometric property obtained through Naitoh's embedding in [**14**] and an algebraic property on cubic equations associated with helices. By use of this result on lengths, we consider lamination structures on a moduli space of helices, which is a set of congruence classes of helices. In the preceding paper [**2**], we give lamination structures on moduli spaces of helices of proper order less than 4 on real space forms. Corresponding to these we give lamination structures on moduli spaces of Killing helices of proper order less than 5 associated with length spectrum of helices on non-flat complex space forms. We then find structures of these moduli spaces for real and complex space forms are quite different from each other.

2. Essential Killing helices.

A smooth curve γ parameterized by its arclength on a Riemannian manifold M is said to be a *helix of proper order* d if it satisfies the following system of ordinary differential equations

$$\nabla_{\dot\gamma} Y_j = -\kappa_{j-1}Y_{j-1} + \kappa_j Y_{j+1}, \quad 1 \le j \le d, \tag{2.1}$$

with positive constants $\kappa_1, \dots, \kappa_{d-1}$ and an orthonormal system $\{Y_1 = \dot\gamma, Y_2, \dots, Y_d\}$ of vector fields along γ. Here we set $\kappa_0 = \kappa_d = 0$ and Y_0, Y_{d+1} to be null vector fields along γ. These constants $\kappa_1, \dots, \kappa_{d-1}$ and the frame field $\{Y_1, \dots, Y_d\}$ are called the *geodesic curvatures* and *Frenet frame* of γ, respectively. A helix of proper order 1 is a geodesic and a helix of proper order 2 is called a circle of positive geodesic curvature.

We say a helix to be *Killing* if it is generated by some Killing vector field. Trivially every helix on a real space form is Killing. On the other hand, on a non-flat complex space form $\boldsymbol{C}M^n$ a helix is not necessarily Killing. For a helix γ of proper order d with Frenet frame $\{Y_1, \dots, Y_d\}$, we define its complex torsions τ_{ij} $(1 \le i < j \le d)$ by $\tau_{ij} = \langle Y_i, JY_j\rangle$, where J is the complex structure on $\boldsymbol{C}M^n$. It was shown in [**13**] that a helix γ is Killing if and only if each of its complex torsions are constant along γ. By (2.1) we see complex torsions of Killing helices of proper order d satisfy the relations

$$\big(\tau_{ij}' = \big)\ -\kappa_{i-1}\tau_{i-1\,j} + \kappa_i\tau_{i+1\,j} - \kappa_{j-1}\tau_{i\,j-1} + \kappa_j\tau_{i\,j+1} = 0, \tag{2.2}$$

where we set $\tau_{ij}=0$ for i,j which do not satisfy $1\le i<j\le d$. It is known that Killing helices are closely related with submanifolds in a complex space form. We here give some examples. We denote by $\boldsymbol{C}P^n(c)$ a complex projective space of constant holomorphic sectional curvature c.

EXAMPLE 1. Let $f:\boldsymbol{C}P^1(2)\to\boldsymbol{C}P^2(4)$ be a Veronese embedding of order 2 which is defined as $f\big([z_0,z_1]\big)=\big[z_0^2,\sqrt{2}z_0z_1,z_1^2\big]$ with homogeneous coordinates. If we consider a circle γ of positive geodesic curvature k on $\boldsymbol{C}P^1(2)$, then the extrinsic shape $f\circ\gamma$ through f is as follows:

(1) When $k=\sqrt{2}/2$, it is a helix of proper order 3 with geodesic curvatures $\kappa_1=\sqrt{6}/2$, $\kappa_2=\sqrt{3}$;
(2) otherwise, it is a helix of proper order 4 with geodesic curvatures $\kappa_1=\sqrt{k^2+1}$, $\kappa_2=3k/\sqrt{k^2+1}$, $\kappa_3=|2k^2-1|/\sqrt{k^2+1}$.

EXAMPLE 2 ([**9**]). Let $\iota:G(r)\to\boldsymbol{C}P^n(4)$ be an isometric embedding of a geodesic sphere of radius r. For a geodesic γ we define its structure torsion τ_γ by $\tau_\gamma=\langle\dot\gamma,-J\mathscr{N}\rangle$ with complex structure J on $\boldsymbol{C}P^n$ and a unit normal $\mathscr{N}$ of $G(r)$ in $\boldsymbol{C}P^n$. Its extrinsic shape $\iota\circ\gamma$ in $\boldsymbol{C}P^n$ is as follows:

(1) When $\tau_\gamma=\pm\cot r$ in the case $\pi/4\le r<\pi/2$, it is a geodesic;
(2) when $\tau_\gamma=\pm1$, it is a circle of geodesic curvature $2|\cot 2r|$;
(3) when $\tau_\gamma=0$, it is a circle of geodesic curvature $\cot r$;
(4) otherwise, it is a helix of proper order 4 with geodesic curvature $\kappa_1=|\cot r-\tau_\gamma^2\tan r|$, $\kappa_2=|\tau_\gamma|\sqrt{1-\tau_\gamma^2}\tan r$, $\kappa_3=\cot r$.

Each of them lies on some totally geodesic $\boldsymbol{C}P^2$.

On a non-flat complex space form $\boldsymbol{C}M^n$, it is clear that a helix of proper order d lies on some totally geodesic $\boldsymbol{C}M^m$ with $m=\min\{n,2d\}$. We shall call a helix on $\boldsymbol{C}M^n$ of proper order either $2d-1$ or $2d$ *essential* if it lies on some totally geodesic $\boldsymbol{C}M^d$. If we borrow a terminology in [**16**], we may say that it is complex d-planner. Clearly, every geodesic on $\boldsymbol{C}M^n$ lies on some totally geodesic $\boldsymbol{C}M^1$ hence is essential. A circle on $\boldsymbol{C}M^n$ is essential if and only if its complex torsion is $\tau_{12}=\pm1$. As we have relations (2.2) and $|\tau_{ij}|\le1$, we obtain the following.

LEMMA 1 ([**5**]).

(1) *A helix of proper order* 3 *on* $\boldsymbol{C}M^n$ *is essential and Killing if and only if its geodesic curvatures and complex torsions satisfy*

$$\tau_{12}=\pm\frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}},\quad \tau_{13}=0,\quad \tau_{23}=\pm\frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2}},$$

where double signs take the same signatures.

(2) *A helix of proper order* 4 *on* $\boldsymbol{C}M^n$ *is essential and Killing if and only if their geodesic curvatures and complex torsions satisfy one of the following;*

i) $\tau_{12} = \tau_{34} = \pm\dfrac{\kappa_1+\kappa_3}{\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}}, \quad \tau_{23} = \tau_{14} = \pm\dfrac{\kappa_2}{\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}},$
$\tau_{13} = \tau_{24} = 0,$

ii) $\tau_{12} = -\tau_{34} = \pm\dfrac{\kappa_1-\kappa_3}{\sqrt{\kappa_2^2+(\kappa_1-\kappa_3)^2}}, \quad \tau_{23} = -\tau_{14} = \pm\dfrac{\kappa_2}{\sqrt{\kappa_2^2+(\kappa_1-\kappa_3)^2}},$
$\tau_{13} = \tau_{24} = 0.$

In each of the above conditions double signs take the same signatures.

As a consequence of this, we find that if γ is an essential Killing helix of proper order 3 on $\boldsymbol{C}M^n$ then the vector fields in its Frenet frame $\{\dot\gamma, Y_2, Y_3\}$ satisfy

$$Y_3 = \left(\frac{\kappa_1}{\kappa_2}\right)\dot\gamma \mp \left(\frac{\sqrt{\kappa_1^2+\kappa_2^2}}{\kappa_2}\right)JY_2.$$

Also, if γ is an essential Killing helix of proper order 4 on $\boldsymbol{C}M^n$, the vector fields in its Frenet frame $\{\dot\gamma, Y_2, Y_3, Y_4\}$ satisfy

$$\begin{cases} \kappa_2 Y_3 = (\kappa_1+\kappa_3)\dot\gamma \mp \sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}\,JY_2, \\ \kappa_2 Y_4 = \mp\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}\,J\dot\gamma - (\kappa_1+\kappa_3)Y_2, \end{cases}$$

in the case i) in Lemma 1, and they satisfy

$$\begin{cases} \kappa_2 Y_3 = (\kappa_1-\kappa_3)\dot\gamma \mp \sqrt{\kappa_2^2+(\kappa_1-\kappa_3)^2}\,JY_2, \\ \kappa_2 Y_4 = \pm\sqrt{\kappa_2^2+(\kappa_1-\kappa_3)^2}\,J\dot\gamma + (\kappa_1-\kappa_3)Y_2, \end{cases}$$

in the case of ii) in Lemma 1.

