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Complete classification of binary normal regular Hermitian lattices

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Abstract. A positive definite Hermitian lattice is called regular if it represents all integers which can be represented locally by the lattice. We investigate binary regular Hermitian lattices over imaginary quadratic fields $\boldsymbol{Q}(\sqrt{-m})$ and provide a complete list of the normal binary regular Hermitian lattices.

1. Introduction.

Dickson first called a positive definite quadratic form f *regular* if $f = n$ has an integral solution for each n such that $f \equiv n \pmod{m}$ has solutions for all positive integers m. He found all regular forms $x^2 + ay^2 + bz^2$, as a generalization of the famous unsolved problem, Euler's idoneal numbers a admitting $x^2 + ay^2$ to be regular [**4**].

The outstanding result about regular quadratic forms was achieved by Watson. He showed that there are finitely many equivalence classes of primitive positive definite regular ternary quadratic forms [**20**], [**21**]. The complete list of 913 regular ternary forms including 22 candidates was given by Jagy, Kaplansky and Schiemann [**11**]. Recently, eight of the candidates were proved to be regular [**16**]. On the contrary, Earnest found an infinite family of regular quaternary forms [**5**] and the first author classified all regular diagonal quaternary forms [**12**].

The regularity of integral quadratic forms is naturally generalized to that of lattices over totally real number fields. Recently the analogue of Watson's finiteness result for regular positive definite ternary quadratic lattices over the ring $\mathscr{O}$ of $\boldsymbol{Q}(\sqrt{5})$ was proved [**2**].

The regular Hermitian lattices over imaginary quadratic fields are defined in a similar way. If a Hermitian lattice represents all positive integers, it is trivially regular. We call such Hermitian lattices *universal*. The universal Hermitian lattices were concentrative subjects studied by many mathematicians including the

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authors in the last couple of decades [**6**], [**9**], [**13**], [**15**].

The finiteness of similar isometry classes of binary *normal* regular Hermitian lattices was proved by Earnest and Khosravani [**7**]. Besides, binary regular diagonal Hermitian lattices including a candidate $\langle 1,14\rangle$ over $\boldsymbol{Q}(\sqrt{-7})$ were listed by Rokicki [**19**]. But her inventory was limited to diagonal lattices $\mathscr{A}_1 v_1 \perp \mathscr{A}_2 v_2$ with two ideals $\mathscr{A}_1, \mathscr{A}_2 \subseteq \mathscr{O}$ and two vectors v_1, v_2.

The obstruction against studying Hermitian lattices was that the matrix presentation was unprovided. The authors, however, developed the formal matrix presentation and were able to delve into universality and regularity of Hermitian lattices. Using this method, we can find all binary regular Hermitian lattices including non-diagonal ones. In addition, we prove the regularity of all these Hermitian lattices including $\langle 1,14\rangle$ over $\boldsymbol{Q}(\sqrt{-7})$. To do this, we developed a method to calculate numbers represented by a quaternary quadratic form which have no ternary sublattice of class number one.

Theorem. *There are* 68 *positive definite binary normal regular Hermitian lattices, including* 9 *non-diagonal ones, up to similar isometry over* $\boldsymbol{Q}(\sqrt{-m})$ *with positive square-free integers* m. *The symbol* † *indicates universal lattices.*

$\boldsymbol{Q}(\sqrt{-1})$: $\langle 1,1\rangle^{\dagger}$, $\langle 1,2\rangle^{\dagger}$, $\langle 1,3\rangle^{\dagger}$, $\langle 1,4\rangle$, $\langle 1,8\rangle$, $\langle 1,16\rangle$,
$\begin{pmatrix} 2 & -1+\omega_1 \\ -1+\overline{\omega}_1 & 3 \end{pmatrix}$, $\begin{pmatrix} 3 & -1+\omega_1 \\ -1+\overline{\omega}_1 & 6 \end{pmatrix}$, $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$\boldsymbol{Q}(\sqrt{-2})$: $\langle 1,1\rangle^{\dagger}$, $\langle 1,2\rangle^{\dagger}$, $\langle 1,3\rangle^{\dagger}$, $\langle 1,4\rangle^{\dagger}$, $\langle 1,5\rangle^{\dagger}$, $\langle 1,8\rangle$, $\langle 1,16\rangle$, $\langle 1,32\rangle$, $\begin{pmatrix} 2 & \omega_2 \\ \overline{\omega}_2 & 5 \end{pmatrix}$

$\boldsymbol{Q}(\sqrt{-3})$: $\langle 1,1\rangle^{\dagger}$, $\langle 1,2\rangle^{\dagger}$, $\langle 1,3\rangle$, $\langle 1,4\rangle$, $\langle 1,6\rangle$, $\langle 1,9\rangle$, $\langle 1,12\rangle$, $\langle 1,36\rangle$, $\langle 2,3\rangle$,
$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$, $\begin{pmatrix} 3 & 1+\omega_3 \\ 1+\overline{\omega}_3 & 5 \end{pmatrix}$, $\begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$

$\boldsymbol{Q}(\sqrt{-5})$: $\langle 1,2\rangle^{\dagger}$, $\langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_5 \\ -1+\overline{\omega}_5 & 3 \end{pmatrix}^{\dagger}$, $\langle 1,8\rangle$, $\langle 1,10\rangle$, $\langle 1,40\rangle$, $\langle 1\rangle \perp 5\begin{pmatrix} 2 & -1+\omega_5 \\ -1+\overline{\omega}_5 & 3 \end{pmatrix}$,
$\begin{pmatrix} 2 & -1+\omega_5 \\ -1+\overline{\omega}_5 & 3 \end{pmatrix} \perp \langle 4\rangle$, $\begin{pmatrix} 2 & -1+\omega_5 \\ -1+\overline{\omega}_5 & 3 \end{pmatrix} \perp \langle 5\rangle$, $\begin{pmatrix} 2 & -1+\omega_5 \\ -1+\overline{\omega}_5 & 3 \end{pmatrix} \perp \langle 20\rangle$

$\boldsymbol{Q}(\sqrt{-6})$: $\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_6 \\ \overline{\omega}_6 & 3 \end{pmatrix}^{\dagger}$, $\langle 1,3\rangle$, $\begin{pmatrix} 2 & \omega_6 \\ \overline{\omega}_6 & 3 \end{pmatrix} \perp 3\begin{pmatrix} 2 & \omega_6 \\ \overline{\omega}_6 & 3 \end{pmatrix}$

$\boldsymbol{Q}(\sqrt{-7})$: $\langle 1,1\rangle^{\dagger}$, $\langle 1,2\rangle^{\dagger}$, $\langle 1,3\rangle^{\dagger}$, $\langle 1,7\rangle$, $\langle 1,14\rangle$, $\begin{pmatrix} 3 & \omega_7 \\ \overline{\omega}_7 & 3 \end{pmatrix}$

$\boldsymbol{Q}(\sqrt{-10})$: $\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{10} \\ \overline{\omega}_{10} & 5 \end{pmatrix}^{\dagger}$, $\langle 1,5\rangle$

$\boldsymbol{Q}(\sqrt{-11})$: $\langle 1,1\rangle^{\dagger}$, $\langle 1,2\rangle^{\dagger}$, $\langle 1,4\rangle$, $\langle 1,11\rangle$, $\langle 1,44\rangle$

$\boldsymbol{Q}(\sqrt{-15})$: $\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{15} \\ \overline{\omega}_{15} & 2 \end{pmatrix}^{\dagger}$, $\langle 1,3\rangle$, $\langle 1,5\rangle$, $\begin{pmatrix} 2 & \omega_{15} \\ \overline{\omega}_{15} & 2 \end{pmatrix} \perp \langle 5\rangle$,

$\begin{pmatrix} 2 & \omega_{15} \\ \overline{\omega}_{15} & 2 \end{pmatrix} \perp 3\begin{pmatrix} 2 & \omega_{15} \\ \overline{\omega}_{15} & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & \omega_{15} \\ \overline{\omega}_{15} & 2 \end{pmatrix} \perp \langle 9\rangle$, $\begin{pmatrix} 2 & \omega_{15} \\ \overline{\omega}_{15} & 2 \end{pmatrix} \perp \langle 15\rangle$

$\boldsymbol{Q}(\sqrt{-19})$: $\langle 1,2\rangle^{\dagger}$

$\boldsymbol{Q}(\sqrt{-23})$: $\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{23} \\ \overline{\omega}_{23} & 3 \end{pmatrix}^{\dagger}$, $\langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_{23} \\ -1+\overline{\omega}_{23} & 3 \end{pmatrix}^{\dagger}$

$\boldsymbol{Q}(\sqrt{-31})$: $\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{31} \\ \overline{\omega}_{31} & 4 \end{pmatrix}^{\dagger}$, $\langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_{31} \\ -1+\overline{\omega}_{31} & 4 \end{pmatrix}^{\dagger}$

Remark. The binary subnormal regular Hermitian lattices will be investigated in our next articles. Binary subnormal regular Hermitian lattices over $\boldsymbol{Q}(\sqrt{-m})$ with norm ideal $2\mathcal{O}$ occur only when

$$m = 1, 2, 5, 6, 10, 13, 14, 17, 22, 29, 34, 37 \text{ and } 38.$$

Also, we found that a binary primitive subnormal regular Hermitian lattice of norm ideal $m\mathcal{O}$ exists over $\boldsymbol{Q}(\sqrt{-m})$. For example, $\left(\begin{smallmatrix} 3 & \sqrt{-3} \\ -\sqrt{-3} & 3 \end{smallmatrix}\right)$ over $\boldsymbol{Q}(\sqrt{-3})$ is a binary subnormal regular Hermitian lattice with norm ideal $3\mathcal{O}$. It is an impossible phenomenon for quadratic lattices over $\boldsymbol{Z}$.

2. Preliminaries.

In this section, we give some notations and terminologies, which are adopted from [**17**]. Let $\mathcal{O}$ be the ring of integers of the imaginary quadratic field $\boldsymbol{Q}(\sqrt{-m})$, where m is a positive square-free integer. We have that $\mathcal{O} = \boldsymbol{Z}[\omega]$ with $\omega := \omega_m = \sqrt{-m}$ if $m \not\equiv 3 \pmod 4$ and $\omega := \omega_m = (1+\sqrt{-m})/2$ if $m \equiv 3 \pmod 4$. A Hermitian space V is a vector space over $\boldsymbol{Q}(\sqrt{-m})$ with a Hermitian map $H : V \times V \to \boldsymbol{Q}(\sqrt{-m})$ satisfying the following conditions:

(1) $H(v,w) = \overline{H(w,v)}$ for $v, w \in V$,
(2) $H(v_1+v_2,w) = H(v_1,w) + H(v_2,w)$ for $v_1, v_2, w \in V$,
(3) $H(av,w) = aH(v,w)$ for $a \in \boldsymbol{Q}(\sqrt{-m})$ and $v, w \in V$.

For brevity, we write $H(v) = H(v,v)$. A Hermitian lattice L is defined as a finitely generated $\mathcal{O}$-module in the Hermitian space V. We will assume that all Hermitian lattices are integral in the sense that $H(v_1,v_2) \in \mathcal{O}$ for all $v_1, v_2 \in L$. From condition (1), we know that

$$H(v) = H(v,v) = \overline{H(v,v)} = \overline{H(v)}.$$

Hence $H(v) \in \boldsymbol{Z}$ for $v \in L$. If $a = H(v)$ for some $v \in L$, we say that a is represented by L and denote it by $a \to L$. If a cannot be represented by L, we denote it by $a \not\to L$. Through this article, we assume that L is *positive definite*, i.e., $H(v) > 0$ for nonzero vectors $v \in L$.

The *localization* of a Hermitian lattice L at a prime p is defined by $L_p = \mathscr{O}_p \otimes_{\mathscr{O}} L$ where $\mathscr{O}_p = \boldsymbol{Z}_p \otimes_{\boldsymbol{Z}} \mathscr{O}$. If $n \to L_p$ for all primes p including ∞, then we write $n \to \operatorname{gen} L$. The regularity of a Hermitian lattice L can be rephrased as follows: if $n \to \operatorname{gen} L$, then $n \to L$. Thus if the class number of L is one, then L is trivially regular.

If a regular Hermitian lattice L is locally universal over $\mathscr{O}_p$ for all primes p, then L is universal. Since all universal Hermitian lattices are already classified [**6**], [**9**], [**15**], we only consider nonuniversal regular lattices through this article.

A lattice can be written as

$$L = \mathscr{A}_1 v_1 + \mathscr{A}_2 v_2 + \cdots + \mathscr{A}_n v_n$$

with ideals $\mathscr{A}_i \subset \mathscr{O}$ and vectors $v_i \in V$. If these vectors are linearly independent over $\boldsymbol{Q}(\sqrt{-m})$, then we say that L is an n-ary lattice and $\operatorname{rank} L = n$.

