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Atomic property of the fundamental groups of the Hawaiian earring and wild locally path-connected spaces

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Abstract. We strengthen previous results on the fundamental groups of the Hawaiian earring and wild Peano continua. Let X be a path-connected, locally path-connected, first countable space which is not locally semi-simply connected at any point. If the fundamental group $\pi_1(X)$ is a subgroup of a free product $*_{j\in J}H_j$, then it is contained in a conjugate subgroup to some H_j.

1. Introduction.

Until recently the Hawaiian earring had been only a typical example of a non-locally simply connected space [**14**], [**19**, p. 59], but now the fundamental group of the Hawaiian earring has called attentions of several authors [**2**], [**10**], [**11**], [**20**]. The Hawaiian earring $\boldsymbol{H}$ is the plane compactum

$$\left\{(x,y) : \left(x+\frac{1}{n+1}\right)^2 + y^2 = \frac{1}{(n+1)^2},\ n<\omega\right\}.$$

We call the fundamental group of the Hawaiian earring as the Hawaiian earring group for short, following [**2**]. Particularly the Hawaiian earring group played a central role in [**10**], where it is shown that homomorphic images of the Hawaiian earring group in the fundamental group of a one-dimensional metric space determine points of the space.

We call a group G *quasi-atomic*, if for each homomorphism h from G to the free product $A*B$ of arbitrary groups A and B there exists a finitely generated subgroup A' of A or B' of B such that $\mathrm{Im}(h)$ is contained in $A'*B$ or $A*B'$.

By definition, finitely generated groups and abelian groups are quasi-atomic, but free products of infinitely generated groups are not quasi-atomic. Every ho-

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momorphic image of a quasi-atomic group is also quasi-atomic. In the following theorem $\mathop{\times\!\!\!\!\times}^{\sigma}_{i\in I} G_i$ is the free σ-product of groups G_i, the definition of which will be given in the next section.

THEOREM 1.1 ([**3**, Theorem 4.1]. (See also [**4**].)). *Let G_i be a finitely generated group for each $i \in I$. Then $\mathop{\times\!\!\!\!\times}^{\sigma}_{i\in I} G_i$ is quasi-atomic. Consequently the Hawaiian earring group is quasi-atomic.*

A topological counterpart to the above is given in the following theorem, where A and B denote arbitrary groups.

THEOREM 1.2 ([**12**, Theorem 1.5]). *Let X be a Peano continuum which is not semi-locally simply connected at any point. For every injective homomorphism $h : \pi_1(X, x_0) \to A * B$, there exists a finitely generated subgroup A_0 of A such that $\mathrm{Im}(h) \leq A_0 * B$ or there exists a finitely generated subgroup B_0 of B such that $\mathrm{Im}(h) \leq A * B_0$.*

In the present paper we strengthen the above results as follows:

THEOREM 1.3. *Let G_i $(i \in I)$ and H_j $(j \in J)$ be groups and $h : \mathop{\times\!\!\!\!\times}^{\sigma}_{i\in I} G_i \to *_{j\in J} H_j$ be a homomorphism from the free σ-product of groups G_i to the free product of groups H_j. Then there exist a finite subset F of I and $j \in J$ such that $h(\mathop{\times\!\!\!\!\times}^{\sigma}_{i\in I\setminus F} G_i)$ is contained in a subgroup which is conjugate to H_j.*

THEOREM 1.4. *Let X be a path-connected, locally path-connected, first countable space which is not semi-locally simply connected at any point and $h : \pi_1(X, x_0) \to *_{j\in J} H_j$ be an injective homomorphism. Then the image of h is contained in a conjugate subgroup to some H_j.*

COROLLARY 1.5. *Let X be a Peano continuum which is not semi-locally simply connected at any point. For every injective homomorphism $h : \pi_1(X, x_0) \to *_{j\in J} H_j$, the image of h is contained in a conjugate subgroup to some H_j.*

"The atomic property" in the title refers to this type of general property. It may not be adequate to single out a specific property for the name "atomic property", since many similar, but slightly different variants could be observed in various situations. Theorems 1.1 and 1.2 play important role in [**10**] and [**3**], where the information of a wild point of a space is recovered from the fundamental group by these properties. Using key lemmas in [**10**], the author has proved that the fundamental groups of the one-dimensional Peano continua determine their homotopy types [**13**]. There, the quasi-atomness of the Hawaiian earring group, i.e. Theorem 1.1, underlies the proofs.

Since Theorem 1.4 does not presume compactness, the result can be applied to asymptotic cones of word-metric spaces. The asymptotic cone of a word-metric space is a path-connected, geodesic metric space, and in particular locally path-connected. M. Bridson [**1**] constructed a finitely presented group Γ which satisfies a polynomial isoperimetric inequality such that the asymptotic cone $\mathrm{Con}_{\mathscr{U}}\,\Gamma$ is not simply connected for each non-principal ultrafilter $\mathscr{U}$, answering a question of M. Gromov. According to [**1**, p. 544] the space $\mathrm{Con}_{\mathscr{U}}\,\Gamma$ satisfies the assumptions of Theorem 1.4 and hence its fundamental group cannot be decomposed to nontrivial free products. This contrasts with the following result due to Drutu and Sapir [**5**, Corollary 7.32]: there exists a finitely generated group such that the fundamental group of whose asymptotic cone is isomorphic to an uncountable free group.

A modified form of Theorem 1.3 (see also Theorem 3.1) will be applied in our forthcoming paper [**6**]. There we construct a space from a manifold by attaching copies of the Hawaiian earring and then recover the manifold from the fundamental group of the constructed space in the way of defining spaces from abstract groups in [**3**].

2. Word theoretic arguments.

Since our argument requires results of [**8**] in detail, we review and reprove some results of [**8**] and [**4**] for the reader's convenience.

Let G_i $(i \in I)$ be groups such that $G_i \cap G_j = \{e\}$ for distinct i and j. A letter is a non-identity element of $\bigcup_{i\in I} G_i$. Two letters are *of the same kind*, if they belong to the same G_i. A *word* W is a function $W : \overline{W} \to \bigcup_{i\in I} G_i$ such that $\overline{W}$ is a linearly ordered set, $W(\alpha)$ is a letter for each $\alpha \in \overline{W}$ and $\{\alpha \in \overline{W} : W(\alpha) \in G_i\}$ is finite for each $i \in I$. A word W is called a *σ-word*, if $\overline{W}$ is countable. We denote the set of all words by $\mathscr{W}(G_i : i \in I)$ and the set of all σ-words by $\mathscr{W}^\sigma(G_i : i \in I)$.

For a word $W \in \mathscr{W}(G_i : i \in I)$ let W^- be the word defined by the following: for $\alpha \in \overline{W}$ there associates a formal symbol α^- with the order such that $\alpha^- \le \beta^-$ if and only if $\beta \le \alpha$. Let $\overline{W^-} = \{\alpha^- : \alpha \in \overline{W}\}$ and define $W^-(\alpha^-) = W(\alpha)^{-1}$ for each $\alpha \in \overline{W}$.

Two words U and V are *the same word*, denoted by $U \equiv V$, if there exists an order preserving bijection $\varphi : \overline{U} \to \overline{V}$ such that $U(\alpha) = V(\varphi(\alpha))$ for all $\alpha \in \overline{U}$. The notation $U = V$ means that the words U and V represent the same element in the inverse limit $\varprojlim(*_{i\in F} G_i, p_{FF'} : F \subseteq F' \Subset I)$. Here $F' \Subset I$ means that F' is a finite subset of I and $p_{FF'} : *_{i\in F'} G_i \to *_{i\in F} G_i$ is the projection. The *free σ-product* $\times^\sigma_{i\in I} G_i$ is the subgroup of $\varprojlim(*_{i\in F} G_i, p_{FF'} : F \subseteq F' \Subset I)$ consisting of all elements which are presented by σ-words. When the index set I is countable, we simply write $\times_{i\in I} G_i$ instead of $\times^\sigma_{i\in I} G_i$.

