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L_∞ models of based mapping spaces

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Abstract. In this paper, for any pointed map $f\colon X \to Y$ between finite type nilpotent CW-complexes, we obtain L_∞ and Lie models of $\mathrm{map}_f^*(X,Y)$, the pointed space of based maps homotopic to f, in terms of Lie algebras constructed from the Quillen models of X and Y. The main advantage of our approach is to allow X to be an infinite dimensional CW-complex, in which case $\mathrm{map}_f^*(X,Y)$ has no longer the homotopy type of a finite type CW-complex.

1. Introduction.

The notion of L_∞ *algebra* or *strongly homotopy Lie algebra* was introduced in the context of deformation theory of algebraic structures as a generalization of classical differential graded Lie algebras [**22**]. Since then, the geometrical translation of algebraic properties of these algebraic structures have been successfully applied in many situations. Interesting examples are the proof by M. Kontsevich of the *Formality Conjecture* on Poisson manifolds [**14**], and the L_∞ structure defined by M. Chas and D. Sullivan on the equivariant free loop space of a manifold [**8**].

In the same spirit as in rational homotopy theory differential graded Lie algebras are realized by rational spaces, L_∞ algebras can also be realized or "integrated", modeling thus the rational homotopy type of a given space. [**10**], [**12**].

On the other hand, based on the work of Haefliger [**11**], models for the rational homotopy type (or for the rational homotopy groups) of the mapping space $\mathrm{map}_f^*(X,Y)$ have been obtained in different contexts [**4**], [**1**], [**6**] when X is a finite complex. By $\mathrm{map}_f^*(X,Y)$ we denote the pointed space of based maps homotopic to a given $f\colon X \to Y$.

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Moreover, starting from [**23**, Section 11], in which Sullivan interprets the rational homotopy type of the classifying space $Baut_1(X)$ in terms of derivations of the model of X, it has been of interest to model, also in terms of derivations, homotopical features of these mapping spaces. See for instance [**5**], [**18**], or [**7**] in which this approach has been successfully applied to the description of the rational homotopy of the fixed and homotopy fixed point set of an S^1 action on a given space X.

However, all this work concerns the case when X is a finite nilpotent CW-complex in which case [**20**] $\mathrm{map}_f^*(X,Y)$ is a nilpotent space of finite type.

In this paper, for a given map $f\colon X \to Y$ between finite type nilpotent CW-complexes (non necessarily finite), we obtain explicit Lie and L_∞ models of $\mathrm{map}_f^*(X,Y)$ (which is no longer of finite type) in terms of derivations between the Quillen models of X and Y.

More precisely, let L be a Quillen minimal model for X and C be a finite type graded differential coalgebra model for X. Then we have a quasi-isomorphism $\varphi\colon L \to \mathscr{L}(C)$ where $\mathscr{L}(C)$ denotes the Quillen functor on C (see Section 2). Now let $\gamma\colon \mathscr{L}(C) \to L'$ be a Quillen model for f. Then, the Lie bracket in L' and the coalgebra structure on C induce a Lie bracket on the graded vector space $s^{-1}\mathscr{D}er_\gamma(\mathscr{L}(C),L')$. We prove (see Theorems 3.2, 4.1 and 5.2 in the text):

Theorem 1. *When X is a finite nilpotent complex, then*:

1. *The differential graded Lie algebra $s^{-1}\mathscr{D}er_\gamma(\mathscr{L}(C),L')$ is a Lie model for* $\mathrm{map}_f^*(X,Y)$.
2. *There is a L_∞ structure on $s^{-1}\mathscr{D}er_{\gamma\varphi}(L,L')$ for which it becomes a L_∞ model for* $\mathrm{map}_f^*(X,Y)$.

When X is a finite type nilpotent (*non necessarily finite*) *CW-complex, then*:

1. $H_*(s^{-1}\mathscr{D}er_{\gamma\varphi}(L,L')) \cong \pi_*\Omega(\mathrm{map}_f^*(X,Y))$ *as graded Lie algebras.*
2. *The universal cover of $s^{-1}\mathscr{D}er_\gamma(\mathscr{L}(C),L')$ is a Lie model for the universal cover of* $\mathrm{map}_f^*(X,Y)$
3. *The universal cover of $s^{-1}\mathscr{D}er_{\gamma\varphi}(L,L')$ is a L_∞ model for the universal cover of* $\mathrm{map}_f^*(X,Y)$.