The *norm ideal* $\mathfrak{n}L$ of L is an $\mathscr{O}$-ideal generated by the set $\{H(v) | v \in L\}$. The *scale ideal* $\mathfrak{s}L$ of L is an $\mathscr{O}$-ideal generated by the set $\{H(v, w) | v, w \in L\}$. It is clear that $\mathfrak{n}L \subseteq \mathfrak{s}L$. If $\mathfrak{n}L = \mathfrak{s}L$, then we call L *normal*. Otherwise, we call L *subnormal*. We investigate normal lattices in this article. The *volume ideal* of L is defined as

$$\mathfrak{v}L = (\mathscr{A}_1 \overline{\mathscr{A}}_1)(\mathscr{A}_2 \overline{\mathscr{A}}_2) \cdots (\mathscr{A}_n \overline{\mathscr{A}}_n) \det(H(v_i, v_j))_{1 \le i,j \le n}.$$

Note that the volume ideals of sublattices of L are contained in $\mathfrak{v}L$.

If a Hermitian lattice L is a free $\mathscr{O}$-module, then we can write $L = \mathscr{O}v_1 + \cdots + \mathscr{O}v_n$. The matrix presentation $M_L = (H(v_i, v_j))_{1 \le i,j \le n}$ is called the Gram matrix of L. If the matrix is diagonal, we denote it by $\langle H(v_1), H(v_2), \ldots, H(v_n) \rangle$. But, if a Hermitian lattice L is not a free $\mathscr{O}$-module, then $L = \mathscr{O}v_1 + \cdots + \mathscr{O}v_{n-1} + \mathscr{A}v_n$ for some ideal $\mathscr{A} \subset \mathscr{O}$ [**17**, 81:5]. Since any ideal in $\mathscr{O}$ is generated by at most two elements, we can write $L = \mathscr{O}v_1 + \cdots + \mathscr{O}v_{n-1} + (\alpha, \beta)\mathscr{O}v_n$ for some $\alpha, \beta \in \mathscr{O}$. Therefore, we can regard the following $(n+1) \times (n+1)$-matrix as a formal Gram matrix for L:

$$M_L = \begin{pmatrix} H(v_1, v_1) & \ldots & H(v_1, v_{n-1}) & H(v_1, \alpha v_n) & H(v_1, \beta v_n) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ H(v_{n-1}, v_1) & \ldots & H(v_{n-1}, v_{n-1}) & H(v_{n-1}, \alpha v_n) & H(v_{n-1}, \beta v_n) \\ H(\alpha v_n, v_1) & \ldots & H(\alpha v_n, v_{n-1}) & H(\alpha v_n, \alpha v_n) & H(\alpha v_n, \beta v_n) \\ H(\beta v_n, v_1) & \ldots & H(\beta v_n, v_{n-1}) & H(\beta v_n, \alpha v_n) & H(\beta v_n, \beta v_n) \end{pmatrix}.$$

Note that this matrix is positive semi-definite, but this represents an n-ary positive definite Hermitian lattice. We identify a lattice L and the Gram matrix M_L of L. A scaled lattice L^a is obtained from the Hermitian map $H_{L^a} = aH_L$ with $0 < a \in \boldsymbol{Z}$. If M is a matrix presentation of a lattice L, we write aM for the matrix presentation of a scaled lattice L^a.

We can regard a Hermitian space (V, H) over $\boldsymbol{Q}(\sqrt{-m})$ as a $2n$-dimensional quadratic space $(\widetilde{V}, B_H)$ such that $B_H(x,y) = (1/2)[H(x,y) + H(y,x)] = (1/2)\operatorname{Tr}_{E/\mathbb{Q}}(H(x,y))$ [**10**]. Through this consideration, we can obtain an *associated* quadratic form in $(\widetilde{V}, B_H)$ from a Hermitian lattice in (V, H). To distinguish the associated quadratic form from the Hermitian lattice, we use the subscript $\boldsymbol{Z}$. For instance, the quadratic form $\langle 1,1,1,1\rangle_{\boldsymbol{Z}}$ is associated with the Hermitian lattice $\langle 1,1\rangle$ over $\boldsymbol{Q}(\sqrt{-1})$.

3. Some definitions and lemmata.

In this section, we determine all the imaginary quadratic fields $\boldsymbol{Q}(\sqrt{-m})$ that admit binary normal regular Hermitian lattices L. Also we describe the volume condition of L to find all candidates for L.

For a positive integer t and a Hermitian lattice L over $\mathscr{O}$ or $\mathscr{O}_p$, let

$$\Lambda_t(L) = \{v \in L \mid H(v+w) \equiv H(w) \pmod{t} \text{ for all } w \in L\}.$$

The *Watson transformation* $\lambda_t(L)$ of L is defined by

$$\lambda_t(L) = \Lambda_t(L)^{1/a},$$

where a is the maximal positive integer which divides $H(v,w)$ for all $v, w \in \Lambda_t(L)$. It is well-known that if L is regular, then $\lambda_t(L)$ is also regular [**22**], [**3**].

LEMMA 1. *Let L be a primitive normal binary Hermitian lattice over $\boldsymbol{Q}(\sqrt{-m})$ and p is an odd prime. Then $\lambda_{p^n}(L_p)$ represents all elements of $\boldsymbol{Z}_p$ for some nonnegative integer n.*

PROOF. Since L is primitive and normal, $L_p \cong \langle \epsilon, \epsilon' p^k\rangle$ for some nonnegative integer k and units ϵ, ϵ' of $\boldsymbol{Z}_p$. We may assume that any unary $\mathscr{O}_p$-lattice is not isotropic. Otherwise L_p represents all elements of $\boldsymbol{Z}_p$.

If $p \nmid m$ and p is inert in $\mathscr{O}$, then $\lambda_{p^{2\ell}}(L_p) \cong \langle \epsilon, \epsilon' p^r\rangle$ where $k = 2\ell + r$ for some $\ell \in \boldsymbol{Z}$ and $r = 0, 1$. The quadratic form $\langle \epsilon, \epsilon m, \epsilon' p^r, \epsilon' m p^r\rangle_{\boldsymbol{Z}_p}$ associated with $\langle \epsilon, \epsilon' p^r\rangle$ represents all elements of $\boldsymbol{Z}_p$.

If $p \nmid m$ and p splits in $\mathscr{O}$, then $\lambda_{p^{k-1}}(L_p) \cong \langle \epsilon, \epsilon' p\rangle$. The quadratic form $\langle \epsilon, \epsilon m, \epsilon' p, \epsilon' m p\rangle_{\boldsymbol{Z}_p}$ associated with $\langle \epsilon, \epsilon' p\rangle$ represents all elements of $\boldsymbol{Z}_p$.

If $p \mid m$, then $\lambda_{p^k}(L_p) \cong \langle \epsilon, \epsilon' \rangle$. The quadratic form $\langle \epsilon, \epsilon m, \epsilon', \epsilon' m \rangle_{\boldsymbol{Z}_p}$ associated with $\langle \epsilon, \epsilon' \rangle$ represents all elements of $\boldsymbol{Z}_p$. □

LEMMA 2. *Let L be a primitive normal binary Hermitian lattice over $\boldsymbol{Q}(\sqrt{-m})$. Then $\lambda_{2^n}(L_2)$ represents all elements of $\boldsymbol{Z}_2$ for some nonnegative integer n.*

PROOF. Since L is primitive and normal, $L_2 \cong \langle \epsilon, \epsilon' 2^k \rangle$ for some nonnegative integer k and units ϵ, ϵ' in $\boldsymbol{Z}_2$.

If $m \equiv 7 \pmod 8$, then the unary Hermitian lattice $\langle \epsilon \rangle$ over $\mathscr{O}_2$ provides the associated quadratic form $\begin{pmatrix} \epsilon & \epsilon/2 \\ \epsilon/2 & (m+1)\epsilon/4 \end{pmatrix}_{\boldsymbol{Z}_2} \cong \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}_{\boldsymbol{Z}_2}$ over $\boldsymbol{Z}_2$. It represents all elements of $\boldsymbol{Z}_2$ and so does L_2.

If $m \equiv 3 \pmod 8$, let $k = 2\ell + r$ for some $\ell \in \boldsymbol{Z}$ and $r = 0, 1$. Then $\lambda_{4^\ell}(L_2) \cong \langle \epsilon, \epsilon' 2^r \rangle$. Since the Hermitian lattice $\langle \epsilon, \epsilon' 2^r \rangle$ over $\mathscr{O}_2$ provides the associated quadratic form $\begin{pmatrix} \epsilon & \epsilon/2 \\ \epsilon/2 & (m+1)\epsilon/4 \end{pmatrix}_{\boldsymbol{Z}_2} \perp \begin{pmatrix} 2^r\epsilon' & \epsilon' 2^{r-1} \\ \epsilon' 2^{r-1} & (m+1)2^{r-2}\epsilon' \end{pmatrix}_{\boldsymbol{Z}_2}$ which is isometric to $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}_{\boldsymbol{Z}_2} \perp \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}_{\boldsymbol{Z}_2}$ or $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}_{\boldsymbol{Z}_2} \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}_{\boldsymbol{Z}_2}$ over $\boldsymbol{Z}_2$, $\lambda_{4^\ell}(L_2)$ represents all elements of $\boldsymbol{Z}_2$.

If $m \equiv 1 \pmod 4$, let $k = 2\ell + r$ for some $\ell \in \boldsymbol{Z}$ and $r = 0, 1$. Then $\lambda_{4^\ell}(L_2) \cong \langle \epsilon, \epsilon' 2^r \rangle$. Since the Hermitian lattice $\langle \epsilon, \epsilon' 2^r \rangle$ over $\mathscr{O}_2$ provides the associated quadratic form $\langle \epsilon, \epsilon m, 2^r \epsilon', 2^r \epsilon' m \rangle_{\boldsymbol{Z}_2}$ over $\boldsymbol{Z}_2$, $\lambda_{4^\ell}(L_2)$ represents all elements of $\boldsymbol{Z}_2$.

If $m \equiv 2 \pmod 4$, then $\lambda_{2^k}(L_2) \cong \langle \epsilon, \epsilon' \rangle$. If $m = 2m'$, the Hermitian lattice $\langle \epsilon, \epsilon' \rangle$ over $\mathscr{O}_2$ provides the associated quadratic form $\langle \epsilon, \epsilon m, \epsilon', \epsilon' m \rangle_{\boldsymbol{Z}_2}$ over $\boldsymbol{Z}_2$. Since quadratic form $\langle \epsilon, \epsilon m, \epsilon', \epsilon' m \rangle_{\boldsymbol{Z}_2} \cong \langle \epsilon, \epsilon', 2m'\epsilon, 2m'\epsilon' \rangle_{\boldsymbol{Z}_2}$, $\lambda_{2^k}(L_2)$ represents all elements of $\boldsymbol{Z}_2$. □

Suppose L is a primitive binary normal regular Hermitian lattice over $\boldsymbol{Q}(\sqrt{-m})$. From Lemmas 1 and 2, we know that there are primes $p_1, p_2, \ldots, p_k$ and positive integers $s_1, s_2, \ldots, s_k$ such that $\widehat{L} = \lambda_{p_1^{s_1}} \circ \lambda_{p_2^{s_2}} \circ \cdots \circ \lambda_{p_k^{s_k}}(L)$ is locally universal, which means $\widehat{L}$ represents all elements of $\boldsymbol{Z}_p$ for all primes p. Since $\widehat{L}$ is regular [**22**], [**3**], $\widehat{L}$ is universal. From the works [**6**], [**9**], [**15**] on binary universal Hermitian lattices, $\boldsymbol{Q}(\sqrt{-m})$ admits binary normal regular Hermitian lattices only when

$$m = 1, 2, 3, 5, 6, 7, 10, 11, 15, 19, 23 \text{ or } 31.$$

Suppose that L_{p_0} does not represent some element of $\boldsymbol{Z}_{p_0}$ for some odd prime p_0 with $p_0 \nmid m$. Let $p_1, p_2, \ldots, p_k$ be all primes different from p_0 such that L_{p_i} does not represent some element of $\boldsymbol{Z}_{p_i}$. Then, for suitable positive integers $s_1, s_2, \ldots, s_k$,

$$\widehat{L} = \lambda_{p_1^{s_1}} \circ \lambda_{p_2^{s_2}} \circ \cdots \circ \lambda_{p_k^{s_k}}(L)$$

represents all elements of $\boldsymbol{Z}_p$ for all primes p except $p = p_0$. Since $\widehat{L}$ is primitive and normal, $\widehat{L}_{p_0} \cong \langle \epsilon, \epsilon' p_0^{\ell} \rangle$ for some nonnegative integer ℓ and units ϵ, ϵ' in $\boldsymbol{Z}_{p_0}$. Since $p_0 \nmid m$, $\widehat{L}$ represents all units of $\boldsymbol{Z}_{p_0}$. So $\widehat{L}$ represents 1 and 2 locally. Since $\widehat{L}$ is regular, $\widehat{L}$ represents 1 and 2 globally, so $\widehat{L} \cong \langle 1 \rangle \perp M$ for some unary lattice M. If $m \neq 1, 2, 7$, then $\langle 1 \rangle$ does not represent 2. So M represents 1 or 2. Thus $\widehat{L}$ contains $\langle 1, 1 \rangle$ or $\langle 1, 2 \rangle$. Therefore $\widehat{L}$ represents all elements of $\boldsymbol{Z}_{p_0}$. This is a contradiction. If $m = 1$ or 7, then $\langle 1 \rangle$ cannot represent 3. Since $\widehat{L}$ is regular, 3 is not a unit of $\boldsymbol{Z}_{p_0}$. So $p_0 = 3$. Similarly, if $m = 2$, then $p_0 = 5$. We conclude that if L_p does not represent some element of $\boldsymbol{Z}_p$, then we have following cases:

(1) $p = 2$,
(2) an odd prime p divides m,
(3) $p = 3$ if $m = 1, 7$; $p = 5$ if $m = 2$.

