

One dimensional lattice random walks with absorption at a point/on a half line

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Abstract. This paper concerns a random walk that moves on the integer lattice and has zero mean and a finite variance. We obtain first an asymptotic estimate of the transition probability of the walk absorbed at the origin, and then, using the obtained estimate, that of the walk absorbed on a half line. The latter is used to evaluate the space-time distribution for the first entrance of the walk into the half line.

Introduction.

Let $S_n^x = x + Y_1 + \cdots + Y_n$ be a random walk on the integer lattice \mathbf{Z} starting at x where the increments Y_j are independent and identically distributed random variables defined on some probability space (Ω, \mathcal{F}, P) and taking values in \mathbf{Z} . Let Y be a random variable having the same law as Y_1 . We suppose throughout the paper that the walk S_n^x is irreducible and satisfies

$$EY = 0 \quad \text{and} \quad \sigma^2 := E|Y|^2 < \infty, \quad (0.1)$$

where E indicates the expectation by P . In this paper we compute an asymptotic form as $n \rightarrow \infty$ of the probability

$$q^n(x, y) = P_x[S_n = y, S_1 \neq 0, S_2 \neq 0, \dots, S_n \neq 0], \quad (0.2)$$

the transition probability of the walk absorbed at the origin, where (and in what follows) P_x denotes the law of the walk $(S_n^x)_{n=0}^\infty$ and under P_x we simply write S_n for S_n^x . The result on q^n will be used to evaluate $q_{(-\infty, 0]}^n(x, y)$, the transition probability of the walk that is absorbed when it enters the negative half line, and the result on the latter in turn to evaluate the space-time distribution for the first entrance of S_n^x into the negative half line.

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The local central limit theorem, which gives a precise asymptotic form of the transition probabilities $p^n(y-x) := P_x[S_n = y]$, plays a fundamental role in both theory and application of random walks, whereas concerning its analogue for $q_{(-\infty,0]}^n(x,y)$ or $q^n(x,y)$, for all its significance, there seem lacking, except for simple random walk case, detailed results such as provide precise asymptotic forms of them [but see ‘Added in Proof’ at the end of the paper]. According to Donsker’s invariance principle their distribution functions $\sum_{z=-\infty}^y q_{(-\infty,0]}^n(x,z)$ under suitable scaling of space variables converge weakly to the corresponding ones of a Brownian motion which are known explicitly. This however does not mean, of course, the convergence of the probabilities $q_{(-\infty,0]}^n(x,y)$ themselves.

In this paper we observe that the asymptotic forms of both q^n and $q_{(-\infty,0]}^n$ are given by the corresponding density of the Brownian motion if space variables x, y as well as n become large in a suitable way, but obviously they fail to be if x and/or y remain in a finite set. In the latter case the order of magnitude of decay (as $n \rightarrow \infty$) does not differ but the coefficients do from the Brownian ones. These coefficients are expressed by means of either the potential function of the walk or a pair of ‘harmonic’ and ‘conjugate harmonic’ functions on the positive half line according as the absorption is made at the origin or on the negative half line.

A primary estimate of q^n is derived by using Fourier analytic method; afterwards we refine it by applying the result on the entrance distribution of $(-\infty, 0]$ mentioned above (under an additional moment condition). Our results concerning $q_{(-\infty,0]}^n$ partly but significantly rest on a profound theory of the random walk on the half line as found in Spitzer’s book [5].

The transition probability q^n may be viewed as the Green function of the space-time walk, an extremal case of two dimensional walks, absorbed on the coordinate axis of the time variable. In a separate paper (cf. [9]) we study the corresponding problem for two-dimensional random walks with zero mean and finite variances. With the help of some of the results obtained here and in [9] the asymptotic estimates of the Green functions of the walks restricted on the upper half space are computed in [10].

We illustrate how fine the estimate obtained is by applying them to a problem on a system of independent random walks. Suppose that the particles are initially placed on the positive half line of \mathbf{Z} (one on each site) and independently move according to the substochastic transition law q^n . Then how does the total number of particles on the negative half line at time n behave for large n ? We shall prove that the expected number of such particles converges to a positive constant if $E[|Y|^3; Y < 0] < \infty$ and diverges to infinity otherwise, provided that the walk is not left continuous; an analytic expression of the constant will be given.

1. Statements of results.

Let S_n^x be the random walk described in Introduction and P_x its probability law. Put $p^n(x) = P[S_n^0 = x]$, $p(x) = p^1(x)$ and define the potential function

$$a(x) = \sum_{n=0}^{\infty} [p^n(0) - p^n(-x)]; \tag{1.1}$$

the series on the right side is convergent and $a(x)/|x| \rightarrow 1/\sigma^2$ as $|x| \rightarrow \infty$ (cf. Spitzer [5, Propositions 28.8 and 29.2]). Denote by d_\circ the period of the walk (namely d_\circ is the smallest positive integer such that $p^{d_\circ n}(0) > 0$ for all sufficiently large n). Put

$$g_n(u) = \frac{e^{-u^2/2n_*}}{\sqrt{2\pi n_*}} \quad \text{where } n_* = \sigma^2 n.$$

The following notation will also be used: $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$ ($a, b \in \mathbf{R}$); for functions g and G of a variable ξ , $g(\xi) = O(G(\xi))$ means that there exists a constant C such that $|g(\xi)| \leq C|G(\xi)|$ whenever ξ ranges over a specified set; $\mathbf{1}(\mathcal{S})$ denotes the indicator of a statement \mathcal{S} , i.e., $\mathbf{1}(\mathcal{S}) = 1$ or 0 according as \mathcal{S} is true or not.

We shall denote by a_\circ an arbitrarily chosen constant that is greater than unity (as in the items (i) and (ii) below), whereas positive constants to be determined independently of variables x, y, n etc. but may depend on the law of Y are denoted by C, C_1, C_2, \dots , whose values do not have to be the same at different occurrences even though the same letter may be used.

1.1.

Let $q^n(x, y)$ denote the transition probability of the walk S_n^x that is absorbed when it hits the origin as defined by (0.2) (which entails that $q^n(x, y) = 0$ if $y = 0, x \neq 0$ and $q^0(x, y) = \mathbf{1}(x = y)$). For convenience sake we put

$$a^*(x) = \mathbf{1}(x = 0) + a(x).$$

THEOREM 1.1. *The following asymptotic estimates of $q^n(x, y)$ as $n \rightarrow \infty$, given in three cases of constraints on x and y , hold true uniformly for x and y subject to the respective constraints.*

- (i) Under $|x| \vee |y| < a_\circ \sqrt{n}$ and $|x| \wedge |y| = o(\sqrt{n})$,

$$q^n(x, y) = \frac{\sigma^4 a^*(x)a(-y) + xy}{n_*} p^n(y - x) + o\left(\frac{(|x| \vee 1)|y|}{n^{3/2}}\right). \tag{1.2}$$

(ii) Under $a_\circ^{-1}\sqrt{n} < |x|, |y| < a_\circ\sqrt{n}$ (both $|x|$ and $|y|$ are between the two extremities),

$$q^n(x, y) = d_\circ \mathbf{1}(p^n(y - x) \neq 0) [\mathbf{g}_n(y - x) - \mathbf{g}_n(y + x)] + o\left(\frac{1}{\sqrt{n}}\right) \tag{1.3}$$

if $xy > 0$,

$$q^n(x, y) = o\left(\frac{1}{\sqrt{n}}\right) \tag{1.4}$$

if $xy < 0$.

(iii) Let $0 < |x| \wedge |y| < \sqrt{n} < |x| \vee |y|$. Then, if $E|Y|^{2+\delta} < \infty$ for some $\delta \geq 0$,

$$q^n(x, y) = O\left(\frac{|x| \wedge |y|}{|x| \vee |y|} \mathbf{g}_{4n}(|x| \vee |y|)\right) + o\left(\frac{|x| \wedge |y|}{(|x| \vee |y|)^{2+\delta}}\right).$$

(\mathbf{g}_{4n} on the right side can be replaced by $\mathbf{g}_{(1+\varepsilon)n}$ with any $\varepsilon > 0$.)

As a simple consequence of (i) and (iii) of Theorem 1.1 we have the bound

$$q^n(x, y) \leq C \frac{|xy|}{n^{3/2}} \tag{1.5}$$

valid for all $x \neq 0, y$ and n . It is noted that if the walk is *left continuous*, i.e. $P[Y \leq -2] = 0$, then $\sigma^2 a(x) = x$ for $x > 0$, hence the leading term in the formula of (i) vanishes for $x > 0, y < 0$ in agreement with the trivial fact that q^n itself does.

If $E[|Y|^3] < \infty$ and $xy < 0$, then the assertion (i) can be refined in two ways: the error term in (1.2) may be replaced by $o((|x| + |y|)n^{-3/2})$ and the resulting formula is valid uniformly for $|x| \vee |y| < a_\circ\sqrt{n}$. Let C^+ be the constant given by

$$C^+ := \lim_{x \rightarrow \infty} (\sigma^2 a(x) - x) \leq \infty. \tag{1.6}$$

We shall show (Corollary 2.1 in Section 2; see also Corollary 7.1) that the limit exists and that it is finite if and only if $E[|Y|^3; Y < 0] < \infty$ and positive unless the walk is left continuous. It follows that

$$\sigma^4 a^*(x)a(-y) + xy = C^+(x - y)(1 + o(1)) \quad \text{as } x \wedge (-y) \rightarrow \infty,$$

provided $E[|Y|^3; Y < 0] < \infty$. In view of this relation and duality the refined version of (i) mentioned above may read as follows.

THEOREM 1.2. *Suppose that $E[|Y|^3; Y < 0] < \infty$. Let $y < 0 < x$. Then uniformly for $x \vee |y| \leq a_\circ \sqrt{n}$, as $x \wedge |y| \rightarrow \infty$*

$$q^n(x, y) = C^+ \frac{x + |y|}{n_*} p^n(y - x) + o\left(\frac{x \vee |y|}{n^{3/2}}\right). \tag{1.7}$$

The proof of Theorem 1.2 requires more delicate analysis than that of Theorem 1.1; it rests on Theorem 1.4 below and will be given after the proof of it.

Given a constant $\alpha \in (0, 1)$, one may consider the absorption which is not absolute but takes place with probability α each time the walk is about to visit the origin. Denote by $q_\alpha^n(x, y)$ the transition probability of the process subject to such absorption. In Section 6 we shall obtain the asymptotic estimates

$$q_\alpha^n(x, y) - q^n(x, y) = \frac{(1 - \alpha)\sigma^2}{\alpha} \cdot \frac{a^*(x) + a^*(-y)}{n} p^n(y - x)(1 + o(1)) \tag{1.8}$$

valid uniformly for $|x| \vee |y| < a_\circ \sqrt{n}$. Note that as $|x| \wedge |y| \rightarrow \infty$ under the same constraint on x, y , the right side divided by $q^n(x, y)$ tends to zero for $xy > 0$, while it is asymptotically a positive constant for $xy < 0$, provided $E[|Y|^3] < \infty$ according to Theorem 1.2.

1.2.

Here we consider the walk absorbed when it enters $(-\infty, 0]$. Let T denote the first entrance time into $(-\infty, 0]$:

$$T = \inf\{n \geq 1 : S_n \leq 0\},$$

and $q_{(-\infty, 0]}^n(x, y)$ the transition probability of the absorbed walk:

$$q_{(-\infty, 0]}^n(x, y) = P_x[S_n = y, S_1 > 0, \dots, S_n > 0] = P_x[S_n = y, n < T] \quad (x, y > 0).$$

The next result states that $q_{(-\infty, 0]}^n(x, y)$ behaves similarly to $q^n(x, y)$ within any parabolic region if both x and y get large.

PROPOSITION 1.1. *Uniformly for $n \geq (x \vee y)^2/a_\circ$, as $x \wedge y \rightarrow \infty$*

$$q_{(-\infty, 0]}^n(x, y) = q^n(x, y)(1 + o(1)).$$

Let $f_+(x)$ (resp. $f_-(x)$) ($x = 1, 2, \dots$) be the positive function on $x > 0$ that is asymptotic to x as $x \rightarrow \infty$ and harmonic with respect to the walk S_n (resp. $-S_n$) absorbed on $(-\infty, 0]$:

$$f_{\pm}(x) = E[f_{\pm}(x \pm Y); x \pm Y > 0] \quad (x \geq 1) \quad \text{and} \quad \frac{\lim_{x \rightarrow \infty} f_{\pm}(x)}{x} = 1, \quad (1.9)$$

each of which exists uniquely (Spitzer [5, Proposition 19.5]). (It is warned that it is not $[1, \infty)$ but $[0, \infty)$ on which the harmonic function is considered in [5].)