A word V is a *subword* of a word W, if there exist words X and Y such that $XVY \equiv W$. A word W is a *reduced* word [**8**], if

(1) for contiguous elements $\alpha, \beta \in \overline{W}$ $W(\alpha)$ and $W(\beta)$ never belong to the same group G_i;
(2) $V \neq e$ for any non-empty subword V of W.

The notion of reduced words is also defined in [**2**] and [**9**, Definition 4.3] in terms of generators. These all generalize the standard notion of reduced words of finite length [**16**], [**18**]. In particular every word has the unique reduced word [**8**, Theorem 1.4].

A word W is *quasi-reduced* if the reduced word of W is obtained by multiplying contiguous elements which are of the same kind so that the multiplication does not cause cancellation. A word W is *cyclically reduced*, if W is either empty, a single letter, or WW is reduced.

For a word W and a non-negative integer n, we define W^0 to be an empty word, $W^{n+1} \equiv W^n W$ and $W^{-(n+1)} \equiv W^{-n} W^-$.

For a word W and a letter g, let $a_W(g)$ be the cardinality of the set $\{\alpha \in \overline{W} : W(\alpha) = g\}$, that is, the number of appearances of g in W. For $a \in *_{j \in J} H_j$, $l(a)$ denotes the length of the reduced word for a. We remark that the reduced word for a is a word of finite length. Under this convention, for a word W of finite length, $l(W)$ is the length of the reduced word of W, but not the length of W itself.

Let W be words and g be a letter. The appearance of g in the word gW is called the *head* of gW. Similarly, the appearance of g in the word Wg is called the *tail* of Wg.

We say that an appearance of a letter g is *stable* in XgY, if the reduced word of XgY is of the form $X'gY'$ where X' and Y' are the reduced words of X and Y respectively. We simply say that the head of W is stable when it is stable in W, and say similarly for the tail. The head of a word W is *quasi-stable*, if the head of the reduced word of W is of the same kind as that of W. The quasi-stability of the tail of a word is defined similarly.

In the present paper the notions "heads" and "tails" are considered only for non-empty words of finite length.

The following lemmas are those for infinitary words, but will be applied for words of finite length.

LEMMA 2.1 ([**8**, Corollary 1.6]). *Let U be a non-empty reduced word such that $U \equiv U^-$. Then there exist a letter u and a word W such that $u^2 = e$ and $U \equiv W^- u W$.*

LEMMA 2.2. *Let U be a non-empty word such that $UU \equiv XU^-Y$ for some words X and Y. Then there exist U_0 and U_1 such that $U \equiv U_0 U_1$, $U_0 \equiv U_0^-$ and $U_1 \equiv U_1^-$.*

Proof. We have U_0 and U_1 such that $U_1U_0 \equiv U^-$, $U \equiv XU_1 \equiv U_0Y$ by assumption. Then $U \equiv U_0^-U_1^-$, which implies $U_0 \equiv U_0^-$ and $U_1 \equiv U_1^-$. □

Lemma 2.3 ([**8**, Lemma 2.2]). *Let $H_j (j \in J)$ be groups, and U and X be reduced words. If the head g or the tail g^{-1} in XUX^- is not stable, then $l(XUX^-) \leq l(U) + 1$.*

Proof. It suffices to deal with the case that the tail g^{-1} is not stable. Let V be the reduced word of XU. Then, $l(V) \leq l(X) + l(U)$. Since the tail g^{-1} in VX^- is not stable, $l(VX^-) \leq l(V) - l(X^-) + 1$. Therefore, $l(XUX^-) \leq l(U) + 1$. □

The following lemma holds and is stated in [**8**, Lemma 2.3] under the weaker assumption "$m+n+2 \leq k$." However the proof in [**8**] contains some inaccuracies. Here we re-prove the lemma under the assumption "$m+n+3 \leq k$." This weaker form, whose proof is less involved, is enough for the present paper and also the other papers [**8**], [**3**], [**12**] in which we used this lemma. But, in Appendix we will give a correct and full proof under the original assumption "$m + n + 2 \leq k$."

Lemma 2.4 ([**8**, Lemma 2.3]). *Let H_j $(j \in J)$ be groups and $m, n, k \in \boldsymbol{N}$ such that $m + n + 3 \leq k$. Also let $y_i, z \in *_{j\in J}H_j$ $(1 \leq i \leq M)$ be elements of the free product of H_j. If the element $u = y_1z^k \cdots y_Mz^k$ satisfies $l(u) \leq m$ and $l(y_i) \leq n$ for all $1 \leq i \leq M$, then one of the following holds*:

(1) *z is a conjugate to an element of some H_j*;
(2) *there exist $j, j' \in J$, $i \in \{2, \dots, M\}$, $f \in H_j$, $g \in H_{j'}$, $x, y \in *_{j\in J}H_j$ and a non-negative integer r such that $f^2 = g^2 = e$, $z = x^{-1}fxy^{-1}gy$, and $y_i = z^rx^{-1}fx$ or $y_i = y^{-1}gyz^r$.*

Proof. It is easy to see that for $z \neq e$ there exist reduced words U and W such that

(a) $z = W^-UW$;
(b) UU is reduced or $l(U) = 1$;
(c) W^-UW is quasi-reduced.

If $l(U) = 1$, then the proof is done. Hence, we assume $l(U) \geq 2$ and that UU is reduced. Let Y_i be the reduced word for y_i for each $1 \leq i \leq M$. Then,

$$u = Y_1W^-U^kWY_2W^- \cdots WY_MW^-U^kW.$$

If $M = 1$, then $u = y_1z^k$ and hence $2k \leq l(z^k) \leq l(u) + l(y_1) \leq m + n$, which is a contradiction. Hence $M \geq 2$. Let p be the least number so that $2p \geq n + 1$. Then

$2p \leq n+2$. Since W^-U^k is reduced and $l(Y_1) \leq n$, the reduced word of $Y_1W^-U^k$ is of the form Z_1U^{p+2} whether the tail of Y_1W^- is stable or not. Let Z_i be the reduced word of WY_iW^- for each $2 \leq i \leq M$. Now we have

$$(*) \quad u = Z_1U^{p+2}Z_2U^k \cdots U^kZ_MU^kW.$$

We are concerned with the reduced word of $U^{p+1}Z_iU^{p+1}$. Suppose that the head and the tail of $U^{p+1}Z_iU^{p+1}$ are quasi-stable for every $i \geq 2$. Since $2p+2 \leq n+4 \leq m+n+3 \leq k$ and U is cyclically reduced, by considering the rightmost U^k we conclude

$$(**) \quad 2m+n \leq 2k-(n+2)-2 \leq 2(k-(p+1)) \leq l(u) \leq m,$$

which is a contradiction. Therefore the head or the tail of $U^{p+1}Z_iU^{p+1}$ is not quasi-stable for some $i \geq 2$. We fix such an i.

Case 1: The tail of $U^{p+1}Z_iU^{p+1}$ is not quasi-stable.