The following Lemma 3 explains the condition on $\mathfrak{v}L$ and gives an efficiency for finding candidates for L.

LEMMA 3. *Let L be a binary Hermitian lattice over $\boldsymbol{Q}(\sqrt{-m})$.*

(1) *Let p be an odd prime. If L_p represents a unit in $\boldsymbol{Z}_p$ and does not represent $p^k\epsilon$ for some nonnegative integer k and for some unit ϵ in $\boldsymbol{Z}_p$ over $\mathscr{O}_p$, then*

$$\mathfrak{v}L \subset p^{k+1}\mathscr{O}.$$

(2) *If L_2 represents a unit in $\boldsymbol{Z}_2$ and does not represent $2^k\epsilon$ for some nonnegative integer k for some unit ϵ in $\boldsymbol{Z}_2$ over $\mathscr{O}_2$, then*

$$\begin{cases} \mathfrak{v}L \subset 2^{k+2}\mathscr{O} & \text{if } m \equiv 1 \pmod 4, \\ \mathfrak{v}L \subset 2^{k+3}\mathscr{O} & \text{if } m \equiv 2 \pmod 4, \\ \mathfrak{v}L \subset 2^{k+1}\mathscr{O} & \text{if } m \equiv 3 \pmod 8. \end{cases}$$

PROOF.

(1) Since L_p represents a unit in $\boldsymbol{Z}_p$, $L_p \cong \langle a, bp^{\ell} \rangle$ for some units $a, b \in \boldsymbol{Z}_p$ and for some nonnegative integer ℓ. If $\langle a \rangle$ is isotropic, then $p^k\epsilon \to \langle a \rangle$ and hence $p^k\epsilon \to L_p$. This is a contradiction. Therefore $\langle a \rangle$ is anisotropic.

Assume that $p \nmid m$. Then the quadratic form associated with L_p is isometric to $\langle a, a', bp^{\ell}, b'p^{\ell} \rangle_{\boldsymbol{Z}_p}$ for some units $a', b' \in \boldsymbol{Z}_p$. If $\ell \leq k$, then $p^k\epsilon \to L_p$.

Now assume that $p \mid m$. Then the associated quadratic form is $\langle a, a'p, bp^{\ell}, b'p^{\ell+1} \rangle_{\boldsymbol{Z}_p}$. If $\ell \leq k$, then $p^k\epsilon \to L_p$. Thus $\ell \geq k+1$ and $\mathfrak{v}L_p =$

$abp^\ell \mathscr{O}_p \subset p^{k+1}\mathscr{O}_p$.

(2) Since L_2 represents a unit in $\boldsymbol{Z}_2$, $L_2 \cong \langle a, 2^\ell b\rangle$ for some units $a, b \in \boldsymbol{Z}_2$ and for some integer ℓ. If $m \equiv 7 \pmod 8$, then $\langle a\rangle$ is isotropic and thus $2^k\epsilon \to \langle a\rangle$.

Suppose $m \equiv 1 \pmod 4$. If $\ell = 0, 1$, then $L_2 = \langle a, 2^\ell b\rangle$ represents all elements of $\boldsymbol{Z}_2$. Hence we have $\ell \geq 2$. Then, $2^{k-\ell+1}\epsilon \to \lambda_{2^{\ell-1}}(L_2) = \langle a, 2b\rangle$ if $\ell \leq k+1$. Hence $(2^{k-\ell+1}\epsilon)2^{\ell-1} = \epsilon 2^k \to L_2$, which is a contradiction. So $\ell \geq k+2$ and $\mathfrak{v}L_2 = ab2^\ell\mathscr{O}_2 \subset 2^{k+2}\mathscr{O}_2$.

Suppose $m \equiv 2 \pmod 4$. If $\ell = 0, 1, 2$, then $L_2 = \langle a, 2^\ell b\rangle$ represents all elements of $\boldsymbol{Z}_2$. Hence we have $\ell \geq 3$. Then, $2^{k-\ell+2}\epsilon \to \lambda_{2^{\ell-2}}(L_2) = \langle a, 4b\rangle$ if $\ell \leq k+2$. Hence $(2^{k-\ell+2}\epsilon)2^{\ell-2} = \epsilon 2^k \to L_2$, which is a contradiction. So $\ell \geq k+3$ and $\mathfrak{v}L_2 = ab2^\ell\mathscr{O}_2 \subset 2^{k+3}\mathscr{O}_2$.

Suppose $m \equiv 3 \pmod 8$. If $\ell = 0, 1$, then $L_2 = \langle a, 2^\ell b\rangle$ represents all elements of $\boldsymbol{Z}_2$. Hence we have $\ell \geq 2$. Then, $2^{k-\ell}\epsilon \to \lambda_{2^\ell}(L_2)$ if $\ell \leq k$. Hence $(2^{k-\ell}\epsilon)2^\ell = \epsilon 2^k \to L_2$, which is a contradiction. So $\ell \geq k+1$ and $\mathfrak{v}L_2 = ab2^\ell\mathscr{O}_2 \subset 2^{k+1}\mathscr{O}_2$. □

4. Candidates for binary normal regular Hermitian lattices.

In this section, we will find all candidates for binary normal regular Hermitian lattices over imaginary quadratic fields $\boldsymbol{Q}(\sqrt{-m})$ with the information of L_p and the volume ideal $\mathfrak{v}L$ by the following strategy. We assume that L is regular but not universal.

(1) Find the minimal number a such that $a \to \operatorname{gen} L$.
(2) Find the minimal number b such that $b \to \operatorname{gen} L$ but $b \not\to \langle a\rangle$.
(3) Find a lattice $\ell = \left(\begin{smallmatrix} a & \alpha \\ \bar{\alpha} & b\end{smallmatrix}\right)$ satisfying the volume condition.
(4) If $\mathfrak{v}\ell$ reaches the volume bound in Lemma 3, then we stop.
(5) If $\mathfrak{v}\ell \subsetneq$ the volume bound, append a suitable vector to ℓ.
(6) Repeat the above steps.

We call these two numbers a and b *essential numbers* (abbr. Ess.#). When a binary Hermitian lattice L is not regular, we will give an integer n such that $n \to \operatorname{gen} L$ but $n \not\to L$. This number is called the *exceptional number* (abbr. Exc.#). We check the exceptional numbers in the range of 2^{10}.

We give Example 1 to explain this strategy. Some notations are needed for convenience. For an odd prime p, 1_p and $\triangle_p$ denote a square unit and a non-square unit in $\boldsymbol{Z}_p$, respectively. 1_2, 3_2, 5_2, 7_2 denote the four types of units in $\boldsymbol{Z}_2$.

Example 1. Suppose that a regular lattice L over $\boldsymbol{Q}(\sqrt{-5})$ satisfies

$$1_2 \to L_2, \quad 3_2 \to L_2, \quad 1_5 \to L_5, \quad \triangle_5 \not\to L_5.$$

Then L_2 represents all elements in $\boldsymbol{Z}_2$ and the volume condition for L is $\mathfrak{v}L \subset 5\mathscr{O}$.

Steps (1) and (2): The essential numbers are 1 and 11.

Step (3): We can find two binary *free* lattices

$$\ell_1 = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \text{ with } \mathfrak{v}\ell_1 = 5\mathscr{O} \quad \text{and} \quad \ell_2 = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix} \text{ with } \mathfrak{v}\ell_2 = 10\mathscr{O}$$

satisfying the volume condition.

Step (4): Since $\mathfrak{v}\ell_1 = 5\mathscr{O}$, we need not to expand ℓ_1.

Step (5): Since $15 \to \operatorname{gen} \ell_2$ but $15 \not\to \ell_2$, ℓ_2 is not a candidate. We consider the following formal Gram matrix for a *binary* lattice

$$\langle 1 \rangle \perp \begin{pmatrix} 10 & \alpha \\ \overline{\alpha} & 5\beta \end{pmatrix} \text{ with } 50\beta - \alpha\overline{\alpha} = 0.$$

Thus $\alpha = -5 + 5\omega$ and $\beta = 3$. This produces a *nonfree* binary lattice

$$\langle 1 \rangle \perp \begin{pmatrix} 10 & -5+5\omega \\ -5+5\overline{\omega} & 15 \end{pmatrix}$$

as a candidate. Since its volume ideal is $5\mathscr{O}$, we stop here.

Through iterative and long process, we find all the candidates. But, instead of giving a detailed proof of finding candidate, we give abridged tables which describe these whole process for each field $\boldsymbol{Q}(\sqrt{-m})$. Here, we give Example 2 which explains how to understand the Table 9 and shows the whole process of finding candidates for $\boldsymbol{Q}(\sqrt{-15})$. The other tables are understandable by similar way. For simplicity, we write

$$[a, \alpha, b] := \begin{pmatrix} a & \alpha \\ \overline{\alpha} & b \end{pmatrix}.$$

Example 2. Let $m = 15$ and refer Table 9. Note that

$$\begin{aligned} 1_2 \to L_2 &\iff 3_2 \to L_2 \iff 5_2 \to L_2 \iff 7_2 \to L_2 \\ &\implies 2{\cdot}1_2, 2{\cdot}3_2, 2{\cdot}5_2, 2{\cdot}7_2 \to L_2. \end{aligned}$$

Since a primitive binary normal regular lattice L over $\boldsymbol{Q}(\sqrt{-15})$ should represent a unit in $\boldsymbol{Z}_2$, L_2 represents all elements in $\boldsymbol{Z}_2$. So we consider the local conditions on L_3 and L_5.

For the case (1), since $1_3, \triangle_3 \to L_3$ and $1_5 \to L_5$, $\triangle_5 \not\to L_5$, the essential numbers are 1, 11 and $\mathfrak{v}L \subset 5\mathscr{O}$. So L contains

$$\begin{pmatrix} 1 & \alpha \\ \overline{\alpha} & 11 \end{pmatrix} \text{ with } 5 \mid (11 - \alpha\overline{\alpha}) \text{ and } \alpha \in \mathscr{O}.$$

So we have $L \cong \langle 1, 5\rangle$ or L contains $\langle 1, 10\rangle$. Since $\mathfrak{v}\langle 1, 5\rangle = 5\mathscr{O}$, we need not to expand it. Since $5 \to \mathrm{gen}\langle 1, 10\rangle$ and $5 \not\to \langle 1, 10\rangle$, $\langle 1, 10\rangle$ is not a candidate as 5 is an exceptional number of $\langle 1, 10\rangle$. Note that $\langle 1, 10\rangle$ infects the given local condition (1). So 5 can be an exceptional number of $\langle 1, 10\rangle$ obeying the new local condition. We consider the formal Gram matrix for a binary lattice

$$\langle 1\rangle \perp \begin{pmatrix} 10 & \beta \\ \overline{\beta} & 5\gamma \end{pmatrix} \text{ with } 50\gamma - \beta\overline{\beta} = 0 \text{ and } \beta, \gamma \in \mathscr{O}.$$

Thus $\beta = 5\omega$, $\gamma = 2$ and this produces $L \cong \langle 1\rangle \perp 5\left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right)$. Since $\mathfrak{v}L = 5\mathscr{O}$, we need not to expand it. However, since 5 is an exceptional number, it is not regular. Similarly, we get results for the cases (3) and (4-2).

For the case (2), since $1_3, \triangle_3 \to L_3$ and $1_5 \not\to L_5$, $\triangle_5 \to L_5$, the essential numbers are 2, 3 and $\mathfrak{v}L \subset 5\mathscr{O}$. So L contains

$$\begin{pmatrix} 2 & \alpha \\ \overline{\alpha} & 3 \end{pmatrix} \text{ with } 5 \mid (6 - \alpha\overline{\alpha}) \text{ and } \alpha \in \mathscr{O}.$$

So we have $L \cong \left(\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}\right)$ or L contains a unimodular lattice $\left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right)$. Since $\mathfrak{v}\left(\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}\right) = 5\mathscr{O}$, we need not expand it. Since 5 is an exceptional number of $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}\right)$, it is not regular. Suppose L contains a lattice $\left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right)$. Since $7 \to L$ and $7 \not\to \left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right)$, we have a candidate $\left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right) \perp \langle 5\rangle$ by comparing volume of L. Similarly, we can get results for the cases (6-2), (8-1-1) and (8-2).