As an example we then describe the homotopy type of $\mathrm{map}_c^*(BS^1_{\boldsymbol{Q}},Y)$ for any rational space Y, with c denoting the constant map. More generally, for any formal space X and any coformal space Y, both finite type 1-connected CW-complexes, $\mathrm{map}_c^*(X,Y)$ is a coformal space whose homotopy Lie algebra is $Hom(H_*(X;\boldsymbol{Q}),\pi_*\Omega Y\otimes\boldsymbol{Q})$.

We also deduce a splitting result for certain mapping spaces which generalizes [**15**, Theorem 1.2]. Suppose that X and Y are finite type nilpotent CW-complexes (non necessarily finite) and let $\alpha \in \pi_*(X)\otimes \boldsymbol{Q}$ be an iterated Whitehead bracket of length $n \geq \operatorname{nil}\pi_*(Y)\otimes \boldsymbol{Q}$. Then, any $f\colon X \to Y$ extend to a map $\overline{f}\colon X\cup_\alpha e^{k+1} \to Y_{\boldsymbol{Q}}$ and we prove the following in which $\widetilde{\operatorname{map}}^*_f(X,Y)$ denotes the universal cover of $\operatorname{map}^*_f(X,Y)$) and $\operatorname{nil}\pi_*(Y)\otimes \boldsymbol{Q}$ is the length of the longest non zero iterated Whitehead bracket in $\pi_*(Y)\otimes \boldsymbol{Q}$.

Proposition 1. *Let $\alpha \in \pi_*(X)\otimes \boldsymbol{Q}$ be a Whitehead bracket of length $n \geq nil\,\pi_*(Y)\otimes \boldsymbol{Q}$. Then*:

(1) $\widetilde{\operatorname{map}}^*_{\overline{f}}(X\cup_\alpha e^{k+1}, Y)_{\boldsymbol{Q}} \simeq \widetilde{\operatorname{map}}^*_f(X,Y)_{\boldsymbol{Q}} \times \Omega^{k+1}Y_{\boldsymbol{Q}}$.

(2) *Moreover, for any $q\geq 1$,*

$$\pi_q \operatorname{map}^*_{\overline{f}}\big(X\cup_\alpha e^{k+1}, Y\big)\otimes \boldsymbol{Q} \cong \big(\pi_q \operatorname{map}^*_f(X,Y)\otimes \boldsymbol{Q}\big)\oplus\big(\pi_q\Omega^{k+1}Y\otimes \boldsymbol{Q}\big).$$

2. L_∞ algebras.

For a basic compendium of known properties of L_∞ algebras we refer to [**16**] or [**17**]. Also, in [**14**], the algebraic behavior of these structures is nicely introduced as a result of their geometrical counterpart. Here, we simply recall the basic facts we shall need.

Definition 1. An L_∞ *algebra* or *sh-Lie algebra* (*sh* stands for strongly homotopy) is a graded vector space L endowed with a system of linear maps ℓ_k (denoted also by $[\,,\dots,\,]$), $k\geq 0$, of degree $k-2$

$$\ell_k = [\,,\dots,\,]\colon \otimes^k L \to L$$

which satisfy:

(1) ℓ_k are skew-symmetric, i.e., for any k-permutation σ,

$$[x_{\sigma(1)},\dots,x_{\sigma(k)}] = \operatorname{sgn}(\sigma)\varepsilon_\sigma[x_1,\dots,x_k],$$

where ε_σ is the sign given by the Koszul convention and $\operatorname{sgn}(\sigma)$ is the signature of σ.

