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Invariant means on bounded vector-valued functions

Dedicated to the late Respectable Professor Sen-Yen Shaw

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Abstract. Shioji and Takahashi proved that for every bounded sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers,

$$\begin{aligned}&\bigl\{\phi(\{a_n\}_{n=0}^{\infty}) \mid \phi \text{ is a Banach limit}\bigr\}\\ &\quad = \bigcap_{j=1}^{\infty} \overline{\mathrm{co}}\biggl\{(n+1)^{-1}\sum_{k=0}^{n} a_{k+m} \mid n \geq j, m \geq 0\biggr\}.\end{aligned}$$

We generalize this result to bounded sequences of vectors and also apply it to bounded measurable functions.

1. Introduction.

Let X be a Banach space over the complex field $\boldsymbol{C}$ and $f : [0, \infty) \to X$ be a locally integrable function. It is well-known that the existence of the Cesáro limit $y := \lim_{t\to\infty} t^{-1}\int_0^t f(s)ds$ implies that the Abel limit $\lim_{\lambda\downarrow 0} \lambda \int_0^{\infty} e^{-\lambda t} f(t)dt$ also exists and equals y. In general, the existence of the Abel limit does not guarantee the existence of the Cesáro limit (cf. [**4**, p. 8] and [**10**]). The discrete case has similar result, too. We ask what will happen if one of these two limits does not exsist.

We denote the dual space of X by X^*, the algebra of all bounded (linear) operators on X by $B(X)$, and $x^*(x)$ by $\langle x, x^*\rangle$ for $x \in X$ and $x^* \in X^*$. For a normed algebra $\mathbf{A}$ with the identity $\mathbf{1}$, we denote by $D(\mathbf{1}, \mathbf{A})$ the state which is the set:

$$D(\mathbf{1}, \mathbf{A}) := \bigl\{F \in \mathbf{A}^* \mid \|F\| = F(\mathbf{1}) = 1\bigr\}.$$

The (algebra) *numerical range* [**1**], [**2**] of an element $a \in \mathbf{A}$ is defined as the

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nonempty compact convex set

$$V(a) := \{\phi(a) \mid \phi \in D(\mathbf{1}, \mathbf{A})\}.$$

If L is a closed linear operator in $\mathbf{A}$ with $L\mathbf{1} = \mathbf{1}$, we define $\pi_L := \{\phi \in D(\mathbf{1}, \mathbf{A}) \mid L^*\phi = \phi\}$ [**7**] and

$$\pi_L(a) := \{\phi(a) \mid \phi \in \pi_L\} \quad \text{for } a \in \mathbf{A}.$$

An element ϕ of $D(\mathbf{1}, \mathbf{A})$ is said to be a *mean* (cf. [**6**]) and $\phi \in \pi_L$ is said to be an *invariant mean* under L^*. If $\sigma : \ell^\infty \to \ell^\infty$ is the operator $\sigma(\{a_n\}_{n=0}^\infty) := \{a_{n+1}\}_{n=0}^\infty$, then π_σ = the set of all Banach limits. Here ℓ^∞ is the space of all bounded sequences in $\boldsymbol{C}$.

In 1948, Lorentz [**13**] first studied Banach limits and defined the so-called σ-limits for bounded sequences in ℓ^∞ as following:

$$\sigma\text{-}\lim a_n := a$$

if for $\{a_n\}_{n=0}^\infty \in \ell^\infty$, $\phi(\{a_n\}_{n=0}^\infty) = a$ for all Banach limits ϕ. Lorentz also showed that $\sigma\text{-}\lim a_n := a$ if and only if $\{a_n\}_{n=0}^\infty$ is *almost convergence*, i.e.,

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} a_{k+m} = a \quad \text{uniformly on } m \geq 0.$$

For related results of almost convergence, we refer to [**3**], [**5**], [**12**], [**14**], [**15**], [**16**], [**17**], [**18**], [**19**], [**20**].

Recently, Naoki Shioji and Wataru Takahashi [**23**] proved that for every bounded sequence $\{a_n\}_{n=0}^\infty$ of real numbers and a real number α, $\phi(\{a_n\}_{n=0}^\infty) \leq \alpha$ for all Banach limits ϕ if and only if for every $\varepsilon > 0$ there is an integer $n_0 \geq 1$ such that

$$(n+1)^{-1} \sum_{k=0}^{n} a_{k+m} \leq \alpha + \varepsilon \text{ for all } n \geq n_0 \text{ and } m \geq 0.$$

In fact, their result implies that for any bounded sequence $\{a_n\}_{n=0}^\infty$ of real numbers,

$$\pi_\sigma\big(\{a_n\}_{n=0}^\infty\big) = \bigcap_{j=1}^{\infty} \overline{\mathrm{co}}\bigg\{(n+1)^{-1} \sum_{k=0}^{n} a_{k+m} \mid n \geq j, m \geq 0\bigg\}.$$

We ask what will happen if the sequence is an arbitrary bounded sequence of vectors in a Banach space X.

In Section 2, we shall give some necessary results. For example, we prove a result (Corollary 2.5) that for a mapping f from a set Ω to a Banach space X, the range of f is relatively weak compact if and only if for any $\phi \in \mathbf{A}^*$ there is a $z \in X$ such that

$$\phi(\langle f(\cdot), x^* \rangle) = \langle z, x^* \rangle \quad \text{for all } x^* \in X^*.$$

In Section 3, we show two general theorems. One of them is a result (Theorem 3.2) that under some conditions, if the range of $f \in \mathbf{A}(X)$ is relatively weak compact, then

$$\Phi_f(\pi_L) = \bigcap_\alpha \overline{\text{co}}(S_\alpha f)(\Omega) = \bigcap_\alpha \overline{\text{co}}\Big[\bigcup_{\beta \geq \alpha} (S_\beta f)(\Omega)\Big].$$