For the case (4-1), since $1_3, 3 \cdot 1_3 \to L_3$, $\triangle_3 \not\to L_3$ and $1_5 \to L_5$, $\triangle_5 \not\to L_5$, the essential numbers are 1, 21 and $\mathfrak{v}L \subset 3 \cdot 5\mathscr{O}$. So L contains

$$\begin{pmatrix} 1 & \alpha \\ \overline{\alpha} & 21 \end{pmatrix} \text{ with } 5 \mid (21 - \alpha\overline{\alpha}) \text{ and } \alpha \in \mathscr{O}.$$

Thus we have a candidate $\langle 1, 15\rangle$. Since 45 is an exceptional number, it is not regular. Similarly, we get results for the cases (5-1-1), (5-2-1) and (7-2-1).

For the case (5-1-2), since $1_3, 3 \cdot 1_3 \to L_3$, $\triangle_3 \not\to L_3$ and $1_5, 5\triangle_5 \not\to L_5$, $\triangle_5 \to L_5$, the essential numbers are 3, 7 and $\mathfrak{v}L \subset 3 \cdot 5^2\mathscr{O}$. Thus L contains

$$\begin{pmatrix} 3 & \alpha \\ \overline{\alpha} & 7 \end{pmatrix} \text{ with } 75 \mid (21-\alpha\overline{\alpha}).$$

So $\alpha\overline{\alpha}=21$. But there is no such α and we have no candidate. Similarly, we can get results for the cases (5-2-2), (7-1-2) and (7-2-2).

For the case (6-1), $1_3 \not\to L_3$, $\triangle_3, 3\triangle_3 \to L_3$ and $1_5, \triangle_5 \to L_5$, the essential numbers are 2, 6 and $\mathfrak{v} \subset 3\mathscr{O}$. So L contains

$$\begin{pmatrix} 2 & \alpha \\ \overline{\alpha} & 6 \end{pmatrix} \text{ with } 3 \mid (12-\alpha\overline{\alpha}).$$

So $\alpha\overline{\alpha}=0,\ 6,\ 9$, i.e. $\alpha=0,\ 1+\omega,\ 3$. If $\alpha=0$, from the condition $3\to L$, L contains a lattice

$$\begin{pmatrix} 2 & 0 & \beta \\ 0 & 6 & \gamma \\ \overline{\beta} & \overline{\gamma} & 3 \end{pmatrix} \text{ with } 3 \mid (6-\beta\overline{\beta}),\quad 3 \mid (18-\gamma\overline{\gamma})$$

and its determinant $36-6\beta\overline{\beta}-2\gamma\overline{\gamma}=0$. So L contains a lattice $\left(\begin{smallmatrix} 2 & 1+\omega \\ 1+\overline{\omega} & 3 \end{smallmatrix}\right) \perp \langle 6\rangle \cong \left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right) \perp \langle 6\rangle$. Since $15 \to L$ and $15 \not\to \left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right) \perp \langle 6\rangle$, we have a candidate $L \cong \left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right) \perp 3\left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right)$, via similar way. Note that L is isometric to the binary *free* lattice $\left(\begin{smallmatrix} 8 & -1+4\omega \\ -1+4\overline{\omega} & 8 \end{smallmatrix}\right)$. If $\alpha=1+\omega$ or 3, then we know that there is no candidate via similar way.

For the case (7-1-1), $1_3 \not\to L_3$, $\triangle_3, 3\triangle_3 \to L_3$ and $1_5, 5\cdot 1_5 \to L_5$, $\triangle_5 \not\to L_5$, the essential numbers are 5, 6 and $\mathfrak{v}L \subset 3\cdot 5\mathscr{O}$. Thus L contains

$$\begin{pmatrix} 5 & \alpha \\ \overline{\alpha} & 6 \end{pmatrix} \text{ with } 15 \mid (30-\alpha\overline{\alpha}).$$

So $\alpha\overline{\alpha}=0, 15$, i.e. $\alpha=0,\ -1+2\omega$. We have L contains $\langle 5,6\rangle$ or $L \cong \left(\begin{smallmatrix} 5 & -1+2\omega \\ -1+\overline{\omega} & 6 \end{smallmatrix}\right)$. Since 9 is an exceptional number of $\left(\begin{smallmatrix} 5 & -1+2\omega \\ -1+\overline{\omega} & 6 \end{smallmatrix}\right)$, it is not regular. Suppose L contains $\langle 5,6\rangle$. Since $9 \to \mathrm{gen}\langle 5,6\rangle$ and $9 \not\to \langle 5,6\rangle$, $L \cong \langle 5\rangle \perp 3\left(\begin{smallmatrix} 2 & 1+\omega \\ 1+\overline{\omega} & 3 \end{smallmatrix}\right)$ by the volume condition. Since 21 is an exceptional number, it is not regular.

For the case (8-1-2), $1_3 \not\to L_3$, $\triangle_3, 3\triangle_3 \to L_3$ and $1_5, 5\triangle_5 \not\to L_5$, $\triangle_5 \to L_5$, the essential numbers are 2, 3 and$\mathfrak{v}L \subset 3\cdot 5\mathscr{O}$. So L contains

$$\begin{pmatrix} 2 & \alpha \\ \overline{\alpha} & 3 \end{pmatrix} \text{ with } 15 \mid (6-\alpha\overline{\alpha}) \text{ and } \alpha \in \mathscr{O}.$$

Hence L contains unimodular lattice $\begin{pmatrix} 2 & \omega \\ \bar{\omega} & 2 \end{pmatrix}$ which splits L. Since $33 \to L$ and $33 \not\to \begin{pmatrix} 2 & \omega \\ \bar{\omega} & 2 \end{pmatrix}$, we get $L \cong \begin{pmatrix} 2 & \omega \\ \bar{\omega} & 2 \end{pmatrix} \perp \langle 15 \rangle$ or L contains $\begin{pmatrix} 2 & \omega \\ \bar{\omega} & 2 \end{pmatrix} \perp \langle 30 \rangle$ by appending a suitable vector to L. Since $15 \to \mathrm{gen}\begin{pmatrix} 2 & \omega \\ \bar{\omega} & 2 \end{pmatrix} \perp \langle 30 \rangle$ but $15 \not\to \begin{pmatrix} 2 & \omega \\ \bar{\omega} & 2 \end{pmatrix} \perp \langle 30 \rangle$, we have $L \cong \begin{pmatrix} 2 & \omega \\ \bar{\omega} & 2 \end{pmatrix} \perp \langle 15 \rangle$ after adding suitable vector and comparing volume condition.

This is the end of whole process of finding candidates for binary normal regular Hermitian lattices over $\boldsymbol{Q}(\sqrt{-15})$.

Case 1. $[m = 1]$ Note that

$$p = 2: \ 1_2 \to L_2 \Longleftrightarrow 5_2 \to L_2 \Longrightarrow 2{\cdot}1_2, 2{\cdot}5_2 \to L_2;$$
$$3_2 \to L_2 \Longleftrightarrow 7_2 \to L_2 \Longrightarrow 2{\cdot}3_2, 2{\cdot}7_2 \to L_2;$$
$$p = 3: \ 1_3 \to L_3 \Longleftrightarrow \triangle_3 \to L_3.$$

We obtain 6 candidates (see Table 1).

Local condition		Volume	Ess.#	Lattice	Exc.#
(1)	$1_2, 3_2 \to L_2$ $3{\cdot}1_3 \not\to L_3$	$\mathfrak{v}L \subset 3^2\mathscr{O}$	$1, 7$	N/A	
(2)	$1_2 \to L_2,\ 3_2 \not\to L_2$ $3{\cdot}1_3 \to L_3$	$\mathfrak{v}L \subset 2^2\mathscr{O}$	$1, 21$	$\langle 1, 4\rangle_{\mathrm{vol}:4}$ $\langle 1, 8\rangle_{\mathrm{vol}:8}$ $\langle 1, 12\rangle_{\mathrm{vol}:12}$ $\langle 1, 16\rangle_{\mathrm{vol}:16}$ $\langle 1, 20\rangle_{\mathrm{vol}:20}$	none none 6 none 6
(3)	$1_2 \to L_2,\ 3_2 \not\to L_2$ $3{\cdot}1_3 \not\to L_3$	$\mathfrak{v}L \subset 2^2{\cdot}3^2\mathscr{O}$	$1, 77$	$\langle 1, 36\rangle_{\mathrm{vol}:36}$ $\langle 1, 72\rangle_{\mathrm{vol}:72}$	14 28
(4-1)	$1_2 \not\to L_2,\ 3_2, 2{\cdot}1_2 \to L_2$ $3{\cdot}1_3 \to L_3$	$\mathfrak{v}L \subset 2^2\mathscr{O}$	$2, 3$	$[2, -1+\omega, 3]_{\mathrm{vol}:4}$	none
(4-2)	$1_2, 2{\cdot}1_2 \not\to L_2,\ 3_2 \to L_2$ $3{\cdot}1_3 \to L_3$	$\mathfrak{v}L \subset 2^3\mathscr{O}$	$3, 7$	$[3, -1+\omega, 6]_{\mathrm{vol}:16}$ $[3, 1, 3]_{\mathrm{vol}:8}$	none none
(5-1)	$1_2 \not\to L_2,\ 2{\cdot}1_2, 3_2 \to L_2$ $3{\cdot}1_3 \not\to L_3$	$\mathfrak{v}L \subset 2^2{\cdot}3^2\mathscr{O}$	$2, 7$	N/A	
(5-2)	$1_2, 2{\cdot}1_2 \not\to L_2,\ 3_2 \to L_2$ $3{\cdot}1_3 \not\to L_3$	$\mathfrak{v}L \subset 2^3{\cdot}3^2\mathscr{O}$	$7, 11$	$[7, -2+\omega, 11]_{\mathrm{vol}:72}$	4

Table 1. Candidates for $m = 1$ ($L_{\mathrm{vol}:a}$ means $\mathfrak{v}L = a\mathscr{O}$).

Case 2. $[m = 2]$ Note that

$$p = 2: \ 1_2 \to L \Longleftrightarrow 3_2 \to L \Longrightarrow 2{\cdot}1_2, 2{\cdot}3_2 \to L_2;$$
$$5_2 \to L \Longleftrightarrow 7_2 \to L \Longrightarrow 2{\cdot}5_2, 2{\cdot}7_2 \to L_2;$$
$$p = 5: \ 1_5 \to L \Longleftrightarrow \triangle_5 \to L_5.$$

We obtain 4 candidates (see Table 2).

	Local condition	Volume	Ess.#	Lattice	Exc.#
(1)	$1_2, 5_2 \to L_2$ $5{\cdot}1_5 \not\to L_5$	$\mathfrak{v}L \subset 5^2\mathscr{O}$	$1, 7$	N/A	
(2)	$1_2 \to L_2,\ 5_2 \not\to L_2$ $5{\cdot}1_5 \not\to L_5$	$\mathfrak{v}L \subset 2^3{\cdot}5^2\mathscr{O}$	$1, 91$	N/A	
(3-1)	$1_2, 2{\cdot}5_2 \to L_2,\ 5_2 \not\to L_2$ $5{\cdot}1_5 \to L_5$	$\mathfrak{v}L \subset 2^3\mathscr{O}$	$1, 10$	$\langle 1, 8\rangle_{\mathrm{vol}:8}$	none
(3-2)	$1_2 \to L_2,\ 5_2, 2{\cdot}5_2 \not\to L_2$ $5{\cdot}1_5 \to L_5$	$\mathfrak{v}L \subset 2^4\mathscr{O}$	$1, 35$	$\langle 1, 16\rangle_{\mathrm{vol}:16}$ $\langle 1, 32\rangle_{\mathrm{vol}:32}$	none none
(4)	$1_2 \not\to L_2,\ 5_2 \to L_2$ $5{\cdot}1_5 \to L_5$	$\mathfrak{v}L \subset 2^3\mathscr{O}$	$5, 7$	$[5, -1+\omega, 7]_{\mathrm{vol}:32}$ $[5, -2+\omega, 6]_{\mathrm{vol}:24}$ $[5, -1+2\omega, 5]_{\mathrm{vol}:16}$ $[4, -2+2\omega, 5]_{\mathrm{vol}:8}$ $[2, \omega, 5]_{\mathrm{vol}:8}$	8 2 4 2 none
(5)	$1_2 \not\to L_2,\ 5_2 \to L_2$ $5{\cdot}1_5 \not\to L_5$	$\mathfrak{v}L \subset 2^3{\cdot}5^2\mathscr{O}$	$7, 13$	N/A	

Table 2. Candidates for $m = 2$ ($L_{\mathrm{vol}:a}$ means $\mathfrak{v}L = a\mathscr{O}$).