The head or the tail of WY_iW^- is not stable, because otherwise $U^{p+1}Z_iU^{p+1}$ is a quasi-reduced word. Hence we have $l(Z_i) = l(WY_iW^-) \leq l(Y_i)+1 \leq n+1$ by Lemma 2.3. The reduced word of Z_iU^{p+1} is of the form $Z_i'X_i'U^{q+1}$ for some $0 \leq q \leq p$ such that $X_iX_i' \equiv U$ for some word X_i and $l(Z'_i) \leq l(Z_i)$. We examine the cancellation that occurs in the rightmost U. Then we have S, T such that $ST \equiv U$ and $S \equiv S^-$ and $T \equiv T^-$. Since UU is reduced, neither S nor T is empty. By Lemma 2.1, $S = x_0^{-1}fx_0$ and $T = y_0^{-1}gy_0$ for some $f \in H_j$ and $g \in H_{j'}$ with $f^2 = g^2 = e$ and $x_0, y_0 \in *_{j\in J}H_j$. Let $x = x_0W$ and $y = y_0W$. Then $z = x^{-1}fxy^{-1}gy$. Moreover $WY_iW^-U^{p-q} = Z_i'X_i' = U^{-l}T$ for some l. Hence

$$y_i = Y_i = W^-U^{-l}TU^{-(p-q)}W = W^-(TS)^lT(TS)^{p-q}W.$$

If $p-q > l$, then we have $y_i = z^{p-q-l-1}x^{-1}fx$ and if $p-q \leq l$, then we have $y_i = y^{-1}gyz^{l-p+q}$.

Case 2: The head of $U^{p+1}Z_iU^{p+1}$ is not quasi-stable.

Since the reduced word of $U^{p+1}Z_i$ is of the form $U^{q+1}X_iZ_i'$ for some $0 \leq q \leq p$ such that $X_iX_i' \equiv U$ for some word X_i' and $l(Z'_i) \leq l(Z_i)$. We observe the cancellation of the left most U and have the same conclusion as in Case 1. $\square$

Let A and B be groups, and let C_1 and C_2 be subsets of $A * B$ defined by: $C_1 = \{x^{-1}ux : u \in A \cup B, x \in A * B\}$ and $C_2 = \{xy : x, y \in C_1\}$.

Note C_2 is closed under conjugacy, that is, $u^{-1}xu \in C_2$ if and only if $x \in C_2$. We consider cyclically reduced words for elements in C_2. A word U is *cyclically* *equivalent* to a word V, if $U = X^{-}VX$ for some X, i.e. U represents an element conjugate to the element represented by V. Now we easily have:

LEMMA 2.5. *Every word W for an element of $C_2(\subseteq A * B)$ is cyclically equivalent to a word which has one of the following forms*:

(1) *empty*;
(2) u_0 *where* $u_0 \in A \cup B$;
(3) $V_0^- u_0 V_0 v_0$ *where* $u_0, v_0 \in A \cup B$ *and* V_0 *is a reduced word.*

We remark the following: if $W \in C_2 \setminus C_1$ for a cyclically reduced word W, then W is of the form $W_0^- w_0 W_0 W_1^- w_1 W_1$.

In the remaining part of this section A and B denote groups. We prove word theoretic lemmas that will be used in the proof of Theorem 3.1.

LEMMA 2.6. *Suppose that every element of A has order 2. Let $a_1, a_2 \in A$ $(a_1 \neq a_2)$ and $b \in B$ be non-trivial elements. Then*

$$u^{-1}a_1uv^{-1}bvu^{-1}a_2uv^{-1}bvu^{-1}(a_1a_2)uv^{-1}bv$$

does not belong to C_2 for any $u, v \in A * B$.

PROOF. Let $a_3 = a_1a_2$. Since

$$\begin{aligned} &vu^{-1}a_1uv^{-1}bvu^{-1}a_2uv^{-1}bvu^{-1}(a_1a_2)uv^{-1}bvv^{-1} \\ &\quad = w^{-1}a_1wbw^{-1}a_2wbw^{-1}a_3wb \end{aligned}$$

where $w = uv^{-1}$, we may assume $v = e$ and moreover that $U^-a_1UbU^-a_2UbU^-(a_1a_2)Ub$ is cyclically reduced for the reduced word U for u. Let $V \equiv U^-a_1UbU^-a_2UbU^-a_3Ub$. We remark $a_3 \neq a_1$ and $a_3 \neq a_2$. Then $a_V(a_1)$, $a_V(a_2)$ and $a_V(a_3)$ are odd, because each of a_1, a_2 and a_3 appearing in U also appears in U^-. Hence we have three distinct letters g for which $a_V(g)$ is odd and $g^2 = e$.

Since V is cyclically reduced, V does not belong to C_1. If $V \in C_2$, then by the remark preceding Lemma V must be of the form $W_0^- w_0 W_0 W_1^- w_1 W_1$. But then we have at most two distinct letters g for which $a_V(g)$ is odd and $g^2 = e$. Hence we conclude $V \notin C_2$. □

LEMMA 2.7. *Let $a \in A$ be an element satisfying $a^2 \neq e$ and $b \in B$ be a non-trivial element. Then $u^{-1}auv^{-1}bvu^{-1}auv^{-1}bvu^{-1}auv^{-1}bv$ does not belong to C_2 for every $u, v \in A * B$.*

PROOF. Let U be the reduced word for u. As in the proof of Lemma 2.6, we may assume that $v = e$ and $W \equiv U^- aUbU^- aUbU^- aUb$ is cyclically reduced. Since the length of W is even, W does not belong to C_1. We see $a_W(a) - a_W(a^{-1}) = 3$. If W is of the form $W_0^- w_0 W_0 W_1^- w_1 W_1$, then we have $a_W(a) - a_W(a^{-1}) \leq 2$. Hence we conclude $W \notin C_2$ again by the remark preceding Lemma 2.6. $\square$

LEMMA 2.8. *Let H be a subgroup of $A * B$ containing $\langle W^- aW \rangle * \langle V \rangle$, where $a \in A \cup B$, $W^- aW$ is a reduced word and V is a cyclically reduced word with $l(V) \geq 2$. There exists $u \in H$ such that $u \notin C_2$.*

PROOF. Since V is cyclically reduced and $l(V) \geq 2$, either $W^- aWV$ or $VW^- aW$ is reduced. Since the argument proceeds symmetrically, we assume that $VW^- aW$ is reduced. Choose k so that $k \cdot l(V) > l(W^- aW)$. Then the tail of $VW^- aW$ is stable in $VW^- aWV^{k+1}$. We claim that the head of $VW^- aW$ is also stable in $VW^- aWV^{k+1}$. To show this by contradiction, suppose that it is unstable. We consider a reduction process of the word $VW^- aWV^{k+1}$. Since $VW^- aW$ and V^{k+1} are reduced words, a reduction process of the word $VW^- aWV^{k+1}$ is a straight road. Let c be the head of V. The head c of $VW^- aW$ is affected under the reduction between $VW^- aW$ and V^{k+1}, i.e. there is a letter c' in V^{k+1} such that the multiplication cc' occurs in the reduction process. Then, we have V_0 and V_1 such that $cV_0 V_1 \equiv V \equiv V_0^- c' V_1^-$, which implies $cV_0 \equiv V_0^- c'$ and hence $c' = c^{-1}$. Now we have $V_0^- c^{-1} \equiv cV_0$, $V_1^- \equiv V_1$ and $W^- aW \equiv V_0^- c^{-1} (V^-)^l$ for some $l \geq 0$. This implies $(W^- aWV^{l+1})^2 = (V_0^- c^{-1} V)^2 = (V_1)^2 = e$, which contradicts that $\langle W^- aW \rangle$ and $\langle V \rangle$ have no relation.

Let W_0 be the reduced word of $VW^- aWV^{k+1}$. Since the head and tail of $VW^- aWV^{k+1}$ are stable, the word $VW_0 V$ is reduced. The reduction process of $VW^- aWV^{k+1}$ stops when the multiplication in A or B produces a non-identity element. By looking at this final step, we have a non-negative integer l, letters u_0, u of the same kind and words U_0, U_1, X such that $U_0 u_0 U_1 \equiv V$, $W_0 \equiv XuU_1 V^l$, and $u \neq u_0$.