In section 4, we show a result (Theorem 4.1) that if $f \in L^\infty([0, \infty), X)$ satisfies that $f[0, \infty)$ is relatively weak compact, then

$$\begin{aligned}
&\bigcap_{t>0} \overline{\text{co}}\Big\{s^{-1}\int_0^s f(r+u)dr \mid s \geq t, u \geq 0\Big\} \\
&\quad = \bigcap_{t>0} \overline{\text{co}}\Big\{t^{-1}\int_0^t f(r+u)dr \mid u \geq 0\Big\} \\
&\quad = \bigcap_{\lambda>0} \overline{\text{co}}\Big\{\lambda\int_0^\infty e^{-\lambda t} f(t+s)dt \mid s \geq 0\Big\} \\
&\quad = \bigcap_{\lambda>0} \overline{\text{co}}\Big\{\mu\int_0^\infty e^{-\mu t} f(t+s)dt \mid 0 < \mu < \lambda, s \geq 0\Big\}.
\end{aligned}$$

In Section 5, we prove that if $\{x_n\}_{n=0}^\infty$ is a bounded sequence in a Banach space X such that the trace $\{x_n \mid n \geq 0\}$ is relatively weak compact, then

$$\begin{aligned}
&\bigcap_{n\geq 1} \overline{\text{co}}\Big\{\frac{1}{j+1}\sum_{k=0}^j x_{k+m} \mid j \geq n, m \geq 0\Big\} \\
&\quad = \bigcap_{n\geq 1} \overline{\text{co}}\Big\{\frac{1}{n+1}\sum_{k=0}^n x_{k+m} \mid m \geq 0\Big\}
\end{aligned}$$

$$= \bigcap_{r>0} \overline{\text{co}}\left\{(1-e^{-r})\sum_{k=0}^{\infty} e^{-kr} x_{k+m} \mid m \geq 0\right\}$$

$$= \bigcap_{r>0} \overline{\text{co}}\left\{(1-e^{-s})\sum_{k=0}^{\infty} e^{-ks} x_{k+m} \mid 0 < s < r, m \geq 0\right\}.$$

2. Preliminaries.

To do our work, we need the following definitions and some basic results.

DEFINITION 2.1. Let A be a closed linear operator in X. A net $\{A_\alpha\}$ of bounded operators on X is called an *A-semi-ergodic net* if it satisfies the following conditions:

(Ea) There is an $M > 0$ such that $\|A_\alpha\| \leq M$ for all α;
(Eb) $N(A) \subset N(A_\alpha - I)$ and $R(A_\alpha - I) \subset \overline{R(A)}$ for all α, where $N(A)$ is the null space of A and $R(A)$ the range of A;
(Ec) $R(A_\alpha) \subset D(A)$ for all α and $s\text{-}\lim_\alpha A_\alpha A x = 0$ for all $x \in D(A)$.

$\{A_\alpha\}$ is called an *A-ergodic net* [**7**], [**21**], [**22**] if it is an A-semi-ergodic net and satisfies

$$w\text{-}\lim_{\alpha} A A_\alpha x = 0 \quad \text{for all } x \in X.$$

The A-ergodic net $\{A_\alpha\}$ is said to be *contractive* if $M = 1$.

EXAMPLE 1. Let $S : [0,\infty) \to B(Y)$ be an integrated semigroup (cf. [**8**]) with generator A, where Y is a Banach space. Suppose $\|S(t+h) - S(t)\| \leq h$ for all $t, h \geq 0$. Thus $\|S(t)\| \leq t$ for all $t \geq 0$. Let $A_t := t^{-1}S(t)$, $t > 0$ and let the resolvent operators of $S(\cdot)$ defined by $R(\lambda)f := \lambda^2 \int_0^\infty e^{-\lambda t} S(t) f dt$ for $f \in Y$ and $\lambda > 0$. (For instance, if $Y = L^\infty([0,\infty), X)$, we can take $[S(t)f](s) := \int_0^t [T(r)f](s) dr$ for all $t, s \geq 0$ and $f \in L^\infty([0,\infty), Y)$, where $T(\cdot)$ is the translation semigroup on $L^\infty([0,\infty), Y)$.) Then we have [**8**], [**9**]

$$S(t)f - tf = A\int_0^t S(r) f dr \quad \text{for all } t \geq 0 \text{ and } f \in Y$$

$$= \int_0^t S(r) A f dr \quad \text{for all } t \geq 0 \text{ and } f \in D(A).$$

It follows from the assumption on $S(\cdot)$ that we have $\|A_t\| \leq 1$ for all $t > 0$ and $\|R(\lambda)\| \leq 1$ for all $\lambda > 0$. Therefore both $\{A_t\}_{t>0}$ and $\{R(\lambda)\}_{\lambda>0}$ satisfy

(Ea). And for every $f \in D(A)$, $\|S'(t)f\| \le \|f\|_\infty$ and $S'(t)f - f = S(t)Af$. This implies

$$\|A_t\| \le 1 \quad \text{for all } t > 0$$

and

$$\|A_t Af\| = \|t^{-1}S(t)Af\| = t^{-1}\|S'(t)f - f\| \le t^{-1}\|f\| \to 0 \quad \text{as } t \to \infty.$$

So, $\{A_t\}_{t>0}(t \to \infty)$ satisfies (Ec). Next, integrating by parts, we have that for every $f \in D(A)$ and $\lambda > 0$,

$$\begin{aligned} R(\lambda)Af &= \lambda^3 \int_0^\infty e^{-\lambda t}\left[\int_0^t S(r)Af dr\right]dt \\ &= \lambda^3 \int_0^\infty e^{-\lambda t}[S(t)f - tf]dt \\ &= \lambda R(\lambda)f - \lambda f \to 0 \text{ as } \lambda \downarrow 0. \end{aligned}$$

So, the $R(\lambda)(\lambda \downarrow 0)$ satisfies (Ec).