In the following sections, we study essential Killing helices of proper orders 3 and 4 on a complex projective space $\boldsymbol{C}P^n(4)$ and on a complex hyperbolic space $\boldsymbol{C}H^n(-4)$ of constant holomorphic sectional curvature -4. We say a smooth curve γ parameterized by its arclength *closed* if there is positive t_c satisfying $\gamma(t+t_c) = \gamma(t)$ for all t. The minimal positive t_c with this property is called the *length* of γ and is denoted by $\mathrm{length}(\gamma)$. When a smooth curve γ is not closed, we say it is open and set $\mathrm{length}(\gamma) = \infty$. On $\boldsymbol{C}P^n(4)$, circles of geodesic curvature κ and of complex torsion ± 1 are closed and have length $2\pi/\sqrt{\kappa^2+4}$ (see [**1**]).

3. Lengths of essential Killing helices on $\boldsymbol{CP}^n$.

In this section we study when essential Killing helices of proper orders 3 and 4 are closed on a complex projective space. When we study curves on a complex projective space, it is a basic idea to use a Hopf fibration $\varpi : S^{2n+1} \to \boldsymbol{CP}^n(4)$ of a standard sphere of radius 1 onto a complex projective space of constant holomorphic sectional curvature 4. We denote by $\mathscr{N}$ the outward unit normal of S^{2n+1} in $\boldsymbol{C}^{n+1}$. The Riemannian connections ∇ and $\overline{\nabla}$ on $\boldsymbol{CP}^n(4)$ and $\boldsymbol{C}^{n+1}$ are related by

$$\overline{\nabla}_X Y = \nabla_X Y - \langle X, Y\rangle \mathscr{N} + \langle X, JY\rangle J\mathscr{N} \tag{3.1}$$

for arbitrary vector fields X, Y on $\boldsymbol{CP}^n(4)$. Here we regard X, Y as horizontal vector fields on S^{2n+1} and we denote the complex structure on $\boldsymbol{C}^{n+1}$ also by J.

For the sake of later use, we here summarize some results in [**5**] which were obtained by two ways; a geometrical way through the isometric immersion given by Naitoh [**14**] and an arithmetical way through the Hopf fibration. We take a circle σ of geodesic curvature $1/\sqrt{2}$ and of complex torsion τ $(0 \le |\tau| < 1)$ on $\boldsymbol{CP}^n(4)$. By the relation (3.1) we find its horizontal lift $\hat{\sigma}$ with respect to the Hopf fibration satisfies the differential equation

$$\hat{\sigma}''' + \left(\frac{3}{2}\right)\hat{\sigma}' - \sqrt{-1}\left(\frac{\tau}{\sqrt{2}}\right)\hat{\sigma} = 0$$

as a curve in $\boldsymbol{C}^{n+1}$. We consider its characteristic equation $\lambda^3 + (3/2)\lambda - \sqrt{-1}\tau/\sqrt{2} = 0$. By putting $\Lambda = -\sqrt{-1}\lambda$ we obtain a cubic equation

$$\Lambda^3 - \left(\frac{3}{2}\right)\Lambda + \frac{\tau}{\sqrt{2}} = 0, \tag{3.2}$$

which has three distinct real solutions a, b, c $(a < b < c)$. Thus we find σ is of the form $\sigma(t) = \varpi(Ae^{\sqrt{-1}at} + Be^{\sqrt{-1}bt} + Ce^{\sqrt{-1}ct})$ with some $A, B, C \in \boldsymbol{C}^{n+1}$. We therefore find the following.

Fact 1 ([**8**]). Let σ be a circle of geodesic curvature $1/\sqrt{2}$ and of complex torsion τ $(0 \le |\tau| < 1)$ on $\boldsymbol{CP}^n(4)$.

(1) If $0 < |\tau| < 1$, it is closed if and only if one of (hence all of) the ratios $a/b, b/c, c/a$ of the solutions of (3.2) is (are) rational. In this case its length is given as

$$\text{length}(\sigma) = 2\pi \times \text{L.C.M.}\big((b-a)^{-1},\,(c-a)^{-1}\big),$$

where L.C.M.(α,β) for positive numbers α,β denotes the minimum number in the set $\{\alpha, 2\alpha, 3\alpha, \dots\} \cap \{\beta, 2\beta, 3\beta, \dots\}$.

(2) If $\tau = 0$, the solutions of (3.2) are $\pm\sqrt{6}/2$ and 0 (i.e. $b = 0$ and $a = -c$), hence σ is closed and $\text{length}(\sigma) = 2\pi/c = 2\sqrt{6}\pi/3$.

On the other hand, we have the following.

FACT 2 ([**8**]). Let σ be a circle of geodesic curvature $1/\sqrt{2}$ and of complex torsion τ on $\boldsymbol{C}P^n(4)$.

(1) If $\tau = \pm 1$, it is closed and of length $2\sqrt{2}\pi/3$.
(2) If $\tau = 0$, it is closed and of length $2\sqrt{6}\pi/3$.
(3) Otherwise, it is closed if and only if $\tau = q(9p^2 - q^2)(3p^2+q^2)^{-3/2}$ with some relatively prime positive integers p, q $(p > q)$. In this case

$$\text{length}(\sigma) = \frac{1}{3}\delta(p,q)\pi\sqrt{2(3p^2+q^2)},$$

where $\delta(p,q) = 1$ when pq is odd and $\delta(p,q) = 2$ when pq is even.

We now study an essential Killing helix γ of proper order 3 on $\boldsymbol{C}P^n(4)$ with geodesic curvatures κ_1, κ_2. We consider its horizontal lift $\hat{\gamma}$ on S^{2n+1} with respect to the Hopf fibration. Since its Frenet frame $\{Y_1 = \dot{\gamma}, Y_2, Y_3\}$ satisfies the relation $\kappa_2 Y_3 = \kappa_1\dot{\gamma} \mp \sqrt{\kappa_1^2+\kappa_2^2}JY_2$, we find by use of (3.1) that

$$\begin{aligned}\overline{\nabla}_{\dot{\gamma}}\overline{\nabla}_{\dot{\gamma}}\dot{\gamma} &= \kappa_1\overline{\nabla}_{\dot{\gamma}}Y_2 - \dot{\gamma} = -\big(\kappa_1^2+1\big)\dot{\gamma} + \kappa_1\kappa_2 Y_3 + \kappa_1\tau_{12}J\mathscr{N} \\ &= -\dot{\gamma} \mp \sqrt{\kappa_1^2+\kappa_2^2}\,J\big(\overline{\nabla}_{\dot{\gamma}}\dot{\gamma} + \mathscr{N}\big) \pm \frac{\kappa_1^2}{\sqrt{\kappa_1^2+\kappa_2^2}}\,J\mathscr{N}.\end{aligned}$$

Thus $\hat{\gamma}$ satisfies the differential equation

$$\hat{\gamma}''' \pm \sqrt{-(\kappa_1^2+\kappa_2^2)}\,\hat{\gamma}'' + \hat{\gamma}' \pm \frac{\sqrt{-1}\kappa_2^2}{\sqrt{\kappa_1^2+\kappa_2^2}}\hat{\gamma} = 0 \tag{3.3}$$

if we regard it as a curve in $\boldsymbol{C}^{n+1}$. Comparing this with the equation of a horizontal lift of a circle we obtain the following.