THEOREM 1.3. *Uniformly for $0 < x, y \leq a_0\sqrt{n}$, as $xy/n \rightarrow 0$*

$$q_{(-\infty, 0]}^n(x, y) = \frac{2f_+(x)f_-(y)}{n_*} p^n(y - x)(1 + o(1)).$$

From Theorem 1.3 one derives an asymptotic form of the space-time distribution of the first entrance into $(-\infty, 0]$, which we denote by $h_x(n, y)$: for $y \leq 0$

$$h_x(n, y) = P_x[S_T = y, T = n].$$

Put

$$H_{\infty}^+(y) = \frac{2}{\sigma^2} E[f_-(y - Y); Y < y] = \frac{2}{\sigma^2} \sum_{j=1}^{\infty} f_-(j)p(y - j) \quad (y \leq 0). \quad (1.10)$$

THEOREM 1.4. *Suppose $E[|Y|^{2+\delta}; Y < 0] < \infty$ for some $\delta \geq 0$ and $d_0 = 1$. Then, uniformly for $y \leq 0 < x \leq a_0\sqrt{n}$, as $n \rightarrow \infty$*

$$h_x(n, y) = \frac{f_+(x)g_n(x)}{n} H_{\infty}^+(y)(1 + o(1)) + \frac{x}{n^{3/2}} \alpha_n(x, y), \quad (1.11)$$

with

$$\alpha_n(x, y) = o((|y| \vee \sqrt{n})^{-1-\delta}),$$

$$\sum_{y \leq 0} |\alpha_n(x, y)| = o(n^{-\delta/2}) \quad \text{and} \quad \sum_{y \leq 0} |\alpha_n(x, y)||y|^{\delta} = o(1);$$

and for $x \geq \sqrt{n}$ and $y \leq 0$

$$h_x(n, y) \leq C \left[\frac{g_{4n}(x)}{\sqrt{n}} + o\left(\frac{1}{x^{2+\delta}}\right) \right] H_{\infty}^+(y) + \frac{C}{\sqrt{n}} P \left[Y < y - \frac{1}{2}x \right], \quad (1.12)$$

and in particular

$$h_x(n, y) \leq CH_\infty^+(y)x^{-1}n^{-1/2}. \tag{1.13}$$

Since $P_x[T = n] = \sum_{y \leq 0} h_x(n, y)$ we have the following corollary of Theorem 1.4.

COROLLARY 1.1. *Uniformly in $x \geq 1$*

$$P_x[T = n] = \frac{f_+(x)g_n(x)}{n} + o\left(\frac{x}{n^{3/2}} \wedge \frac{1}{x\sqrt{n}}\right).$$

REMARK.

- (a) H_∞^+ is the probability on $(-\infty, 0]$ that arises as the limit as $x \rightarrow \infty$ of the first entrance distribution $H_x^+(\cdot) = \sum_n h_x(n, \cdot)$ ([5, Proposition 19.4]). This in particular gives the identity $\sum_{j=1}^\infty f_-(j)P[Y \leq -j] = \sigma^2/2$.
- (b) If the walk is right continuous (i.e., $P[Y \geq 2] = 0$) as well as in the case when it is left continuous we have $q_{(-\infty, 0]}^n(x, y) = q^n(x, y)$ for $x, y > 0$.
- (c) The formula (1.11) holds true also in the periodic case (i.e., $d_o > 1$), if the leading term on its right side is multiplied by $d_o \mathbf{1}(p^n(y - x) \neq 0)$ as in (1.3).
- (d) The function f_- may be given by the formula

$$f_-(x) = f_-(1)(1 + E_0[\text{the number of ascending ladder points} \in [1, x - 1]])$$

and its dual formula for f_+ ([5, pp. 201–203]). Under our normalization of f_\pm the initial value $f_-(1)$ (resp. $f_+(1)$) equals the expectation of the strictly ascending (resp. descending) ladder height:

$$f_-(1) = E_0[S_{\tau([1, \infty))}] \quad \text{and} \quad f_+(1) = -E_0[S_{\tau((-\infty, -1])}], \tag{1.14}$$

where $\tau(B)$ denotes the first entrance time into a set B , in view of the renewal theorem.

- (e) If the starting point is 1, the Baxter-Spitzer identity gives

$$\sum_{n=0}^\infty r^n \sum_{y \leq 0} z^{1-y} h_1(n, y) = 1 - \exp\left(-\sum_{k=1}^\infty \frac{r^k}{k} E_0[z^{-S_k}; S_k < 0]\right) \tag{1.15}$$

(|z| ≤ 1, |r| < 1)

and a similar formula for $q_{(-\infty, 0]}^k(1, y)$ ([1, Theorem 8.4.2], [2, Lemmas 1 and

2 of Section XVIII.3], [5, Proposition 17.5]). We shall use these identities not directly but via certain fundamental results (including those on f_{\pm} and found in [5]) that are based on them. Taking $z = 1$ the above formula reduces to

$$1 - E_1[r^T] = \sqrt{1 - r} \exp \left[\sum_{k=1}^{\infty} \frac{r^k}{k} \left(\frac{1}{2} - P_0[S_k < 0] \right) \right],$$

and, applying Karamata's Tauberian theorem, one can readily find an asymptotic formula of $P_1[T \geq n]$, which is also obtained from Corollary 1.1 and (1.14). It would however be difficult to derive directly from the formula (1.15) such fine estimates of $h_1(n, y)$ as given in Theorem 1.4.

1.3.

For $x \in \mathbf{Z}$, let $Q_x^+(n)$ denote the probability that the walk starting at x is found in the negative half line at time n without having hit the origin before n :

$$Q_x^+(n) = \sum_{y=-\infty}^{-1} q^n(x, y).$$

PROPOSITION 1.2. As $x/\sqrt{n} \rightarrow 0$

$$Q_x^+(n) = \frac{\sigma^2 a^*(x) - x}{\sqrt{2\pi n_*}} + o\left(\frac{|x| + 1}{\sqrt{n}}\right); \tag{1.16}$$

and uniformly in n , as $x \rightarrow \infty$

$$Q_{-x}^+(n) = \int_{-x}^x g_n(u) du [1 + o(1)]. \tag{1.17}$$

If $E[|Y|^3; Y < 0] < \infty$, then for $x > 0$, the error term in (1.16) can be replaced by $o(1/\sqrt{n})$.

The formula (1.17) follows from (1.16) if $x/\sqrt{n} \rightarrow 0$ so that it signifies only in the case $x > a_0^{-1}\sqrt{n}$.

Let C^+ be the same constant as introduced in Subsection 1.1 (just before Theorem 1.2). C^+ is finite if and only if $E[|Y|^3; Y < 0] < \infty$ as remarked there.

THEOREM 1.5. Let $\nu_n = \sum_{x=1}^{\infty} Q_x^+(n)$. Then $\lim_{n \rightarrow \infty} \nu_n = (1/2)C^+$.

One can extend Theorem 1.5 as follows. We are concerned with the parti-

cles each of which performs random walk according to the transition law $q^n(x, y)$ independently of the other ones. Consider an experiment such that at a time n that is determined prior to the experiment the experimenter counts the number of particles lying in any interval of the negative half line for the system of our particles in which at time 0 the particles are randomly placed at each site $x > 0$ whose mean number, denoted by $m_n(x)$, may depend on n as well as x . Let $N_n(\ell)$, $\ell > 0$, denote the number of particles that are found in the interval $[-\ell\sqrt{n_*}, -1]$ at the time n . The following extension is a corollary of the proofs of Proposition 1.2 and Theorem 1.5.

COROLLARY 1.2. *Suppose $m_n(x) = 0$ for $x < 0$, $m_n(x)$ is uniformly bounded and for each $K > 0$, $m_n(x) = 1 + o(1)$ as $n \rightarrow \infty$ uniformly for $0 < x < K\sqrt{n}$. Then for each positive number ℓ ,*

$$\lim_{n \rightarrow \infty} E[N_n(\ell)] = \frac{C^+}{\sqrt{2\pi}} \int_0^\ell e^{-t^2/2} dt. \tag{1.18}$$

That $\nu_n = \sum_{x=1}^\infty Q_x^+(n)$ is bounded if and only if $E[|Y|^3; Y < 0] < \infty$ is easy to prove. Indeed

$$\nu_n = \sum_{k=1}^n \sum_{w=1}^\infty \sum_{y=-\infty}^{-1} Q_w^{*+}(k-1)p(y-w)Q_y^+(n-k),$$

where $Q_w^{*+}(k) = \sum_{x=1}^\infty q_{(-\infty, 0]}^k(x, w)$. Crude applications of Theorem 1.1 and Proposition 1.1 give

$$C_1 \mathbf{1}\left(-1 < \frac{y}{\sqrt{n}} < 0\right) \frac{|y|}{\sqrt{n}} \leq Q_y^+(n) \leq C_2 \frac{|y|}{\sqrt{n}} \quad \text{for } y < 0 \tag{1.19}$$

and similar bounds for $Q_u^{*+}(n)$, respectively, whereupon, noting $\sum (k(n-k))^{-1/2} \sim \int_0^1 (t(1-t))^{-1/2} dt$, one finds that ν_n is bounded if and only if $\sum_{w=-\infty}^{-1} \sum_{y=-\infty}^{-1} p(y+w)wy < \infty$, but the latter condition is equivalent to $E[|Y|^3; Y < 0] < \infty$.

The rest of the paper is organized as follows. In Section 2 we give some preliminary lemmas. The estimation of q^n and that of $q_{(-\infty, 0]}^n$ and $h_x(n, y)$ are carried out in Section 3 and Section 4, respectively. Further detailed estimation of $q^n(x, y)$ for $xy < 0$ that leads to the proof of Theorem 1.2 is made in the end of Section 4. In Section 5 $Q_x^+(n)$ is dealt with. In Section 6 we briefly discuss on $q_\alpha^n(x, y)$ and prove (1.8).

2. Preliminary lemmas.

This section is divided into four subsections. In the first one we give some terminologies and notation as well as some fundamental results from Spitzer's book [5] in addition to those given in Section 1. Both the second and the third ones depend in an essential way on the classical results given in the first subsection but self-contained otherwise.

2.1.

Let B be a subset of \mathbf{Z} . Denote by τ_B the first time when S_n enters B after time 0; $\tau_B = \inf\{n \geq 1 : S_n \in B\}$. For a point $x \in \mathbf{Z}$ write τ_x for $\tau_{\{x\}}$. For typographical reason we sometimes write $\tau(B)$ for τ_B .

A function $\varphi(x)$ on $\mathbf{Z} \setminus B$ that is bounded from below is said to be *harmonic* on $\mathbf{Z} \setminus B$ if $E_x[\varphi(S_1); S_1 \notin B] = \varphi(x)$ for all $x \notin B$. From this property with the help of Fatou's lemma one infers that for any Markov time τ , $E_x[\varphi(S_\tau); \tau < \tau_B] \leq \varphi(x)$ ($x \notin B$). The functions $f_+(x)$ and $a(x)$ introduced in Section 1 (see (1.9) and (1.1)) are harmonic on $[1, \infty)$ and on $\mathbf{Z} \setminus \{0\}$, respectively ([5, Theorem 29.1]). The function $f_-(x)$ is harmonic also on $[1, \infty)$ but for the *dual* walk, namely the walk determined by the probability law $p^*(x) = p(-x)$.

Let $g_{(-\infty, 0]}(x, y)$ ($x, y > 0$) denote the Green function of the walk S_n absorbed on $(-\infty, 0]$: $g_{(-\infty, 0]}(x, y) = \sum_{n=1}^{\infty} g_{(-\infty, 0]}^n(x, y) = \sum_{n=0}^{\infty} P_x[S_n = y, n < T]$, where $T = \tau_{(-\infty, 0]}$ as in Section 1. It follows from the Propositions 18.8, 19.3, 19.5 of [5] that the increments

$$u^\pm(y) := f_\pm(y) - f_\pm(y-1) \quad (y = 1, 2, \dots), \quad f_\pm(0) := 0$$

are all positive and have limits $\lim_{y \rightarrow \infty} u^\pm(y) = 1$ and with them and with $u^\pm(0) := 0$ the function $g_{(-\infty, 0]}$ is expressed as

$$g_{(-\infty, 0]}(x, y) = \frac{2}{\sigma^2} \sum_{z=0}^{x \wedge y} u^+(x-z)u^-(y-z) \quad (x, y > 0). \quad (2.1)$$

Similarly let $g_{\{0\}}(x, y)$ be the Green function of the process S_n absorbed at the origin: $g_{\{0\}}(x, y) = \sum_{n=0}^{\infty} q^n(x, y)$. Then, according to Spitzer [5, Proposition 29.4],

$$g_{\{0\}}(x, y) = a(x) + a(-y) - a(x-y) \quad (x, y \in \mathbf{Z} \setminus \{0\}). \quad (2.2)$$

The results given in the following Subsections 2.1 and 2.2 are easy consequences of (2.1) and (2.2).