(2) The following generalized Jacobi identities hold:

$$\sum_{i+j=n+1}\ \sum_{\sigma\in S(i,n-i)} \operatorname{sgn}(\sigma)\varepsilon_\sigma(-1)^{i(j-1)}\ell_j\big(\ell_i(x_{\sigma(1)},\dots,x_{\sigma(i)}),x_{\sigma(i+1)},\dots,x_{\sigma(n)}\big)=0.$$

By $S(i,n-i)$ we denote the $(i,n-i)$ *shuffles* whose elements are permutations σ such that $\sigma(1)<\dots<\sigma(i)$ and $\sigma(i+1)<\dots<\sigma(n)$.

In particular (L, ℓ_1) is a differential vector space and $\ell_2 = [\,,]$ is a skew-symmetric operation for which ℓ_1 satisfies the usual Leibniz rule and the Jacobi identity up to the homotopy given by ℓ_3. Thus a differential graded Lie algebra is the same as an L_∞ algebra for which $\ell_k = 0$ for $k \geq 3$.

The lower central series of an L_∞ algebra L is, as in the classical setting, defined inductively by $F^1L = L$ and, for $i > 1$,

$$F^iL = \sum_{i_1+\cdots+i_k=i} \big[F^{i_1}L, \dots, F^{i_k}L\big].$$

We say that L is *nilpotent* if $F^iL = 0$ for $i > i_0$ for some i_0.

On the other hand, recall that the free commutative algebra ΛV generated by the graded vector space V has a structure of cocommutative graded coalgebra whose comultiplication Δ is defined as the unique morphism of algebras for which every generator $v \in V$ is primitive, i.e., $\Delta(v) = v \otimes 1 + 1 \otimes v$. Explicitly, the reduced diagonal is given by

$$\overline{\Delta}(v_1 \wedge \cdots \wedge v_n) = \sum_{j=1}^{n-1} \sum_{\sigma \in S_n} \varepsilon_\sigma \big(v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(j)}\big) \otimes \big(v_{\sigma(j+1)} \wedge \cdots \wedge v_{\sigma(n)}\big).$$

This structure is naturally augmented by $\varepsilon\colon \Lambda V \to \boldsymbol{Q}$, $\varepsilon(\Lambda^+V) = 0$, $\varepsilon(1) = 1$, and coaugmented by $\boldsymbol{Q} = \Lambda^0 V$. It is the *cofree cocommutative coalgebra* generated by V. This terminology is due to the following:

For a general graded coalgebra C augmented by $\varepsilon\colon C \to \boldsymbol{Q}$ and coaugmented by $\boldsymbol{Q} \to C$, define the n-th reduced diagonal inductively by $\overline{\Delta}^{(n)} = (\overline{\Delta} \otimes 1_{\overline{C}} \otimes \cdots \otimes 1_{\overline{C}})\overline{\Delta}^{(n-1)}\colon \overline{C} \to \otimes^{n+1}\overline{C}$ with $\overline{\Delta}^{(0)} = 1_{\overline{C}}$ and $\overline{\Delta}^{(1)} = \overline{\Delta}$. Call C *primitively cogenerated* if $\overline{C} = \cup_n \ker \overline{\Delta}^{(n)}$.

Then, given a primitively cogenerated cocommutative coalgebra C and a degree zero linear map $f\colon \overline{C} \to V$ there exists a unique morphism of coalgebras $\varphi\colon C \to \Lambda V$ such that $\pi\varphi|_{\overline{C}} = f$. Here, $\pi\colon \Lambda V \to V$ denotes the obvious projection. Indeed, if we consider the linear maps $f_k\colon \otimes^k \overline{C} \to \Lambda^k V$, $f_k(c_1 \otimes \cdots \otimes c_k) = 1/k! f(c_1) \wedge \cdots \wedge f(c_k)$, define $\varphi(1) = 1$ and $\varphi(c) = \sum_{k\geq 0} f_{k+1}\overline{\Delta}^{(k)}(c)$, for $c \in \overline{C}$.