Finally, if $f \in N(A)$, the null space of A, then $0 = \int_0^t S(r)Af dr = S(t)f - tf$. So, we have $A_t f = f$ for all $t > 0$ and

$$R(\lambda)f = \lambda^2 \int_0^\infty e^{-\lambda t} S(t) f dt = \lambda^2 \int_0^\infty e^{-\lambda t} t f dt = f.$$

On the other hand, we have that for every $f \in Y$, $A_t f - f = t^{-1}A\int_0^t S(r)f dr \in R(A)$ and the closedness of A implies

$$\begin{aligned} R(\lambda)f - f &= \lambda^2 \int_0^\infty e^{-\lambda t}[S(t)f - tf]dt \\ &= \lambda^2 \int_0^\infty e^{-\lambda t} A\left[\int_0^t S(r)f dr\right]dt \\ &= \lambda^2 A \int_0^\infty e^{-\lambda t}\left[\int_0^t S(r)f dr\right]dt \in D(A). \end{aligned}$$

Therefore both $\{A_t\}_{t>0}(t \to \infty)$ and $\{R(\lambda)\}_{\lambda>0}(\lambda \downarrow 0)$ satisfy (Eb) and then they are all A-semi-ergodic nets on Y.

Lemma 2.2. *Let $\mathbf{A}$ be a complex unital normed algebra and let L be a closed linear operator on $\mathbf{A}$ with $L\mathbf{1} = \mathbf{1}$. Suppose that $\{A_\alpha\}$ is a contractive $(L - I)$-semi-ergodic net on $\mathbf{A}$.*

(i) *If $\phi_\alpha \in D(\mathbf{1}, \mathbf{A})$ for all α and ψ is a weakly* limiting point of $\{A_\alpha^*\phi_\alpha\}$, then $\psi \in \pi_L$.*
(ii) *If $\phi \in \pi_L$, then $A_\alpha^*\phi = \phi$ for all α.*

Proof. Since $L\mathbf{1} = \mathbf{1}$, it is immediate that (ii) follows from the second part of (Eb). We show (i). The assumption $\|A_\alpha\| \leq 1$ and Alaoglu's theorem imply that there is a weakly* convergent subnet $\{A_\beta^*\phi\}$ of $\{A_\alpha^*\phi\}$ such that $\psi = w^*$-$\lim_\beta A_\beta^*\phi$ and $\|\psi\| \leq 1$ for some $\psi \in \mathbf{A}^*$. Since $(L-I)\mathbf{1} = 0$, the first part of (Eb) implies $A_\alpha\mathbf{1} = \mathbf{1}$ for all α. Therefore, we have

$$\begin{aligned}
|\psi(\mathbf{1}) - 1| &= \lim_\beta \left|A_\beta^*\phi_\beta(\mathbf{1}) - \phi_\beta(\mathbf{1})\right| \\
&= \lim_\beta |\phi(A_\beta\mathbf{1} - \mathbf{1})| = 0
\end{aligned}$$

and hence $\psi \in D(\mathbf{1}, \mathbf{A})$. Since $\{A_\alpha\}$ is an $(L-I)$-semi-ergodic net, by (Ec) we have $\lim_\alpha A_\alpha(L-I)a = 0$ for all $a \in D(L)$. It follows that

$$\begin{aligned}
|\psi(La - a)| &= \lim_\beta \left|A_\beta^*\phi(La - a)\right| = \lim_\beta |\phi_\beta(A_\beta(La - a))| \\
&\leq \limsup_\beta \|A_\beta(La - a)\| = 0
\end{aligned}$$

for all $a \in D(L)$. This means $\psi \in \pi_L$ and then (i) holds. The proof is complete.

Lemma 2.2(i) shows that π_L can not be empty. Since $D(\mathbf{1}, \mathbf{A})$ is weakly* compact and convex, it is easy to see from the definition of π_L that π_L is also weakly* compact and convex.

Lemma 2.3. *Let $T : X \to Y$ be a bounded linear operator, where X and Y are two Banach spaces.*

(a) *If F is a nonempty subset of X, then $T(\overline{\text{co}}F) \subset \overline{T(\text{co}F)}$. If, in addition, $\overline{\text{co}}F$ is weakly compact, then $T(\overline{\text{co}}F) = \overline{T(\text{co}F)}$.*
(b) *If $\{F_\alpha\}$ is a decreasing net of nonempty weakly compact sets in X and $F := \bigcap_\alpha F_\alpha$, then $TF = \bigcap_\alpha TF_\alpha$.*
(c) *If $\{F_\alpha\}$ is a decreasing net of nonempty compact sets in X and $F = \bigcap_\alpha F_\alpha$, then*

$$\lim_{\alpha} \sup_{x \in F_\alpha} \operatorname{dist}(F, x) = 0,$$

where $\operatorname{dist}(K, x) = \inf\{\|y - x\| \mid y \in K\}$ *for* $\emptyset \neq K \subset X$*. In particular, if* $F = \{x\}$*, then* $\lim_\alpha \operatorname{dist}(F_\alpha, x) = 0$

Proof. (a) Since T is continuous, $T(\overline{\text{co}}F) \subset \overline{T(\text{co}F)}$ is immediate. Now, suppose that $\overline{\text{co}}F$ is weakly compact. Since T is a bounded linear operator, T is weakly continuous. This implies that $T(\overline{\text{co}}F)$ is weakly compact and so $T(\overline{\text{co}}F)$ is closed. Since

$$T(\text{co}F) \subset T(\overline{\text{co}}F) \subset \overline{T(\text{co}F)},$$

we must have that $T(\overline{\text{co}}F) = \overline{T(\text{co}F)}$. This proves (a).

(b) Since $\{F_\alpha\}$ is a decreasing net of weakly compact sets in X, $F \subset F_\alpha$ for all α. So, we have $TF \subset \bigcap_\alpha TF_\alpha$. Conversely, put a $y \in \bigcap_\alpha TF_\alpha$ and fix an arbitrary α_0. Then for every α there is an $x_\alpha \in F_\alpha$ such that $y = Tx_\alpha$. Since $x_\alpha \in F_{\alpha_0}$ for all $\alpha \geq \alpha_0$ and F_{α_0} is weakly compact, $\{x_\alpha\}$ has a weakly convergent subnet $\{x_\beta\}$ which is independent of the choice of α_0, say $x := w\text{-}\lim_\beta x_\beta$. Since for every $\alpha \geq \alpha_0$, F_α is weakly compact and $x_\beta \in F_{\alpha_0}$ for all $\beta \geq \alpha_0$, we must have $x \in F_{\alpha_0}$. Since the choice of α_0 is arbitrary, we must have $x \in F$. Since T is weakly continuous, this also implies

$$y = \lim_\beta Tx_\beta = Tw\text{-}\lim_\beta x_\beta = Tx \in TF.$$

Therefore $\bigcap_\alpha TF_\alpha \subset TF$ and hence the equality holds. This proves (b).