THEOREM 1. *Let γ be an essential Killing helix of proper order 3 on $\boldsymbol{C}P^n(4)$ with geodesic curvatures κ_1, κ_2 and complex torsion $\tau_{12} = \pm\kappa_1/\sqrt{\kappa_1^2+\kappa_2^2}$. We set*

$$\tau_P(\kappa_1,\kappa_2)=\frac{2(\kappa_1^2+\kappa_2^2)^2+9(\kappa_1^2-2\kappa_2^2)}{2(\kappa_1^2+\kappa_2^2+3)^{3/2}\sqrt{\kappa_1^2+\kappa_2^2}}.$$

(1) *When* $0<\kappa_1\le\sqrt{6}/2$ *and* $2\kappa_2^2=9-2\kappa_1^2\pm3\sqrt{3(3-2\kappa_1^2)}$*, then* $\tau_P(\kappa_1,\kappa_2)=0$ *and* γ *is closed of length* $2\sqrt{3}\pi/\sqrt{\kappa_1^2+\kappa_2^2+3}$*.*

(2) *When* $\tau_P(\kappa_1,\kappa_2)=\pm q(9p^2-q^2)(3p^2+q^2)^{-3/2}$ *with some relatively prime positive integers* p,q *with* $p>q$*, then* γ *is closed of length*

$$\delta(p,q)\pi\frac{\sqrt{3p^2+q^2}}{\sqrt{\kappa_1^2+\kappa_2^2+3}},$$

where $\delta(p,q)=1$ *when* pq *is odd and* $\delta(p,q)=2$ *when* pq *is even.*

(3) *Otherwise, it is open.*

Proof. We study the characteristic equation

$$\lambda^3\pm\sqrt{-(\kappa_1^2+\kappa_2^2)}\,\lambda^2+\lambda\pm\frac{\sqrt{-1}\,\kappa_2^2}{\sqrt{\kappa_1^2+\kappa_2^2}}=0 \tag{3.4}$$

for the differential equation (3.3) on a horizontal lift of γ. By putting $\theta=\{-3\sqrt{-1}\lambda\pm\sqrt{\kappa_1^2+\kappa_2^2}\}/\sqrt{2(\kappa_1^2+\kappa_2^2+3)}$, we find it turns to

$$\theta^3-\frac{3}{2}\theta\pm\frac{2(\kappa_1^2+\kappa_2^2)^2+9(\kappa_1^2-2\kappa_2^2)}{2\sqrt{2}(\kappa_1^2+\kappa_2^2+3)^{3/2}\sqrt{\kappa_1^2+\kappa_2^2}}=0. \tag{3.5}$$

If we denote by $\sqrt{-1}a_i$ $(i=1,2,3,\ a_1\le a_2\le a_3)$ the solutions of (3.4), then the solutions for (3.5) are $\hat{a}_i=(3a_i\pm\sqrt{\kappa_1^2+\kappa_2^2})/\sqrt{2(\kappa_1^2+\kappa_2^2+3)}$ $(i=1,2,3)$. Since γ is expressed as

$$\begin{aligned}\gamma(t)&=\varpi\big(A_1e^{\sqrt{-1}a_1t}+A_2e^{\sqrt{-1}a_2t}+A_3e^{\sqrt{-1}a_3t}\big)\\&=\varpi\big(A_1+A_2e^{\sqrt{-1}(a_2-a_1)t}+A_3e^{\sqrt{-1}(a_3-a_1)t}\big)\end{aligned}$$

with some $A_1,A_2,A_3\in\boldsymbol{C}^{n+1}$, we see γ is closed if and only if $(a_2-a_1)/(a_3-a_1)$ is rational. This condition is equivalent to the condition that the number $(\hat{a}_2-\hat{a}_1)/(\hat{a}_3-\hat{a}_1)$ is rational. As $\hat{a}_1+\hat{a}_2+\hat{a}_3=0$, we find that this condition holds when $\tau_P(\kappa_1,\kappa_2)=0$ and that this condition is equivalent to the condition that one of (hence all of) $\hat{a}_2/\hat{a}_1$, $\hat{a}_3/\hat{a}_2$, $\hat{a}_1/\hat{a}_3$ is (are) rational when $\tau_P(\kappa_1,\kappa_2)\neq0$. In the latter case its length is given as

$$\begin{aligned}\operatorname{length}(\gamma) &= 2\pi \times \text{L.C.M.}\big((a_2-a_1)^{-1},(a_3-a_1)^{-1}\big)\\ &= \frac{6\pi}{\sqrt{2(\kappa_1^2+\kappa_2^2+3)}} \times \text{L.C.M.}\big((\hat{a}_2-\hat{a}_1)^{-1},(\hat{a}_3-\hat{a}_1)^{-1}\big),\end{aligned}$$

We here compare two cubic equations (3.2) and (3.5). We find $\hat{a}_1,\hat{a}_2,\hat{a}_3$ are the solutions for (3.2) with $\tau = \tau_P(\kappa_1,\kappa_2)$. By direct computation we see $|\tau_P(\kappa_1,\kappa_2)| < 1$. Since Fact 1 and 2 combine algebraic and geometric conditions, we get the conclusion. $\square$

Next we study essential Killing helices of proper order 4 on $\boldsymbol{C}P^n(4)$. We first consider an essential Killing helix γ of proper order 4 with geodesic curvatures $\kappa_1,\kappa_2,\kappa_3$ whose complex torsions satisfy the relations in Lemma 1 (2-i). By use of (3.1) we find its horizontal lift $\hat{\gamma}$ satisfies the differential equation

$$\begin{aligned}&\hat{\gamma}''' \pm \sqrt{-\{\kappa_2^2+(\kappa_1+\kappa_3)^2\}}\,\hat{\gamma}''\\ &\qquad + (1-\kappa_1\kappa_3)\hat{\gamma}' \pm \frac{\sqrt{-1}\,\{\kappa_2^2+\kappa_3(\kappa_1+\kappa_3)\}}{\sqrt{\{\kappa_2^2+(\kappa_1+\kappa_3)^2\}}}\hat{\gamma} = 0. \qquad (3.6)\end{aligned}$$

Its characteristic equation with variable λ turns to

$$\theta^3 - \left(\frac{3}{2}\right)\theta \pm \frac{\tau_P^+(\kappa_1,\kappa_2,\kappa_3)}{\sqrt{2}} = 0$$

with

$$\begin{aligned}&\tau_P^+(\kappa_1,\kappa_2,\kappa_3)\\ &\quad = \frac{2\{\kappa_2^2+(\kappa_1+\kappa_3)^2\}^2 - 9(2+\kappa_1\kappa_3)\{\kappa_2^2+(\kappa_1+\kappa_3)^2\} + 27\kappa_1(\kappa_1+\kappa_3)}{2\{\kappa_2^2+(\kappa_1+\kappa_3)^2+3(1-\kappa_1\kappa_3)\}^{3/2}\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}},\end{aligned}$$

if we put

$$\theta = \frac{-3\sqrt{-1}\lambda \pm \sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}}{\sqrt{2\{\kappa_2^2+(\kappa_1+\kappa_3)^2+3-3\kappa_1\kappa_3\}}}.$$

Along the same lines as in the proof of Theorem 1, we obtain the following.

THEOREM 2. *Let γ be an essential Killing helix of proper order 4 on $\boldsymbol{C}P^n(4)$ with geodesic curvatures $\kappa_1,\kappa_2,\kappa_3$ and complex torsion $\tau_{12} = \pm(\kappa_1+\kappa_3)/\sqrt{(\kappa_1+\kappa_3)^2+\kappa_2^2}$.*

(1) *When its geodesic curvatures satisfy the relation*

$$4\kappa_2^2 = 9(2+\kappa_1\kappa_3) - 4(\kappa_1+\kappa_3)^2 \pm 3\sqrt{3(12-8\kappa_1^2+4\kappa_1\kappa_3+3\kappa_1^2\kappa_3^2)},$$

then $\tau_P^+(\kappa_1,\kappa_2,\kappa_3)=0$, *and* γ *is closed and is of length*

$$\frac{2\sqrt{3}\pi}{\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2+3-3\kappa_1\kappa_3}}.$$

(2) *When* $\tau_P^+(\kappa_1,\kappa_2,\kappa_3) = \pm q(9p^2-q^2)(3p^2+q^2)^{-3/2}$ *with some relatively prime positive integers* p, q *with* $p>q$, *then* γ *is closed and is of length*

$$\frac{\delta(p,q)\pi\sqrt{3p^2+q^2}}{\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2+3-3\kappa_1\kappa_3}}.$$

(3) *Otherwise, it is open.*

We next consider an essential Killing helix γ of proper order 4 with geodesic curvatures $\kappa_1,\kappa_2,\kappa_3$ whose complex torsions satisfy the relations in Lemma 1 (2-ii). We find its horizontal lift $\hat\gamma$ of γ satisfies

$$\begin{aligned}\hat\gamma''' &\pm \sqrt{-\{\kappa_2^2+(\kappa_1-\kappa_3)^2\}}\,\hat\gamma'' \\ &+ (1+\kappa_1\kappa_3)\hat\gamma' \pm \frac{\sqrt{-1}\,\{\kappa_2^2-\kappa_3(\kappa_1-\kappa_3)\}}{\sqrt{\{\kappa_2^2+(\kappa_1-\kappa_3)^2\}}}\hat\gamma = 0. \qquad (3.7)\end{aligned}$$