It is also worth remarking that if M is a bi-comodule over ΛV then there is a natural isomorphism of graded vector spaces $\operatorname{Coder}(M, \Lambda V) \cong Hom(M, V)$, given by $\theta \mapsto \pi\theta$. The inverse, for the special case $M = \Lambda V$ is given as follows: decompose any linear map of arbitrary degree $h\colon \Lambda V \to V$ as $\sum_k h^{(k)}$, $h^{(k)}\colon \Lambda^k V \to V$. For each k consider the coderivation $\theta_k\colon \Lambda V \to \Lambda V$:

$$\theta_k(v_1 \wedge \cdots \wedge v_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} \pm h^{(k)}(v_{i_1} \wedge \cdots \wedge v_{i_k}) \wedge v_1 \wedge \ldots \widehat{v}_{i_1} \ldots \widehat{v}_{i_k} \ldots \wedge v_n.$$

Then, in the isomorphism above the map h is sent to the coderivation $\sum_k \theta_k$.

PROPOSITION 2 ([**16**], [**17**]). *L_∞ algebra structures on a graded vector space L are in bijective correspondance with codifferentials on the coalgebra ΛsL.*

PROOF. As this is standard, we only sketch this equivalence:

On the one hand, a codifferential on ΛsL is just a degree -1 coderivation D which, as stated, corresponds to a degree -1 linear map $D\colon \Lambda sL \to sL$. Such a map is the sum of linear maps $D^{(k)}\colon \Lambda^k sL \to sL$ which, in turn, correspond to a collection of skew-symmetric operations ℓ_k of degree $k-2$,

$$\ell_k = s^{-1} \circ D^{(k)} \circ s^{\otimes k}\colon \otimes^k L \to L.$$

Here, $s^{\otimes k}\colon \otimes^k L \to \otimes^k sL$ denotes the obvious map of degree k. The equation $D^2 = 0$ implies the generalized Jacobi identities.

On the other hand given k-ary maps ℓ_k as above, define degree -1 linear maps $D^{(k)}\colon \otimes^k sL \to sL$ by

$$D^{(k)} = (-1)^{(k(k-1))/2} s \circ \ell_k \circ (s^{-1})^{\otimes k}.$$

These maps are symmetric (in the graded sense) so they factor as $D^{(k)}\colon \Lambda^k sL \to sL$. Finally the map $D = \sum_k D^{(k)}$ determines a degrere -1 coderivation on ΛsL which again satisfy $D^2 = 0$. $\square$

Observe that, if L is a finite type graded vector space, an L_∞ structure on L is then equivalent to a CDGA (commutative differential graded algebra) structure on $(\Lambda sL)^\vee \cong \Lambda(sL)^\vee \cong \Lambda s^{-1}L^\vee$. We shall denote by $\mathscr{C}^\infty(L)$ this structure and call it the *Cartan-Chevalley-Eilenberg algebra* on L.

DEFINITION 2. Given two L_∞ algebras L and L', a morphism of L_∞ algebras is a morphism of differential graded coalgebras $f\colon (\Lambda sL, D) \to (\Lambda sL', D')$.

REMARK 3. Observe that an L_∞ morphism does not correspond, in general to a degree zero map $f\colon L \to L'$ commuting with all the k-ary bracket ($f \circ \ell_k = \ell'_k \circ f^{\otimes k}$). In fact a morphism $f\colon (\Lambda sL, D) \to (\Lambda sL', D')$ of differential graded coalgebras is determined by $\widetilde{f} = \pi f\colon \Lambda sL \to sL'$ which is the sum of maps $\widetilde{f}^{(k)}\colon \Lambda^k sL \to sL'$. This produces a system of skew-symmetric maps of degree $1-k$

$$f^{(k)}\colon \otimes^k L \longrightarrow L'.$$

Therefore, morphisms of differential coalgebras correspond to systems of maps $f^{(k)}$ that satisfy a sequence of equations involving the brackets ℓ_k and ℓ'_k, $k \geq 0$.