(c) Clearly, the $\sup_{x \in F_\alpha} \operatorname{dist}(F, x)$ decrease. Fix an arbitrary α_0. Suppose that there is a positive number $\varepsilon > 0$ such that

$$\lim_\alpha \sup_{x \in F_\alpha} \operatorname{dist}(F, x) > \varepsilon.$$

Then for every $\alpha \geq \alpha_0$ there is an $x_\alpha \in F_\alpha$ such that $\operatorname{dist}(F, x_\alpha) > \varepsilon$. Since F_{α_0} is compact and F_α decrease, $\{x_\alpha\}$ has a convergent subnet $\{x_\beta\}$ in F_{α_0}. Say $y = \lim_\beta x_\beta$. Thus we have $y \in F_{\alpha_0}$. Since α_0 is arbitrary, this implies

$$y \in \bigcap_\alpha F_\alpha = F.$$

Therefore

$$0 = \operatorname{dist}(F, y) = \lim_\beta \operatorname{dist}(F, x_\beta) \geq \varepsilon.$$

This is impossible and the proof is complete.

Now, we consider a unital normed algebra $\mathbf{A}$ consisting of bounded functions from a nonempty set Ω to $\boldsymbol{C}$ equipped with the sup-norm $\|\cdot\|_\infty$. Define

$$\mathbf{A}(X) := \{f : \Omega \to X \mid \langle f(\cdot), x^* \rangle \in \mathbf{A} \text{ for all } x^* \in X^*\},$$

where X^* is the dual space of X. Let $f \in \mathbf{A}(X)$. Then for any $\phi \in D(\mathbf{1}, \mathbf{A})$, there is an $x^{**} \in X^{**} = (X^*)^*$ such that

$$\phi(\langle f(\cdot), x^* \rangle) = \langle x^*, x^{**} \rangle \text{ for all } x^* \in X^*.$$

In general, such x^{**} may not be in X, where X is considered as the canonical subspace of X^{**}.

Lemma 2.4. *Let $f \in \mathbf{A}(X)$.*

(a) *If $\phi \in \mathbf{A}^*$ and the range $f(\Omega)$ of f is relatively weak compact in X, then there is a vector $z \in X$ such that*

$$\phi(\langle f(\cdot), x^* \rangle) = \langle z, x^* \rangle \text{ for all } x^* \in X^*.$$

In particular, $\|z\| \leq \|\phi\| \cdot \|f\|_\infty$, where $\|f\|_\infty := \sup_{w \in \Omega} \|f(w)\|$. Such z is unique and will be denoted by $\Phi_f(\phi)$.

(b) *Suppose that for every $\phi \in \mathbf{A}^*$, there is an $\Phi_f(\phi) \in X$ such that*

$$\phi(\langle f(\cdot), x^* \rangle) = \langle \Phi_f(\phi), x^* \rangle \text{ for all } x^* \in X^* \tag{1}$$

Then $\Phi_f(D(\mathbf{1}, \mathbf{A})) = \overline{\mathrm{co}} f(\Omega)$.

Proof. (a) Let the range $f(\Omega)$ of $f \in \mathbf{A}(X)$ be relatively weak compact in X. By Theorem 31.1 of [**2**], we have $\mathbf{A}^*$ = the linear span of $D(\mathbf{1}, \mathbf{A})$. Such mean case was shown by Kido and Takahashi [**6**] for another situation. We show the mean case by applying Kido and Takahashi's method as following. Assume ϕ is a mean on $\mathbf{A}$. Then there is some $h \in X^{**}$ such that

$$\phi(\langle f(\cdot), x^* \rangle) = \langle x^*, h \rangle \text{ for all } x^* \in X^*.$$

Since $f(\Omega)$ is relatively weak compact, $\overline{\mathrm{co}} f(\Omega)$ is a weakly compact subset of X, and so the strong and weak closed subset $\overline{\mathrm{co}} f(\Omega)$ is also a weakly compact subset of X. This subset of X can also be written as $\sigma(X^{**}, X^*)$-clco$f(\Omega)$ when it is considered as a subset of X^{**}. We show that $h \in \sigma(X^{**}, X^*)$-clco$f(\Omega)$. If it is not, then by the Hahn-Banach separation theorem and the property of a mean, there would exist an $x^* \in X^*$ such that

$$\begin{aligned}\mathrm{Re} h(x^*) &< \inf \mathrm{Re}\big\{\langle x^*, x^{**}\rangle \mid x^{**} \in \sigma(X^{**}, X^*)\text{-clco} f(\Omega)\big\} \\ &= \inf \mathrm{Re}\{\langle f(s), x^*\rangle ; s \in \Omega\} \\ &\le \mathrm{Re}\phi(\langle f(\cdot), x^*\rangle) = \mathrm{Re} h(x^*).\end{aligned}$$

This is a contradiction. Therefore $h \in \overline{\mathrm{co}} f(\Omega)$ and this proves the existence of z.

Since $z \in X$ satisfies that

$$\phi(\langle f(\cdot), x^*\rangle) = \langle z, x^*\rangle \quad \text{for all } x^* \in X^*,$$

it is clear that such z is unique by the Hahn-Banach separation theorem and

$$\begin{aligned}\|z\| &= \sup\big\{|\phi(\langle f(\cdot), x^*\rangle)| \mid x^* \in X^*, \|x^*\| \le 1\big\} \\ &= \|\phi\| \cdot \sup\big\{|\langle f(w), x^*\rangle| \mid w \in \Omega, x^* \in X^*, \|x^*\| \le 1\big\} \\ &\le \|\phi\| \cdot \|f\|_\infty.\end{aligned}$$

This proves (a).