2.2.

Let $H_x^+(y)$ denote (as in Remark (a)) the hitting distribution of $(-\infty, 0]$ for the walk S_n^x :

$$H_x^+(y) := P_x[S_T = y] \quad (x > 0, y \leq 0), \tag{2.3}$$

which may be expressed as

$$H_x^+(y) = \sum_{w=1}^{\infty} g_{(-\infty, 0]}(x, w)p(y - w). \tag{2.4}$$

In view of (2.1) we have $g_{(-\infty, 0]}(x, w) \leq Cf_-(w)$, hence

$$H_x^+(y) \leq CH_{\infty}^+(y). \tag{2.5}$$

LEMMA 2.1. For $x > 0$ and $y \leq 0$,

(a) $\sum_{z=-\infty}^0 H_x^+(z)a(z - y) = a(x - y) - \sigma^{-2}f_+(x).$

(b) $\sum_{z=-\infty}^0 H_{\infty}^+(z)a(z) = \lim_{x \rightarrow \infty} [a(x) - \sigma^{-2}f_+(x)].$

(Both sides of (b) may be infinite simultaneously.)

PROOF. With $y \leq 0$ fixed define $\varphi(x) = \sum_{z=-\infty}^0 H_x^+(z)a(z - y)$ for $x > 0$ and $\varphi(x) = a(x - y)$ for $x \leq 0$. Owing to (2.1) and (2.4)

$$H_x^+(z) \leq C \sum_{w=1}^{\infty} (x \wedge w)p(z - w) \leq CxP[Y < z] \quad (z < 0 < x), \tag{2.6}$$

which combined with $\sum_{z \leq 0} |z|P[Y < z] \leq \sigma^2 < \infty$ shows that $\varphi(x)$ takes a finite value; moreover, by dominated convergence, $\varphi(x)/x \rightarrow 0$ as $x \rightarrow \infty$. It is observed that $\sum_{z=-\infty}^{\infty} p(z - x)\varphi(z) = \varphi(x)$ for $x > 0$ and $\sum_{z=-\infty}^{\infty} p(z - x)a(z - y) = a(x - y)$ for $x \neq y$. Hence $a(x - y) - \varphi(x)$, vanishing on $x \leq 0$, is harmonic on $x > 0$ and asymptotic to x/σ^2 as $x \rightarrow \infty$. We may now conclude that $a(x - y) - \varphi(x)$ agrees with $\sigma^{-2}f_+(x)$ for $x > 0$ since a harmonic function on $x > 0$ that is bounded below is unique apart from a constant factor. Thus (a) has been verified. Let $y = 0$ and $x \rightarrow \infty$ in (a). If the left side of (b) is infinite, so is the right side in view of Fatou's lemma. If it is finite, the dominated convergence theorem may

apply owing to (2.5). □

The proof of Lemma 2.1 may be repeated word for word but with $a(z - y)$ replaced by z to yield

$$\sum_{z=-\infty}^{-1} H_x^+(z)|z| = f_+(x) - x \tag{2.7}$$

and

$$\sum_{z=-\infty}^{-1} H_\infty^+(z)|z| = \lim_{x \rightarrow \infty} [f_+(x) - x]. \tag{2.8}$$

Taking $y = 0$ in (a) of Lemma 2.1 and combining it with (2.7) we obtain

$$\sum_{z=-\infty}^{-1} H_x^+(z)(\sigma^2 a(z) - z) = \sigma^2 a(x) - x \quad (x > 0). \tag{2.9}$$

Here we advance a corollary of Lemma 2.1 that involves the constant C^+ introduced in Subsection 1.1. It is convenient to define it by

$$C^+ = \sum_{y=-\infty}^{-1} H_\infty^+(y)[\sigma^2 a(y) + |y|]$$

rather than by (1.6). The relation (1.6) then ensues as stated in the corollary below. According to this definition it is clear that C^+ is finite if and only if $E[|Y|^3; Y < 0] < \infty$, and positive unless the walk is left continuous. Define $H_{-\infty}^-(y)$ and C^- analogously to H_∞^+ and C^+ :

$$H_{-\infty}^-(y) = \frac{2}{\sigma^2} E[f_+(Y - y); Y > y] \quad (y \geq 0) \quad \text{and} \quad C^- = \sum_{y=1}^{\infty} H_{-\infty}^-(y)(\sigma^2 a(y) + y).$$

It holds that $C^- < \infty$ if and only if $E[|Y|^3; Y > 0] < \infty$.

COROLLARY 2.1. $C^+ = \lim_{x \rightarrow +\infty} (\sigma^2 a(x) - x)$ and $C^- = \lim_{x \rightarrow -\infty} (\sigma^2 a(x) - |x|)$.

PROOF. From (2.8) and (b) of Lemma 2.1 one deduces the first relation of the corollary. The second one is its dual. □

2.3.

The results in this subsection are somewhat different in nature from and independent of those of the preceding one (except for the use of (2.5)) although machinery for the proof is essentially the same. Recall that T is written for $\tau_{(-\infty,0]}$. We shall show that $P_x[\tau_{[N,\infty)} < \tau_0]$ and $P_x[\tau_N < \tau_0]$ are asymptotically equivalent (see Proposition 2.1). For the moment we obtain the following

LEMMA 2.2. *Uniformly in $0 < x < N$, as $N \rightarrow \infty$*

$$\frac{a(x)}{a(N)} \geq P_x[\tau_{[N,\infty)} < \tau_0] \geq P_x[\tau_N < \tau_0] = \frac{\sigma^2 a(x) + x}{2N} (1 + o(1)).$$

PROOF. We have $P_x[\tau_N < \tau_0] = g_{\{0\}}(x, N)/g_{\{0\}}(N, N)$ and the last relation of the lemma follows from (2.2) together with

$$g_{\{0\}}(N, N) = a(N) + a(-N) = \frac{2N}{\sigma^2} (1 + o(1)),$$

$$\lim_{y \rightarrow \infty} [a(-y) - a(1 - y)] = \frac{1}{\sigma^2}$$

([5, Proposition 29.2]). Since $a(x)$ is positive and harmonic on $\mathbf{Z} \setminus \{0\}$ (i.e. $E[a(Y + x)] = a(x)$, $x \neq 0$) and non-decreasing for $x > 0$ large enough,

$$a(x) \geq E_x[a(S_{\tau_{([N,\infty))}}); \tau_{[N,\infty)} < \tau_0] \geq a(N)P_x[\tau_{[N,\infty)} < \tau_0].$$

Thus $P_x[\tau_{[N,\infty)} < \tau_0] \leq a(x)/a(N)$ provided N is large enough, verifying the first inequality of the lemma. The second one is trivial. \square

LEMMA 2.3. *As $N \rightarrow \infty$*

$$\sup_{z > N} P_z[\tau_N > T] \asymp \left[N^{-1} \sum_{y=1}^{N-1} y H_\infty^+(-y) + \sum_{y=N}^\infty H_\infty^+(-y) \right] \rightarrow 0.$$

PROOF. We use the decomposition

$$P_z[\tau_N > T] = \sum_{w < N} P_z[S_{\tau_{((-\infty, N])}} = w] (P_w[T < \tau_N] \mathbf{1}(w > 0) + \mathbf{1}(w \leq 0)).$$

Writing the first probability under the summation sign by means of $H_x^+(y)$ (defined in (2.3)) and using the bound $H_x^+(y) \leq C H_\infty^+(y)$ (see (2.5)) together with Lemma

2.2 we obtain

$$P_z[\tau_N > T] \leq \frac{C}{a(-N)} \sum_{-N < y < 0} a(y)H_\infty^+(y) + \sum_{y \leq -N} H_\infty^+(y),$$

the right side approaching zero. The lower bound is obtained by an application of Fatou's lemma. □

PROPOSITION 2.1. *Uniformly in $0 < x < N$, as $N \rightarrow \infty$*

(a) $P_x[\tau_{[N,\infty)} < T] - P_x[\tau_N < T] = o\left(\frac{x}{N}\right);$

(b) $P_x[\tau_{[N,\infty)} < \tau_0] - P_x[\tau_N < \tau_0] = o\left(\frac{x}{N}\right).$

PROOF. The difference on the left side of (a) is expressed as

$$\sum_{z > N} P_x[\tau_{[N,\infty)} < T, S_{\tau([N,\infty))} = z] P_z[\tau_N > T], \tag{2.10}$$

and hence (a) follows from the preceding two lemmas (and the inequality $T \leq \tau_0$).

The proof of (b) is similar: one has only to replace T by τ_0 in (2.10). □

The next Proposition refines Theorem 22.1 of [5] where the problem is treated by a quite different method from the present one.

PROPOSITION 2.2. *Uniformly for $1 \leq x < N$, as $N \rightarrow \infty$*

$$P_x[\tau_{[N,\infty)} < T] = \frac{f_+(x)}{N} + o\left(\frac{x}{N}\right).$$

PROOF. By Proposition 2.1

$$P_x[\tau_{[N,\infty)} < T] = P_x[\tau_N < T] + o\left(\frac{x}{N}\right) = \frac{g_{(-\infty,0]}(x, N)}{g_{(-\infty,0]}(N, N)} + o\left(\frac{x}{N}\right).$$

It is readily inferred that as $N \rightarrow \infty$

$$g_{(-\infty,0]}(x, N) = f_+(x) \left(\frac{2}{\sigma^2} + o(1) \right) \quad \text{uniformly for } 1 \leq x \leq N,$$

in particular, $g_{(-\infty,0]}(N, N) = 2\sigma^{-2}N + o(N)$ and substitution leads to the desired relation. \square

PROPOSITION 2.3. *Uniformly for $1 \leq x < N$, as $N \rightarrow \infty$*

$$\frac{1}{x} E_x [S_{\tau_{([N,\infty))}}; \tau_{[N,\infty)} < T] = \frac{N}{x} P_x [\tau_{[N,\infty)} < T] + o(1),$$

or, what is the same thing, $E_x [S_{\tau_{([N,\infty))}} - N \mid \tau_{[N,\infty)} < T] = o(N)$.

PROOF. That f_+ is non-negative and harmonic on $[1, \infty)$ implies that for $x > 0$,

$$E_x [f_+(S_{\tau_{([N,\infty))}}); \tau_{[N,\infty)} < T] \leq f_+(x).$$

Hence, employing Proposition 2.2, one first observes that uniformly for $1 \leq x < N$

$$\begin{aligned} 0 &\leq E_x [f_+(S_{\tau_{([N,\infty))}}) - f_+(N); \tau_{[N,\infty)} < T] \\ &\leq f_+(x) - f_+(N) P_x [\tau_{[N,\infty)} < T] \\ &= f_+(x) \left(1 - \frac{f_+(N)}{N} \right) + o(x) \end{aligned}$$

and then use $\lim_{x \rightarrow \infty} f_+(x)/x = 1$ to find the formula of the proposition. \square

LEMMA 2.4. *Uniformly for $0 < x < N$, as $N \rightarrow \infty$*

$$P_x [\tau_{[N,\infty)} < \tau_0] = \frac{\sigma^2 a(x) + x}{2N} (1 + o(1)); \tag{2.11}$$

and

$$\begin{aligned} \frac{\sigma^2 [a(x) + a(N) - a(x + N)]}{2N} (1 + o(1)) &\leq P_x [\tau_{(-\infty, -N]} < \tau_0] \\ &\leq \frac{\sigma^2 a(x) - x}{2N} (1 + o(1)). \end{aligned}$$

PROOF. The first relation (2.11) follows from (b) of Proposition 2.1 and the last equality in Lemma 2.2. The lower bound of the second relation is obtained in the same way as the second inequality in Lemma 2.2. For the upper bound we apply (2.11) (or rather its dual) and (2.9) in turn to see

$$\begin{aligned}
 P_x[\tau_{(-\infty, -N]} < \tau_0] &= \sum_{y=-N+1}^{-1} H_x^+(y) P_y[\tau_{(-\infty, -N]} < \tau_0] + \sum_{y \leq -N} H_x^+(y) \\
 &\leq \sum_{y=-\infty}^{-1} H_x^+(y) \frac{\sigma^2 a(y) - y}{2N} (1 + o(1)) = \frac{\sigma^2 a(x) - x}{2N} (1 + o(1)).
 \end{aligned}$$

The proof of the lemma is complete. □

PROPOSITION 2.4. *Uniformly for $0 < |x| < N$, as $N \rightarrow \infty$*

$$P_x[\tau_{\mathbf{Z} \setminus (-N, N)} < \tau_0] = \frac{\sigma^2 a(x)}{N} (1 + o(1)).$$

PROOF. Use Lemma 2.4 first to infer that for $0 < |x| < N$,

$$\begin{aligned}
 &P_x[\tau_{(-\infty, -N]} \vee \tau_{[N, \infty)} < \tau_0] \\
 &\leq C \frac{|x|}{N} \sup_{z > N} (P_z[\tau_{(-\infty, -N]} < \tau_0] + P_{-z}[\tau_{[N, \infty)} < \tau_0]) = o\left(\frac{x}{N}\right);
 \end{aligned}$$

and then, by employing the inclusion-exclusion formula, to obtain the relation of the lemma. □

2.4.