Let $U \equiv W_0 V^{2k+2} W_0 V^{6k+8} W_0 V^{14k+18}$ which is the reduced word of

$$VW^- aWV^{k+1} V^{2k+2} VW^- aWV^{k+1} V^{6k+8} VW^- aWV^{k+1} V^{14k+18}.$$

To show $U \notin C_2$ by contradiction, suppose $U \in C_2$. Since U is cyclically reduced, U does not belong to C_1 and hence is of the form $X_0^- x_0 X_0 X_1^- x_1 X_1$. Remark the inequality $l(W_0) < (2k+2) l(V)$. From this we see that the indicated appearance of x_1 is located in V^{14k+18}.

Case 1: The rightmost $W_0 V$ is located in $X_1^- x_1 X_1$. Then it is located in X_1^- because $l(V^{14k+17}) > l(W_0 V^{2k+2} W_0 V^{6k+8} W_0 V)$. Since X_1 is a subword of

V^{14k+18}, we have V_0 and V_1 such that $V_0V_1 \equiv V$, $V_0^- \equiv V_0$, $V_1^- \equiv V_1$ and $W_0V \equiv XuU_1V^lV \equiv YV_0V_1V_0V^{-l}V_1$ for some word Y. This implies $XuU_1 \equiv YV_0V_1 \equiv YV \equiv YU_0u_0U_1$, which contradicts $u \neq u_0$.

Case 2: The rightmost W_0V is not located in $X_1^-x_1X_1$. In this case we examine the leftmost W_0V and the middle W_0V. They are located in X_0^-, because $l(V^{6k+7}) > l(W_0V^{2k+2}W_0V)$. Since $l(V^{2k+2}) > l(W_0)$, we see the length of the word between the leftmost W_0 and the middle W_0 is greater than the length of the rightmost W_0. Thus for the leftmost or the middle W_0V, a similar argument to Case 1 is used to deduce a contradiction. $\square$

Lemma 2.9. *Let H be a subgroup of $A * B$ such that*

(1) *H contains a non-trivial element which is conjugate to an element of A or B;*
(2) *H is not contained in any conjugate subgroup to A nor B; and*
(3) *H is not contained in any subgroup of the form $\langle u_0\rangle * \langle u_1\rangle$ with $u_0^2 = u_1^2 = e$.*

Then, H contains an element $u \notin C_2$.

Proof. By the Kurosh subgroup theorem [**17**], H is of the form $*_{i\in I}u_i^{-1}H_iu_i * *_{j\in J}v_j^{-1}\langle V_j\rangle v_j$, where H_i's are subgroups of A or B and V_j's are cyclically reduced words and $l(V_j) \geq 2$. Under the condition (1)–(3) H contains

(a) a subgroup $u^{-1}\langle a\rangle u * v^{-1}\langle b\rangle v$ for some non-trivial elements $a, b \in A \cup B$ with $a^2 \neq e$; or
(b) a subgroup $\langle w^{-1}aw\rangle * \langle v^{-1}Vv\rangle$, where $a \in A \cup B$ and V is a non-empty cyclically reduced word with $l(V) \geq 2$.

When H contains a subgroup of type (a), Lemma 2.7 implies the conclusion. When H contains a subgroup of type (b), vHv^{-1} contains a subgroup $\langle (wv^{-1})^{-1}awv^{-1}\rangle * \langle V\rangle$. Let W be the reduced word for wv^{-1}. If $W^{-1}aW$ is a reduced word, we can apply Lemma 2.8 to vHv^{-1}. Otherwise, $W \equiv a_0W_0$ with a_0 being of the same kind as a. Let $a_1 = a_0^{-1}aa_0$. Then, ${a_1}^2 = e$ and $W_0^-a_1W_0$ is a reduced word. Hence we can apply Lemma 2.8 to vHv^{-1}. In any case we have an element $u \in vHv^{-1}$ satisfying $u \notin C_2$. We have $v^{-1}uv \in H$ and $v^{-1}uv \notin C_2$. $\square$

Lemma 2.10. *Let $x \in (A * B) \setminus C_2$. Then $x^m \notin C_2$ for every integer $m \geq 4$. For given $x_1, \dots, x_n$, there exists a positive integer $m \geq 4$ such that $x_ix^m \notin C_1$ for every $1 \leq i \leq n$.*

Proof. Let V be the reduced word for x. Obviously $l(V) \geq 2$.

First we assume that V is cyclically reduced and prove the lemma. To show the first statement by contradiction, suppose that $V^m \in C_2$. Then we have letters w_0, w_1 and words W_0, W_1 such that $V^m \equiv W_0^-w_0W_0W_1^-w_1W_1$. Since $l(V) \leq$

$l(W_0)$ or $l(V) \leq l(W_1)$, V is a subword of W_0^- or W_1. By Lemma 2.2 we have V_0, V_1 such that $V_0V_1 \equiv V$, $V_0 \equiv V_0^-$ and $V_1 \equiv V_1^-$ and consequently $V \in C_2$, which is a contradiction. Next we show the second statement. Let m_0 be a natural number such that $l(x_i) < m_0 l(V)$ for every $1 \leq i \leq n$ and let $m = m_0+3$. To show $x_iV^m \notin C_1$ by contradiction, suppose that $x_iV^m \in C_1$. Then the reduced word for x_iV^m is of the form XV^{k+3} where $l(X) < l(V^{k+1})$. Examining the leftmost V, by a similar argument to the above we conclude $V \in C_2$, which is a contradiction.

In a general case we have u such that the reduced word for $u^{-1}xu$ is cyclically reduced. Since C_2 is closed under conjugacy, we have the first statement on x from the one on $u^{-1}xu$. To see the second statement, we choose m for $u^{-1}xu$ and $u^{-1}x_iu$ $(1 \leq i \leq n)$ so that, for every $1 \leq i \leq n$, $u^{-1}x_ix^mu = u^{-1}x_iu(u^{-1}xu)^m \notin C_1$. Since C_1 is closed under conjugacy, we have $x_ix^m \notin C_1$. □

The following lemma seems to be a folklore-result. We prove it for completeness.

Lemma 2.11. *Let H be a non-trivial subgroup of $\langle a\rangle * \langle b\rangle$ where $a^2 = b^2 = e$. If H is not conjugate to $\langle a\rangle$ nor $\langle b\rangle$, then H contains an element of the form $(ab)^k$ for an arbitrarily large even $k > 0$.*

Proof. Every non-empty reduced word of even length is of the form $(ab)^k$ or $(ba)^k$ for some $k > 0$ and every reduced word of odd length is of the form W^-aW or W^-bW for some word W. The reduced word of the concatenation of two words of odd length is of even length and $(ba)^k$ is the inverse of $(ab)^k$. Hence, if $(ab)^k$ does not belong to H for any even $k > 0$ and H is not trivial, then H is a conjugate to $\langle a\rangle$ or $\langle b\rangle$. In case H contains an element $(ab)^k$ for some $k > 0$, it contains an element $(ab)^k$ for arbitrary large even k. □

Lemma 2.12. *Let $u, w_0 \in *_{j\in J}H_j$ and $e \neq h \in H_{j_0}$. If $u^{-1}w_0^{-1}hw_0u \in w_0^{-1}H_{j_0}w_0$, then $u \in w_0^{-1}H_{j_0}w_0$.*

Proof. Under the assumption we obtain $(w_0uw_0^{-1})^{-1}hw_0uw_0^{-1} \in H_{j_0}$. Since H_{j_0} is a free factor of the free product, $w_0uw_0^{-1} \in H_{j_0}$, that is, $u \in w_0^{-1}H_{j_0}w_0$. □

3. Proofs of Theorems 1.3 and 1.4.

The following theorem strengthens a part of [**3**, Theorem 4.1] (see also [**4**]) and is a special case of Theorem 1.3. In what follows, $\boldsymbol{Z}_n$ denotes a copy of $\boldsymbol{Z}$ indexed by $n < \omega$ and a generator of $\boldsymbol{Z}_n$ is denoted by δ_n.