(b) Clearly, Φ_f is linear. Since $D(\mathbf{1}, \mathbf{A})$ is weakly* compact and convex in $\mathbf{A}^*$, $\Phi_f(D(\mathbf{1}, \mathbf{A}))$ is closed and convex. If for $w \in \Omega$, $\delta_w \in \mathbf{A}^*$ is defined by

$$\delta_w(h) := h(w) \quad \text{for all } h \in \mathbf{A},$$

then $\delta_w \in D(\mathbf{1}, \mathbf{A})$ and

$$\langle \Phi_f(\delta_w), x^*\rangle = \delta_w(\langle f(\cdot), x^*\rangle) = \langle f(w), x^*\rangle$$

for all $x^* \in X^*$. Therefore $f(w) = \Phi_f(\delta_w)$ and so $\overline{\mathrm{co}} f(\Omega) \subset \Phi_f(D(\mathbf{1}, \mathbf{A}))$.

Conversely, suppose $\phi \in D(\mathbf{1}, \mathbf{A})$ and $\Phi_f(\phi) \notin \overline{\mathrm{co}} f(\Omega)$. By the Hahn-Banach separation theorem, there is an $x^* \in X^*$ such that

$$\begin{aligned}\sup_{w \in \Omega} \mathrm{Re}\langle f(w), x^*\rangle &< \mathrm{Re}\langle \Phi_f(\phi), x^*\rangle \\ &= \mathrm{Re}\phi(\langle f(\cdot), x^*\rangle) \\ &\le \sup_{w \in \Omega} \mathrm{Re}\langle f(w), x^*\rangle.\end{aligned}$$

This is a contradiction and so $\Phi_f(\phi) = \overline{\text{co}} f(\Omega)$.

From Lemma 2.4, we see that if $f \in \mathbf{A}(X)$ satisfies (1), then $\Phi_f : \mathbf{A}^* \to X$ is a bounded linear operator and $\|\Phi_f\| \leq \|f\|_\infty$.

Corollary 2.5. *Let $f \in \mathbf{A}(X)$. Then f satisfies* (1) *if and only if $f(\Omega)$ is relatively weak compact.*

Proof. "If" part. follows from Lemma 2.4(a).

We prove "Only if" part. Suppose f satisfies (1). By Lemma 2.4, it suffices to show that $\Phi_f(D(\mathbf{1}, \mathbf{A}))$ is weakly compact. Let $\{z_\alpha\}$ be an arbitrary net in $\Phi_f(D(\mathbf{1}, \mathbf{A}))$. Then we have that for every α, there is an $\phi_\alpha \in D(\mathbf{1}, \mathbf{A})$ such that

$$\phi_\alpha(\langle f(\cdot), x^*\rangle) = \langle z_\alpha, x^*\rangle \quad \text{for all } x^* \in X^*.$$

Since $D(\mathbf{1}, \mathbf{A})$ is weakly* compact, $\{\phi_\alpha\}$ has a weakly* convergent subnet $\{\phi_\beta\}$. Say, $\{\phi_\beta\}$ converges to ψ weakly*. Then we have $\psi \in D(\mathbf{1}, \mathbf{A})$ and there is a unique $z \in \Phi_f(D(\mathbf{1}, \mathbf{A}))$ by the assumption of (1) such that

$$\psi(\langle f(\cdot), x^*\rangle) = \langle z, x^*\rangle \quad \text{for all } x^* \in X^*.$$

Therefore we have for every $x^* \in X^*$,

$$\begin{aligned} \langle z, x^*\rangle &= \psi(\langle f(\cdot), x^*\rangle) \\ &= \lim_\beta \phi_\beta(\langle f(\cdot), x^*\rangle) \\ &= \lim_\beta \langle z_\beta, x^*\rangle. \end{aligned}$$

This proves that the subnet $\{z_\beta\}$ of $\{z_\alpha\}$ converges weakly to z and so $\Phi_f(D(\mathbf{1}, \mathbf{A}))$ is weakly compact.

Definition 2.6. Let T and S be two bounded linear operators in $\mathbf{A}$ and $\mathbf{A}(X)$, respectively. Then (T, S) is said to be a *corresponding pair* in $(\mathbf{A}, \mathbf{A}(X))$ if for every $f \in \mathbf{A}(X)$ and for all $x^* \in X^*$,

$$T\langle f(\cdot), x^*\rangle = \langle (Sf)(\cdot), x^*\rangle \quad \text{for all } x^* \in X^*.$$

The following lemma is immediately from Definition 2.6 and (1).

Lemma 2.7. *Let $f \in \mathbf{A}(X)$ satisfy* (1).

(a) *If S is a bounded linear operator on $\mathbf{A}(X)$, then Sf also satisfy* (1).
(b) *Let T and S be two bounded linear operators on $\mathbf{A}$ and $\mathbf{A}(X)$, respectively. If (T, S) is a corresponding pair, then*

$$\Phi_f(T^*\phi) = \Phi_{Sf}(\phi) \quad \textit{for all } \phi \in \mathbf{A}^*.$$

3. General results.

We show the following main theorems.

THEOREM 3.1. *Let $\mathbf{A}$ be a complex normed algebra with identity $\mathbf{1}$ and let L be a closed linear operator on $\mathbf{A}$ with $L\mathbf{1} = \mathbf{1}$. Suppose that $\{A_\alpha\}$ is a contractive $(L - I)$-semi-ergodic net on $\mathbf{A}$. If $a \in \mathbf{A}$, then*

$$\pi_L(a) = \bigcap_\alpha V(A_\alpha a) = \bigcap_\alpha \overline{\mathrm{co}}\Big(\bigcup_{\beta \geq \alpha} V(A_\beta a)\Big).$$

PROOF. Since $\|A_\alpha a\| \leq \|a\|$ for all α, it follows from the definition of numerical range that $V(A_\alpha a) \subset \{x \in \boldsymbol{C} \mid |z| \leq \|a\|\}$ for all α. If $\phi \in \pi_L$, then by Lemma 2.2(ii) we have that for every α,

$$\phi(a) = (A_\alpha^*\phi)(a) = \phi(A_\alpha a) \in V(A_\alpha a).$$

So, we have

$$\pi_L(a) \subset V(A_\alpha a) \text{ for all } \alpha,$$

that is, $\pi_L(a) \subset \bigcap_\alpha V(A_\alpha a)$. We show

$$\bigcap_\alpha \overline{\bigcup_{\beta \geq \alpha} V(A_\beta a)} \subset \pi_L(a).$$