In the following two lemmas we suppose that the walk S_n is aperiodic (i.e., $d_o = 1$).

LEMMA 2.5. *Let $d_o = 1$. Then uniformly for $x, y \in \mathbf{Z}$, as $n \rightarrow \infty$*

$$\begin{aligned}
 &p^n(y - x) - p^n(-x) - p^n(y) + p^n(0) \\
 &= \mathbf{g}_n(y - x) - \mathbf{g}_n(-x) - \mathbf{g}_n(y) + \mathbf{g}_n(0) + o(xyn^{-3/2}). \tag{2.12}
 \end{aligned}$$

PROOF. Let $\phi(l)$ denote the characteristic function of Y : $\phi(l) = Ee^{ilY}$, $l \in \mathbf{R}$. As in the usual proof of the local central limit theorem choose a positive constant ε so that $|\phi(l) - 1| \geq \sigma^2 l^2 / 4$ for $|l| < \varepsilon$ and set $\eta = \sup_{\varepsilon \leq |l| \leq \pi} |1 - \phi(l)| < 1$. Then the error in (2.12) that we are to show to be $o(xyn^{-3/2})$ is written as

$$(2\pi)^{-1/2} \int_{-\varepsilon}^{\varepsilon} ([\phi(l)]^n - e^{-n\sigma^2 l^2 / 2}) K_{x,y}(l) dl + O(e^{-n\sigma^2 \varepsilon^2 / 2} + \eta^n)$$

where $K_{x,y}(l) = e^{-i(y-x)l} - e^{ixl} - e^{-iy l} + 1$. Since $K_{x,y}(l) = (e^{ixl} - 1)(e^{-iy l} - 1)$, we have $|K_{x,y}(l)| \leq |xy|l^2$ and, scaling l by $\sqrt{n_*}$ and applying the dominated convergence theorem, we deduce that the integral above is $o(xyn^{-3/2})$ as required. \square

LEMMA 2.6. *Let $d_o = 1$. Then uniformly in $y \in \mathbf{Z}$, $p^n(y) - p^n(0) = g_n(y) - g_n(0) + o(y/n)$ as $n \rightarrow \infty$; in particular $|p^n(y) - p^n(0)| \leq C|y|/n$.*

PROOF. The proof is similar to the preceding one. We have only to use $|1 - e^{ixl}| \leq |xl|$ in place of the bound of $K_{x,y}(l)$. \square

3. Estimation of $q^n(x, y)$.

In this section we prove Theorem 1.1. The proof relies on the asymptotic estimate of the hitting-time distribution

$$f_x^{\{0\}}(k) = P_x[\tau_0 = k] \quad (k = 1, 2, \dots)$$

as $k \rightarrow \infty$, where τ_0 denotes, as in Section 2, the first time that S_n^x hits the origin after time 0. The following theorem is essentially proved in [8].

THEOREM A. *Under the basic assumption of this paper, as $|x| \vee k \rightarrow \infty$*

$$f_x^{\{0\}}(k) = \frac{\sigma a^*(x)e^{-x^2/2\sigma^2k}}{\sqrt{2\pi}k^{3/2}} + o\left(\frac{|x| + 1}{k^{3/2}} \wedge \frac{1}{|x|^2 + 1}\right). \tag{3.1}$$

PROOF. Immediate from Theorems 1.1 and 1.2 of [8]. \square

We have only to consider the case $0 < |y| \leq x$ in view of the duality of $q^n(x, y)$ and $q^n(y, x)$ (i.e., transformed to each other by time-reversal).

Let $\phi(l)$ denote the characteristic function of Y as in the proof of Lemma 2.5. In what follows we suppose that the walk S_n is aperiodic so that $|\phi(l)| < 1$ for $|l| \leq \pi$. We shall use the representation

$$q^n(x, y) = p^n(y - x) - \sum_{k=1}^n f_x^{\{0\}}(n - k)p^k(y) \tag{3.2}$$

and its Fourier version

$$q^n(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\pi_{y-x}(t) - \rho(t)\pi_{-x}(t)\pi_y(t)]e^{-int} dt \quad (x \neq 0) \tag{3.3}$$

and

$$q^n(0, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(t)\pi_y(t)e^{-int} dt, \tag{3.4}$$

where

$$\pi_x(t) = \lim_{r \uparrow 1} \sum_{n=0}^{\infty} p^n(x)e^{itn}r^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-ixl}}{1 - e^{it}\phi(l)} dl \quad (t \neq 0)$$

and $\rho(t) = 1/\pi_0(t)$; it holds that

$$\rho(t) = \sigma\sqrt{-2it}(1 + o(1)) \quad \text{as } t \rightarrow 0$$

(cf. [8, Section 2]). Note that $q^n(0, y) = f_{-y}^{\{0\}}(n)$ by duality (or by coincidence of the Fourier coefficients), so that Theorem 1.1 in the case $x = 0$ is immediate from Theorem A.

The supposition that the walk S_n is aperiodic gives rise to no essential loss of generality. To see this let $d_o > 1$ and put $\omega = 2\pi/d_o$. Then one can find a number ξ among $1, \dots, d_o - 1$ such that $p(x + \xi) = 0$ for all $x \notin d_o\mathbf{Z}$. Hence for all $l \in (-\pi, \pi]$,

$$\phi(l + \omega) = \sum e^{i(x+\xi)(l+\omega)}p(x + \xi) = e^{i\xi\omega} \phi(l).$$

Owing to the irreducibility of the walk there exists an integer k such that $k\xi = 1(\text{mod}(d_o))$. Noting $\phi(l + k\omega) = e^{i\omega} \phi(l)$, one observes that

$$\pi_x(t - \omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-ix(l+k\omega)}}{1 - e^{it}e^{-i\omega}\phi(l + k\omega)} dl = \pi_x(t)e^{-ixk\omega};$$

in particular $\rho(t - \omega) = \rho(t)$. It accordingly follows that the integrand of the integral on the right side of (3.3) is invariant by a shift of t by ω if (and only if) $(y - x)k\omega - n\omega \in 2\pi\mathbf{Z}$, namely $(y - x)k = n(\text{mod}(d_o))$ (the only case when $q^n(x, y) \neq 0$), hence the general case is reduced to the case $d_o = 1$ since all our estimation of $q^n(x, y)$ is based on (3.3).

THEOREM 3.1. *Uniformly for $0 < |y| \leq x < a_o\sqrt{n}$, as $n \rightarrow \infty$ and $|y|/\sqrt{n} \rightarrow 0$*

$$q^n(x, y) = \mathbf{g}_n(x) \frac{\sigma^4 a(x)a(-y) + xy}{n_*} + o\left(\frac{xy}{n^{3/2}}\right).$$

PROOF. First consider the case when not only y but also x is $o(\sqrt{n})$. Of the integrand in (3.3) make the decomposition

$$\begin{aligned} \pi_{y-x}(t) - \rho(t)\pi_{-x}(t)\pi_y(t) &= \pi_{y-x} - \pi_{-x} - \pi_y + \pi_0 + a(x)a(-y)\rho \\ &\quad - \rho e_x e_{-y} + a(x)\rho e_{-y} + a(-y)\rho e_x, \end{aligned}$$

where

$$e_x = e_x(t) = \pi_{-x}(t) - \pi_0(t) + a(x).$$

Noting that $e^{-(\xi-\eta)^2} - e^{-\xi^2} - e^{-\eta^2} + 1 = e^{-\xi^2-\eta^2}(e^{2\xi\eta} - 1) + O(\xi^2\eta^2) = 2\xi\eta + o(\xi\eta)$ as $\xi, \eta \rightarrow 0$, we apply Theorem A and Lemma 2.5 to see

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\pi}^{\pi} [\pi_{y-x} - \pi_{-x} - \pi_y + \pi_0 + a(x)a(-y)\rho] e^{-int} dt \\ &= p^n(y-x) - p^n(-x) - p^n(y) + p^n(0) + a(x)a(-y)f_0^{\{0\}}(n) \\ &= g_n(0) \frac{\sigma^4 a(x)a(-y) + xy}{n_*} + o\left(\frac{xy}{n^{3/2}}\right). \end{aligned}$$

In [8, Section 3] we have made decomposition $(2\pi) e_x(t) = c_x(t) + is_x(t)$, where

$$\begin{aligned} c_x(t) &= \int_{-\pi}^{\pi} \left(\frac{1}{1 - e^{it}\phi(l)} - \frac{1}{1 - \phi(l)} \right) (\cos xl - 1) dl \\ s_x(t) &= \int_{-\pi}^{\pi} \left(\frac{1}{1 - e^{it}\phi(l)} - \frac{1}{1 - \phi(l)} \right) \sin xl dl \end{aligned}$$

and verified the estimates given in the following two lemmas.

LEMMA B1. *There exists a constant C such that*

$$|c_x(t)| \leq Cx^2\sqrt{|t|}, \quad |c'_x(t)| \leq \frac{Cx^2}{\sqrt{|t|}}, \quad |c''_x(t)| \leq \frac{Cx^2}{|t|^{3/2}}.$$

LEMMA B2. *Suppose that $E|Y|^{2+\delta} < \infty$ for some $0 \leq \delta < 1$. Then, uniformly in $x \neq 0$, as $t \rightarrow 0$*

$$\frac{|s_x(t)|}{|x|} = o(|t|^{\delta/2}), \quad \frac{|s'_x(t)|}{|x|} = o\left(\frac{|t|^{\delta/2}}{|t|}\right), \quad \frac{|s''_x(t)|}{|x|} = o\left(\frac{|t|^{\delta/2}}{|t|^2}\right).$$

By a simple change of variables we derive the bounds

$$|\pi_{-x}^{(j)}(t)| \leq C|t|^{-1/2-j} \quad (j = 0, 1, 2), \tag{3.5}$$

which in particular give $|\rho^{(j)}(t)| \leq C|t|^{1/2-j}$, where the superscript (j) indicates the derivative of j -th order. With the help of the bounds given above as well as of Lemmas B1 and B2 we can readily infer that each of the contributions of $\rho e_x e_{-y}$, $a(x)\rho e_{-y}$, and $a(-y)\rho e_x$ to the integral in (3.3) is $o(xy/n^{3/2})$. E.g., writing $g(t) = (\rho c_x c_{-y})(t)$, integrating by parts and observing that $|g'(t)| \leq Cx^2y^2\sqrt{|t|}$ we obtain

$$\int_{-\pi}^{\pi} g(t)e^{-int} dt = \frac{1}{in} \int_{-\pi}^{\pi} g'(t)e^{-int} dt = O\left(\frac{x^2y^2}{n^2\sqrt{n}}\right) + \int_{1/n < |t| < \pi} g'(t)e^{-int} dt. \tag{3.6}$$

Integrate by parts once more and apply the bound $|g''(t)| \leq Cx^2y^2/\sqrt{|t|}$ to evaluate the last integral to be $O(x^2y^2/n^2\sqrt{n})$, which is $o(xy/n^{3/2})$ since $x = o(\sqrt{n})$.

It remains to consider the case $\varepsilon\sqrt{n} < x < a_\circ\sqrt{n}$. This time we use the decomposition

$$\pi_{y-x} - \rho\pi_{-x}\pi_y = \pi_{y-x} - \pi_{-x} + a(-y)\rho\pi_{-x} + \rho\pi_{-x}e_{-y}.$$

Owing to Lemma 2.6 and the present assumption on x, y , $p^n(y) - p^n(0) = o(y/n) = o(xy/n^{3/2})$. Hence, again by Lemma 2.5 and Theorem A,

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} [\pi_{y-x} - \pi_{-x} + a(-y)\rho\pi_{-x}] e^{-int} dt \\ &= p^n(y-x) - p^n(-x) + a(-y)f_x^{\{0\}}(n) \\ &= \mathbf{g}_n(x) \frac{\sigma^4 a(x)a(-y) + xy}{n_*} + o\left(\frac{xy}{n^{3/2}}\right). \end{aligned}$$

In the same way as argued at (3.6) the contribution of $\rho\pi_{-x}e_{-y}$ to the integral in (3.3) can be evaluated to be $O(y^2/n^{3/2}) + o(y/n)$, which is $o(xy/n^{3/2})$.

Theorem 3.1 has been proved. □

Theorem 3.1 implies (i) of Theorem 1.1 (in view of the local central limit theorem).