Case 3. $[m = 3]$ Note that

$$p = 2:\ 1_2 \to L_2 \iff 3_2 \to L_2 \iff 5_2 \to L_2 \iff 7_2 \to L_2;$$

$$p = 3:\ 1_3 \to L_3 \Longrightarrow 3{\cdot}1_3 \to L_3; \quad \triangle_3 \to L_3 \Longrightarrow 3\triangle_3 \to L_3.$$

We obtain 11 candidates (see Table 3).

	Local condition	Volume	Ess.#	Lattice	Exc.#
(1)	$2{\cdot}1_2 \to L_2$ $1_3 \to L_3,\ \triangle_3 \not\to L_3$	$\mathfrak{v}L \subset 3\mathscr{O}$	$1, 10$	$\langle 1, 3\rangle_{\mathrm{vol}:3}$ $\langle 1, 6\rangle_{\mathrm{vol}:6}$ $\langle 1, 9\rangle_{\mathrm{vol}:9}$	none none none
(2-1)	$2{\cdot}1_2 \to L_2$ $1_3 \not\to L_3,\ \triangle_3, 3{\cdot}1_3 \to L_3$	$\mathfrak{v}L \subset 3\mathscr{O}$	$2, 3$	$\langle 2, 3\rangle_{\mathrm{vol}:6}$ $[2, 1, 2]_{\mathrm{vol}:3}$	none none
(2-2)	$2{\cdot}1_2 \to L_2$ $1_3, 3{\cdot}1_3 \not\to L_3,\ \triangle_3 \to L_3$	$\mathfrak{v}L \subset 3^2\mathscr{O}$	$2, 5$	$[2, 1, 5]_{\mathrm{vol}:9}$	none
(3)	$2{\cdot}1_2 \not\to L_2$ $1_3, \triangle_3 \to L_3$	$\mathfrak{v}L \subset 2^2\mathscr{O}$	$1, 10$	$\langle 1, 4\rangle_{\mathrm{vol}:4}$	none
(4)	$2{\cdot}1_2 \not\to L_2$ $1_3 \to L_3,\ \triangle_3 \not\to L_3$	$\mathfrak{v}L \subset 2^2{\cdot}3\mathscr{O}$	$1, 55$	$\langle 1, 12\rangle_{\mathrm{vol}:12}$ $\langle 1, 24\rangle_{\mathrm{vol}:24}$ $\langle 1, 36\rangle_{\mathrm{vol}:36}$ $\langle 1, 48\rangle_{\mathrm{vol}:48}$	none 15 none 15
(5-1)	$2{\cdot}1_2 \not\to L_2$ $1_3 \not\to L_3,\ \triangle_3, 3{\cdot}1_3 \to L_3$	$\mathfrak{v}L \subset 2^2{\cdot}3\mathscr{O}$	$3, 5$	$[3, 1+\omega, 5]_{\mathrm{vol}:12}$	none
(5-2)	$2{\cdot}1_2 \not\to L_2$ $1_3, 3{\cdot}1_3 \not\to L_3,\ \triangle_3 \to L_3$	$\mathfrak{v}L \subset 2^2{\cdot}3^2\mathscr{O}$	$5, 11$	$[5, 2, 8]_{\mathrm{vol}:36}$	none

Table 3. Candidates for $m = 3$ ($L_{\mathrm{vol}:a}$ means $\mathfrak{v}L = a\mathscr{O}$).

Case 4. $[m=5]$ Note that

$$\begin{aligned} p=2: &\ 1_2 \to L_2 \Longleftrightarrow 5_2 \to L_2 \Longrightarrow 2{\cdot}3_2, 2{\cdot}7_2 \to L_2;\\ &\ 3_2 \to L_2 \Longleftrightarrow 7_2 \to L_2 \Longrightarrow 2{\cdot}1_2, 2{\cdot}5_2 \to L_2;\\ p=5: &\ 1_5 \to L_5 \Longrightarrow 5{\cdot}1_5 \to L_5; \quad \triangle_5 \to L_5 \Longrightarrow 5\triangle_5 \to L_5. \end{aligned}$$

We obtain 7 candidates (see Table 4).

Local condition		Volume	Ess.#	Lattice	Exc.#
(1)	$1_2, 3_2 \to L_2$ $1_5 \to L_5,\ \triangle_5 \not\to L_5$	$\mathfrak{v}L \subset 5\mathscr{O}$	$1, 11$	$\langle 1, 5\rangle_{\mathrm{vol:5}}$ $\langle 1, 10\rangle_{\mathrm{vol:10}}$ $\langle 1\rangle \perp 5[2, -1+\omega, 3]_{\mathrm{vol:5}}$	15 none none
(2-1)	$1_2, 3_2 \to L_2$ $1_5 \not\to L_5,\ \triangle_5, 5{\cdot}1_5 \to L_5$	$\mathfrak{v}L \subset 5\mathscr{O}$	$2, 3$	$[2, 1, 3]_{\mathrm{vol:5}}$ $[2, -1+\omega, 3] \perp \langle 5\rangle_{\mathrm{vol:5}}$	11 none
(2-2)	$1_2, 3_2 \to L_2$ $1_5, 5{\cdot}1_5 \not\to L_5,\ \triangle_5 \to L_5$	$\mathfrak{v}L \subset 5^2\mathscr{O}$	$2, 3$	N/A	
(3-1)	$1_2, 2{\cdot}1_2 \to L_2,\ 3_2 \not\to L_2$ $1_5, \triangle_5 \to L_5$	$\mathfrak{v}L \subset 2^3\mathscr{O}$	$1, 2$	N/A	
(3-2)	$1_2 \to L_2,\ 3_2, 2{\cdot}1_2 \not\to L_2$ $1_5, \triangle_5 \to L_5$	$\mathfrak{v}L \subset 2^3\mathscr{O}$	$1, 13$	$\langle 1, 8\rangle_{\mathrm{vol:8}}$	none
(4-1)	$1_2, 2{\cdot}1_2 \to L_2,\ 3_2 \not\to L_2$ $1_5 \to L_5,\ \triangle_5 \not\to L_5$	N/A ($\lambda_{5^k}(L)$ cannot be regular by (3-1))			
(4-2-1)	$1_2 \to L_2,\ 3_2, 2{\cdot}1_2 \not\to L_2$ $1_5, 5\triangle_5 \to L_5,\ \triangle_5 \not\to L_5$	$\mathfrak{v}L \subset 2^3{\cdot}5\mathscr{O}$	$1, 65$	$\langle 1, 40\rangle_{\mathrm{vol:40}}$	none
(4-2-2)	$1_2 \to L_2,\ 3_2, 2{\cdot}1_2 \not\to L_2$ $1_5 \to L_5,\ \triangle_5, 5\triangle_5 \not\to L_5$	$\mathfrak{v}L \subset 2^3{\cdot}5^2\mathscr{O}$	$1, 209$	$\langle 1, 200\rangle_{\mathrm{vol:200}}$	44
(5-1)	$1_2, 2{\cdot}1_2 \to L_2,\ 3_2 \not\to L_2$ $1_5 \not\to L_5,\ \triangle_5 \to L_5$	N/A ($\lambda_{5^k}(L)$ cannot be regular by (3-1))			
(5-2-1)	$1_2 \to L_2,\ 3_2, 2{\cdot}1_2 \not\to L_2$ $1_5 \not\to L_5,\ \triangle_5, 5{\cdot}1_5 \to L_5$	$\mathfrak{v}L \subset 2^3{\cdot}5\mathscr{O}$	$5, 13$	$\langle 5, 8\rangle_{\mathrm{vol:40}}$	12
(5-2-2)	$1_2 \to L_2,\ 3_2, 2{\cdot}1_2 \not\to L_2$ $1_5, 5{\cdot}1_5 \not\to L_5,\ \triangle_5 \to L_5$	$\mathfrak{v}L \subset 2^3{\cdot}5^2\mathscr{O}$	$13, 17$	$[13, 4+\omega, 17]_{\mathrm{vol:200}}$	8
(6-1)	$1_2 \not\to L_2,\ 3_2, 2{\cdot}3_2 \to L_2$ $1_5, \triangle_5 \to L_5$	$\mathfrak{v}L \subset 2^2\mathscr{O}$	$2, 3$	$[2, -1+\omega, 3] \perp \langle 4\rangle_{\mathrm{vol:4}}$	none
(6-2)	$1_2, 2{\cdot}3_2 \not\to L_2,\ 3_2 \to L_2$ $1_5, \triangle_5 \to L_5$	$\mathfrak{v}L \subset 2^3\mathscr{O}$	$2, 3$	$[2, -1+\omega, 3] \perp \langle 8\rangle_{\mathrm{vol:8}}$	8
(7-1)	$1_2 \not\to L_2,\ 3_2, 2{\cdot}3_2 \to L_2$ $1_5 \to L_5,\ \triangle_5 \not\to L_5$	$\mathfrak{v}L \subset 2^2{\cdot}5\mathscr{O}$	$4, 6$	$\begin{pmatrix} 4 & -2+2\omega & -2 \\ -2+2\overline{\omega} & 6 & 1+\omega \\ -2 & 1+\overline{\omega} & 11 \end{pmatrix}_{\mathrm{vol:20}}$	10
(7-2)	$1_2, 2{\cdot}3_2 \not\to L_2,\ 3_2 \to L_2$ $1_5 \to L_5,\ \triangle_5 \not\to L_5$	N/A ($\lambda_{5^k}(L)$ cannot be regular by (6-2))			
(8-1)	$1_2 \not\to L_2,\ 3_2, 2{\cdot}3_2 \to L_2$ $1_5 \not\to L_5,\ \triangle_5 \to L_5$	$\mathfrak{v}L \subset 2^2{\cdot}5\mathscr{O}$	$2, 3$	$[2, -1+\omega, 3] \perp \langle 20\rangle_{\mathrm{vol:20}}$	none
(8-2)	$1_2, 2{\cdot}3_2 \not\to L_2,\ 3_2 \to L_2$ $1_5 \not\to L_5,\ \triangle_5 \to L_5$	N/A ($\lambda_{5^k}(L)$ cannot be regular by (6-2))			

Table 4. Candidates for $m=5$ ($L_{\mathrm{vol}:a}$ means $\mathfrak{v}L = a\mathscr{O}$).

Case 5. $[m=6]$ Note that

$$p=2:\ 1_2\to L_2 \Longleftrightarrow 7_2\to L_2 \Longrightarrow 2{\cdot}3_2, 2{\cdot}5_2\to L_2;$$
$$3_2\to L_2 \Longleftrightarrow 5_2\to L_2 \Longrightarrow 2{\cdot}1_2, 2{\cdot}7_2\to L_2;$$
$$p=3:\ 1_3\to L_3 \Longrightarrow 3\triangle_3\to L_3;\quad \triangle_3\to L_3\Longrightarrow 3{\cdot}1_3\to L_3.$$

We obtain 2 candidates (see Table 5).

	Local condition	Volume	Ess.#	Lattice	Exc.#
(1-1)	$1_2, 3_2\to L_2$ $1_3, 3{\cdot}1_3\to L_3,\ \triangle_3\not\to L_3$	$\mathfrak{v}L\subset 3\mathscr{O}$	1, 3	$\langle 1,3\rangle_{\mathrm{vol:3}}$	none
(1-2)	$1_2, 3_2\to L_2$ $1_3\to L_3,\ \triangle_3, 3{\cdot}1_3\not\to L_3$	$\mathfrak{v}L\subset 3^2\mathscr{O}$	1, 13	$\langle 1,9\rangle_{\mathrm{vol:9}}$	27
(2-1)	$1_2, 3_2\to L_2$ $1_3, 3\triangle_3\not\to L_3,\ \triangle_3\to L_3$	$\mathfrak{v}L\subset 3^2\mathscr{O}$	2, 3	$[2,\omega,3]\perp\langle 9\rangle_{\mathrm{vol:9}}$	26
(2-2)	$1_2, 3_2\to L_2$ $1_3\not\to L_3,\ \triangle_3, 3\triangle_3\to L_3$	$\mathfrak{v}L\subset 3\mathscr{O}$	2, 3	$[2,0,3]_{\mathrm{vol:6}}$ $[2,\omega,3]\perp\langle 3\rangle_{\mathrm{vol:3}}$ $[2,\omega,3]\perp 3[2,\omega,3]$ $\cong[9,4\omega,11]_{\mathrm{vol:3}}$	6 9 none
(3-1)	$1_2, 2{\cdot}1_2\to L_2,\ 3_2\not\to L_2$ $1_3, \triangle_3\to L_3$	$\mathfrak{v}L\subset 2^3\mathscr{O}$	1, 2	N/A	
(3-2)	$1_2\to L_2,\ 2{\cdot}1_2, 3_2\not\to L_2$ $1_3, \triangle_3\to L_3$	$\mathfrak{v}L\subset 2^4\mathscr{O}$	1, 17	$\langle 1,16\rangle_{\mathrm{vol:16}}$	7
(4)	$1_2\to L_2,\ 3_2\not\to L_2$ $1_3\to L_3,\ \triangle_3\not\to L_3$	N/A ($\lambda_{3^k}(L)$ cannot be regular by (3-1), (3-2), (6))			
(5)	$1_2\to L_2,\ 3_2\not\to L_2$ $1_3\not\to L_3,\ \triangle_3\to L_3$	N/A ($\lambda_{3^k}(L)$ cannot be regular by (3-1), (3-2), (6))			
(6)	$1_2\not\to L_3,\ 3_2\to L_2$ $1_3, \triangle_3\to L_3$	$\mathfrak{v}L\subset 2^3\mathscr{O}$	2, 3	$[2,\omega,3]\perp\langle 8\rangle_{\mathrm{vol:8}}$	6
(7)	$1_2\not\to L_2,\ 3_2\to L_2$ $1_3\to L_3,\ \triangle_3\not\to L_3$	N/A ($\lambda_{3^k}(L)$ cannot be regular by (3-1), (3-2), (6))			
(8)	$1_2\not\to L_2,\ 3_2\to L_2$ $1_3\not\to L_3,\ \triangle_3\to L_3$	N/A ($\lambda_{3^k}(L)$ cannot be regular by (3-1), (3-2), (6))			

Table 5. Candidates for $m=6$ ($L_{\mathrm{vol}:a}$ means $\mathfrak{v}L=a\mathscr{O}$).