Let $F := \bigcap_\alpha \overline{\bigcup_{\beta \geq \alpha} V(A_\beta a)}$. We have shown $\pi_L(a) \subset F$. Let $\varepsilon > 0$ be arbitrary. Suppose $F \setminus \overline{N(\pi_L(a);\varepsilon)} \neq \emptyset$, where $N(\pi_L(a);\varepsilon) = \{z \in \boldsymbol{C} \mid \mathrm{dist}(\pi_L(a), z) < \varepsilon\}$. Then for every α, there is a

$$\lambda_\alpha \in \bigcup_{\beta \geq \alpha} V(A_\beta a) \setminus \overline{N(\pi_L(a);\varepsilon)}.$$

Thus $\lambda_\alpha = \phi_\alpha(A_{r_\alpha} a)$ for some $\phi_\alpha \in D(\mathbf{1}, \mathbf{A})$ and for some $r_\alpha \geq \alpha$. By Alaoglu's theorem, $\{A^*_{r_\alpha}\phi_\alpha\}$ has a convergent subnet $\{A^*_{r_\beta}\phi_\beta\}$. Say, $A^*_{r_\beta}\phi_\beta \to \psi$ weakly*. Therefore $\psi \in \pi_L$ by Lemma 2.2(i) and

$$\psi(a) = \lim_\beta \big(A^*_{r_\beta}\phi_\beta\big)(a) = \lim_\beta \phi_\beta(A_{r_\beta}a) \notin N(\pi_L(a);\varepsilon).$$

This is impossible because $\psi(a) \in \pi_L(a)$. We have shown $F \subset \overline{N(\pi_L(a);\varepsilon)}$ for any $\varepsilon > 0$. Since $\pi_L(a)$ is a compact set in $\boldsymbol{C}$, this implies

$$\bigcap_\alpha \bigcup_{\beta\geq\alpha} V(A_\beta a) \supset \pi_L(a) = \bigcap_{\varepsilon>0} \overline{N(\pi_L(a);\varepsilon)} \supset F \supset \pi_L(a).$$

Therefore these sets are all equal. By Lemma 2.3(c), we have that for every $\varepsilon > 0$,

$$\bigcup_{\beta\geq\alpha} V(A_\beta a) \subset \overline{N(\pi_L(a);\varepsilon)} \;\text{ for sufficiently large } \alpha.$$

Since $\overline{N(\pi_L(a);\varepsilon)}$ is also compact and convex, this implies

$$\overline{\text{co}} \bigcup_{\beta\geq\alpha} V(A_\beta a) \subset \overline{N(\pi_L(a);\varepsilon)}$$

and hence

$$\pi_L(a) = \bigcap_\alpha \overline{\text{co}} \bigcup_{\beta\geq\alpha} V(A_\beta a).$$

The proof is complete.

Theorem 3.2. *Let A be a closed linear operator in $\mathbf{A}(X)$, let L be a closed linear operator in $\mathbf{A}$ with $L\mathbf{1} = \mathbf{1}$ and let $f \in \mathbf{A}(X)$ satisfy* (1). *Suppose that $\{(A_\alpha, S_\alpha)\}$ is a net in $B(\mathbf{A}) \times B(\mathbf{A}(X))$ satisfying the following conditions*:

(1*) *$\{A_\alpha\}$ is a contractive $(L-I)$-semi-ergodic net on $\mathbf{A}$ such that $L\mathbf{1} = \mathbf{1}$;*
(2*) *$\{S_\alpha\}$ is a contractive (A)-semi-ergodic net on $\mathbf{A}(X)$;*
(3*) *for every α, the pair (A_α, S_α) is a corresponding pair in $(\mathbf{A}, \mathbf{A}(X))$.*

Then

$$\Phi_f(\pi_L) = \bigcap_\alpha \overline{\text{co}}(S_\alpha f)(\Omega) = \bigcap_\alpha \overline{\text{co}}\Big[\bigcup_{\beta\geq\alpha} (S_\beta f)(\Omega)\Big].$$

PROOF. If $\phi \in \pi_L$, then we have that for every α, $\phi = A_\alpha^* \phi$ by (1*). By Lemma 2.4(b) and Lemma 2.7(b), we have

$$\Phi_f(\phi) = \Phi_f(A_\alpha^*\phi) = \Phi_{S_\alpha f}(\phi) \in \overline{\mathrm{co}}(S_\alpha f)(\Omega).$$

This means that $\Phi_f(\pi_L) \subset \bigcap_\alpha \overline{\mathrm{co}}(S_\alpha f)(\Omega)$. It suffices to show $\bigcap_\alpha \overline{\mathrm{co}}[\bigcup_{\beta \geq \alpha}(S_\beta f)(\Omega)] \subset \Phi_f(\pi_L)$. Since $A_\alpha^* D(\mathbf{1}, \mathbf{A}) \subset D(\mathbf{1}, \mathbf{A})$ for every α, we have

$$\begin{aligned}
\overline{\mathrm{co}}(S_\alpha f)(\Omega) &= \Phi_{S_\alpha f}(D(\mathbf{1}, \mathbf{A})) && \text{by Lemma 2.4(b)} \\
&= \Phi_f(A_\alpha^* D(\mathbf{1}, \mathbf{A})) \subset \Phi_f(D(\mathbf{1}, \mathbf{A})) && \text{by Lemma 2.7(b) and (1*)} \\
&= \overline{\mathrm{co}} f(\Omega) && \text{by Lemma 2.4(b) again.}
\end{aligned}$$