PROPOSITION 3.1. *Uniformly for x, y such that both x and $|y|$ are between $a_\circ^{-1}\sqrt{n}$ and $a_\circ\sqrt{n}$, as $n \rightarrow \infty$*

$$\begin{aligned}
 q^n(x, y) &= \mathbf{g}_n(y - x) - \mathbf{g}_n(y + x) + o\left(\frac{1}{\sqrt{n}}\right) \quad \text{if } y > 0, \\
 &= o\left(\frac{1}{\sqrt{n}}\right) \quad \text{if } y < 0.
 \end{aligned}$$

PROOF. We prove the second relation first. To this end we introduce an auxiliary walk. Let $\tilde{p}(x)$ be any probability law on \mathbf{Z} of zero mean and variance σ^2 such that its third absolute moment is finite and the random walk determined by \tilde{p} is left continuous, namely $\tilde{p}(y) = 0$ for $y \leq -2$. Let $\tilde{p}^n(y - x)$ and $\tilde{q}^n(x, y)$ denote the corresponding n -th step transition probabilities and let $y < 0 < x$. From the assumed left continuity it follows that $\tilde{q}^n(x, y) = 0$, and hence

$$\begin{aligned}
 q^n(x, y) &= p^n(y - x) - \tilde{p}^n(y - x) - \sum_{k=1}^n \left(f_x^{\{0\}}(k) p^{n-k}(y) - \tilde{f}_x^{\{0\}}(k) \tilde{p}^{n-k}(y) \right) \\
 &= U_n(x, y) + V_n(x, y) + W_n(x, y),
 \end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
 U_n(x, y) &= p^n(y - x) - \tilde{p}^n(y - x), \\
 V_n(x, y) &= - \sum_{k=1}^{n-1} \left(f_x^{\{0\}}(k) - \tilde{f}_x^{\{0\}}(k) \right) p^{n-k}(y)
 \end{aligned}$$

and

$$W_n(x, y) = - \sum_{k=1}^{n-1} \tilde{f}_x^{\{0\}}(k) (p^{n-k}(y) - \tilde{p}^{n-k}(y)).$$

By the local limit theorem $U_n = o(1/\sqrt{n})$. We apply Theorem A to see that $\sup_{k \geq 1} |f_x^{\{0\}}(k) - \tilde{f}_x^{\{0\}}(k)| = o(1/x^2) = o(1/n)$, which combined with the trite bound $\sup_z p^k(z) \leq C/\sqrt{k}$ shows $V_n = o(1/\sqrt{n})$. The bound $W_n = o(1/\sqrt{n})$ is verified e.g. by observing that $\sup_{1 \leq k \leq n} |p^k(y) - \tilde{p}^k(y)|\sqrt{k} \rightarrow 0$ as $n \rightarrow \infty$ uniformly in y .

For the proof of the first relation we write for $y > 0$

$$\begin{aligned}
 q^n(x, y) &= p^n(y - x) - \sum_{k=1}^n f_{-x}^{\{0\}}(k) p^{n-k}(y) + \sum_{k=1}^n [f_{-x}^{\{0\}}(k) - f_x^{\{0\}}(k)] p^{n-k}(y) \\
 &= p^n(y - x) - p^n(y + x) + q^n(-x, y) + r_n(x, y),
 \end{aligned}$$

where $r_n(x, y) = \sum_{k=1}^n [f_{-x}^{\{0\}}(k) - f_x^{\{0\}}(k)]p^{n-k}(y)$. In view of what has been shown above as well as the local central limit theorem it suffices to show $r_n(x, y) = o(1/\sqrt{n})$. There exists a positive integer N such that for $0 < \varepsilon < 1/2$ and $n > N$,

$$\sum_{1 \leq k < \varepsilon n} f_{\pm x}^{\{0\}}(k)p^{n-k}(y) \leq \frac{P_{\pm x}[\tau_0 < \varepsilon n]}{\sqrt{\pi n_*}}$$

and in view of Donsker’s invariance principle the probability on the right side above tends to zero as $\varepsilon \downarrow 0$ uniformly for $x > \sqrt{n}/a_o$. Now the required estimate follows from Theorem A, according to which $f_{-x}^{\{0\}}(k) - f_x^{\{0\}}(k) = o(x/k^{3/2})$ as $x \wedge k \rightarrow \infty$. □

PROPOSITION 3.2. *Suppose $E|Y|^{2+\delta} < \infty$ for some $\delta \geq 0$. Then, uniformly for $|x| < a_o\sqrt{n}$ and $|y| > a_o^{-1}\sqrt{n}$, as $n \rightarrow \infty$*

$$q^n(x, y) = O\left(\frac{x}{y}g_{4n}(y)\right) + o\left(\frac{x}{|y|^{2+\delta}}\right).$$

PROOF. Suppose $y/2 > \sqrt{n_*}$ for simplicity. Put $\tau = \tau_0 \wedge \tau_{(y/4, \infty)}$. Then

$$\begin{aligned} q^n(x, y) &= P_x[\tau \leq n < \tau_0, S_n = y] \\ &= P_x\left[\frac{y}{4} < S_\tau < \frac{y}{2}, n < \tau_0, S_n = y\right] + P_x\left[S_\tau \geq \frac{y}{2}, n < \tau_0, S_n = y\right] \\ &= I + II \quad (\text{say}). \end{aligned}$$

We employ the inequality

$$I \leq \sum_{k=1}^n \sum_{y/4 < z < y/2} P_x[\tau = n - k, S_\tau = z]P_z[S_k = y].$$

The following less familiar version of local central limit theorem is found in [7] (see its Corollary 6): under the assumption of Proposition 3.2

$$P_0[S_n = x] = g_n(x)[1 + P^{n,\nu}(x)] + o\left(\frac{1}{\sqrt{n}^{1+\delta}} \wedge \frac{\sqrt{n}}{|x|^{2+\delta}}\right), \tag{3.8}$$

$(n + |x| \rightarrow \infty)$, where $\nu = [\delta]$ (the largest integer that does not exceeds δ), $P^{n,0} \equiv 0$ and $P^{n,\nu}(x) = \frac{1}{\sqrt{n}}P_1\left(\frac{x}{\sqrt{n}}\right) + \dots + \frac{1}{\sqrt{n}^\nu}P_\nu\left(\frac{x}{\sqrt{n}}\right)$ if $\nu \geq 1$ with the same

real polynomials P_j of degree j as those associated with the Edgeworth expansion. From (3.8) one deduces

$$\max_{1 \leq k \leq n} \max_{y/4 < z < y/2} P_z[S_k = y] = O(\mathbf{g}_{4n}(y)) + o\left(\frac{\sqrt{n}}{y^{2+\delta}}\right)$$

(use $(y/2)^2 > n_*$ for evaluation of the maximum over k). On the other hand

$$\sum_{k=1}^n \sum_{y/4 < z < y/2} P_x[\tau = n - k, S_\tau = z] \leq P_x[\tau_{(y/4, \infty)} < \tau_0] = O\left(\frac{x}{y}\right).$$

Hence

$$I = O\left(\frac{x}{y} \mathbf{g}_{4n}(y)\right) + o\left(\frac{x\sqrt{n}}{|y|^{3+\delta}}\right). \tag{3.9}$$

For evaluation of II we begin with

$$II \leq \sum_{k=1}^n P_x\left[S_k \geq \frac{y}{2}, \tau = k, S_n = y\right].$$

Under $S_k \geq y/2$ we have $\{\tau = k\} = \{\tau > k - 1\}$, hence the sum on the right side equals

$$\sum_{k=1}^n E_x\left[P_{S_{k-1}}\left[S_1 \geq \frac{y}{2}, S_{n-k+1} = y\right]; \tau > k - 1\right].$$

Since $P_x[\tau > k - 1] \leq P_x[\tau_0 > k - 1] = O(x/\sqrt{k})$ and for $z < y/4$,

$$P_z\left[S_1 \geq \frac{y}{2}, S_{n-k+1} = y\right] \leq \sum_{w \geq y/2} p(w - z)p^{n-k}(y - w) = o\left(\frac{1}{y^{2+\delta}\sqrt{n - k + 1}}\right),$$

we get

$$II = \sum_{k=1}^n P_x[\tau > k - 1] \times o\left(\frac{1}{y^{2+\delta}\sqrt{n - k + 1}}\right) = o\left(\frac{x}{y^{2+\delta}}\right).$$

This together with (3.9) shows the estimate of the proposition. □

4. Estimation of $q_{(-\infty,0]}^n$ and $h_x(n, y)$.

PROOF OF PROPOSITION 1.1. In view of (i) and (ii) of Theorem 1.1 it suffices to prove that uniformly in n ,

$$q^n(x, y) - q_{(-\infty,0]}^n(x, y) = o\left(\frac{xy}{n^{3/2}}\right) \text{ as } x \wedge y \rightarrow \infty.$$

This difference may be written as

$$\sum_{k=1}^n \sum_{z < 0} h_x(k, z) q^{n-k}(z, y). \tag{4.1}$$

Employing the identity (2.7) one observes that

$$\sum_{1 \leq k < n/2} \sum_{z < 0} h_x(k, z) |z| \leq \sum_{z < 0} H_x^+(z) |z| = f_+(x) - x = o(x) \text{ as } x \rightarrow \infty.$$

Combined with the simple bound (1.5) this shows that the sum over $k \leq n/2$ in (4.1) is $o(xy/n^{3/2})$. The other half of the sum is less than the probability that the time-reversed walk starting at y enters $(-\infty, 0]$ by the time $n/2$ and ends in x at the time n and hence estimated also to be $o(xy/n^{3/2})$. \square

LEMMA 4.1. *For each $x = 1, 2, \dots$, uniformly for $n \geq y^2/a_\circ$, as $y \rightarrow \infty$*

$$q_{(-\infty,0]}^n(x, y) = \frac{2f_+(x)y}{n_*} \mathbf{g}_n(y) (1 + o(1)).$$

PROOF. Given a positive integer x , take an integer $N > x$ and put $\tau = T \wedge \tau_{[N,\infty)}$, the first leaving time from $[1, N - 1]$. Then

$$q_{(-\infty,0]}^n(x, y) = E_x [q_{(-\infty,0]}^{n-\tau}(S_\tau, y); \tau < T \wedge (n + 1)]. \tag{4.2}$$

Let α be any positive number less than 1. For each $\varepsilon > 0$ we can choose N large enough that for all k, n, z and y that satisfy $0 \leq k < n^\alpha$, $2N < y \leq \sqrt{a_\circ n}$ and $N \leq z \leq \sqrt{n}/N$, the following three bounds hold:

$$\left| q_{(-\infty,0]}^{n-k}(z, y) - \frac{2zy}{n_*} \mathbf{g}_n(y) \right| < \frac{\varepsilon zy}{n_*^{3/2}}, \tag{4.3}$$

$$\left| P_x[\tau < T] - \frac{f_+(x)}{N} \right| \leq \frac{\varepsilon x}{N}, \tag{4.4}$$

$$E_x[S_\tau - N; \tau < T] \leq \varepsilon x, \tag{4.5}$$

according to (i) of Theorem 1.1 and Proposition 1.1 for (4.3), to Proposition 2.2 for (4.4) and to Proposition 2.3 for (4.5). Since τ equals the sum of the sojourn times of sites w in the interval $[1, N - 1]$ spent by the walk before leaving it, we have $E_x[\tau] = \sum_{w=1}^{N-1} \sum_{k=0}^{\infty} P_x[S_k = w, k < \tau] \leq \sum_{w=1}^{N-1} g_{(-\infty, 0]}(x, w) \leq CxN$, and on using this

$$\begin{aligned} P_x \left[S_\tau > \frac{\sqrt{n}}{N}, \tau < T \right] &\leq \sum_{k=1}^{\infty} P_x \left[S_k > \frac{\sqrt{n}}{N}, \tau = k \right] \\ &\leq \sum_{k=1}^{\infty} P_x \left[Y_k > \frac{\sqrt{n}}{N} - S_{k-1}, \tau > k - 1 \right] \\ &\leq \sum_{k=0}^{\infty} P_x[\tau > k] P \left[Y > \frac{\sqrt{n}}{N} - N \right] = xN^3 \times o\left(\frac{1}{n}\right) \end{aligned} \tag{4.6}$$

as $n \rightarrow \infty$. Since $q_{(-\infty, 0]}^{n-k}(\cdot, y) \leq C'/\sqrt{n}$ if $k < n^\alpha$, this entails that

$$E_x \left[q_{(-\infty, 0]}^{n-\tau}(S_\tau, y); S_\tau > \frac{\sqrt{n}}{N}, \tau < T \wedge n^\alpha \right] = o(n^{-3/2}) \tag{4.7}$$

as $n \rightarrow \infty$ (with N, x fixed). On the other hand, using (4.3) we obtain

$$\begin{aligned} E_x \left[q_{(-\infty, 0]}^{n-\tau}(S_\tau, y); S_\tau \leq \frac{\sqrt{n}}{N}, \tau < T \wedge n^\alpha \right] \\ = E_x \left[\frac{2Ny}{n_*} \mathbf{g}_n(y); S_\tau \leq \frac{\sqrt{n}}{N}, \tau < T \wedge n^\alpha \right] \\ + E_x \left[\frac{2(S_\tau - N)y}{n_*} \mathbf{g}_n(y); S_\tau \leq \frac{\sqrt{n}}{N}, \tau < T \wedge n^\alpha \right] + r(n, x, y) \end{aligned} \tag{4.8}$$

with $|r(n, x, y)| \leq \varepsilon y E_x[S_\tau; \tau < T]/n_*^{3/2}$. Writing $E_x[S_\tau; \tau < T] = NP_x[\tau < T] + E_x[S_\tau - N; \tau < T]$ we apply (4.5) and (4.4) to see that the remainder r as well as the second expectation on the right side in (4.8) is dominated in absolute value by $3\varepsilon xy/n_*^{3/2}$. We also have $P_x[\tau > n^\alpha] = O(e^{-\kappa n^\alpha})$ with some $\kappa = \kappa_N > 0$, and hence, owing to (4.6),

$$P_x \left[S_\tau \leq \frac{\sqrt{n}}{N}, \tau < T \wedge n^\alpha \right] = P_x[\tau < T] + o\left(\frac{1}{n}\right).$$