For instance, an L_∞ morphism between DGL's (differential graded Lie algebras) (L, ∂) and (L', ∂') is just a morphism between their Cartan-Chevalley-Eilenberg constructions (see next section) $f\colon (\Lambda sL, D = d_1 + d_2) \to (\Lambda sL', D' = d'_1 + d'_2)$. In this particular case, the equations satisfied by the $f^{(k)}$'s only involve the differentials and the Lie brackets on L and L'. Explicitly, the first two equations are:

(1) $\partial' f^{(1)} = f^{(1)}\partial$, i.e., $f^{(1)}\colon L \to L'$ is a differential map.
(2) $f^{(1)}[x, y] = [f^{(1)}(x), f^{(1)}(y)] + \partial' f^{(2)}(x \otimes y) - f^{(2)}(\partial x \otimes y - (-1)^{|x|} x \otimes \partial y)$.

Hence, an L_∞ morphism between DGL's is not in general a DGL morphism.

A *quasi-isomorphism* between L_∞ algebras L and L' is an L_∞ morphism $f\colon (\Lambda sL, D) \to (\Lambda sL', D')$ such that $f^{(1)}$ is a quasi-isomorphism of differential vector spaces.

3. Sullivan, Quillen and L_∞ models of a space.

Our results heavily depend on known facts and techniques arising from rational homotopy theory. All of them, and with the same notation we use, can be found in [**9**]. Here we simply recall the following:

In [**23**] (see also [**2**]), Sullivan introduces a couple of adjoint functors,

$$\text{SimplSets} \underset{\langle\,\rangle}{\overset{A_{PL}}{\rightleftarrows}} \text{CDGA}$$

between the homotopy categories of commutative differential graded algebras (CDGA henceforth) and simplicial sets which turns out to be an equivalence when considering 1-connected (more generally, nilpotent) simplicial sets of finite type (over $\boldsymbol{Q}$) and 1-connected rational CDGA's of finite type. In fact, for every space (or simplicial set) M of this kind there is a CDGA $(\Lambda V, d)$ which algebraically models the rational homotopy type of M and is unique up to isomorphism. This is the *Sullivan minimal model* of M. By ΛV we mean the free commutative graded algebra generated by the vector space V, i.e., $\Lambda V = TV/I$ in which TV is the tensor algebra over V and I is the ideal generated by $v \otimes w - (-1)^{|v||w|} w \otimes v$, $v, w \in V$. The differential d satisfies a *minimal* condition which, in the simply connected case, is equivalent to say that, for any generator $v \in V$, dv is a "polynomial" in ΛV with no linear term.

As the dual of a cocommutative differential graded coalgebra (CDGC henceforth) is a CDGA, a *coalgebra model* of a nilpotent space X of finite type is a CDGC such that its dual is a CDGA model of X.

The bridge between the categories of CDGC's and that of DGL's is provided by the Quillen and Cartan-Chevalley-Eilenberg functors:

$$\mathrm{DGL} \underset{\mathscr{L}}{\overset{\mathscr{C}}{\rightleftarrows}} \mathrm{CDGC}$$

On one hand, the *Cartan-Chevalley-Eilenberg* construction on a given DGL (L, d_L) is the CDGC $\mathscr{C}(L, d_L) = (\Lambda sL, d = d_1 + d_2)$ in which, as usual, $(sL)_k = L_{k-1}$ and

$$d_1(sx_1 \wedge \cdots \wedge sx_k) = -\sum_{i=1}^{k}(-1)^{n_i} sx_1 \wedge \cdots \wedge sd_L x_i \wedge \cdots \wedge sx_k,$$

$$d_2(sx_1 \wedge \cdots \wedge sx_k) = \sum_{1\le i<j\le k} (-1)^{n_{ij}} s[x_i, x_j] \wedge sx_1 \cdots \widehat{sx_i} \cdots \widehat{sx_j} \cdots \wedge sx_k.$$

Here, $n_i = \sum_{j<i} |sx_j|$, and

$$sx_1 \wedge \cdots \wedge sx_k = (-1)^{n_{ij}} sx_i \wedge sx_j \wedge sx_1 \cdots \widehat{sx_i} \cdots \widehat{sx_j} \cdots \wedge sx_k.$$

The dual of the construction above is the CDGA

$$\mathscr{C}^*(L, d_L) = Hom(\mathscr{C}(L, d_L), \boldsymbol{Q}).$$

If L is of finite type, then $\mathscr{C}^*(L, d_L) = (\Lambda V, d)$ where V and sL are dual graded vector spaces and $d = d_1 + d_2$ in which: $\langle d_1 v; sx\rangle = (-1)^{|v|}\langle v; sd_L x\rangle$ and $\langle d_2 v; sx \wedge sy\rangle = (-1)^{|y|+1}\langle v; s[x, y]\rangle$.