Its characteristic equation with variable λ turns to

$$\theta^3 - \left(\frac{3}{2}\right)\theta \pm \frac{\tau_P^-(\kappa_1,\kappa_2,\kappa_3)}{\sqrt{2}} = 0$$

with

$$\begin{aligned}&\tau_P^-(\kappa_1,\kappa_2,\kappa_3) \\ &\quad = \frac{2\{\kappa_2^2+(\kappa_1-\kappa_3)^2\}^2 - 9(2-\kappa_1\kappa_3)\{\kappa_2^2+(\kappa_1-\kappa_3)^2\} + 27\kappa_1(\kappa_1-\kappa_3)}{2\{\kappa_2^2+(\kappa_1-\kappa_3)^2+3(1+\kappa_1\kappa_3)\}^{3/2}\sqrt{\kappa_2^2+(\kappa_1-\kappa_3)^2}}.\end{aligned}$$

if we put

$$\theta = \frac{-3\sqrt{-1}\lambda \pm \sqrt{\kappa_2^2 + (\kappa_1 - \kappa_3)^2}}{\sqrt{2\{\kappa_2^2 + (\kappa_1 - \kappa_3)^2 + 3 + 3\kappa_1\kappa_3\}}}.$$

We can hence conclude the following.

THEOREM 3. *Let γ be an essential Killing helix of proper order 4 on $\boldsymbol{C}P^n(4)$ with geodesic curvatures $\kappa_1, \kappa_2, \kappa_3$ and complex torsion $\tau_{12} = \pm(\kappa_1 - \kappa_3)/\sqrt{(\kappa_1 - \kappa_3)^2 + \kappa_2^2}$.*

(1) *When its geodesic curvatures satisfy the relation*

$$4\kappa_2^2 = 9(2 - \kappa_1\kappa_3) - 4(\kappa_1 - \kappa_3)^2 \pm 3\sqrt{3(12 - 8\kappa_1^2 - 4\kappa_1\kappa_3 + 3\kappa_1^2\kappa_3^2)},$$

then $\tau_P^-(\kappa_1, \kappa_2, \kappa_3) = 0$, and γ is closed and is of length

$$\frac{2\sqrt{3}\pi}{\sqrt{\kappa_2^2 + (\kappa_1 - \kappa_3)^2 + 3 + 3\kappa_1\kappa_3}}.$$

(2) *When $\tau_P^-(\kappa_1, \kappa_2, \kappa_3) = \pm q(9p^2 - q^2)(3p^2 + q^2)^{-3/2}$ with some relatively prime positive integers p, q with $p > q$, then γ is closed and is of length*

$$\frac{\delta(p,q)\pi\sqrt{3p^2 + q^2}}{\sqrt{\kappa_2^2 + (\kappa_1 - \kappa_3)^2 + 3 + 3\kappa_1\kappa_3}}.$$

(3) *Otherwise, it is open.*

4. Behavior of essential Killing helices on $\boldsymbol{C}H^n$.

In this section we study some properties of essential Killing helices on a complex hyperbolic space. We call a smooth curve γ unbounded in both directions if both of the sets $\gamma((-\infty, 0]), \gamma([0, \infty))$ are unbounded. As a complex hyperbolic space $\boldsymbol{C}H^n$ is an example of a Hadamard manifold, we can consider its ideal boundary $\partial\boldsymbol{C}H^n$ with respect to the cone topology. For a smooth curve γ which is unbounded in both directions, we set $\gamma(\infty) = \lim_{t\to\infty}\gamma(t)$, $\gamma(-\infty) = \lim_{t\to-\infty}\gamma(t) \in \partial\boldsymbol{C}H^n$ if they exist. We call them the points at infinity of γ. We shall call a smooth curve γ *horocyclic* if the following two conditions hold:

i) $\gamma(\infty) = \gamma(-\infty)$;
ii) if γ and a geodesic σ satisfying $\sigma(\infty) = \gamma(\infty)$ cross at some point, then they cross orthogonally at that point.

Let $\varpi : H_1^{2n+1} \to \boldsymbol{C}H^n(-4)$ denote a canonical fibration of an anti de-Sitter space $H_1^{2n+1}(\subset \boldsymbol{C}^{n+1})$ to a complex hyperbolic space of constant holomorphic sectional curvature -4. We consider a Hermitian form $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\boldsymbol{C}^{n+1}$ defined by $\langle\!\langle z, w\rangle\!\rangle = -z_0\bar{w}_0 + z_1\bar{z}_1 + \cdots + z_n\bar{w}_n$ for $z = (z_0, \ldots, z_n)$, $w = (w_0, \ldots, w_n) \in \boldsymbol{C}^{n+1}$. The space H_1^{2n+1} is given as $H_1^{2n+1} = \{z \in \boldsymbol{C}^{n+1} \mid \langle\!\langle z, z\rangle\!\rangle = -1\}$. The Riemannian connections ∇ and $\overline{\nabla}$ on $\boldsymbol{C}H^n(-4)$ and on $\boldsymbol{C}^{n+1}$ are related by

$$\overline{\nabla}_X Y = \nabla_X Y + \langle X, Y\rangle \mathscr{N} - \langle X, JY\rangle J\mathscr{N} \tag{4.1}$$

for arbitrary vector fields X, Y on $\boldsymbol{C}H^n(-4)$. Here we regard X, Y as horizontal vector fields on H_1^{2n+1} and $\mathscr{N}$ denotes a normal vector field on H_1^{2n+1} in $\boldsymbol{C}^{n+1}$ with $\langle\!\langle \mathscr{N}, \mathscr{N}\rangle\!\rangle = -1$.

We take an essential Killing helix γ of proper order 3 with geodesic curvatures κ_1, κ_2 on $\boldsymbol{C}H^n(-4)$. Since its Frenet frame $\{\dot{\gamma}, Y_2, Y_3\}$ satisfies the relation $\kappa_2 Y_3 = \kappa_1\dot{\gamma} \mp \sqrt{\kappa_1^2 + \kappa_2^2}\, JY_2$, by use of (4.1) we find its horizontal lift $\hat{\gamma}$ on H_1^{2n+1} satisfies the differential equation

$$\hat{\gamma}''' \pm \sqrt{-(\kappa_1^2 + \kappa_2^2)}\, \hat{\gamma}'' - \hat{\gamma}' \mp \frac{\sqrt{-1}\,\kappa_2^2}{\sqrt{\kappa_1^2 + \kappa_2^2}}\, \hat{\gamma} = 0 \tag{4.2}$$

if we regard it as a curve in $\boldsymbol{C}^{n+1}$.

Theorem 4. *Let γ be an essential Killing helix of proper order 3 on $\boldsymbol{C}H^n(-4)$ with geodesic curvatures κ_1, κ_2 and complex torsion $\tau_{12} = \pm\kappa_1/\sqrt{\kappa_1^2 + \kappa_2^2}$.*

(1) *It is bounded if and only if its geodesic curvatures satisfy one of the following conditions*:
  i) $0 < \kappa_2 < 1/2$ *and* $(1 - 4\kappa_2^2)\kappa_1^2 > 2\{2\kappa_2^4 - 5\kappa_2^2 + 1 + (1 - 3\kappa_2^2)^{3/2}\}$;
  ii) $\kappa_2 = 1/2$ *and* $\kappa_1 > 5\sqrt{2}/4$;
  iii) $1/2 < \kappa_2 < 1/\sqrt{3}$ *and*

$$\begin{aligned} &2\big\{-2\kappa_2^4 + 5\kappa_2^2 - 1 - (1 - 3\kappa_2^2)^{3/2}\big\} \\ &\qquad < (4\kappa_2^2 - 1)\kappa_1^2 < 2\big\{-2\kappa_2^4 + 5\kappa_2^2 - 1 + (1 - 3\kappa_2^2)^{3/2}\big\}. \end{aligned}$$

(2) *When γ is not bounded, it is unbounded in both directions. In the cases*
  i) $(1 - 4\kappa_2^2)\kappa_1^2 = 2\big\{2\kappa_2^4 - 5\kappa_2^2 + 1 + (1 - 3\kappa_2^2)^{3/2}\big\}$ *with* $0 < \kappa_2 \le 1/2$,
  ii) $(4\kappa_2^2 - 1)\kappa_1^2 = 2\big\{-2\kappa_2^4 + 5\kappa_2^2 - 1 \pm (1 - 3\kappa_2^2)^{3/2}\big\}$ *with* $1/2 < \kappa_2 \le 1/\sqrt{3}$,
  it has single point at infinity. In particular, if $\kappa_1 = 2\sqrt{6}/3$, $\kappa_2 = 1/\sqrt{3}$, it is

horocyclic. In other cases, it has two distinct points at infinity.