Combining these bounds with (4.7) and (4.2) shows that for all sufficiently large y and for $n > y^2/a_\circ$,

$$\left| q_{(-\infty,0]}^n(x, y) - \frac{2Ny}{n_*} \mathbf{g}_n(y) P_x[\tau < T] \right| < \frac{7\varepsilon xy}{n_*^{3/2}}$$

and substitution from (4.4) completes the proof of Lemma 4.1. □

LEMMA 4.2. *For each $x, y = 1, 2, \dots$, as $n \rightarrow \infty$*

$$q_{(-\infty,0]}^n(x, y) = \frac{2f_+(x)f_-(y)}{n_*} \mathbf{g}_n(0)(1 + o(1)).$$

PROOF. Applying Lemma 4.1 to the time-reversed walk we have

$$\left| q_{(-\infty,0]}^n(z, y) - \frac{2z\mathbf{g}_n(z)f_-(y)}{n_*} \right| < \frac{\varepsilon zy}{n^{3/2}}$$

(valid for all $z \geq N$) in place of (4.3) and we can proceed as in the proof of Lemma 4.1. □

Theorem 1.3 follows from Proposition 1.1 and Lemmas 4.1 and 4.2 given above.

PROOF OF THEOREM 1.4. The probability $h_x(n, y)$ is represented as

$$h_x(n, y) = \sum_{z>0} q_{(-\infty,0]}^{n-1}(x, z)p(y - z). \tag{4.9}$$

Write $F(x, n) = 2f_+(x)\mathbf{g}_n(x)/n_*$. In view of Theorem 1.1 and a local limit theorem, for each $\varepsilon > 0$ we can then choose $\eta > 0$ such that for all sufficiently large n ,

$$\left| q_{(-\infty,0]}^{n-1}(x, z) - F(x, n)f_-(z) \right| \leq \varepsilon F(x, n)f_-(z)$$

whenever $0 < z \leq \eta\sqrt{n}$ and $0 < x < a_\circ\sqrt{n}$. Hence, on using the second expression of H_∞^+ in (1.10), the difference $|h_x(n, y) - F(x, n)H_\infty^+(y)|$ is at most

$$\varepsilon F(x, n)H_{\infty}^{+}(y) + \sum_{z > \eta\sqrt{n}} |q_{(-\infty, 0]}^{n-1}(x, z) - F(x, n)f_{-}(z)|p(y - z).$$

Owing to (1.5) the summand of the last sum is at most a constant multiple of $n^{-3/2}xzp(y - z)$, so that if $\alpha_n(y) = \sum_{z > \eta\sqrt{n}} zp(y - z)$, then this sum is at most $n^{-3/2}x\alpha_n(y)$, hence

$$|h_x(n, y) - F(x, n)H_{\infty}^{+}(y)| \leq \varepsilon F(x, n)H_{\infty}^{+}(y) + n^{-3/2}x\alpha_n(y).$$

The proof is now finished by observing that if $E[|Y|^{2+\delta}; Y < 0] < \infty$, then

$$\alpha_n(y) = o\left(\frac{1}{(\sqrt{n} + |y|)^{1+\delta}}\right), \quad \sum_{y \leq 0} \alpha_n(y) = o(n^{-\delta/2}) \quad \text{and} \quad \sum_{y \leq 0} \alpha_n(y)|y|^{\delta} = o(1).$$

The first half of Theorem 1.4 has been verified.

For the second half we verify that for $y \leq 0$ and $x \geq \sqrt{n}$,

$$\sum_{z=1}^{\infty} q^n(x, z)p(y-z) \leq C \left[\frac{\mathbf{g}_{4n}(x)}{n^{1/2}} + o\left(\frac{1}{x^{2+\delta}}\right) \right] H_{\infty}^{+}(y) + \frac{C}{n^{1/2}} P\left[Y < y - \frac{1}{2}x\right]. \quad (4.10)$$

Since $h_x(n, y)$ is not larger than the sum on the left side, this implies (1.12). For verification of (4.10) we break the range of summation into three parts $0 < z \leq \sqrt{n} \wedge (1/2)x$, $\sqrt{n} < z \leq x/2$ and $z > x/2$, and denote the corresponding sums by *I*, *II* and *III*, respectively. It is immediate from (iii) of Theorem 1.1 that $I = (O(\mathbf{g}_{4n}(x)x^{-1}) + o(x^{-2-\delta}))H_{\infty}^{+}(y)$. The local limit theorem estimate (3.8) gives that $q^n(x, z) \leq p^n(z - x) \leq 2\mathbf{g}_n(x/2) + o(n^{1/2}x^{-2-\delta})$ for $\sqrt{n} < z \leq x/2$ and we apply this bound as well as the bound

$$\sum_{z > \sqrt{n}} p(y - z) \leq \frac{1}{\sqrt{n}} \sum_{z=1}^{\infty} zp(y - z) \leq \frac{2}{\sigma^2\sqrt{n}} \left[\sup_{z \geq 1} \frac{f_{-}(z)}{z} \right] H_{\infty}^{+}(y) \quad (4.11)$$

to have $II = [O(\mathbf{g}_{4n}(x)n^{-1/2}) + o(x^{-2-\delta})]H_{\infty}^{+}(y)$. Finally $III \leq Cn^{-1/2} \sum_{z < y-x/2} p(z)$. These estimates together verify (4.10). As in (4.11) we derive $P[Y < y - (1/2)x] \leq C_1 H_{\infty}^{+}(y)/x$. We also have $\mathbf{g}_{4n}(x) \leq C_1/x$. Hence the last relation of the theorem follows from (1.12). The proof of Theorem 1.4 is complete. \square

The proof of Theorem 1.2 is based on Theorem 1.4. Put

$$\Phi_\xi(t) = \frac{|\xi|e^{-\xi^2/2t}}{\sqrt{2\pi}t^{3/2}} \quad (t > 0, \xi \neq 0).$$

Then Theorem 1.2 (under $d_o = 1$) may be restated as follows.

THEOREM 1.2. *Suppose that $E[|Y|^3; Y < 0] < \infty$. Let $y < 0 < x$. Then uniformly for $x, |y| \leq a_o\sqrt{n}$, as $n \rightarrow \infty$ and $x \wedge |y| \rightarrow \infty$*

$$q^n(x, y) = C^+ \Phi_{x+|y|}(n_*) + o\left(\frac{x + |y|}{n^{3/2}}\right). \tag{4.12}$$

PROOF. We use the representation

$$q^n(x, y) = \sum_{k=1}^n \sum_{z < 0} h_x(k, z) q^{n-k}(z, y).$$

Break the right side into three parts by partitioning the range of the first summation as follows

$$1 \leq k < \varepsilon n; \quad \varepsilon n \leq k \leq (1 - \varepsilon)n; \quad (1 - \varepsilon)n < k \leq n \tag{4.13}$$

and call the corresponding sums *I*, *II* and *III*, respectively. Here ε is a positive constant that will be chosen small.

Consider the limit procedure as indicated in the theorem. First suppose that $(x \wedge |y|)/\sqrt{n}$ is bounded away from zero. Then by (1.13), the last relation of Theorem 1.4, and (1.5) we have

$$I \leq \sum_{1 \leq k < \varepsilon n} \frac{C}{k^{1/2}x} \sum_{z < 0} H_\infty^+(z) \frac{|zy|}{n^{3/2}} \leq C' \frac{\sqrt{\varepsilon}}{n},$$

and similarly $III \leq C'' \sqrt{\varepsilon}/n$ (see also (4.9)). By the first half of Theorem 1.4 and (1.5)

$$II = \sum_{\varepsilon n \leq k \leq (1-\varepsilon)n} \frac{f_+(x)g_k(x)}{k} \sum_{z=-x}^{-1} H_\infty^+(z) q^{n-k}(z, y) (1 + o_\varepsilon(1)) + o_\varepsilon\left(\frac{1}{n}\right).$$

Here (and in the rest of the proof) the estimate indicated by o_ε may depend on ε but is uniform in the limit under consideration once ε is fixed. We substitute from (i) of Theorem 1.1 for q^{n-k} and observe, on replacing $f_+(x)$ and $a(-y)$, respectively, by x and $-y/\sigma^2$,

$$II = \sum_{\varepsilon n \leq k \leq (1-\varepsilon)n} \frac{x|y|g_k(x)g_{n-k}(y)}{\sigma^2 k(n-k)} \sum_{z=-x}^{-1} H_\infty^+(z)(\sigma^2 a(z) - z)(1 + o_\varepsilon(1)) + o_\varepsilon\left(\frac{1}{n}\right).$$

Noting $xg_k(x)/k = \Phi_{x/\sigma}(k)$, we see

$$\begin{aligned} & \sum_{\varepsilon n \leq k \leq (1-\varepsilon)n} \frac{x|y|g_k(x)g_{n-k}(y)}{\sigma^2 k(n-k)} \\ &= \frac{1}{n\sigma^2} \int_0^1 \Phi_{x/\sqrt{n_*}}(t)\Phi_{y/\sqrt{n_*}}(1-t)dt + O\left(\frac{\varepsilon}{n}\right) + o\left(\frac{1}{n}\right). \end{aligned}$$

Here we have used the assumption that $(x \wedge |y|)/\sqrt{n}$ is bounded away from zero as well as from infinity. Since Φ_ξ is the density of a Brownian passage-time distribution, we have

$$\int_0^1 \Phi_\xi(t)\Phi_\eta(1-t)dt = \Phi_{|\xi|+|\eta|}(1).$$

Hence

$$II = \frac{1}{n\sigma^2} \Phi_{(x+|y|)/\sqrt{n_*}}(1) \sum_{z=-x}^{-1} H_\infty^+(z)(\sigma^2 a(z) - z) + O\left(\frac{\varepsilon}{n}\right) + o_\varepsilon\left(\frac{1}{n}\right). \quad (4.14)$$

Recalling that $C^+ = \sum_{z < 0} H_\infty^+(z)(\sigma^2 a(z) - z)$ we then see $\sigma^2 nII - C^+ \Phi_{(x+|y|)/\sqrt{n_*}}(1) \rightarrow 0$ (as well as $nI + nIII \rightarrow 0$) as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$ in this order. Thus (4.12) is obtained.