Theorem 3.1. *Let A, B be arbitrary groups and $h : \circledast_{n<\omega}\boldsymbol{Z}_n \to A * B$ be a* *homomorphism. Then there exist $m < \omega$ and $u \in A * B$ such that $h(\times_{n\geq m}\boldsymbol{Z}_n) \leq u^{-1}Au$ or $h(\times_{n\geq m}\boldsymbol{Z}_n) \leq u^{-1}Bu$.*

The first lemma is a very special case of the theorem.

LEMMA 3.2. *Theorem* 3.1 *holds, if $A = B = \boldsymbol{Z}/2\boldsymbol{Z}$.*

PROOF. Let $a \in A$ and $b \in B$ be the non-identity elements. To show the conclusion by contradiction we suppose $h(\times_{n\geq m}\boldsymbol{Z}_n)$ is not a subgroup of any conjugate of A or B for any natural number m. We construct $x_m \in \times_{n\geq m}\boldsymbol{Z}_n$ and positive integers k_m by induction. The subgroup $h(\times_{n\geq 0}\boldsymbol{Z}_n)$ contains an element of the form $h(x_0) = (ab)^{k_0}$ with $k_0 > 0$ by Lemma 2.11. For m, we choose x_m and even k_m so that $h(x_m) = (ab)^{k_m}$ and $k_m > \Sigma_{i=0}^{m-1} k_i$.

The following construction of an element of $\times_{n<\omega}\boldsymbol{Z}_n$ is a modification of that in the proofs of [**8**, Theorem 2.1 and etc.], [**7**, Theorem 1.1], [**3**, Theorem 4.1] and [**12**, Theorem 1.5]. A similar construction will appear in the proof of Theorem 3.1. We start with recalling some notions.

Let Seq be the set of all finite sequences of natural numbers and denote the length of $s \in Seq$ by $lh(s)$. An element $s \in Seq$ of the length $n = lh(s)$ is written as $\langle s_0, \ldots, s_{n-1}\rangle$ where $s_k \in \boldsymbol{N}$ $(0 \leq k < n)$. The lexicographical ordering $\prec$ on Seq is defined as follows: for $s, t \in Seq$, $s \prec t$, if $s_i < t_i$ for the minimal i with $s_i \neq t_i$ or t is an extension of s.

Let $W_m \in \mathscr{W}(\boldsymbol{Z}_n : n \geq m)$ be the reduced word for x_m, so $W_m = x_m$. Let

$$\overline{V} = \left\{(s,p) : s \in Seq, 0 \leq s_i < k_i \text{ for } 0 \leq i < lh(s), p \in \overline{W_{lh(s)}}\right\}$$

be endowed with the lexicographical ordering and define a word $V \in \mathscr{W}(\boldsymbol{Z}_n : n < \omega)$ by $V(s,p) = W_{lh(s)}(p)$. We remark $h(W_{lh(s)}) = h(x_{lh(s)}) = (ab)^{k_{lh(s)}}$. Then V is a word in $\mathscr{W}(\boldsymbol{Z}_n : n < \omega)$. Let

$$\overline{V_m} = \overline{V} \cap \left\{(s,p) : lh(s) \geq m, s_i = 0 \text{ for } 0 \leq i < m, p \in \overline{W_{lh(s)}}\right\}$$

and V_m be the restriction of V to $\overline{V_m}$. We remark $V \equiv V_0 \equiv (W_0V_1)^{k_0}$ and $V_m \equiv (W_mV_{m+1})^{k_m}$.

We consider $h(V) \in \langle a\rangle * \langle b\rangle$ and take $m > 0$ so that $m > l(h(V))$. First we assume $l(h(W_{m+1}V_{m+2}))$ is odd, then $h(W_{m+1}V_{m+2})^2 = e$ and hence $h(V_{m+1}) = h((W_{m+1}V_{m+2})^{k_{m+1}}) = e$. Therefore $h(V) = (ab)^k$, where

$$k = \Sigma_{i=0}^{m} k_i \Pi_{j=0}^{i} k_j \geq k_m \geq m > l(h(V)) = 2k,$$

which is a contradiction.

Next we assume $l(h(W_{m+1}V_{m+2}))$ is even. Then, $h(W_{m+1}V_{m+2}) = (ab)^p$ for some $p \geq 0$ or $(ba)^p$ for some $p > 0$. Since $h(W_{lh(s)}) = (ab)^{k_{lh(s)}}$, in the former case we deduce a contradiction similarly to the odd case. In the latter case, we have $h(V) = (ba)^k$, where

$$\begin{aligned} k &= pk_{m+1}\Pi_{j=0}^{m} k_j - \Sigma_{i=0}^{m} k_i \Pi_{j=0}^{i} k_j \\ &\geq \big(pk_{m+1} - \Sigma_{i=0}^{m} k_i\big)\Pi_{j=0}^{m} k_j > m > l(h(V)) = 2k, \end{aligned}$$

which is a contradiction. Now we have shown the lemma. $\square$

PROOF OF THEOREM 3.1. Let $h : \times_{n<\omega} \boldsymbol{Z}_n \to A * B$ be a homomorphism. We consider the subgroups $h(\times_{n\geq m} \boldsymbol{Z}_n)$ for $m < \omega$. By the Kurosh subgroup theorem a subgroup of $A * B$ is of the form $*_{i\in I} u_i^{-1} H_i u_i * *_{j\in J} v_j^{-1}\langle V_j\rangle v_j$, where H_i's are subgroups of A or B and V_j's are cyclically reduced and $l(V_j) \geq 2$. We remark that $v_j^{-1}\langle V_j\rangle v_j$'s are free subgroups.

If there exists $m < \omega$ such that $h(\times_{n\geq m} \boldsymbol{Z}_n)$ is contained in a free subgroup, we have $m_0 \geq m$ such that $h(\times_{n\geq m_0} \boldsymbol{Z}_n)$ is trivial by the Higman theorem [**15**] (see also [**8**, Corollary 3.7]) and we are done. So, we may assume that, for each m, $h(\times_{n\geq m} \boldsymbol{Z}_n)$ has a free factor $u^{-1}Hu$ for some non-trivial subgroup H of A or B. If, further, there exists $m_0 < \omega$, $u_0, u_1 \in A \cup B$ and $w_0, w_1 \in A * B$ such that $u_0^2 = u_1^2 = e$, $h(\times_{n\geq m} \boldsymbol{Z}_n) \leq w_0^{-1}\langle u_0\rangle w_0 * w_1^{-1}\langle u_1\rangle w_1$, then the conclusion for $w_0^{-1}\langle u_0\rangle w_0 * w_1^{-1}\langle u_1\rangle w_1$ follows from Lemma 3.2 and so does for $A * B$.

Therefore, in the following argument we assume that $h(\times_{n\geq m} \boldsymbol{Z}_n)$ has a free factor $u^{-1}Hu$ for some non-trivial subgroup H of A or B and is not a subgroup of $w_0^{-1}\langle u_0\rangle w_0 * w_1^{-1}\langle u_1\rangle w_1$ for any $m < \omega$, $u_0, u_1 \in A \cup B$ with $u_0^2 = u_1^2 = e$ and $w_0, w_1 \in A * B$.