(3) *When γ is bounded, we set*

$$\tau_H(\kappa_1,\kappa_2)=\frac{2(\kappa_1^2+\kappa_2^2)^2-9(\kappa_1^2-2\kappa_2^2)}{2(\kappa_1^2+\kappa_2^2-3)^{3/2}\sqrt{\kappa_1^2+\kappa_2^2}}.$$

1) *When $0<\kappa_1<3/\sqrt{2}$ and $2\kappa_2^2=3\sqrt{6\kappa_1^2+9}-2\kappa_1^2-9$, then $\tau_H(\kappa_1,\kappa_2)=0$, and γ is closed and is of length $2\sqrt{3}\pi/\sqrt{\kappa_1^2+\kappa_2^2-3}$.*

2) *When $\tau_H(\kappa_1,\kappa_2)=\pm q(9p^2-q^2)(3p^2+q^2)^{-3/2}$ with some relatively prime positive integers p,q with $p>q$, then γ is closed and is of length*

$$\delta(p,q)\pi\frac{\sqrt{3p^2+q^2}}{\sqrt{\kappa_1^2+\kappa_2^2-3}}.$$

3) *Otherwise, it is open.*

PROOF. We consider the characteristic equation

$$\lambda^3\pm\sqrt{-(\kappa_1^2+\kappa_2^2)}\,\lambda^2-\lambda\mp\frac{\sqrt{-1}\,\kappa_2^2}{\sqrt{\kappa_1^2+\kappa_2^2}}=0. \tag{4.3}$$

of the differential equation (4.2). It turns to

$$\Theta^3-\frac{1}{3}\big(\kappa_1^2+\kappa_2^2-3\big)\Theta\pm\frac{2(\kappa_1^2+\kappa_2^2)^2-9(\kappa_1^2-2\kappa_2^2)}{27\sqrt{\kappa_1^2+\kappa_2^2}}=0 \tag{4.4}$$

if we put $\Theta=-\sqrt{-1}\lambda\pm\sqrt{\kappa_1^2+\kappa_2^2}/3$. When $\kappa_1^2+\kappa_2^2\le 3$, it is clear that this cubic equation has only one real solution, except the case $\kappa_1^2=8/3$, $\kappa_2^2=1/3$. In the exceptional case, 0 is the triple solution of this equation. Thus when $\kappa_1^2+\kappa_2^2\le 3$ we find γ is unbounded in both directions. When $\kappa_1^2+\kappa_2^2>3$, by putting $\theta=3\Theta/\sqrt{2(\kappa_1^2+\kappa_2^2-3)}$ we find the cubic equation (4.4) turns to

$$\theta^3-\left(\frac{3}{2}\right)\theta\pm\frac{\tau_H(\kappa_1,\kappa_2)}{\sqrt{2}}=0.$$

It is clear that it has only one real solution and two imaginary solutions when $|\tau_H(\kappa_1,\kappa_2)|>1$ and that it has one double real solution when $|\tau_H(\kappa_1,\kappa_2)|=1$. In these cases γ is also unbounded in both directions. Here, by direct computation we have $|\tau_H(\kappa_1,\kappa_2)|\ge 1$ if and only if

$$\big(4\kappa_2^2-1\big)\kappa_1^4+4\big(2\kappa_2^4-5\kappa_2^2+1\big)\kappa_1^2+4\kappa_2^2\big(\kappa_2^2+1\big)^2\geq 0.$$

When $|\tau_H(\kappa_1,\kappa_2)|<1$, comparing this cubic equation with (3.2), we obtain our conclusions on bounded essential Killing helices of proper order 3.

What we have to show is the asymptotic behavior of unbounded essential Killing helices. We represent $\boldsymbol{C}H^n$ as a unit ball $D_n(\boldsymbol{C})=\{w=(w_1,\ldots,w_n)\in \boldsymbol{C}^n \mid \sum_{i=1}^n w_i\overline{w_i}<1\}$. By use of homogeneous coordinate the identification $\boldsymbol{C}H^n\to B_n(\boldsymbol{C})$ is given by $[z_0,\ldots,z_n]\mapsto (z_1/z_0,\ldots,z_n/z_0)$. The ideal boundary of $\boldsymbol{C}H^n$ corresponds to the topological boundary of $D_n(\boldsymbol{C})$. When (4.3) has solutions of type $\sqrt{-1}\alpha+\beta$ and $\sqrt{-1}\alpha-\beta$ with real α,β $(\beta>0)$, we see γ is of the form

$$\gamma(t)=\varpi\big(Ae^{\sqrt{-1}at}+Be^{\sqrt{-1}\alpha t+\beta t}+Ce^{\sqrt{-1}\alpha t-\beta t}\big)$$

with some real a and vectors $A,B,C\in\boldsymbol{C}^{n+1}$. If we consider it on the ball model $D_n(\boldsymbol{C})$, its points at infinity are $(B_1/B_0,\ldots,B_n/B_0)$ and $(C_1/C_0,\ldots,C_n/C_0)$. Since $\hat{\gamma}$ does not satisfy differential equation of order less than 3, we find these point do not coincide. Hence we see γ has two distinct points at infinity in this case.

We hence consider the case $\kappa_1^2=8/3$, $\kappa_2^2=1/3$ and the case $\kappa_1^2+\kappa_2^2>3$ and $\tau_H(\kappa_1,\kappa_2)=1$. In the former case, under the condition $\gamma(0)=\varpi(z)$, $\dot{\gamma}(0)=d\varpi(z,u)$, $\nabla_{\dot{\gamma}}\dot{\gamma}(0)=\kappa_1 d\varpi(z,v)$ we have

$$\gamma(t)=\varpi\Bigg(z+t\bigg(u\pm\sqrt{-\frac{1}{3}}\,z\bigg)+t^2\bigg(\sqrt{\frac{2}{3}}\,v\pm\sqrt{-\frac{1}{3}}\,u+\frac{z}{3}\bigg)\Bigg),$$

hence it has single point at infinity. Further more, if we take a geodesic σ with $\sigma(0)=\varpi(z)$, $\dot{\sigma}(0)=d\varpi(z,\sqrt{6}\,v\pm\sqrt{-3}\,u)$, it satisfies $\sigma(\infty)=\gamma(\infty)$. As $\dot{\sigma}(0)$ is orthogonal to $\dot{\gamma}(0)$, we find γ is horocyclic. In the latter case, the solutions of (4.3) are $\pm\sqrt{-1}(K+L)$, $\pm\sqrt{-1}(K+L)$, $\pm\sqrt{-1}(-2K+L)$ with $K=\sqrt{\kappa_1^2+\kappa_2^2-3}/3$, $L=-\sqrt{\kappa_1^2+\kappa_2^2}/3$. Hence γ is of the form

$$\gamma(t)=\varpi\big(e^{\pm\sqrt{-1}Lt}\big((A+tB)e^{\pm\sqrt{-1}Kt}+Ce^{\mp 2\sqrt{-1}Kt}\big)\big).$$

Hence we find it has single point at infinity. □

We here note that the first assertion of the above theorem was shown in [**7**].

Remark 1. When $\kappa_1=5/2\sqrt{2}$, $\kappa_2=1/2$, an essential unbounded Killing helix γ in Theorem 4 is of the form

$$
\begin{aligned}
&\gamma(t) = \varpi\big((A+tB)e^{\mp\sqrt{-6}t/6} + Ce^{\mp 5\sqrt{-6}t/12}\big),\\
&\qquad A = \frac{29}{9}z \pm \frac{8\sqrt{6}}{9}Ju + \frac{10\sqrt{2}}{3}v, \quad B = \mp\frac{7\sqrt{6}}{18}Jz + \frac{7}{3}u \mp \frac{5\sqrt{3}}{3}Jv,\\
&\qquad C = -\frac{20}{9}z \mp \frac{8\sqrt{6}}{9}Ju - \frac{10\sqrt{2}}{3}v,
\end{aligned}
$$

under initial condition $\gamma(0) = \varpi(z)$, $\dot\gamma(0) = d\varpi(z,u)$, $\nabla_{\dot\gamma}\dot\gamma(0) = \kappa_1 d\varpi(z,v)$. As $\|5\sqrt{3}v \pm 7Ju\|^2 = 124 \mp 70\sqrt{3}\tau_{12} = 22/3$ does not coincide with $(7/\sqrt{6})^2$, we find γ is not horocyclic.