Next suppose $x \wedge |y| = o(\sqrt{n})$. By duality one may suppose that $x = o(\sqrt{n})$. From Theorem 1.4 (with $\delta = 1$) and from the bound $H_x^+(y) \leq CH_\infty^+(y)$ (see (2.5)) one deduces, respectively,

$$\text{i) } \sum_{k \geq \varepsilon n} \sum_{z < 0} h_x(k, z)|z| \leq \frac{M_\varepsilon x}{\sqrt{n}} \quad \text{and} \quad \text{ii) } \sum_{z < -x} H_x^+(z)z = o(1). \quad (4.15)$$

Here (and below) M_ε indicates a constant that may depend on ε but not on the other variables. On using i) above with the help of the bound $q^k(z, y) \leq C|zy|k^{-3/2}$

$$II \leq \frac{M_\varepsilon xy}{n^2} = o_\varepsilon(yn^{-3/2})$$

(as $n \rightarrow \infty$ under the supposed constraints on x, y); similarly on using Theorem 1.1 (i) together with ii) above

$$I = \sum_{1 \leq k < \varepsilon n} \sum_{z=-x}^{-1} h_x(k, z) \cdot \frac{\sigma^4 a(z)a(-y) + zy}{\sigma^2(n-k)} \mathbf{g}_{n-k}(y)(1 + o_\varepsilon(1)) + o\left(\frac{y}{n^{3/2}}\right).$$

For the evaluation of the last double sum we may replace $(n-k)^{-1} \mathbf{g}_{n-k}$ by $n^{-1} \mathbf{g}_n(1 + O(\varepsilon))$. Since $x \wedge |y|$ is supposed to go to infinity, we may also replace $a(-y)$ by $|y|/\sigma^2$ and in view of (4.15) we may extend the range of the double summation in the above expression of I to the whole quadrant $k \geq 1, z < 0$; moreover the sum $\sum_{z=-\infty}^{-1} H_x^+(z)[\sigma^2 a(z) - z]$ that accordingly comes out and equals $\sigma^2 a(x) - x$ may be replaced by C^+ (see Lemma 2.1, Corollary 2.1 and (2.7)). This leads to

$$I = C^+ |y| \mathbf{g}_n(y) n_*^{-1} (1 + O(\varepsilon)) + o_\varepsilon(yn^{-3/2}).$$

As to *III* first observe that

$$\sum_{k=1}^{\varepsilon n} q^k(z, y) = g_{\{0\}}(z, y) - r_n \leq C(|z| \wedge |y|) \quad \text{with} \quad 0 \leq r_n \leq \frac{C|zy|}{\sqrt{\varepsilon n}},$$

as follows from (1.5) and (2.2). If y/\sqrt{n} is bounded away from zero, then *III* = $O(x/n^{3/2}) = o(y/n^{3/2})$. On the other hand, applying Theorem 1.4 we find that if $y = o(\sqrt{n})$,

$$III = f_+(x) \mathbf{g}_n(x) n_*^{-1} \sum_{z < 0} H_\infty^+(z) g_{\{0\}}(z, y) (1 + O(\varepsilon)) + o_\varepsilon(xn^{-3/2}),$$

hence in view of $g_{\{0\}}(z, y) = a(z) - \sigma^{-2}z(1 + o(1))$ (as $y \rightarrow -\infty$ uniformly for $z < 0$)

$$III = C^+ x \mathbf{g}_n(x) n_*^{-1} (1 + O(\varepsilon)) + o_\varepsilon(xn^{-3/2}).$$

Adding these contributions yields the desired formula. □

5. Estimation of Q_n^+ .

PROOF OF PROPOSITION 1.2. We apply (i) and (iii) of Theorem 1.1. On noting that $\sum_{y < 0} (\sigma^2 a(-y) + y) \mathbf{g}_n(y) = o(\sqrt{n})$, as $|x|/\sqrt{n} \rightarrow 0$,

$$Q_x^+(n) = \frac{\sigma^2 a(x) - x}{n_*} \sum_{-a_o \sqrt{n} < y < 0} (-y) \mathbf{g}_n(y) [1 + o(1)] + O\left(x \sum_{y \leq -a_o \sqrt{n}} \frac{\mathbf{g}_{4n}(y)}{-y}\right) + o\left(\frac{x}{n^{1/2}}\right),$$

which shows the first assertion of Proposition 1.2 since a_o is arbitrary and $\int_0^\infty ue^{-u^2/2} du = 1$.

For the second one, the case $x = o(\sqrt{n})$ follows from what has just been proved. In view of Proposition 1.1 and (i) of Theorem 1.1, it therefore suffices to show that $\sum_y q_{(-\infty, 0]}^n(x, y) = \int_{-x}^x \mathbf{g}_n(t) dt (1 + o(1))$ uniformly for $|x| \geq a_o^{-1} \sqrt{n}$, which however follows from Donsker's invariance principle together with the reflection principle.

The last assertion of the proposition follows from Theorem 1.2, (iii) of Theorem 1.1 and the fact that $\int_0^{a_o} \Phi_{|\xi|+\eta}(1) d\eta \rightarrow (2\pi)^{-1/2}$ as $\xi \rightarrow 0, a_o \rightarrow \infty$. \square

LEMMA 5.1. *Suppose $E[|Y|^3; Y < 0] < \infty$. Then uniformly for $x > a_o^{-1} \sqrt{n}$,*

$$Q_x^+(n) \leq C \mathbf{g}_{4n}(x) + o\left(\frac{\sqrt{n}}{x^3}\right) + C \sum_{y < 0} |y| P\left[Y < y - \frac{1}{2}x\right]; \tag{5.1}$$

in particular $\sum_{x > M\sqrt{n}} Q_x^+(n) \rightarrow 0$ as $M \rightarrow \infty$ uniformly in n .

PROOF. As in the proof of Theorem 1.2 we use the representation

$$Q_x^+(n) = \sum_{k=1}^n \sum_{z < 0} h_x(k, z) Q_z^+(n - k), \tag{5.2}$$

By Proposition 1.2 $Q_z^+(n - k) \leq C|z|/\sqrt{n - k}$ for all z since $Q_x^+(n) \leq 1$. From the assumption $E[|Y|^3; Y < 0] < \infty$ it follows that $\sum_{z < 0} H_\infty^+(z)|z| < \infty$. Note that $\sum_{k=1}^{n-1} k^{-1/2}(n - k)^{-1/2}$ is bounded. Substitution from (1.12) then leads to the first estimate (5.1). The second relation is immediate from it if one notes that $\sum_{x=0}^\infty \sum_{y < 0} |y| P[Y < y - (1/2)x]$ is dominated by a constant multiple of $E[|Y|^3; Y < 0]$. \square

PROOF OF THEOREM 1.5. Suppose $E[|Y|^3; Y < 0] < \infty$. Then in view of Lemma 5.1 we have only to evaluate $\sum_{1 \leq x \leq M\sqrt{n}} Q_x^+(n)$ for each M . Now apply Theorem 1.2 with the observation that $\int_{\xi > 0} d\xi \int_{\eta > 0} \Phi_{\xi+\eta}(1) d\eta = (2\pi)^{-1/2} \cdot \int_0^\infty e^{-\xi^2/2} d\xi = 1/2$, and you immediately find the first formula of the theorem.

If $E[|Y|^3; Y < 0] = \infty$, one has only to look at the relation (4.14) which is valid without the third moment condition; in fact, $\sum_{x=1}^{\infty} Q_x^+(n)$ is bounded below by the sum of II in (4.14) over $1 \leq x < \sqrt{n}$, $-\sqrt{n} < y < 0$ and the latter diverges to $+\infty$. \square

PROOF OF COROLLARY 1.2. Lemma 5.1 shows that if one writes

$$E[N_n(\ell)] = \sum_{1 \leq x \leq M\sqrt{n_*}} \sum_{-\ell\sqrt{n_*} \leq y \leq -1} m_n(x)q^n(x, y) + \varepsilon_M(n),$$

then $\varepsilon_M(n) \rightarrow 0$ as $M \rightarrow \infty$ uniformly in n . According to Theorem 1.2 and the assumption on $m_n(x)$ the double sum on the right side is asymptotically equal to

$$C^+ \int_0^M d\xi \int_0^\ell \Phi_{\xi+\eta}(1) d\eta = C^+ \frac{1}{\sqrt{2\pi}} \int_0^M (e^{-\xi^2/2} - e^{-(\ell+\xi)^2/2}) d\xi.$$

Since M is arbitrary, we may let $M \rightarrow \infty$ to find the desired formula. \square

6. Absorption at the origin with probability $\alpha \in (0, 1)$.

Let $\alpha \in (0, 1)$ and consider the walk that is absorbed with probability α and continues to walk with probability $1 - \alpha$ every time when it is about to visit the origin (thus the walk visits the origin if it is not absorbed, while it does not and disappears if it is). Let $q_\alpha^n(x, y)$ be the n -th step transition probability of it (set $q_\alpha^0(x, y) = \mathbf{1}(x = y)$ as usual) and denote by $r_\alpha^n(x, y)$ the probability that this walk starting at x has visited the origin but not been absorbed by the time n when it is at y , so that

$$q_\alpha^n(x, y) = q^n(x, y) + r_\alpha^n(x, y).$$

PROPOSITION 6.1. *Let $d_o = 1$. Then uniformly for $|x| \vee |y| < a_o\sqrt{n}$, as $n \rightarrow \infty$*

$$r_\alpha^n(x, y) = \frac{1 - \alpha}{\alpha} \cdot \frac{\sigma^2[a^*(x) + a^*(-y)]}{n} \mathbf{g}_n(|x| + |y|)(1 + o(1)). \quad (6.1)$$

PROOF. Set $f_x^{(1)}(k) = f_x^{\{0\}}(k)$ and, for $j = 2, 3, \dots$, inductively define $f_x^{(j)}(k) = \sum_{l=1}^k f_x^{(j-1)}(k-l)f_0^{(1)}(l)$ (the probability that the j -th visit of the origin occurs at time k). Then for $n = 1, 2, \dots$,

$$r_\alpha^n(x, y) = \sum_{j=1}^{\infty} \sum_{k=1}^n (1 - \alpha)^j f_x^{(j)}(k) q^{n-k}(0, y).$$

(This is valid even for $y = 0$ when the second sum concentrate on $k = n$.) We have $\hat{f}_x^{(j)}(t) := \sum_k f_x^{(j)}(k) e^{ikt} = \hat{f}_x^{\{0\}}(t) [\hat{f}_0^{\{0\}}(t)]^{j-1}$. One can readily derive

$$\hat{f}_x^{\{0\}}(t) = \pi_{-x}(t) \rho(t) \quad (x \neq 0) \quad \text{and} \quad \hat{f}_0^{\{0\}}(t) = 1 - \rho(t) \tag{6.2}$$

(cf. [8]). We also have $q^k(0, y) = f_{-y}^{\{0\}}(k)$ as previously noted. Now, employing the second identity in (6.2) we see

$$\sum_k r_\alpha^k(x, y) e^{ikt} = \frac{\gamma \hat{f}_x^{\{0\}}(t) \hat{f}_{-y}^{\{0\}}(t)}{1 + \gamma \rho(t)} \quad \text{with} \quad \gamma = \frac{1 - \alpha}{\alpha}.$$

Hence, for all $x, y \in \mathbf{Z}$

$$r_\alpha^n(x, y) = \frac{\gamma}{2\pi} \int_{-\pi}^{\pi} \frac{\hat{f}_x^{\{0\}}(t) \hat{f}_{-y}^{\{0\}}(t)}{1 + \gamma \rho(t)} e^{-int} dt.$$

Since $\hat{f}_x^{\{0\}}(t) \hat{f}_{-y}^{\{0\}}(t)$ is the characteristic function of the convolution $f_x^{\{0\}} * f_{-y}^{\{0\}}$,

$$r_\alpha^n(x, y) = \gamma f_x^{\{0\}} * f_{-y}^{\{0\}}(n) - \frac{\gamma^2}{2\pi} \int_{-\pi}^{\pi} \frac{\hat{f}_x^{\{0\}}(t) \hat{f}_{-y}^{\{0\}}(t) \rho(t)}{1 + \gamma \rho(t)} e^{-int} dt.$$

Under the constraints $|x| \vee |y| < a_\circ \sqrt{n}$ and $\varepsilon \sqrt{n} < |x| \wedge |y|$ with $\varepsilon > 0$ we apply Theorem A and, making scaling argument, infer that the first term on the right side above agrees with that of the formula (6.1) with the factor $(|x| + |y|) \sigma^2 [a^*(x) a^*(-y)] / (a^*(x) + a^*(-y)) |xy|$ multiplied, which factor may be replaced by 1 since $|x| \wedge |y| \rightarrow \infty$. The second term is readily evaluated to be negligible (see (6.3) below and the argument following it as well as (3.5), which provides relevant properties of ρ).