As in the proof of Lemma 3.2 we suppose the negation of the conclusion. Then by Lemma 2.9 we have an element $x_m \in \times_{n\geq m} \boldsymbol{Z}_n$ such that $h(x_m) \notin C_2$. Then we choose natural numbers k_m by induction. Let $k_0 = 1$ and k_m be a natural number which meets the following requirements:

(1) $k_m \geq 4$ and

$$\max\big\{l(h(x_i^{k_i} \cdots x_{m-1}^{k_{m-1}})) : 0 \leq i \leq m-1\big\} + m + 2 \leq k_m;$$

(2) $h(x_i^{k_i} \cdots x_{m-1}^{k_{m-1}})h(x_m^{k_m}) \notin C_1$ for every $0 \leq i \leq m-1$.

The existence of k_m and also $h(x_m^{k_m}) \notin C_2$ are assured by Lemma 2.10. Now we modify the proof of Lemma 3.2. Let $W_m \in \mathscr{W}(\boldsymbol{Z}_n : n \geq m)$ be a reduced word for $x_m^{k_m}$. Let

$$\overline{V} = \{(s,p) : s \in Seq, 0 \leq s_i < k_i \text{ for } 0 \leq i < lh(s), p \in \overline{W_{lh(s)}}\}$$

be endowed with the lexicographical ordering and define a word $V \in \mathscr{W}(\boldsymbol{Z}_n : n \geq m)$ by $V(s,p) = W_{lh(s)}(p)$. We remark $h(W_{lh(s)}) = h(x_{lh(s)}^{k_{lh(s)}})$. A subword V_m of V is defined by a restriction as before with a slightly different domain. Let

$$\begin{aligned}\overline{V_m} = \{(s,p) : s \in Seq,\ lh(s) \geq m, s_i = 0 \text{ for } 0 \leq i < m,&\\ 0 \leq s_i < k_i \text{ for } m \leq i < lh(s), p \in \overline{W_{lh(s)}}\}& \quad (1)\end{aligned}$$

and V_m be the restriction of V to $\overline{V_m}$. We remark $V_m = x_m^{k_m} V_{m+1}^{k_{m+1}}$.

Finally choose m such that $l(h(V)) \leq m$. We apply Lemma 2.4 to $h(V)$. Here, $n = \max\{l(h(x_j^{k_j} \cdots x_{m-1}^{k_{m-1}})) : 0 \leq j \leq m-1\}$, $z = V_m$, $k = k_m$, $M = \Pi_{j=0}^{m-1} k_j$. Each y_i is $h(x_j^{k_j} \cdots x_{m-1}^{k_{m-1}})$ for some $0 \leq j < m$. To be precise, let

$$\begin{aligned} y_1 &= h(x_0^{k_0} \cdots x_{m-1}^{k_{m-1}}),\\ y_2 &= h(x_{m-1}^{k_{m-1}}), \ldots, y_{k_{m-1}} = h(x_{m-1}^{k_{m-1}}),\\ y_{k_{m-1}+1} &= h(x_{m-2}^{k_{m-2}} x_{m-1}^{k_{m-1}}), \ldots,\\ y_{\Pi_{j=2}^{m-1} k_j + 1} &= h(x_1^{k_1} \cdots x_{m-1}^{k_{m-1}}), \ldots, y_{\Pi_{j=1}^{m-1} k_j} = h(x_{m-1}^{k_{m-1}}) \end{aligned}$$

and so on.

We claim that $V_m \in C_1$. To show this, suppose that Lemma 2.4 (2) holds. Then there exist $j, j' \in J$, $i \in \{2, \ldots, M\}$, $f \in H_j$, $g \in H_{j'}$, $x, y \in *_{j \in J} H_j$ and a non-negative integer r such that $f^2 = g^2 = e$, $z = x^{-1} f x y^{-1} g y$, and $y_i = z^r x^{-1} f x$ or $y_i = y^{-1} g y z^r$. Then y_i is conjugate to g or f, which implies $y_i \in C_1$ for some $2 \leq i \leq M$. But, according to our construction $h(x_j^{k_j} \cdots x_{m-1}^{k_{m-1}})$ does not belong to C_1 for any $0 \leq j < m$. Therefore, Lemma 2.4 (1) holds, i.e. $V_m \in C_1$.

We apply the above argument also to $m+1$, then we have $V_{m+1} \in C_1$ and consequently $h(x_m^{k_m}) = h(V_m V_{m+1}^{-k_{m+1}}) \in C_2$, which contradicts our construction. □

The following is a part of [**8**, Proposition 1.9].

LEMMA 3.3. *Let g_n $(n < \omega)$ be elements of $\times_{i \in I}^{\sigma} G_i$ such that*

$$\{n < \omega : \text{ the reduced word of } g_n \text{ contains a letter of } G_i\}$$

is finite for each $i \in I$.

Then, there exists a homomorphism $\varphi : \text{✕}_{n<\omega}\boldsymbol{Z}_n \to \text{✕}^\sigma_{i\in I}G_i$ *such that* $\varphi(\delta_n) = g_n$ *for* $n < \omega$.

LEMMA 3.4 ([**8**, Theorem 2.1]). *Let* G_i $(i \in I)$ *and* H_j $(j \in J)$ *be groups and* $h : \text{✕}^\sigma_{i\in I}G_i \to *_{j\in J}H_j$ *be a homomorphism to the free product of groups* H_j*'s. Then there exist finite subsets* F *of* I *and* E *of* J *such that* $h(\text{✕}^\sigma_{i\in I\setminus F}G_i)$ *is contained in* $*_{j\in E}H_j$.

This lemma is strengthened in Theorem 1.3.

PROOF OF THEOREM 1.3. The proof is an application of Theorem 3.1. First we show Theorem 1.3 when $J = \{0, 1\}$. Let $H_0 = A$ and $H_1 = B$. To show by contradiction as before, suppose that

($*$) for any finite subsets F of I neither $h(\text{✕}^\sigma_{i\in I\setminus F}G_i)$ is contained in any conjugate of A nor B.

We claim that for each finite subset F of I there exists $x \in \text{✕}^\sigma_{i\in I\setminus F}G_i$ such that $h(x) = w^{-1}aw$ or $w^{-1}bw$ for some $a \in A$, $b \in B$ and $w \in A * B$. This follows by a similar argument to the first half of the proof of Theorem 3.1. That is, by the Kurosh subgroup Theorem $h(\text{✕}^\sigma_{i\in I\setminus F}G_i)$ is a free product of a free group and conjugates of subgroups of A or B. If $h(\text{✕}^\sigma_{i\in I\setminus F}G_i)$ is a free group, then by [**8**, Proposition 3.5] we reach a contradiction. Hence, we suppose that $h(\text{✕}^\sigma_{i\in I\setminus F}G_i)$ contains a conjugate to a non-trivial subgroup of A or B.

Now we construct $x_m \in \text{✕}^\sigma_{i\in I}G_i$, $w_m \in A * B$, $u_m \in A \cup B$, a finite subset F_m of I and an at most countable subset K_m of I by induction so that the following conditions hold:

(1) $x_m \in \text{✕}^\sigma_{i\in K_m}G_i$ where $K_m \cap F_m = \emptyset$, $I^* = \bigcup_{m<\omega} K_m = \bigcup_{m<\omega} F_m$ and $F_m \subseteq F_{m+1}$;
(2) $h(x_m) = w_m^{-1}u_m w_m$ with $u_m \neq e$;
(3) if $w_m = w_{m+1}$, then $u_m \in A$ if and only if $u_{m+1} \in B$.

Assuming that x_m, u_m and w_m are constructed, we choose a countable subset K_m of I such that $x_m \in \text{✕}^\sigma_{i\in K_m}G_i$ and enumerate K_m so that $\{p(m,n) : n < \omega\} = K_m$. Then we let $F_{m+1} = \{p(k,n) : n \leq m, k \leq m\}$. This is the standard book-keeping method. By our assumption ($*$) we can continue the construction of x_m, u_m, w_m, K_m and F_m satisfying (1)–(3).