This proves that $\overline{\mathrm{co}}[\bigcup_{\beta \geq \alpha}(S_\beta f)(\Omega)]$ is weakly compact for all α. Let $x^* \in X^*$ be arbitrary. We have

$$\begin{aligned}
\langle \Phi_f(\pi_L), x^* \rangle &= \big\{ \langle \Phi_f(\phi), x^* \rangle \mid \phi \in \pi_L \big\} \\
&= \big\{ \phi(\langle f(\cdot), x^* \rangle) \mid \phi \in \pi_L \big\} \\
&= \bigcap_\alpha \overline{\mathrm{co}} \bigcup_{\beta \geq \alpha} \big\{ \phi(A_\beta \langle f(\cdot), x^* \rangle) \mid \phi \in D(\mathbf{1}, \mathbf{A}) \big\} && \text{by Theorem 3.1} \\
&= \bigcap_\alpha \overline{\mathrm{co}} \bigcup_{\beta \geq \alpha} \big\{ \phi(\langle S_\beta f(\cdot), x^* \rangle) \mid \phi \in D(\mathbf{1}, \mathbf{A}) \big\} && \text{by (3*)} \\
&= \bigcap_\alpha \overline{\mathrm{co}} \Big\langle \bigcup_{\beta \geq \alpha} \Phi_{S_\beta f}(D(\mathbf{1}, \mathbf{A})), x^* \Big\rangle \\
&= \bigcap_\alpha \overline{\mathrm{co}} \Big\langle \bigcup_{\beta \geq \alpha} (S_\beta f)(\Omega), x^* \Big\rangle && \text{by Lemma 2.4(b)} \\
&= \Big\langle \bigcap_\alpha \overline{\mathrm{co}} \bigcup_{\beta \geq \alpha} (S_\beta f)(\Omega), x^* \Big\rangle && \text{by Lemma 2.3(a) and (b).}
\end{aligned}$$

Since $\Phi_f(\pi_L)$ and $\bigcap_\alpha \overline{\mathrm{co}}(S_\alpha f)(\Omega)$ are closed and convex in X, it follows from the Hahn-Banach separation theorem that

$$\Phi_f(\pi_L) = \bigcap_\alpha \overline{\mathrm{co}} \bigcup_{\beta \geq \alpha} (S_\beta f)(\Omega).$$

This completes the proof.

4. Continuous case.

The following theorem is deduced from Theorem 3.2.

THEOREM 4.1. *Let X be a Banach space and let the range $f[0,\infty)$ of $f \in L^\infty([0,\infty),X)$ be relatively weak compact. Then*

$$\begin{aligned}
&\bigcap_{t>0} \overline{\mathrm{co}}\left\{ s^{-1}\int_0^s f(r+u)dr \mid s \geq t, u \geq 0 \right\} \\
&\quad = \bigcap_{t>0} \overline{\mathrm{co}}\left\{ t^{-1}\int_0^t f(r+u)dr \mid u \geq 0 \right\} \\
&\quad = \bigcap_{\lambda>0} \overline{\mathrm{co}}\left\{ \lambda\int_0^\infty e^{-\lambda t} f(t+s)dt \mid s \geq 0 \right\} \\
&\quad = \bigcap_{\lambda>0} \overline{\mathrm{co}}\left\{ \mu\int_0^\infty e^{-\mu t} f(t+s)dt \mid 0 < \mu < \lambda, s \geq 0 \right\}.
\end{aligned}$$

PROOF. Consider the integrated semigroup $S(\cdot)$ and its resolvent operators $R(\lambda)$ defined on $L^\infty([0,\infty),X)$ as in Example 1. By Example 1, both $\{t^{-1}S(t)\}_{t>0}(t \to \infty)$ and $\{R(\lambda)\}_{\lambda>0}(\lambda \downarrow 0)$ are all A-semi-ergodic nets on $L^\infty([0,\infty),X)$, where A is the generator of $S(\cdot)$. Whenever $X = \boldsymbol{C}$, we denote $S(\cdot)$ and $R(\cdot)$ by $S_0(\cdot)$ and $R_0(\cdot)$, respectively. Let $L - I$ be the generator of $S_0(\cdot)$. Then we have that for every $g \in L^\infty([0,\infty),X)$ and $x^* \in X^*$,

$$S_0(t)\langle g(\cdot), x^*\rangle = \langle (S(t)g)(\cdot), x^*\rangle.$$

So, $(S_0(t), S(t))$ is a corresponding pair for all $t \geq 0$. Similarly, $(R_0(\lambda), R(\lambda))$ is also a corresponding pair for all $\lambda > 0$. Now, we assume that the range $f[0,\infty)$ of $f \in L^\infty([0,\infty),X)$ is relatively weak compact. By Theorem 3.2, we have

$$\begin{aligned}
&\bigcap_{t>0} \overline{\mathrm{co}}\left\{ s^{-1}\int_0^s f(r+u)dr \mid s \geq t, u \geq 0 \right\} \\
&\quad = \bigcap_{t>0} \overline{\mathrm{co}} \bigcup_{s\geq t} (s^{-1}S(s)f)[0,\infty) \\
&\quad = \bigcap_{t>0} \overline{\mathrm{co}}[t^{-1}S(t)f][0,\infty) = \bigcap_{t>0} \overline{\mathrm{co}}\left\{ t^{-1}\int_0^t f(r+u)dr \mid u \geq 0 \right\}
\end{aligned}$$

$$= \Phi_f(\pi_L)$$

$$= \bigcap_{\lambda>0} \overline{\mathrm{co}}[R(\lambda)f][0,\infty) \equiv \bigcap_{\lambda>0} \overline{\mathrm{co}}\Big\{\lambda \int_0^\infty e^{-\lambda t} f(t+s)dt \mid s \geq 0\Big\}$$

by integrating by parts.

$$= \bigcap_{\lambda>0} \overline{\mathrm{co}} \bigcup_{0<\mu<\lambda} [R(\mu)f][0,\infty)$$

$$= \bigcap_{\lambda>0} \overline{\mathrm{co}}\Big\{\mu^2 \int_0^\infty e^{-\mu t} \int_0^t [T(r)f](s)drdt \mid 0<\mu<\lambda, s \geq 0\Big\},$$

$$= \bigcap_{\lambda>0} \overline{\mathrm{co}}\Big\{\mu \int_0^\infty e^{-\mu t} f(t+s)dt \mid 0<\mu<\lambda, s \geq 0\Big\}$$ by integrating by parts,

where $T(\cdot)$ is the translation semigroup on $L^\infty([0,\infty),X)$. This completes the proof.