Case 6. $[m=7]$ Note that

$$p=2:\ 1_2\to L_2 \Longleftrightarrow 3_2\to L_2 \Longleftrightarrow 5_2\to L_2 \Longleftrightarrow 7_2\to L_2$$
$$\Longrightarrow 2{\cdot}1_2, 2{\cdot}3_2, 2{\cdot}5_2, 2{\cdot}7_2\to L_2;$$
$$p=3:\ 1_3\to L_3 \Longleftrightarrow \triangle_3\to L_3;$$
$$p=7:\ 1_7\to L_7 \Longrightarrow 7{\cdot}1_7\to L_7;\quad \triangle_7\to L_7\Longrightarrow 7\triangle_7\to L_7.$$

We obtain 3 candidates (see Table 6).

	Local condition	Volume	Ess.#	Lattice	Exc.#
(1)	$1_7 \to L_7,\ \triangle_7 \not\to L_7$ $3{\cdot}1_3 \to L_3$	$\mathfrak{v}L \subset 7\mathscr{O}$	$1, 15$	$\langle 1, 7\rangle_{\mathrm{vol}:7}$ $\langle 1, 14\rangle_{\mathrm{vol}:14}$	none none
(2)	$1_7 \not\to L_7,\ \triangle_7 \to L_7$ $3{\cdot}1_3 \to L_3$	$\mathfrak{v}L \subset 7\mathscr{O}$	$3, 5$	$[3, 1, 5]_{\mathrm{vol}:14}$ $[3, \omega, 3]_{\mathrm{vol}:7}$	7 none
(3)	$1_7 \to L_7,\ \triangle_7 \not\to L_7$ $3{\cdot}1_3 \not\to L_3$	$\mathfrak{v}L \subset 7{\cdot}3^2\mathscr{O}$	$1, 65$	$\langle 1, 63\rangle_{\mathrm{vol}:63}$	35
(4)	$1_7 \not\to L_7,\ \triangle_7 \to L_7$ $3{\cdot}1_3 \not\to L_3$	$\mathfrak{v}L \subset 7{\cdot}3^2\mathscr{O}$	$5, 13$	$[5, \omega, 13]_{\mathrm{vol}:63}$	7

Table 6. Candidates for $m = 7$ ($L_{\mathrm{vol}:a}$ means $\mathfrak{v}L = a\mathscr{O}$).

Case 7. $[m = 10]$ Note that

$$p = 2:\ 1_2 \to L_2 \Longleftrightarrow 3_2 \to L_2 \Longrightarrow 2{\cdot}5_2, 2{\cdot}7_2 \to L_2;$$
$$5_2 \to L_2 \Longleftrightarrow 7_2 \to L_2 \Longrightarrow 2{\cdot}1_2, 2{\cdot}3_2 \to L_2;$$
$$p = 5:\ 1_5 \to L_5 \Longrightarrow 5\triangle_5 \to L_5; \quad \triangle_5 \to L_5 \Longrightarrow 5{\cdot}1_5 \to L_5.$$

We obtain only 1 candidate (see Table 7).

	Local condition	Volume	Ess.#	Lattice	Exc.#
(1)	$1_2, 5_2 \to L_2$ $1_5 \to L_5,\ \triangle_5 \not\to L_5$	$\mathfrak{v}L \subset 5\mathscr{O}$	$1, 6$	$\langle 1, 5\rangle_{\mathrm{vol}:5}$	none
(2)	$1_2, 5_2 \to L_2$ $1_5 \not\to L_5,\ \triangle_5 \to L_5$	$\mathfrak{v}L \subset 5\mathscr{O}$	$2, 3$	$[2, 1, 3]_{\mathrm{vol}:5}$	5
(3)	$1_2 \to L_2,\ 5_2 \not\to L_2$ $1_5, \triangle_5 \to L_5$	$\mathfrak{v}L \subset 2^3\mathscr{O}$	$1, 3$	N/A	
(4)	$1_2 \to L_2,\ 5_2 \not\to L_2$ $1_5 \to L_5,\ \triangle_5 \not\to L_5$	N/A ($\lambda_{5^k}(L)$ cannot be regular by (3), (6))			
(5)	$1_2 \to L_2,\ 5_2 \not\to L_2$ $1_5 \not\to L_5,\ \triangle_5 \to L_5$	N/A ($\lambda_{5^k}(L)$ cannot be regular by (3), (6))			
(6)	$1_2 \not\to L_2,\ 5_2 \to L_2$ $1_5, \triangle_5 \to L_5$	$\mathfrak{v}L \subset 2^3\mathscr{O}$	$2, 5$	N/A	
(7)	$1_2 \not\to L_2,\ 5_2 \to L_2$ $1_5 \to L_5,\ \triangle_5 \not\to L_5$	N/A ($\lambda_{5^k}(L)$ cannot be regular by (3), (6))			
(8)	$1_2 \not\to L_2,\ 5_2 \to L_2$ $1_5 \not\to L_5,\ \triangle_5 \to L_5$	N/A ($\lambda_{5^k}(L)$ cannot be regular by (3), (6))			

Table 7. Candidates for $m = 10$ ($L_{\mathrm{vol}:a}$ means $\mathfrak{v}L = a\mathscr{O}$).

CASE 8. $[m = 11]$ Note that

$$p = 2: \; 1_2 \to L_2 \Longleftrightarrow 3_2 \to L_2 \Longleftrightarrow 5_2 \to L_2 \Longleftrightarrow 7_2 \to L_2;$$

$$p = 11: \; 1_{11} \to L_{11} \Longrightarrow 11{\cdot}1_{11} \to L_{11}; \quad \triangle_{11} \to L_{11} \Longrightarrow 11\triangle_{11} \to L_{11}.$$

We obtain 3 candidates (see Table 8).

	Local condition	Volume	Ess.#	Lattice	Exc.#
(1)	$2{\cdot}1_2 \to L_2$ $1_{11} \to L_{11},\ \triangle_{11} \not\to L_{11}$	$\mathfrak{v}L \subset 11\mathscr{O}$	$1, 14$	$\langle 1, 11\rangle_{\mathrm{vol}:11}$	none
(2)	$2{\cdot}1_2 \to L_2$ $1_{11} \not\to L_{11},\ \triangle_{11} \to L_{11}$	$\mathfrak{v}L \subset 11\mathscr{O}$	$2, 7$	$[2, \omega, 7]_{\mathrm{vol}:11}$	11
(3)	$2{\cdot}1_2 \not\to L_2$ $1_{11}, \triangle_{11} \to L_{11}$	$\mathfrak{v}L \subset 2^2\mathscr{O}$	$1, 7$	$\langle 1, 4\rangle_{\mathrm{vol}:4}$	none
(4)	$2{\cdot}1_2 \not\to L_2$ $1_{11} \to L_{11},\ \triangle_{11} \not\to L_{11}$	$\mathfrak{v}L \subset 2^2{\cdot}11\mathscr{O}$	$1, 91$	$\langle 1, 44\rangle_{\mathrm{vol}:44}$ $\langle 1, 88\rangle_{\mathrm{vol}:88}$	none 77
(5)	$2{\cdot}1_2 \not\to L_2$ $1_{11} \not\to L_{11},\ \triangle_{11} \to L_{11}$	$\mathfrak{v}L \subset 2^2{\cdot}11\mathscr{O}$	$7, 13$	$[7, \omega, 13]_{\mathrm{vol}:88}$ $[7, 2\omega, 8]_{\mathrm{vol}:44}$	8 11

Table 8. Candidates for $m = 11$ ($L_{\mathrm{vol}:a}$ means $\mathfrak{v}L = a\mathscr{O}$).

CASE 9. $[m = 15]$ Note that

$$p = 2: \; 1_2 \to L_2 \Longleftrightarrow 3_2 \to L_2 \Longleftrightarrow 5_2 \to L_2 \Longleftrightarrow 7_2 \to L_2$$
$$\Longrightarrow 2{\cdot}1_2, 2{\cdot}3_2, 2{\cdot}5_2, 2{\cdot}7_2 \to L_2;$$

$$p = 3: \; 1_3 \to L_2 \Longrightarrow 3\triangle_3 \to L_3; \quad \triangle_3 \to L_2 \Longrightarrow 3{\cdot}1_3 \to L_3;$$

$$p = 5: \; 1_5 \to L_5 \Longrightarrow 5\triangle_5 \to L_5; \quad \triangle_5 \to L_5 \Longrightarrow 5{\cdot}1_5 \to L_5.$$

We obtain 6 candidates (see Table 9).