Let δ_n be a generator of $\boldsymbol{Z}_n$. By Lemma 3.3 we have a homomorphism $\varphi : \text{✕}_{n<\omega}\boldsymbol{Z}_n \to \text{✕}^\sigma_{i\in I}G_i$ such that $h(\delta_m) = x_m$ for $m < \omega$. The above book-keeping method assures that the sequence $(x_m : m < \omega)$ satisfies the condition of Lemma 3.3. By Theorem 3.1 we have $n_0 < \omega$ such that $h \circ \varphi(\text{✕}_{n\geq n_0}\boldsymbol{Z}_n)$ is contained in a conjugate subgroup to A or B. But, it never occurs that the both

$h(x_{n_0})$ and $h(x_{n_0+1})$ belong to the same conjugate subgroup to A or B by (3), which is a contradiction. Now we have shown the case that $J = \{0, 1\}$.

For a general case let $h : \times^\sigma_{i\in I} G_i \to *_{j\in J} H_j$ be a homomorphism. By Lemma 3.4 there exist finite subsets F of I and E of J such that $h(\times^\sigma_{i\in I\setminus F} G_i)$ is contained in $*_{j\in E} H_j$. Now the restriction of h to $\times^\sigma_{i\in I\setminus F} G_i$ maps into $*_{j\in E} H_j$. Since $u^{-1}(*_{j\in E'} H_j)u = *_{j\in E'}\langle u^{-1} H_j u\rangle$ for $E' \subseteq I$, by successive use of the case of $J = \{0, 1\}$ we have the conclusion. $\square$

Next we prove Theorem 1.4. We recall some notions about loops. For a path $f : [0, 1] \to X$, f^- denotes the path defined by: $f^-(s) = f(1-s)$ for $0 \le s \le 1$. For paths $f : [0, 1] \to X$ and $g : [0, 1] \to X$ with $f(1) = g(0)$ we denote the concatenation of the paths f and g by fg.

For the Hawaiian earring $\boldsymbol{H}$ (see Introduction), let $\boldsymbol{e}_n(t) = ((\cos 2\pi t - 1)/(n+1), \sin 2\pi t/(n+1))$ for $n < \omega, 0 \le t \le 1$. Here, $\boldsymbol{e}_n$ refers to the n-th earring. Since $\pi_1(\boldsymbol{H}, (0, 0))$ is isomorphic to $\times_{n<\omega} \boldsymbol{Z}_n$ [**8**, Theorem A.1] and the loop $\boldsymbol{e}_n$ corresponds to δ_n under this isomorphism, we identify δ_n and the homotopy class of $\boldsymbol{e}_n$. For a path $p : [0, 1] \to X$, we denote the base-point-change isomorphism from $\pi_1(X, p(0))$ to $\pi_1(X, p(1))$ by φ_p.

LEMMA 3.5. *Let X be a path-connected, locally path-connected, first countable space which is not semi-locally simply connected at any point and $h : \pi_1(X, x_0) \to *_{j\in J} H_j$ be an injective homomorphism. For each point $x \in X$ and a path p from x to x_0 there exists a path-connected open neighborhood U of x satisfying: there exist $w_x \in *_{j\in J} H_j$ and $j(x) \in I$ such that for every loop l in U with base point x, $h \circ \varphi_p([l]) \in w_x^{-1} H_{j(x)} w_x$.*

PROOF. Let $\{U_n : n < \omega\}$ be a neighborhood base of x consisting of path-connected open sets such that $U_{n+1} \subseteq U_n$. To show this by contradiction, suppose that the desired neighborhood does not exist for a point x and a path p. We inductively construct loops l_n in U_n with base point x as follows.

Let l_0 be an essential loop in U_0 with base point x. Suppose that we have constructed a loop l_{n-1} in U_{n-1} with base point x. If $h \circ \varphi_p([l_{n-1}])$ is conjugate to an element of some H_j, then we have $w_{n-1} \in *_{j\in J} H_j$ such that $h \circ \varphi_p([l_{n-1}]) \in w_{n-1}^{-1} H_j w_{n-1}$. By the assumption we have an essential loop l_n in U_n with base point x such that $h \circ \varphi_p([l_{n-1}]) \notin w_{n-1}^{-1} H_j w_{n-1}$. If $h \circ \varphi_p([l_{n-1}])$ is not conjugate to any element of any H_j, we choose an arbitrary essential loop l_n in U_n with base point x.

Since the images $l_n([0, 1])$ converge to x, we can define a continuous map $f : \boldsymbol{H} \to X$ so that $f((0, 0)) = x$ and $f \circ \boldsymbol{e}_n = l_n$. By Theorem 1.3 there exists m such that $h \circ \varphi_p \circ f_*(\times_{n\ge m} \boldsymbol{Z}_n)$ is contained in a subgroup conjugate to some H_j. Then, $h \circ \varphi_p \circ f_*(\delta_m)$ belongs to the subgroup conjugate to H_j, but $h \circ \varphi_p \circ f_*(\delta_{m+1})$

does not belong to the subgroup by our construction, which is a contradiction. □

PROOF OF THEOREM 1.4. Let p be an arbitrary path from x to x_0. For each $s \in [0,1]$ let p_s be the path defined by: $p_s(t) = p((1-s)t)$. Then p_s is a path from $p(s)$ to x_0. By Lemma 3.5 we have a path-connected open neighborhood U_s of $p(s)$, $w_s \in *_{j\in J} H_j$ and $j_s \in J$ such that, for any loop l in U_s with base point $p(s)$, $h \circ \varphi_{p_s}([l]) \in w_s^{-1} H_{j_s} w_s$. Since p_1 is a constant path, for any loop l in U_1 with base point $x_0 = p(1)$, $h([l])$ is contained in $w_1^{-1} H_{j_1} w_1$.

Considering an open interval which contains s and is contained in $p^{-1}(U_s)$ for each s, we have $0 = s_n \leq t_n \leq s_{n-1} \leq \cdots \leq s_1 \leq t_1 \leq s_0 = 1$ such that $p([0,t_n]) \subseteq U_{s_n} = U_0$, $p([t_1,1]) \subseteq U_{s_0} = U_1$ and $p([t_{i+1},t_i]) \subseteq U_{s_i}$ for $1 \leq i \leq n-1$.

Define paths $q_i : [0,1] \to X$ for $1 \leq i \leq n$ and $r_i : [0,1] \to X$ for $0 \leq i \leq n-1$ by: $q_i(t) = p(s_i(1-t) + t_i t)$ and $r_i(t) = p(t_{i+1}(1-t) + s_i t)$ respectively. Then q_i is a path from $p(s_i)$ to $p(t_i)$ and r_i is a path from $p(t_{i+1})$ to $p(s_i)$, which are restrictions of p. We have an essential loop l_i in $U_{s_i} \cap U_{s_{i-1}}$ with base point $p(t_i)$ for $1 \leq i \leq n$. Now

$$\begin{aligned}
\varphi_{p_{s_i}}\big([q_i^- l_i q_i]\big) &= \big[(q_i p_{s_i})^- l_i (q_i p_{s_i})\big] \\
&= \big[p_{t_i}^- l_i p_{t_i}\big] \\
&= \varphi_{p_{s_{i-1}}}\big([r_{i-1}^- l_i r_{i-1}]\big).
\end{aligned}$$

We remark that $q_i^- l_i q_i$ is a loop in U_{s_i} with base points $p(s_i)$ and $r_{i-1}^- l_i r_{i-1}$ is a loop in $U_{s_{i-1}}$ with base point $p(s_{i-1})$.