Next we study essential Killing helices of proper order 4 on $\boldsymbol{C}H^n(-4)$. We first consider an essential Killing helix γ of proper order 4 with geodesic curvatures $\kappa_1, \kappa_2, \kappa_3$ whose complex torsions satisfy the relations in Lemma 1 (2-i). By use of (4.1) we find its horizontal lift $\hat\gamma$ on H_1^{2n+1} satisfies the differential equation

$$
\begin{aligned}
&\hat\gamma''' \pm \sqrt{-\{\kappa_2^2 + (\kappa_1+\kappa_3)^2\}}\,\hat\gamma''\\
&\qquad - (1+\kappa_1\kappa_3)\hat\gamma' \mp \frac{\sqrt{-1}\,\{\kappa_2^2 + \kappa_3(\kappa_1+\kappa_3)\}}{\sqrt{\{\kappa_2^2 + (\kappa_1+\kappa_3)^2\}}}\hat\gamma = 0 \qquad (4.5)
\end{aligned}
$$

as a curve in $\boldsymbol{C}^{n+1}$.

THEOREM 5. *Let γ be an essential Killing helix of proper order 4 on $\boldsymbol{C}H^n(-4)$ with geodesic curvatures $\kappa_1, \kappa_2, \kappa_3$ and complex torsion $\tau_{12} = \pm(\kappa_1 + \kappa_3)/\sqrt{(\kappa_1+\kappa_3)^2 + \kappa_2^2}$.*

(1) *It is unbounded in both directions if and only if its geodesic curvatures satisfy one of the following conditions*:
  i) $(3/4)\kappa_1^2 + \kappa_2^2 + \{\kappa_3 - (\kappa_1/2)\}^2 \le 3$;
  ii) $(3/4)\kappa_1^2 + \kappa_2^2 + \{\kappa_3 - (\kappa_1/2)\}^2 > 3$ *and* $|\tau_H^+(\kappa_1,\kappa_2,\kappa_3)| \ge 1$, *where*

$$
\begin{aligned}
&\tau_H^+(\kappa_1,\kappa_2,\kappa_3)\\
&\quad = \frac{2\{\kappa_2^2 + (\kappa_1+\kappa_3)^2\}^2 + 9(2-\kappa_1\kappa_3)\{\kappa_2^2 + (\kappa_1+\kappa_3)^2\} - 27\kappa_1(\kappa_1+\kappa_3)}{2\{\kappa_2^2 + (\kappa_1+\kappa_3)^2 - 3(1+\kappa_1\kappa_3)\}^{3/2}\sqrt{\kappa_2^2 + (\kappa_1+\kappa_3)^2}}.
\end{aligned}
$$

Otherwise it is bounded.

(2) *In the cases*
  i) $(3/4)\kappa_1^2 + \kappa_2^2 + \{\kappa_3 - (\kappa_1/2)\}^2 = 3$,
  ii) $(3/4)\kappa_1^2 + \kappa_2^2 + \{\kappa_3 - (\kappa_1/2)\}^2 > 3$ *and* $\tau_H^+(\kappa_1,\kappa_2,\kappa_3) = \pm 1$,

it has single point at infinity. If γ is not bounded and its geodesic curvatures do not satisfy above, it has two distinct points at infinity.

(3) *Suppose γ is bounded. Hence $(3/4)\kappa_1^2 + \kappa_2^2 + \{\kappa_3 - (\kappa_1/2)\}^2 > 3$ and $|\tau_H^+(\kappa_1, \kappa_2, \kappa_3)| < 1$.*

1) *When $\tau_H^+(\kappa_1, \kappa_2, \kappa_3) = 0$, then γ is closed and is of length*

$$\frac{2\sqrt{3}\pi}{\sqrt{\kappa_2^2 + (\kappa_1 + \kappa_3)^2 - 3(1 + \kappa_1\kappa_3)}}.$$

2) *When $\tau_H^+(\kappa_1, \kappa_2, \kappa_3) = \pm q(9p^2 - q^2)(3p^2 + q^2)^{-3/2}$ with some relatively prime positive integers p, q with $p > q$, then γ is closed and is of length*

$$\frac{\delta(p,q)\pi\sqrt{3p^2 + q^2}}{\sqrt{\kappa_2^2 + (\kappa_1 + \kappa_3)^2 - 3(1 + \kappa_1\kappa_3)}}.$$

3) *Otherwise, it is open.*

Proof. The characteristic equation of (4.5) with variable λ turns to

$$\begin{aligned}
\Theta^3 &- \frac{1}{3}\big\{\kappa_2^2 + (\kappa_1 + \kappa_3)^2 - 3(1 + \kappa_1\kappa_3)\big\}\Theta \\
&\pm \frac{1}{27}\big[2\{\kappa_2^2 + (\kappa_1 + \kappa_3)^2\}^2 \\
&\qquad - 9(1 + \kappa_1\kappa_3)\{\kappa_2^2 + (\kappa_1 + \kappa_3)^2\} + 27\{\kappa_2^2 + \kappa_3(\kappa_1 + \kappa_3)\}\big] \\
&\quad \times \{\kappa_2^2 + (\kappa_1 + \kappa_3)^2\}^{-1/2} \\
&= 0.
\end{aligned}$$

if we put $\Theta = -\sqrt{-1}\lambda \pm (1/3)\sqrt{\kappa_2^2 + (\kappa_1 + \kappa_3)^2}$. When $\kappa_2^2 + (\kappa_1 + \kappa_3)^2 \leq 3(1 + \kappa_1\kappa_3)$, it has only one real solution, hence we find γ is unbounded in both directions. When $\kappa_2^2 + (\kappa_1 + \kappa_3)^2 > 3(1 + \kappa_1\kappa_3)$, we set $\theta = 3\Theta/\sqrt{2\{\kappa_2^2 + (\kappa_1 + \kappa_3)^2 - 3(1 + \kappa_1\kappa_3)\}}$. We then find the characteristic equation turns to $\theta^3 - (3/2)\theta \pm \tau_H^+(\kappa_1, \kappa_2, \kappa_3)/\sqrt{2} = 0$. We hence get the conclusion. □

We next consider an essential Killing helix γ of proper order 4 with geodesic curvatures $\kappa_1, \kappa_2, \kappa_3$ whose complex torsions satisfy the relations in Lemma 1 (2-ii). By use of (4.1) we find its horizontal lift $\hat{\gamma}$ on H_1^{2n+1} satisfies the differential equation

$$\begin{aligned}\hat\gamma''' &\pm \sqrt{-\{\kappa_2^2+(\kappa_1-\kappa_3)^2\}}\,\hat\gamma'' \\ &- (1-\kappa_1\kappa_3)\hat\gamma' \mp \frac{\sqrt{-1}\,\{\kappa_2^2-\kappa_3(\kappa_1-\kappa_3)\}}{\sqrt{\kappa_2^2+(\kappa_1-\kappa_3)^2}}\hat\gamma = 0 \qquad (4.6)\end{aligned}$$

as a curve in $\boldsymbol{C}^{n+1}$.

THEOREM 6. *Let γ be an essential Killing helix of proper order 4 on $\boldsymbol{C}H^n(-4)$ with geodesic curvatures $\kappa_1,\kappa_2,\kappa_3$ and complex torsion $\tau_{12}=\pm(\kappa_1-\kappa_3)/\sqrt{(\kappa_1-\kappa_3)^2+\kappa_2^2}$.*

(1) *It is unbounded in both directions if and only if its geodesic curvatures satisfy one of the following conditions*:
  i) $(3/4)\kappa_1^2+\kappa_2^2+\{\kappa_3+(\kappa_1/2)\}^2\le 3$;
  ii) $(3/4)\kappa_1^2+\kappa_2^2+\{\kappa_3+(\kappa_1/2)\}^2 > 3$ *and* $|\tau_H^-(\kappa_1,\kappa_2,\kappa_3)|\ge 1$, *where*

$$\begin{aligned}&\tau_H^-(\kappa_1,\kappa_2,\kappa_3)\\ &\quad=\frac{2\{\kappa_2^2+(\kappa_1-\kappa_3)^2\}^2+9(2+\kappa_1\kappa_3)\{\kappa_2^2+(\kappa_1-\kappa_3)^2\}-27\kappa_1(\kappa_1-\kappa_3)}{2\{\kappa_2^2+(\kappa_1-\kappa_3)^2-3(1-\kappa_1\kappa_3)\}^{3/2}\sqrt{\kappa_2^2+(\kappa_1-\kappa_3)^2}}.\end{aligned}$$

Otherwise it is bounded.