By the first identity in (6.2) we have

$$\hat{f}_x^{\{0\}}(t) \hat{f}_{-y}^{\{0\}}(t) = \rho^2(t) \pi_{-x}(t) \pi_y(t) \quad \text{if} \quad x \neq 0, y \neq 0. \tag{6.3}$$

As in Section 3 we write $\pi_{-x}(t) = e_x(t) + \pi_0(t) - a(x)$. Then

$$\begin{aligned} \rho^2(t)\pi_{-x}(t)\pi_y(t) &= \rho^2e_{-x}e_y + a(x)a(-y)\rho^2 - (a(-y)e_x + a(x)e_{-y})\rho^2 - 1 \\ &\quad + (\pi_{-x} + \pi_y)\rho. \end{aligned}$$

The contribution to $r_\alpha^n(x, y)$ of the last term is $\gamma(f_x^{\{0\}}(n) + f_{-y}^{\{0\}}(n))$. Those of the other terms are all $o((|x| \vee |y|)/n^{3/2})$ if $|x| \wedge |y| = o(\sqrt{n})$: for the proof we need the estimates of $c_x'''(t)$ and $s_y'''(t)$ (which require no further moment condition) in addition to those given in Lemmas B1 and B2. Now we may conclude that for $x \neq 0, y \neq 0$,

$$r_\alpha^n(x, y) = \frac{1 - \alpha}{\alpha} (f_x^{\{0\}}(n) + f_{-y}^{\{0\}}(n))(1 + o(1)) \quad \text{as} \quad \frac{|x| \wedge |y|}{\sqrt{n}} \rightarrow 0, \quad (6.4)$$

which agrees with (6.1) owing to Theorem A again. If $x = 0$, then $\hat{f}_x^{\{0\}}\hat{f}_{-y}^{\{0\}} = \rho(1 - \rho)\pi_y = \rho\pi_y - \rho - \rho^2e_y + \rho^2a(-y)$ and we readily obtain (6.4). The case $y = 0$ is similar. □

7. Appendix.

Under the basic assumption of this paper Hoeffding [3] shows that $\Re\{(1 - e^{ixl})\phi^k(l)\}$ is summable on $\{k = 1, 2, \dots\} \times \{-\pi < l < \pi\}$, and hence the series that defines $a(x)$ is absolutely convergent; and, as a consequence of it,

$$a(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re\left\{ \frac{1 - e^{ixl}}{1 - \phi(l)} \right\} dl \quad (7.1)$$

(see [4] for more information on $a(x)$). In this appendix we derive an asymptotic estimate of $a(n)$ under the moment condition $E|Y|^{2+\delta} < \infty$ ($0 \leq \delta \leq 2$). We also include a proof of (7.1) for reader's convenience.

Put for $0 < |l| \leq \pi$,

$$\phi_c(l) = \Re\phi(l) = E[\cos lY] \quad \text{and} \quad \phi_s(l) = \Im\phi(l) = E[\sin lY].$$

Then

$$\Re\left\{ \frac{1 - e^{ixl}}{1 - \phi(l)} \right\} = \frac{1 - \phi_c(l)}{|1 - \phi(l)|^2} (1 - \cos xl) + \frac{\phi_s(l)}{|1 - \phi(l)|^2} \sin xl. \quad (7.2)$$

Noticing $\phi_s(l) = E[\sin Yl - Yl]$, one infers that

$$\int_{-\pi}^{\pi} \frac{|\phi_s(l)l|}{|1-\phi(l)|^2} \leq E \int_{-\pi}^{\pi} \frac{|\sin Yl - Yl|}{|1-\phi(l)|^2} |l| dl \leq CE \left[Y^2 \int_0^{\pi|Y|} \frac{|\sin u - u|}{u^3} du \right] < \infty; \tag{7.3}$$

hence by the dominated convergence theorem

$$\lim_{|x| \rightarrow \infty} \frac{1}{x} \int_{-\pi}^{\pi} \frac{|\phi_s(l)|}{|1-\phi(l)|^2} |\sin xl| dl = 0. \tag{7.4}$$

From this together with the equality $\int_{-\infty}^{\infty} (1 - \cos u)u^{-2} du = \pi$ we conclude the following result.

LEMMA 7.1. As $|x| \rightarrow \infty$

$$\int_{-\pi}^{\pi} \left| \Re \left\{ \frac{1 - e^{ixl}}{1 - \phi(l)} \right\} \right| dl = O(x) \quad \text{and} \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re \left\{ \frac{1 - e^{ixl}}{1 - \phi(l)} \right\} dl = \frac{|x|}{\sigma^2} + o(x).$$

The next proposition in particular implies the identity (7.1).

PROPOSITION 7.1. With a uniformly bounded term $o_b(1)$ that tends to zero as $K \rightarrow \infty$

$$\sum_{k=0}^K [p^k(0) - p^k(-x)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re \left\{ \frac{1 - e^{ixl}}{1 - \phi(l)} \right\} dl (1 + o_b(1)).$$

PROOF. Since we have the Tauberian condition $[p^k(0) - p^k(-x)] = o(1/k)$ (with x fixed) as is assured by Lemma 2.6, owing to the corresponding Tauberian theorem it suffices (apart from the boundedness of convergence) to show that the Abelian sum

$$\frac{1}{x} \sum_{k=0}^{\infty} [p^k(0) - p^k(-x)] r^k = \frac{1}{2\pi x} \int_{-\pi}^{\pi} \Re \left\{ \frac{1 - e^{ixl}}{1 - r\phi(l)} \right\} dl \tag{7.5}$$

converges as $r \uparrow 1$ to $1/x$ times the right side of (7.1). The proof of this convergence is routine and omitted. From (7.3) we see that the convergence above is bounded uniformly in x , which combined with the bound $|p^k(0) - p^k(-x)| \leq C|x/k|$ (cf. Lemma 2.6) implies that the error term $o_b(1)$ is uniformly bounded (see the proof of Tauber's theorem given in Section 1.23 of [6]). \square

REMARK. The remainder term $o_b(1)$ in Proposition 7.1 does not uniformly (in x) approach zero since the Abelian sum in (7.5) tends to zero as $x \rightarrow \infty$ for

each $r < 1$.

If $E|Y|^3 < \infty$, put

$$\lambda_3 = \frac{1}{3\sigma^2}E[Y^3] \quad \text{and} \quad C^* = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{\sigma^2}{1 - \phi(l)} - \frac{1}{1 - \cos l} \right] dl, \quad (7.6)$$

where the integral of the imaginary part is understood to vanish because of skew symmetry. Since $\int_0^\pi [(1 - \cos l)^{-1} - 2l^{-2}] dl = 2/\pi$, the constant C^* may alternatively be given by

$$C^* = \frac{\sigma^2}{2\pi} \int_{-\pi}^{\pi} \left[\Re \left\{ \frac{1}{1 - \phi(l)} \right\} - \frac{2}{\sigma^2 l^2} \right] dl - \frac{2}{\pi^2}. \quad (7.7)$$

The integral in (7.7) as well as the real part of the integral in (7.6) is absolutely convergent (under $E[|Y|^3] < \infty$). In fact, in the expression

$$\Re \left\{ \frac{1}{1 - \phi(l)} \right\} = \frac{-\phi_s^2}{|1 - \phi|^2(1 - \phi_c)} + \frac{1}{1 - \phi_c} \quad (7.8)$$

the first term on the right side is bounded and, since $\phi_c - 1 + (1/2)\sigma^2 l^2 = E[\cos Yl - 1 + (1/2)(Yl)^2]$,

$$\int_{-\pi}^{\pi} \left[\frac{1}{1 - \phi_c(l)} - \frac{2}{\sigma^2 l^2} \right] dl \leq CE \left[|Y|^3 \int_0^{\pi|Y|} \frac{\cos u - 1 + \frac{1}{2}u^2}{u^4} du \right] < \infty. \quad (7.9)$$

We write $\text{sign } t = t/|t|$ ($t \neq 0$). Suppose that $E|Y|^{2+\delta} < \infty$ for some $0 \leq \delta \leq 2$.

PROPOSITION 7.2. *If $0 \leq \delta < 1$, then $\sigma^2 a(x) = |x| + o(|x|^{1-\delta})$ (as $|x| \rightarrow \infty$) where the error term is bounded if and only if $E[|Y|^3] < \infty$. If $1 \leq \delta < 2$, then*

$$\sigma^2 a(x) = |x| + C^* - (\text{sign } x)\lambda_3 + o(|x|^{1-\delta}).$$

If $\delta = 2$, this formula is valid with $o(|x|^{1-\delta})$ replaced by $O(1/x)$.

PROOF. In the case $\delta = 0$ the assertion is proved both in [3] and in [5], and in fact immediate from (7.1) and Lemma 7.1. The integral in (7.1) may be written as

$$\sigma^2 a(x) = \frac{\sigma^2}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1}{1 - \phi(l)} - \frac{2}{\sigma^2 l^2} \right] (1 - e^{ixl}) dl + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2}{l^2} (1 - e^{ixl}) dl. \quad (7.10)$$

The second term on the right side of (7.10) equals

$$\frac{4}{2\pi} \left[|x| \int_0^{\infty} \frac{1 - \cos l}{l^2} dl - \int_{\pi}^{\infty} \frac{1 - \cos xl}{l^2} dl \right] = |x| - \frac{2}{\pi^2} + O\left(\frac{1}{x^2}\right). \quad (7.11)$$

Let $0 < \delta < 1$. The first term is then $o(|x|^{1-\delta})$. In fact, since $\phi(l) - 1 + (1/2)\sigma^2 l^2 = o(|l|^{2+\delta})$, for any $\varepsilon > 0$ there exists a constant M such that

$$\int_0^{\pi} \frac{\left| \phi(l) - 1 + \frac{1}{2}\sigma^2 l^2 \right|}{|1 - \phi(l)| l^2} |1 - e^{ixl}| dl \leq \varepsilon |x|^{1-\delta} \int_0^{\infty} u^{-2+\delta} |1 - e^{iu}| du + M.$$

The assertion concerning boundedness follows from Corollary 2.1.

Let $1 \leq \delta < 2$. Then by (7.10), (7.7) and (7.11)

$$\sigma^2 a(x) = |x| + C^* + I_c + I_s + O(x^{-2}),$$

where

$$I_c = -\frac{\sigma^2}{2\pi} \int_{-\pi}^{\pi} \left[\Re \left\{ \frac{1}{1 - \phi(l)} \right\} - \frac{2}{\sigma^2 l^2} \right] \cos xl \, dl \quad \text{and}$$

$$I_s = \frac{\sigma^2}{2\pi} \int_{-\pi}^{\pi} \frac{\phi_s}{|1 - \phi|^2} \sin xl \, dl.$$

Recall $\phi_s(l) = -(1/2)\sigma^2 \lambda_3 l^3 + o(|l|^{2+\delta})$. Then, employing a truncation argument with a smooth cut-off function $w(t)$ (i.e., w vanishes outside $(-\pi, \pi)$ and equals 1 in a neighborhood of zero) along with integration by parts one infers that

$$I_s = -\lambda_3 \frac{\sigma^4}{2\pi} \int_0^{\infty} \frac{\sin xl}{\frac{\sigma^4 l}{4}} dl + \int_{-\pi}^{\pi} r(l) w(l) \sin xl \, dl + o\left(\frac{1}{x^3}\right)$$

$$= -\lambda_3 \operatorname{sign} x + o(|x|^{1-\delta}).$$

Here the remainder term $r(l) = o(|l|^{\delta-2})$ with $r'(l) = o(|l|^{\delta-3})$. Similarly we obtain $I_c = o(|x|^{1-\delta})$ if $\delta > 1$. If $\delta = 1$, an application of Riemann-Lebesgue lemma with (7.8) and (7.9) taken into account shows $I_c = o(1)$. The case $\delta = 2$ is similar and

omitted. □

From Subsection 2.1 we extract the following result.

PROPOSITION 7.3. *Suppose that the walk is not left continuous. Then both $f_+(x) - x$ and $\sigma^2 a(x) - f_+(x)$ are positive for all $x > 0$ and tend to extended positive numbers as $x \rightarrow \infty$, which are finite if and only if $E[Y^3; Y > 0] < \infty$. Moreover if $E[Y^3; Y > 0] = \infty$, then*

$$\lim_{x \rightarrow \infty} \frac{f_+(x) - x}{\sigma^2 a(x) - x} = \frac{1}{2}.$$

PROOF. The assertions follow on combining Lemma 2.1 with (2.7) and (2.8). □

COROLLARY 7.1. *Suppose that $E|Y|^3 < \infty$. Then $\lim_{x \rightarrow \pm\infty} (\sigma^2 a(x) - |x|) = C^* \mp \lambda_3 \geq 0$; in particular $C^* = \lambda_3$ (resp. $-\lambda_3$) if and only if the walk is left (resp. right) continuous.*

PROOF. The first half follows from Proposition 7.2 and the second one from the proposition above. □

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Added in Proof. Some asymptotic estimates of the transition probability of the walk killed on a half line are obtained in [12], [13], [14], [15] and [11]. In the first four papers the problem is considered for wider classes of random walks: the variance may be infinite with the law of Y in the domain of attraction of a stable law [15], [14] or of the normal law [12], [13]; Y is not necessarily restricted to the arithmetic variables. The very recent paper [14] describes the asymptotic behavior of $p_{(-\infty, 0]}^n(x, y)$ valid uniformly within the region of stable deviation of the space-time variables and it in particular contains Theorem 1.3 and Corollary 1.1 as a special case. The result for the region $a_\circ^{-1}\sqrt{n} < x, y < a_\circ\sqrt{n}$ is contained in [12]. Theorem 1.3 with $x = 1$ and $y = o(\sqrt{n})$ is also a special case of Proposition 1 of [12], Theorem 5 of [15] and readily derived from Theorem 2 of [13]. Similar results are proved in [11], where Corollary 1.1 is also obtained.

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