We have $h([p_{t_1}^- l_1 p_{t_1}]) \in w_1^{-1} H_{j_1} w_1$ and also

$$h\big([p_{t_1}^- l_1 p_{t_1}]\big) = h\big(\varphi_{p_{s_1}}([q_1^- l_1 q_1])\big) \in w_{s_1}^- H_{j_{s_1}} w_{s_1}.$$

Since h is injective and l_1 is essential, $h([p_{t_1}^- l_1 p_{t_1}])$ is a non-trivial element and hence $w_{s_1} = w_1$ and $j_{s_1} = j_1$. Generally we have

$$\begin{aligned}
h\big(\varphi_{p_{s_{i-1}}}([r_{i-1}^- l_i r_{i-1}])\big) &\in w_{s_{i-1}}^- H_{j_{s_{i-1}}} w_{s_i} \\
= h\big(\varphi_{p_{s_i}}([q_i^- l_i q_i])\big) &\in w_{s_i}^- H_{j_{s_i}} w_{s_i}
\end{aligned}$$

and by the same reasoning as above $w_{s_i} = w_{s_{i-1}}$ and $j_{s_i} = j_{s_{i-1}}$. Hence $w_0 = w_{s_n} = w_1$ and $j_0 = j_1$. Since p_1 is the degenerate path on x_0, w_1 and j_1 are determined only by x_0 and so a choice of a path p does not effect to this equality.

We apply this to the case for $x = x_0$ and an arbitrary loop p with base point x_0. We have an open neighborhood U_0 of $p(0) = x = x_0$. Choose an essential loop

l in $U_0 \cap U_1$ with base point x_0. Then we have

$$e \neq h([p])^{-1}h([l])h([p]) = h([p^- lp]) = h(\varphi_p([l])) \in w_1^- H_{j_1} w_1$$

and also $h([l]) \in w_1^{-1} H_{j_1} w_1$ by the choice of U_1. Now Lemma 2.12 implies $h([p]) \in w_1^{-1} H_{j_1} w_1$. □

REMARK.

(1) As is shown in [**12**, Remark 4.6], the injectivity hypothesis of Theorem 1.4 cannot be dropped, yet can be weakened as follows. For a non-empty open set U and a path from a point $p(0)$ in U to x_0, let $H_U^p = \{[p^- lp] \mid l \text{ is a loop in } U\} \subseteq \pi_1(X, x_0)$. Then the injectivity can be replaced with: $h(H_U^p)$ is non-trivial for each non-empty open set U and for each path from a point $p(0)$ in U to x_0.

(2) The free σ-product $\times^{\sigma}_{i\in I} G_i$ is realized as the fundamental group of the one point union of spaces. To be more precise, let X_i be a space locally strongly contractible at x_i with $\pi_1(X_i, x_i) = G_i$ and identify all x_i's with one point x^*. We assume that $X_i \cap X_j = \emptyset$ for $i \neq j$. Let $\widetilde{\bigvee}_{i\in I}(X_i, x_i)$ be the space with its base set $\{x^*\} \cup \bigcup_{i\in I} X_i \setminus \{x_i\}$. The topology of each $X_i \setminus \{x_i\}$ is the same as in X_i. A neighborhood base of x^* is of the form $\{x^*\} \cup \bigcup_{i\in I} O_i \setminus \{x_i\}$ where each O_i is a neighborhood of x_i and all but finite many O_i's are the whole spaces X_i. Then $\pi_1(\widetilde{\bigvee}_{i\in I}(X_i, x_i))$ is isomorphic to $\times^{\sigma}_{i\in I} G_i$ [**8**, Theorem A.1].

Appendix.

We remark that the condition $m+n+2 \leq k$ is sufficient in Lemma 2.4 instead of $m+n+3 \leq k$. We have stated the lemma of the form of $m+n+2 \leq k$ in some published papers [**8**], [**3**] and [**12**]. Their uses are similar to the use in the proof of Theorem 3.1 and so the condition $m+n+3 \leq k$ is sufficient for their proofs. Actually the condition $m+n+100 \leq k$ is also sufficient. Since the original statement is still true, we present an additional proof which assures the statement with $m+n+2 \leq k$.

We follow the proof of Lemma 2.4. The fact that $M \geq 2$ is proved precisely the same. Then, instead of $(*)$, we have

$$u = Z_1 U^{p+1} Z_2 U^k \cdots U^k Z_M U^k W$$

such that $Z_1 U^{p+1}$ is reduced. After the 17-th line of the proof we considered $U^{p+1} Z_i U^{p+1}$. Now we consider $U^p Z_i U^{p+1}$ instead. In case the head and the tail of $U^p Z_i U^{p+1}$ is quasi-stable for every $2 \leq i \leq M$, we have the conclusion by the same reasoning as that in the proof of Lemma 2.4. We remark that the inequality

$(**)$ holds even in this case.

If there exists $2 \leq i \leq M$ such that the tail of $U^p Z_i U^{p+1}$ is not quasi-stable, then we have the conclusion as in the proof again. Hence, we consider the case that for every $2 \leq i \leq M$ the tail of $U^p Z_i U^{p+1}$ is quasi-stable but, for some $2 \leq i_0 \leq M$,

($\dagger$) the head of $U^p Z_{i_0} U^{p+1}$ is not quasi-stable.

In addition if the reduced word of $U^p Z_{i_0}$ is of the form UX, then we get the conclusion as in the case when the tail of $U^p Z_i U^{p+1}$ is not quasi-stable for some i. Otherwise, since $2p = n + 1$ or $n + 2$, we have $l(Z_{i_0}) > 2(p - 1)$ and $l(U) = 2$. Moreover, if $2p = n + 1$, then $2p + 2 \leq k$ and hence this case is reduced to the proof of Lemma 2.4. Now, to get the conclusion, we may assume $2p = n + 2$ and hence $2p - 1 = l(Z_{i_0})$. Let $U \equiv ab$. We show $a^2 = b^2 = e$, which completes our proof. Since the reduced word of $U^p Z_{i_0}$ is not of the form UX, Z_{i_0} is of the form $(b^- a^-)^{p-1} c$ and moreover we have $c = b^-$ by ($\dagger$). Then $U^p Z_{i_0} U^{p+1} = a(ab)^{p+1}$ and again by ($\dagger$) $a^2 = e$ and $U^p Z_{i_0} U^{p+1} = b(ab)^p$.

Next we consider $U^p Z_j U^{p+1}$ for $j \neq i_0$. By our assumption the tail is quasi-stable. If the head is not quasi-stable, then the reduced word of $U^p Z_j U^{p+1}$ is $b(ab)^p$ as in the case of $U^p Z_{i_0} U^{p+1}$. Since $2p = n + 2$, we have $lh(Z_j) \leq 2p - 1$ and hence the reduced word of $Z_j U^p$ is of the form Xb.

If the tail of $U^p Z_j U^p$ is not quasi-stable for some $j \neq i_0$, then we have $b^2 = e$, since $U^p \equiv (ab)^p$. Otherwise, i.e. the tail of $U^p Z_j U^p$ is quasi-stable for every $j \neq i_0$, the reduced word of $U^p Z_j U^{p+1}$ is of the form Xab. Hence the tail of the reduced word of $U^p Z_i U^{p+1}$ is b for every $2 \leq i \leq M$, while the head of it is b, a or of the same kind as a. Hence, unless $b^2 = e$, the rightmost $(ab)^{k-p-1}$ remains in the reduced word for u and $l(u) \geq l((ab)^{k-p-1}) = 2k - 2p - 2 \geq 2m + n > l(u)$, which is a contradiction. Now we conclude $a^2 = b^2 = e$.

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