(2) *In the cases*
  i) $(3/4)\kappa_1^2+\kappa_2^2+\{\kappa_3+(\kappa_1/2)\}^2=3$,
  ii) $(3/4)\kappa_1^2+\kappa_2^2+\{\kappa_3+(\kappa_1/2)\}^2>3$ *and* $\tau_H^-(\kappa_1,\kappa_2,\kappa_3)=\pm1$,

it has single point at infinity. If γ is not bounded and its geodesic curvatures do not satisfy above, it has two distinct points at infinity.

(3) *Suppose γ is bounded. Hence $(3/4)\kappa_1^2+\kappa_2^2+\{\kappa_3+(\kappa_1/2)\}^2>3$ and $|\tau_H^-(\kappa_1,\kappa_2,\kappa_3)|<1$.*
  1) *When $\tau_H^-(\kappa_1,\kappa_2,\kappa_3)=0$, then γ is closed and is of length*

$$\frac{2\sqrt3\pi}{\sqrt{\kappa_2^2+(\kappa_1-\kappa_3)^2-3(1-\kappa_1\kappa_3)}}.$$

  2) *When $\tau_H^-(\kappa_1,\kappa_2,\kappa_3)=\pm q(9p^2-q^2)(3p^2+q^2)^{-3/2}$ with some relatively prime positive integers p,q with $p>q$, then γ is closed and is of length*

$$\frac{\delta(p,q)\pi\sqrt{3p^2+q^2}}{\sqrt{\kappa_2^2+(\kappa_1-\kappa_3)^2-3(1-\kappa_1\kappa_3)}}.$$

  3) *Otherwise, it is open.*

5. Lamination on the moduli space of essential Killing helices.

We devote this section for constructing lamination structures on moduli spaces of essential Killing helices of order less than 5 on non-flat complex space forms. We call two smooth curves γ_1, γ_2 on a Riemannian manifold M which are parameterized by their arclengths *congruent* to each other if there exist an isometry φ of M and a constant t_0 with $\gamma_2(t+t_0) = \varphi\circ\gamma_1(t)$ for all t. We denote by $\mathscr{E}\mathscr{H}_d(\boldsymbol{C}M^n)$ the set of all congruence classes of essential Killing helices of porper order d on $\boldsymbol{C}M^n$. It is known that two helices γ_1, γ_2 are congruent to each other if and only if they satisfy the following conditions:

i) they are of the same proper order d;
ii) their geodesic curvatures coincide, i.e. $\kappa_j^{(1)} = \kappa_j^{(2)}$ for $1 \le j \le d-1$;
iii) there is t_0 satisfying either $\tau_{ij}^{(1)}(t_0) = \tau_{ij}^{(2)}(0)$ for $1 \le i < j \le d$ or $\tau_{ij}^{(1)}(t_0) = -\tau_{ij}^{(2)}(0)$ for $1 \le i < j \le d$.

Here $\kappa_j^{(\ell)}$ and $\tau_{ij}^{(\ell)}$ denote the geodesic curvature and the complex torsion of γ_ℓ. By Lemma 1 we find that the moduli spaces of essential Killing helices are as follows:

$$\begin{aligned} &\mathscr{E}\mathscr{H}_1(\boldsymbol{C}M^n) \cong \{0\}, && \mathscr{E}\mathscr{H}_2(\boldsymbol{C}M^n) \cong (0,\infty), \\ &\mathscr{E}\mathscr{H}_3(\boldsymbol{C}M^n) \cong (0,\infty)^2, && \mathscr{E}\mathscr{H}_4(\boldsymbol{C}M^n) \cong (0,\infty)^2 \times \big(\boldsymbol{R}\setminus\{0\}\big). \end{aligned}$$

Here, for a point $(\kappa_1,\kappa_2,\kappa_3) \in (0,\infty)^2 \times \big(\boldsymbol{R}\setminus\{0\}\big)$, it corresponds to the congruence class of Killing helices with geodesic curvatures $\kappa_1, \kappa_2, \kappa_3$ and complex torsions in the condition (2-i) in Lemma 1 if $\kappa_3 > 0$, and it corresponds to the congruence class of Killing helices with geodesic curvatures $\kappa_1, \kappa_2, -\kappa_3$ and complex torsions in the condition (2-ii) in Lemma 1 if $\kappa_3 < 0$. Set theoretically it seems they form a "building structure". In this section we consider them from the viewpoint of lengths of helices.

We define $\mathscr{L} : \mathscr{E}\mathscr{H}_d(\boldsymbol{C}M^n) \to (0,\infty]$ by $\mathscr{L}([\gamma]) = \mathrm{length}(\gamma)$, where $[\gamma]$ denotes the congruence class containing a helix γ. We call $\mathscr{L}$ the length spectrum of essential Killing helices. First we consider helices on $\boldsymbol{C}P^n$. On $\mathscr{E}\mathscr{H}_3(\boldsymbol{C}P^n) \cup \mathscr{E}\mathscr{H}_4(\boldsymbol{C}P^n) \cong (0,\infty)^2 \times \boldsymbol{R}$ we consider a foliation $\mathscr{G} = \{\mathscr{G}_\mu\}_{\mu\in(-1,1)}$ given by $\mathscr{G}_\mu = \{[\gamma(\kappa_1,\kappa_2,\kappa_3)] \mid \tau_P(\kappa_1,\kappa_2,\kappa_3) = \mu\}$, where $[\gamma(\kappa_1,\kappa_2,\kappa_3)]$ denotes the congruence class of helices of proper order 3 or 4 corresponding to $(\kappa_1,\kappa_2,\kappa_3) \in (0,\infty)^2 \times \boldsymbol{R}$, and

$$\tau_P(\kappa_1,\kappa_2,\kappa_3) = \begin{cases} \tau_P^+(\kappa_1,\kappa_2,\kappa_3), & \text{if } \kappa_3 > 0, \\ \tau_P(\kappa_1,\kappa_2), & \text{if } \kappa_3 = 0, \\ \tau_P^-(\kappa_1,\kappa_2,-\kappa_3), & \text{if } \kappa_3 < 0. \end{cases}$$

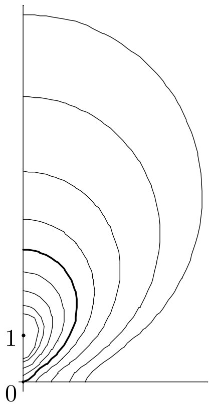

Figure 1. Foliation on $\mathscr{E}\mathscr{H}_3(\boldsymbol{C}P^n)$.

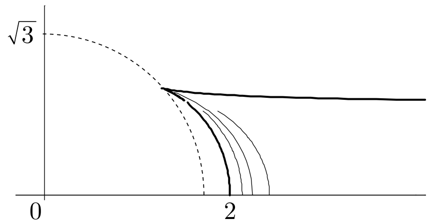

Figure 2. Foliation on $\mathscr{B}\mathscr{E}\mathscr{H}_3(\boldsymbol{C}H^n)$.

If we induce the canonical topology and differential structure on $\mathscr{E}\mathscr{H}_3(\boldsymbol{C}P^n) \cup \mathscr{E}\mathscr{H}_4(\boldsymbol{C}P^n)$ as a subset of $\boldsymbol{R}^3$, Theorems 1, 2 and 3 guarantee the following.