	Local condition	Volume	Ess.#	Lattice	Exc.#
(1)	$1_3, \triangle_3 \to L_3$ $1_5 \to L_5,\ \triangle_5 \not\to L_5$	$\mathfrak{v}L \subset 5\mathscr{O}$	1, 11	$\langle 1,5\rangle_{\text{vol:5}}$ $\langle 1,10\rangle_{\text{vol:10}}$ $\langle 1\rangle \perp 5[2,\omega,2]_{\text{vol:5}}$	none 5 5
(2)	$1_3, \triangle_3 \to L_3$ $1_5 \not\to L_5,\ \triangle_5 \to L_5$	$\mathfrak{v}L \subset 5\mathscr{O}$	2, 3	$[2,1,3]_{\text{vol:5}}$ $[2,\omega,2] \perp \langle 5\rangle_{\text{vol:5}}$	5 none
(3)	$1_3 \to L_3,\ \triangle_3 \not\to L_3$ $1_5, \triangle_5 \to L_5$	$\mathfrak{v}L \subset 3\mathscr{O}$	1, 7	$\langle 1,3\rangle_{\text{vol:3}}$ $\langle 1,6\rangle_{\text{vol:6}}$	none 3
(4-1)	$1_3, 3{\cdot}1_3 \to L_3,\ \triangle_3 \not\to L_3$ $1_5 \to L_5,\ \triangle_5 \not\to L_5$	$\mathfrak{v}L \subset 3{\cdot}5\mathscr{O}$	1, 21	$\langle 1,15\rangle_{\text{vol:15}}$	45
(4-2)	$1_3 \to L_3,\ \triangle_3, 3{\cdot}1_3 \not\to L_3$ $1_5 \to L_5,\ \triangle_5 \not\to L_5$	$\mathfrak{v}L \subset 3^2{\cdot}5\mathscr{O}$	1, 91	$\langle 1,45\rangle_{\text{vol:45}}$ $\langle 1,90\rangle_{\text{vol:90}}$	17 45
(5-1-1)	$1_3, 3{\cdot}1_3 \to L_3,\ \triangle_3 \not\to L_3$ $1_5 \not\to L_5,\ \triangle_5, 5\triangle_5 \to L_5$	$\mathfrak{v}L \subset 3{\cdot}5\mathscr{O}$	3, 7	$[3,1+\omega,7]_{\text{vol:15}}$	15
(5-1-2)	$1_3, 3{\cdot}1_3 \to L_3,\ \triangle_3 \not\to L_3$ $1_5, 5\triangle_5 \not\to L_5,\ \triangle_5 \to L_5$	$\mathfrak{v}L \subset 3{\cdot}5^2\mathscr{O}$	3, 7	N/A	
(5-2-1)	$1_3 \to L_3,\ \triangle_3, 3{\cdot}1_3 \not\to L_3$ $1_5 \not\to L_5,\ \triangle_5, 5\triangle_5 \to L_5$	$\mathfrak{v}L \subset 3^2{\cdot}5\mathscr{O}$	7, 11	$[7,2,7]_{\text{vol:45}}$	13
(5-2-2)	$1_3 \to L_3,\ \triangle_3, 3{\cdot}1_3 \not\to L_3$ $1_5, 5\triangle_5 \not\to L_5,\ \triangle_5 \to L_5$	$\mathfrak{v}L \subset 3^2{\cdot}5^2\mathscr{O}$	7, 13	N/A	
(6-1)	$1_3 \not\to L_3,\ \triangle_3, 3\triangle_3 \to L_3$ $1_5, \triangle_5 \to L_5$	$\mathfrak{v}L \subset 3\mathscr{O}$	2, 6	$[2,\omega,2] \perp \langle 6\rangle_{\text{vol:6}}$ $[2,\omega,2] \perp 3[2,\omega,2]$ $\cong [8,-1+4\omega,8]_{\text{vol:3}}$	15 none
(6-2)	$1_3, 3\triangle_3 \not\to L_3,\ \triangle_3 \to L_3$ $1_5, \triangle_5 \to L_5$	$\mathfrak{v}L \subset 3^2\mathscr{O}$	2, 3	$[2,\omega,2] \perp \langle 9\rangle_{\text{vol:9}}$	none
(7-1-1)	$1_3 \not\to L_3,\ \triangle_3, 3\triangle_3 \to L_3$ $1_5, 5{\cdot}1_5 \to L_5,\ \triangle_5 \not\to L_5$	$\mathfrak{v}L \subset 3{\cdot}5\mathscr{O}$	5, 6	$[5,-1+2\omega,6]_{\text{vol:15}}$ $\langle 5\rangle \perp 3[2,\omega,2]_{\text{vol:5}}$	9 21
(7-1-2)	$1_3 \not\to L_3,\ \triangle_3, 3\triangle_3 \to L_3$ $1_5 \to L_5,\ \triangle_5, 5{\cdot}1_5 \not\to L_5$	$\mathfrak{v}L \subset 3{\cdot}5^2\mathscr{O}$	6, 11	N/A	
(7-2-1)	$1_3, 3\triangle_3 \not\to L_3,\ \triangle_3 \to L_3$ $1_5, 5{\cdot}1_5 \to L_5,\ \triangle_5 \not\to L_5$	$\mathfrak{v}L \subset 3^2{\cdot}5\mathscr{O}$	5, 11	$[5,2+\omega,11]_{\text{vol:45}}$	9
(7-2-2)	$1_3, 3\triangle_3 \not\to L_3,\ \triangle_3 \to L_3$ $1_5 \to L_5,\ \triangle_5, 5{\cdot}1_5 \not\to L_5$	$\mathfrak{v}L \subset 3^2{\cdot}5^2\mathscr{O}$	11, 14	N/A	
(8-1-1)	$1_3 \not\to L_3,\ \triangle_3, 3\triangle_3 \to L_3$ $1_5 \not\to L_5,\ \triangle_5, 5\triangle_5 \to L_5$	$\mathfrak{v}L \subset 3{\cdot}5\mathscr{O}$	2, 3	$[2,\omega,2] \perp \langle 15\rangle_{\text{vol:15}}$	none
(8-1-2)	$1_3 \not\to L_3,\ \triangle_3, 3\triangle_3 \to L_3$ $1_5, 5\triangle_5 \not\to L_5,\ \triangle_5 \to L_5$	$\mathfrak{v}L \subset 3{\cdot}5\mathscr{O}$	2, 3	$[2,\omega,2] \perp \langle 30\rangle_{\text{vol:30}}$ $[2,\omega,2] \perp \langle 15\rangle_{\text{vol:15}}$	15 none
(8-2)	$1_3, 3\triangle_3 \not\to L_3,\ \triangle_3 \to L_3$ $1_5 \not\to L_5,\ \triangle_5 \to L_5$	$\mathfrak{v}L \subset 3^2{\cdot}5\mathscr{O}$	2, 3	$[2,\omega,2] \perp \langle 45\rangle_{\text{vol:45}}$	35

Table 9. Candidates for $m = 15$ ($L_{\text{vol}:a}$ means $\mathfrak{v}L = a\mathscr{O}$).

Case 10. $[m = 19]$ Note that

$$p = 2:\ 1_2 \to L_2 \Longleftrightarrow 3_2 \to L_2 \Longleftrightarrow 5_2 \to L \Longleftrightarrow 7_2 \to L_2;$$

$$p = 19:\ 1_{19} \to L_{19} \Longrightarrow 19{\cdot}1_{19} \to L_{19}; \quad \triangle_{19} \to L_{19} \Longrightarrow 19\triangle_{19} \to L_{19}.$$

There is no candidate (see Table 10).

	Local condition	Volume	Ess.#	Lattice	Exc.#
(1-1)	$2{\cdot}1_2 \to L_2$ $1_{19} \to L_{19}$, $\triangle_{19} \not\to L_{19}$	$\mathfrak{v}L \subset 19\mathscr{O}$	1, 6	N/A	
(1-2)	$2{\cdot}1_2 \not\to L_2$ $1_{19} \to L_{19}$, $\triangle_{19} \not\to L_{19}$	$\mathfrak{v}L \subset 2^2{\cdot}19\mathscr{O}$	1, 39	N/A	
(2-1)	$2{\cdot}1_2 \to L_2$ $1_{19} \not\to L_{19}$, $\triangle_{19} \to L_{19}$	$\mathfrak{v}L \subset 19\mathscr{O}$	2, 3	N/A	
(2-2)	$2{\cdot}1_2 \not\to L_2$ $1_{19} \not\to L_{19}$, $\triangle_{19} \to L_{19}$	$\mathfrak{v}L \subset 2^2{\cdot}19\mathscr{O}$	3, 13	N/A	

Table 10. Candidates for $m = 19$ ($L_{\mathrm{vol}:a}$ means $\mathfrak{v}L = a\mathscr{O}$).

Case 11. $[m = 23]$ Note that

$$p = 2: \; 1_2 \to L_2 \iff 3_2 \to L_2 \iff 5_2 \to L_2 \iff 7_2 \to L_2 \\ \implies 2{\cdot}1_2, 2{\cdot}3_2, 2{\cdot}5_2, 2{\cdot}7_2 \to L_2;$$

$$p = 23: \; 1_{23} \to L_{23} \implies 23{\cdot}1_3 \to L_{23}; \quad \triangle_{23} \to L_{23} \implies 23\triangle_{23} \to L_{23}.$$

There is no candidate (see Table 11).

	Local condition	Volume	Ess.#	Lattice	Exc.#
(1)	$1_{23} \to L_{23}$, $\triangle_{23} \not\to L_{23}$	$\mathfrak{v}L \subset 23\mathscr{O}$	1, 2	N/A	
(2)	$1_{23} \not\to L_{23}$, $\triangle_{23} \to L_{23}$	$\mathfrak{v}L \subset 23\mathscr{O}$	5, 7	$[5, 2+\omega, 7]_{\mathrm{vol}:23}$	10

Table 11. Candidates for $m = 23$ ($L_{\mathrm{vol}:a}$ means $\mathfrak{v}L = a\mathscr{O}$).

Case 12. $[m = 31]$ Note that

$$p = 2: \; 1_2 \to L_2 \iff 3_2 \to L_2 \iff 5_2 \to L_2 \iff 7_2 \to L_2 \\ \implies 2{\cdot}1_2, 2{\cdot}3_2, 2{\cdot}5_2, 2{\cdot}7_2 \to L_2;$$

$$p = 31: \; 1_{31} \to L_{31} \implies 31{\cdot}1_{31} \to L_{31}; \quad \triangle_{31} \to L_{31} \implies 31\triangle_{31} \to L_{31}.$$

There is no candidate (see Table 12).

	Local condition	Volume	Ess.#	Lattice	Exc.#
(1)	$1_{31} \to L_{31}$, $\triangle_{31} \not\to L_{31}$	$\mathfrak{v}L \subset 31\mathscr{O}$	1, 2	N/A	
(2)	$1_{31} \not\to L_{31}$, $\triangle_{31} \to L_{31}$	$\mathfrak{v}L \subset 31\mathscr{O}$	3, 6	N/A	

Table 12. Candidates for $m = 31$ ($L_{\mathrm{vol}:a}$ means $\mathfrak{v}L = a\mathscr{O}$).

5. Proofs for binary regular Hermitian lattices.

From the previous section 4, we get the 43 candidates for binary normal regular Hermitian lattices L over all imaginary quadratic fields $\boldsymbol{Q}(\sqrt{-m})$. Among the candidates, the class numbers of the following 13 Hermitian lattices are one, so their regularity follows.

$$\boldsymbol{Q}(\sqrt{-1}) : \langle 1,4\rangle, \begin{pmatrix} 2 & -1+\omega_1 \\ -1+\overline{\omega}_1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & -1+\omega_1 \\ -1+\overline{\omega}_1 & 6 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\boldsymbol{Q}(\sqrt{-3}) : \langle 1,3\rangle, \langle 1,4\rangle, \langle 1,6\rangle, \langle 2,3\rangle, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 1+\omega_3 \\ 1+\overline{\omega}_3 & 5 \end{pmatrix}, \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$$

$$\boldsymbol{Q}(\sqrt{-7}) : \begin{pmatrix} 3 & \omega_7 \\ \overline{\omega}_7 & 3 \end{pmatrix}$$

We confirm that all the remaining 30 candidates are actually regular. Since the regularity of all candidates are proved by a lot of computation, it is too long to be described here. The proofs for $\langle 1,36\rangle$ over $\boldsymbol{Q}(\sqrt{-3})$ and $\left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right) \perp 3\left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right)$ over $\boldsymbol{Q}(\sqrt{-15})$ are typical and the proofs for the other candidates are quite similar except $\langle 1,14\rangle$ over $\boldsymbol{Q}(\sqrt{-7})$. Since all ternary sublattices of the associated quadratic form of $\langle 1,14\rangle$ have big class numbers, the proof for $\langle 1,14\rangle$ is impregnable against known methods. So we develop a new arithmetic method utilizing ternary sublattices of class number bigger than one. So we provide three kinds of proofs as follow.

Proof for $L=\langle 1,36\rangle$ over $\boldsymbol{Q}(\sqrt{-3})$. Note that

$$H(\operatorname{gen} L) = \{n \in \boldsymbol{Z} \mid n \equiv 0,1,3 \pmod 4 \text{ and } n \equiv 0,1 \pmod 3\}.$$

The lattices $\langle 1,9\rangle$ and $\langle 1,12\rangle$ are regular. In fact, $\langle 1,9\rangle$ represents all positive integers $n \equiv 0,1,3,4,7 \pmod 9$ and $\langle 1,12\rangle$ represents all positive integers $n \equiv 0,1,3,4,7,9 \pmod{12}$. Since L contains sublattices $\langle 4,36\rangle = 4\langle 1,9\rangle$ and $\langle 3,36\rangle = 3\langle 1,12\rangle$, L represents all positive integers n such that $n \equiv 0 \pmod 4$ and $n \equiv 0,1 \pmod 3$, or $n \equiv 0 \pmod 3$ and $n \equiv 1 \pmod 2$. So it suffices to show that L represents all positive integers n such that n is odd and $n \equiv 1 \pmod 3$. The quadratic lattice associated with L is

$$x\overline{x} + 36y\overline{y} = x_1^2 + x_1x_2 + x_2^2 + 36y_1^2 + 36y_1y_2 + 36y_2^2$$

and it contains a sublattice isometric to $\langle 1,3,36,108\rangle_{\boldsymbol{Z}} = \langle 1\rangle_{\boldsymbol{Z}} \perp 3\langle 1,12,36\rangle_{\boldsymbol{Z}}$.

Since $\langle 1,12,36\rangle_{\boldsymbol{Z}}$ is regular [**11**], $3\langle 1,12,36\rangle_{\boldsymbol{Z}} = \langle 3,36,108\rangle_{\boldsymbol{Z}}$ represents all positive integers $n \equiv 3,12 \pmod{36}$. If $n \equiv 1 \pmod{12}$ and $n \geq 49$, then $n - a^2 \equiv 12 \pmod{36}$ for $a = 1,5,7$ and hence L represents n. If $n \equiv 7 \pmod{12}$ and $n \geq 64$, then $n - a^2 \equiv 3 \pmod{36}$ for $a = 2,4,8$ and hence L represents n. We check that $n \to L$ for $n = 1,7,13,19,25,31,37,43,49,55$ by direct computation. Therefore L is regular. □

Proof for $L = \langle 1,14\rangle$ over $\boldsymbol{Q}(\sqrt{-7})$. Note that

$$H(\operatorname{gen} L) = \{n \in \boldsymbol{N}_0 \mid n \equiv 0,1,2,4 \pmod 7\}.$$

Since $7\langle 1,2\rangle = \langle 7,14\rangle$ is a sublattice of L and $\langle 1,2\rangle$ is universal [**6**], $n \to L$ for all positive integers $n \equiv 0 \pmod 7$. So we may assume $n \equiv 1,2,4 \pmod 7$.