The following result is an analogue of the proposition in [**21**].

Corollary 4.2. *Let X be a Banach space and let the range $f[0,\infty)$ of $f \in L^\infty([0,\infty),X)$ is relatively weak compact. If $y \in X$, then*

$\lim_{t\to\infty} t^{-1} \int_0^t f(r+s)dr = y$ *weakly uniformly on* $s \geq 0$

if and only if

$\lim_{\lambda\downarrow 0} \lambda \int_0^\infty e^{-\lambda t} f(t+s)dt = y$ *weakly uniformly on* $s \geq 0$.

From Lemma 2.3(c), if $f[0,\infty)$ is relatively compact, the convergence in Corollary 4.2 is strongly.

5. Discrete case.

Example 2. Let $\ell^\infty(X)$ be the space of all bounded sequences in X with sup-norm $\|\cdot\|_\infty$. Let $\hat\sigma$ be the bounded operator on $\ell^\infty(X)$ defined by $\hat\sigma\{x_n\}_{n=0}^\infty = \{x_{n+1}\}_{n=0}^\infty$. Define $C_m := 1/(m+1)\sum_{k=0}^m \hat\sigma^k$, $m = 1,2,\ldots$, and

$$A_r := (1-e^{-r})\sum_{k=0}^n e^{-kr}\hat\sigma^k \quad \text{for all } r > 0.$$

We show that both the C_m and the A_r are $(\hat\sigma - I)$-ergodic nets on $\ell^\infty(X)$. Since $\|\hat\sigma\| \leq 1$, we have that $\|C_m\| \leq 1$ for all $m \geq 1$ and $\|A_r\| \leq 1$ for all $r > 0$. So, both C_m and A_r satisfy (Ea).

If $\{x_n\}_{n=0}^{\infty} \in N(\hat{\sigma} - I)$, then we have that

$$C_m\{x_n\}_{n=0}^{\infty} = \{x_n\}_{n=0}^{\infty} \text{ and } A_r\{x_n\}_{n=0}^{\infty} = (1-e^{-r})\sum_{k=0}^{n} e^{-kr}\{x_n\}_{n=0}^{\infty} = \{x_n\}_{n=0}^{\infty}.$$

Since $\hat{\sigma}^k - I = (\hat{\sigma} - I)\sum_{j=0}^{k-1}\hat{\sigma}^j$ for all $k = 1, 2, \ldots$, it is easy to see that

$$R(C_m - I) \subset \overline{R(\hat{\sigma} - I)} \text{ for all } m = 1, 2, \ldots$$

and

$$R(A_r - I) \subset \overline{R(\hat{\sigma} - I)} \text{ for all } r > 0.$$

Therefore both C_m and A_r satisfy (Eb).

Finally, we have

$$C_m(\hat{\sigma} - I) = \frac{1}{m+1}(\hat{\sigma}^{m+1} - I) \to 0 \text{ as } m \to \infty$$

and

$$A_r(\hat{\sigma} - I) = (e^r - 1)A_r - e^r(1 - e^{-r}) \to 0 \text{ as } r \downarrow 0.$$

Therefore both C_m and A_r satisfy (Ec) and hence they are all $(\hat{\sigma} - I)$-ergodic nets.

The proof of the following theorem is similar to Theorem 4.1. So, we ommited it.

Theorem 5.1. *Let the trace $\{x_n; n \geq 0\}$ of $\{x_n\}_{n=0}^{\infty} \in \ell^{\infty}(X)$ is relatively weak compact. Then*

$$\bigcap_{n\geq 1} \overline{\text{co}}\left\{\frac{1}{j+1}\sum_{k=0}^{j} x_{k+m} \mid j \geq n, m \geq 0\right\}$$
$$= \bigcap_{n\geq 1} \overline{\text{co}}\left\{\frac{1}{n+1}\sum_{k=0}^{n} x_{k+m} \mid m \geq 0\right\}$$

$$= \bigcap_{r>0} \overline{\mathrm{co}} \Big\{ (1-e^{-r}) \sum_{k=0}^{\infty} e^{-kr} x_{k+m} \mid m \geq 0 \Big\}$$

$$= \bigcap_{r>0} \overline{\mathrm{co}} \Big\{ (1-e^{-s}) \sum_{k=0}^{\infty} e^{-ks} x_{k+m} \mid 0 < s < r, m \geq 0 \Big\}.$$

In Theorem 5.1, we see from Lemma 2.3(c) that the convergence is strongly whenever the trace $\{x_n \mid n \geq 0\}$ is compact. If $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence in X, we say that $\{x_n\}_{n=0}^{\infty}$ is *weakly almost convergent* to some x, written as $\sigma\text{-}\lim_{n\to\infty} x_n = x$ or $\sigma\text{-}\lim x_n = x$ (see [**8**]) if

$$\phi\big(\{\langle x_n, x^*\rangle\}_{n=0}^{\infty}\big) = \langle x, x^*\rangle \quad \text{for all } \phi \in \pi_\sigma \text{ and for all } x^* \in X^*.$$

The following corollary is an analogue of [**8**, Theorem 3.2(d)].

Corollary 5.2. *If $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence in X such that the trace $\{x_n \mid n \geq 0\}$ is relatively weak compact and $x \in X$, then $\sigma\text{-}\lim x_n = x$ if and only if for every $x^* \in X^*$,*

$$\lim_{r\downarrow 0}(1-e^{-r}) \sum_{k=0}^{\infty} e^{-kr} \langle x_{k+m}, x^*\rangle = \langle x, x^*\rangle \quad \textit{uniformly on } m \geq 0.$$

Example 3. For every noninteger real number x,

$$\lim_{r\downarrow 0}(1-e^{-r}) \sum_{k=0}^{\infty} e^{-kr} \cos(2(k+m)\pi x) = 0 \quad \text{uniformly on } m \geq 0$$

since $\sigma\text{-}\lim_{n\to\infty} \cos(2n\pi x) = 0$ for all noninteger real number x (see [**12**, Theorem 3.1]).

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