PROPOSITION 1. *The length spectrum $\mathscr{L} : \mathscr{E}\mathscr{H}_3(\boldsymbol{C}P^n) \cup \mathscr{E}\mathscr{H}_4(\boldsymbol{C}P^n) \to (0,\infty]$ is smooth on each leaf $\mathscr{G}_\mu$. Each leaf is maximal with respect to the continuity of $\mathscr{L}$.*

We here give a figure of the foliation $\mathscr{G}|_{\mathscr{E}\mathscr{H}_3(\boldsymbol{C}P^n)}$ restricted on $\mathscr{E}\mathscr{H}_3(\boldsymbol{C}P^n)$ (Fig. 1). The reader should pay attention on the behaviour of leaves near the κ_1-axis. Their behavior is different from the behavior of leaves of canonical foliation on the moduli space of helices on a standard sphere (see [**3**]). If we consider the topological closure of each leaf $\mathscr{G}_\mu|_{\mathscr{E}\mathscr{H}_3(\boldsymbol{C}P^n)}$ with $\mu > 0$ in $[0,\infty) \times [0,\infty)$, then it has an end point in κ_1-axis. From the viewpoint of the length spectrum, $\mathscr{E}\mathscr{H}_1(\boldsymbol{C}P^n) \cup \mathscr{E}\mathscr{H}_2(\boldsymbol{C}P^n) \cong [0,\infty)$ and $\mathscr{E}\mathscr{H}_3(\boldsymbol{C}P^n) \cup \mathscr{E}\mathscr{H}_4(\boldsymbol{C}P^n) \cong (0,\infty)^2 \times \boldsymbol{R}$ do not form a "building structure", because $\mathscr{L} : \mathscr{E}\mathscr{H}_1(\boldsymbol{C}P^n) \cup \mathscr{E}\mathscr{H}_2(\boldsymbol{C}P^n) \to \boldsymbol{R}$ is continuous and bounded.

We here make mention of our foliation a bit more. We here consider all circles on $\boldsymbol{C}P^n$ (see [**15**] for some basic properties of circles). Let $\mathscr{H}_2(\boldsymbol{C}P^n)$ denote the moduli space of circles of positive geodesic curvature on $\boldsymbol{C}P^n$, which is congruent to the set $(0,\infty) \times [0,1]$. As we see in [**2**], we have a foliation $\mathscr{F} = \{\mathscr{F}_\mu\}_{\mu \in [0,1)}$ on $\mathscr{H}_2(\boldsymbol{C}P^n) \setminus \mathscr{E}\mathscr{H}_2(\boldsymbol{C}P^n)$. For $(\kappa,\tau) \in (0,\infty) \times [0,1]$, we denote by $[\sigma(\kappa,\tau)]$ the congruence class of circles containing a circle of geodesic curvature κ and complex trosion τ. Leaves on $\mathscr{H}_2(\boldsymbol{C}P^n) \setminus \mathscr{E}\mathscr{H}_2(\boldsymbol{C}P^n)$ are given as

$$\mathscr{F}_\mu = \begin{cases} \{[\sigma(\kappa,0)] \mid \kappa > 0\}, & \text{if } \mu = 0, \\ \big\{[\sigma(\kappa,\tau)] \bigm| 3\sqrt{3}\kappa\tau(\kappa^2+1)^{-3/2} = 2\mu,\ 0 < \tau < 1\big\}, & \text{if } \mu > 0. \end{cases}$$

Theorems 1, 2 and 3 show that we have a surjective map $\Phi : \mathscr{E}\mathscr{H}_3(\boldsymbol{C}P^n) \cup \mathscr{E}\mathscr{H}_4(\boldsymbol{C}P^n) \to \mathscr{H}_2(\boldsymbol{C}P^n) \setminus \mathscr{E}\mathscr{H}_2(\boldsymbol{C}P^n)$ which satisfies the following properties.

1) It is continuous with respect to induced Euclidean topology;
2) It preserves the foliation structure, $\Phi(\mathscr{G}_\mu) = \mathscr{F}_{|\mu|}$;
3) $\Phi([\gamma(\kappa_1, 1, \kappa_3)]) = [\sigma(1/\sqrt{2}, \tau_P(\kappa_1, \kappa_2, \kappa_3))]$.

Next we consider the moduli space of Killing helices on a complex hyperbolic space. On $\boldsymbol{C}H^n$ we have both bounded and unbounded essential Killing helices of proper orders 3 and 4. We hence consider the moduli space $\mathscr{B}\mathscr{E}\mathscr{H}_d(\boldsymbol{C}H^n)$ of bounded essential Killing helices of proper order d. On $\mathscr{B}\mathscr{E}\mathscr{H}_3(\boldsymbol{C}H^n) \cup \mathscr{B}\mathscr{E}\mathscr{H}_4(\boldsymbol{C}H^n) \subset (0,\infty)^2 \times \boldsymbol{R}$ we have a foliation $\mathscr{G} = \{\mathscr{G}_\mu\}_{\mu\in(-1,1)}$ given by

$$\mathscr{G}_\mu = \bigl\{[\gamma(\kappa_1, \kappa_2, \kappa_3)] \mid \kappa_2^2 + (\kappa_1+\kappa_3)^2 > 3(1+\kappa_1\kappa_3),\ \tau_H(\kappa_1, \kappa_2, \kappa_3) = \mu\bigr\},$$

where $\tau_H(\kappa_1, \kappa_2, \kappa_3)$ is defined just the same way as for $\boldsymbol{C}P^n$.

Proposition 2. *The length spectrum $\mathscr{L} : \mathscr{B}\mathscr{E}\mathscr{H}_3(\boldsymbol{C}H^n) \cup \mathscr{B}\mathscr{E}\mathscr{H}_4(\boldsymbol{C}H^n) \to (0,\infty]$ is smooth on each leaf $\mathscr{G}_\mu$. Each leaf is maximal with respect to the continuity of $\mathscr{L}$.*

We denote by $\mathscr{B}\mathscr{H}_2(\boldsymbol{C}H^n)$ the moduli space of bounded circles of positive geodesic curvature on $\boldsymbol{C}H^n$. On this space we have a foliation $\mathscr{F} = \{\mathscr{F}_\mu\}_{\mu\in[0,1)}$ whose leaves are given as

$$\mathscr{F}_\mu = \begin{cases} \{[\sigma(\kappa, 0)] \mid \kappa > 1\}, & \text{if } \mu = 0, \\ \bigl\{[\sigma(\kappa, \tau)] \bigm| 3\sqrt{3}\kappa\tau(\kappa^2-1)^{-3/2} = 2\mu,\ 0<\tau<1, \kappa>1\bigr\}, & \text{if } \mu > 0. \end{cases}$$

Theorems 4, 5 and 6 show that we have a surjective map $\Phi : \mathscr{B}\mathscr{E}\mathscr{H}_3(\boldsymbol{C}H^n) \cup \mathscr{B}\mathscr{E}\mathscr{H}_4(\boldsymbol{C}H^n) \to \mathscr{B}\mathscr{H}_2(\boldsymbol{C}H^n) \setminus \mathscr{B}\mathscr{E}\mathscr{H}_2(\boldsymbol{C}H^n)$ which satisfies the following properties.

1) It is continuous with respect to induced Euclidean topology;
2) It preserves the foliation structure, $\Phi(\mathscr{G}_\mu) = \mathscr{F}_{|\mu|}$;
3) $\Phi\bigl([\gamma(3\sqrt{2}/2, \kappa_2, 0)]\bigr) = \bigl[\sigma(2, \tau_H(\kappa_1, \kappa_2))\bigr]$.

We also point out that on $\mathscr{E}\mathscr{H}_3(\boldsymbol{C}H^n) \cup \mathscr{E}\mathscr{H}_4(\boldsymbol{C}H^n)$ we can consider a lamination $\{\mathscr{G}_\mu\}_{\mu\in\boldsymbol{R}} \cup \{\mathscr{G}_\nu^{(u)}\}_{\nu\in\boldsymbol{R}}$ which is given by

$$\mathscr{G}_{\nu}^{(u)} = \left\{ [\gamma(\kappa_1,\kappa_2,\kappa_3)] \;\middle|\; \begin{aligned} &\kappa_2^2 + (\kappa_1+\kappa_3)^2 \le 3(1+\kappa_1\kappa_3), \\ &\frac{1}{2}\big[2\{\kappa_2^2 + (\kappa_1+\kappa_3)^2\}^2 \\ &\quad + 9(2-\kappa_1\kappa_3)\{\kappa_2^2+(\kappa_1+\kappa_3)^2\} - 27\kappa_1(\kappa_1+\kappa_3)\big] \\ &\qquad \times \{3(1+\kappa_1\kappa_3) - \kappa_2^2 + (\kappa_1+\kappa_3)^2\}^{-3/2} \\ &\qquad \times \{\kappa_2^2 + (\kappa_1+\kappa_3)^2\}^{-1/2} \\ &= \nu \end{aligned} \right\}.$$

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