The quadratic lattice $\widetilde{L}$ associated with L is

$$x_1^2 + x_1x_2 + 2x_2^2 + 14y_1^2 + 14y_1y_2 + 28y_2^2 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 2 \end{pmatrix}_{\boldsymbol{Z}} \perp \begin{pmatrix} 14 & 7 \\ 7 & 28 \end{pmatrix}_{\boldsymbol{Z}}.$$

Note that $\langle 1,7\rangle_{\boldsymbol{Z}}$ and $\langle 2,14\rangle_{\boldsymbol{Z}}$ are sublattices of $\left(\begin{smallmatrix} 1 & 1/2 \\ 1/2 & 2 \end{smallmatrix}\right)_{\boldsymbol{Z}}$.

(i) $n \equiv 0 \pmod 2$: Consider a ternary quadratic lattice $K = \langle 2\rangle_{\boldsymbol{Z}} \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix}\right)_{\boldsymbol{Z}}$ whose class number is one [**1**]. Then, $k \to K$ if $7 \nmid k$. Note that $\widetilde{L}$ has a sublattice

$$\langle 2,14\rangle_{\boldsymbol{Z}} \perp \begin{pmatrix} 14 & 7 \\ 7 & 28 \end{pmatrix}_{\boldsymbol{Z}} = \langle 2\rangle_{\boldsymbol{Z}} \perp 7\left(\langle 2\rangle_{\boldsymbol{Z}} \perp \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}_{\boldsymbol{Z}}\right) = \langle 2\rangle_{\boldsymbol{Z}} \perp K^7.$$

Suppose $n \geq 72$. Then $n \to \langle 2\rangle_{\boldsymbol{Z}} \perp K^7$ from the following identities.

$$n \equiv 1 \pmod 7 \Longrightarrow n = 14k + 8 = \begin{cases} 2\cdot 2^2 + 7\cdot 2k & \text{if } 7 \nmid k \\ 2\cdot 5^2 + 7\cdot 2(k-3) & \text{if } 7 \mid k \end{cases}$$

$$n \equiv 2 \pmod 7 \Longrightarrow n = 14k + 2 = \begin{cases} 2\cdot 1^2 + 7\cdot 2k & \text{if } 7 \nmid k \\ 2\cdot 6^2 + 7\cdot 2(k-5) & \text{if } 7 \mid k \end{cases}$$

$$n \equiv 4 \pmod 7 \Longrightarrow n = 14k + 4 = \begin{cases} 2\cdot 3^2 + 7\cdot 2(k-1) & \text{if } 7 \nmid (k-1) \\ 2\cdot 4^2 + 7\cdot 2(k-2) & \text{if } 7 \mid (k-1) \end{cases}$$

(ii) $n \equiv 1 \pmod 4$: Consider a ternary quadratic lattice $N = \langle 1\rangle_{\boldsymbol{Z}} \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix}\right)_{\boldsymbol{Z}}$. The class number of N is two [**1**] and $k \to \operatorname{gen} N$ if $7 \nmid k$. Since N_2 is isotropic over $\boldsymbol{Z}_2$,

$4k \to N$ if $7 \nmid k$ [**8**]. Note that $\widetilde{L}$ has a sublattice

$$\langle 1,7\rangle_{\boldsymbol{Z}} \perp \begin{pmatrix} 14 & 7 \\ 7 & 28 \end{pmatrix}_{\boldsymbol{Z}} = \langle 1\rangle_{\boldsymbol{Z}} \perp 7\left(\langle 1\rangle_{\boldsymbol{Z}} \perp \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}_{\boldsymbol{Z}}\right) = \langle 1\rangle_{\boldsymbol{Z}} \perp N^7.$$

Suppose $n \geq 169$. Then $n \to \langle 1\rangle_{\boldsymbol{Z}} \perp N^7$ from the following identities.

$$n \equiv 1 \pmod 7 \Longrightarrow n = 28k+1 = \begin{cases} 1^2 + 7\cdot 4k & \text{if } 7 \nmid k \\ 13^2 + 7\cdot 4(k-6) & \text{if } 7 \mid k \end{cases}$$

$$n \equiv 2 \pmod 7 \Longrightarrow n = 28k+9 = \begin{cases} 3^2 + 7\cdot 4k & \text{if } 7 \nmid k \\ 11^2 + 7\cdot 4(k-4) & \text{if } 7 \mid k \end{cases}$$

$$n \equiv 4 \pmod 7 \Longrightarrow n = 28k+25 = \begin{cases} 5^2 + 7\cdot 4k & \text{if } 7 \nmid k \\ 9^2 + 7\cdot 4(k-2) & \text{if } 7 \mid k \end{cases}$$

(iii) $n \equiv 3 \pmod 8$: Consider a ternary quadratic lattice $M = \langle 11\rangle_{\boldsymbol{Z}} \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix}\right)_{\boldsymbol{Z}}$. The class number of M is five [**1**] and $k \to \operatorname{gen} M$ if $7 \nmid k$. Since M_2 is isotropic over $\boldsymbol{Z}_2$, $4^4k \to M$ if $7 \nmid k$ [**8**]. Note that $\widetilde{L}$ has a sublattice

$$\langle 11,77\rangle_{\boldsymbol{Z}} \perp \begin{pmatrix} 14 & 7 \\ 7 & 28 \end{pmatrix}_{\boldsymbol{Z}} = \langle 11\rangle_{\boldsymbol{Z}} \perp 7\left(\langle 11\rangle_{\boldsymbol{Z}} \perp \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}_{\boldsymbol{Z}}\right) = \langle 11\rangle_{\boldsymbol{Z}} \perp M^7.$$

There are $a \in \{1,3,5\}$ and $0 \leq b \leq 4^3 - 1$ such that $n \equiv 11a^2 \pmod 7$ and $n \equiv 11(a+14b)^2 \pmod{4^4}$. Put

$$k = \frac{n - 11(a+14b)^2}{7\cdot 4^4} \qquad \text{and}$$

$$\ell = \frac{n - 11(a+14b-7\cdot 2^7)^2}{7\cdot 4^4} = k - 11(a+14b) + 11\cdot 7\cdot 2^6.$$

Then k and ℓ are positive integers if $n \geq 11(7\cdot 2^7)^2 = 8{,}830{,}976$. Note that not both k and ℓ are divisible by 7. Thus $n \to \langle 11\rangle_{\boldsymbol{Z}} \perp M^7$, since

$$n = \begin{cases} 11(a+14b)^2 + 7\cdot 4^4 k & \text{if } 7 \nmid k, \\ 11(a+14b-7\cdot 2^7)^2 + 7\cdot 4^4\ell & \text{if } 7 \mid k. \end{cases}$$

(iv) $n \equiv 7 \pmod 8$: Consider a ternary quadratic lattice $R = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix}\right)_{\boldsymbol{Z}} \perp \langle 23\rangle_{\boldsymbol{Z}}$.

Then $\widetilde{L}$ contains $\langle 23, 161\rangle_{\boldsymbol{Z}} \perp \left(\begin{smallmatrix} 14 & 7 \\ 7 & 28 \end{smallmatrix}\right)_{\boldsymbol{Z}} = \langle 23\rangle_{\boldsymbol{Z}} \perp R^7$. The class number of R is nine and $k \to \operatorname{gen} R$ if $7 \nmid k$. The genus of R consists of nine lattices [**1**]

$$R_1 = R = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}_{\boldsymbol{Z}} \perp \langle 23\rangle_{\boldsymbol{Z}}, \quad R_2 = \langle 1, 7, 23\rangle_{\boldsymbol{Z}}, \quad R_3 = \begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix}_{\boldsymbol{Z}} \perp \langle 7\rangle_{\boldsymbol{Z}},$$

$$R_4 = \begin{pmatrix} 2 & 1 \\ 1 & 12 \end{pmatrix}_{\boldsymbol{Z}} \perp \langle 7\rangle_{\boldsymbol{Z}}, \quad R_5 = \begin{pmatrix} 3 & 1 \\ 1 & 8 \end{pmatrix}_{\boldsymbol{Z}} \perp \langle 7\rangle_{\boldsymbol{Z}}, \quad R_6 = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 15 \end{pmatrix}_{\boldsymbol{Z}},$$

$$R_7 = \langle 1\rangle_{\boldsymbol{Z}} \perp \begin{pmatrix} 11 & 2 \\ 2 & 15 \end{pmatrix}_{\boldsymbol{Z}}, \quad R_8 = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 11 \end{pmatrix}_{\boldsymbol{Z}}, \quad R_9 = \langle 1\rangle_{\boldsymbol{Z}} \perp \begin{pmatrix} 14 & 7 \\ 7 & 15 \end{pmatrix}_{\boldsymbol{Z}}.$$

Let f_{R_i} be the quadratic form corresponding to R_i. Then

$$f_R(x, y, z) = 2x^2 + 2xy + 4y^2 + 23z^2.$$

From the following identities $4^7 k \to R$ if $7 \nmid k$:

$$\begin{aligned}
&f_R(4y, -x-y, 2z) = 4 f_{R_2}(x, y, z), \\
&f_R(8z, -4x-y-2z, 2y) = 4^2 f_{R_3}(x, y, z), \\
&f_R(16y, -2x-8y-4z, 4x) = 4^3 f_{R_4}(x, y, z), \\
&f_R(16y, 5x-4y-6z, 2x+4z) = 4^3 f_{R_5}(x, y, z), \\
&f_R(16y+32z, 10x+7y-20z, 4x-2y+8z) = 4^4 f_{R_6}(x, y, z), \\
&f_R(16x-16y+64z, 7x+33y-40z, -2x+18y+16z) = 4^5 f_{R_7}(x, y, z), \\
&f_R(48x-64y-32z, -47x-28y+66z, 18x+8y+36z) = 4^6 f_{R_8}(x, y, z), \\
&f_R(48x-240y-304z, -47x+123y-65z, 18x+70y+14z) = 4^7 f_{R_9}(x, y, z).
\end{aligned}$$

There are $a \in \{1, 3, 5\}$ and $0 \le b \le 4^6 - 1 = 4095$ such that $n \equiv 23a^2 \pmod 7$ and $n \equiv 23(a + 14b)^2 \pmod{4^7}$. Put

$$\begin{aligned}
k &= \frac{n - 23(a+14b)^2}{7 \cdot 4^7} \quad \text{and} \\
\ell &= \frac{n - 23(a + 14b - 7 \cdot 4^7)^2}{7 \cdot 4^7} = k - 23 \cdot 2(a+14b) + 23 \cdot 7 \cdot 4^7.
\end{aligned}$$

Then k and ℓ are positive integers if $n \geq 23(7 \cdot 2^{14})^2 = 302,526,758,912$. Note that not both k and ℓ are divisible by 7. Thus $n \to \langle 23 \rangle_{\boldsymbol{Z}} \perp R^7$, since

$$n = \begin{cases} 23(a+14b)^2 + 7 \cdot 4^7 k & \text{if } 7 \nmid k \\ 23(a+14b-7\cdot 4^7)^2 + 7 \cdot 4^7 \ell & \text{if } 7 \mid k. \end{cases}$$

We check that $n \to \widetilde{L}$ for all $n \leq 302,526,758,912$ such that $n \equiv 1,2,4 \pmod 7$ by direct computation. Therefore L is regular. □

Proof for $L = \left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right) \perp 3\left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right)$ over $\boldsymbol{Q}(\sqrt{-15})$. Note that

$$H(\operatorname{gen} L) = \{n \in \boldsymbol{N}_0 \mid n \equiv 0, 2 \pmod 3\}.$$

Since $3 \to \left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right)$, $3\left(\langle 1 \rangle \perp \left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right)\right)$ is a sublattice of L. Since the lattice $\langle 1 \rangle \perp \left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right)$ is universal [**9**], L represents all positive integers $n \equiv 0 \pmod 3$. Suppose $n \equiv 2 \pmod 3$. If $n = n_1 + n_2$ such that $n_1 \to \langle 1 \rangle$ and $n_2 \to \left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right)$, then we have $n_1 \equiv 0, 1 \pmod 3$ and $n_2 \equiv 0, 2 \pmod 3$. Since $n \equiv 2 \pmod 3$, $n_1 \equiv 0 \pmod 3$. Then $n_1 = (x_1 + x_2\omega)\overline{(x_1 + x_2\omega)}$ has an integral solution with $x_1 \equiv x_2 \pmod 3$. Since $n_1 = 6\alpha\overline{\alpha} + 3\omega\alpha\overline{\beta} + 3\overline{\omega\alpha}\beta + 6\beta\overline{\beta}$ with $\alpha = (x_1 - y_1)/3$ and $\beta = -(x_1 + 2x_2)/3$, $n_1 \to 3\left(\begin{smallmatrix} 2 & \omega \\ \overline{\omega} & 2 \end{smallmatrix}\right)$ and L is regular. □

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