From now on we shall write, for convenience, $\mathscr{C}(L)$ and $\mathscr{C}^*(L)$ instead of $\mathscr{C}(L, d_L)$ and $\mathscr{C}^*(L, d_L)$.

On the other hand, the *Quillen* functor is constructed for any CDGC C augmented by $\varepsilon\colon C \to \boldsymbol{Q}$ and coaugmented by $\boldsymbol{Q} \to C$. Denote by $\overline{C} = \ker\varepsilon$ and consider the reduced diagonal $\overline{\Delta}\colon \overline{C} \to \overline{C} \otimes \overline{C}$. Then $\mathscr{L}(C, d) = (\boldsymbol{L}(s^{-1}\overline{C}), \partial)$ in which:

(i) $\boldsymbol{L}(s^{-1}\overline{C})$ is the *free* Lie algebra generated by $s^{-1}\overline{C}$, i.e., the sub Lie algebra of the tensor Lie algebra (bracket is the commutator) $T(s^{-1}\overline{C})$ generated by $s^{-1}\overline{C}$.

(ii) $\partial = \partial_1 + \partial_2$ where $\partial_1(s^{-1}c) = -s^{-1}dc$ and

$$\partial_2(s^{-1}c) = \frac{1}{2}\sum_i (-1)^{|a_i|}\left[s^{-1}a_i, s^{-1}b_i\right]$$

being $\overline{\Delta}c = \sum a_i \otimes b_i$.

Observe that for an L_∞ algebra L, $\mathscr{C}^\infty(L)$ is a natural generalization of the Cartan-Chevalley-Eilenberg construction on a differential graded Lie algebra. In fact, the following can be easily proved:

Proposition 3. *Suppose L is a nilpotent L_∞ algebra of finite type concentrated in non negative degrees. Then, $\mathscr{C}^\infty(L) = (\Lambda V, d)$ is a Sullivan model in which V and sL are dual graded vector spaces and $d = \sum_{j\geq 1} d_j$ in which $\langle d_j v; sx_1 \wedge \cdots \wedge sx_j\rangle = \epsilon\langle v; s[x_1,\ldots,x_j]\rangle$ where ϵ is the appropriate sign given by the Koszul convention.*

Conversely, suppose $(\Lambda V, d)$ is an arbitrary Sullivan algebra of finite type. Then, a nilpotent L_∞ algebra L is determined uniquely by the condition $(\Lambda V, d) = \mathscr{C}^\infty(L)$.

The relation of the functors above with the homotopy category is the following:

In [**21**], Quillen associates to any 1-connected space X (non necessarily of finite type!) a DGL (free as Lie algebra) $\lambda(X)$ which determines an equivalence between the homotopy categories of rational 1-connected spaces and that of reduced ($L_{\leq 0} = 0$) DGL's over $\boldsymbol{Q}$. If moreover, X is of finite type then $\mathscr{C}^*\lambda(X)$ is quasi-isomorphic to the Sullivan model of X [**19**]. On the other hand, whenever X is nilpotent and of finite type and C is a coalgebra model of X, the association $X \rightsquigarrow \mathscr{L}(C)$ extends the mentioned equivalence to nilpotent spaces (but finite type is required).

Definition 4. Given a 1-connected space X, an L_∞ *model of* X is an L_∞ algebra quasi-isomorphic, as L_∞ algebra, to $\lambda(X)$.

Note that, by Proposition 3, if X is nilpotent and of finite type, there is an L_∞ structure on $\pi_*(\Omega X)\otimes\boldsymbol{Q}$ modeling X. The Eckmann-Hilton dual of this assertion is also true and well known: in [**13**], $H^*(X;\boldsymbol{Q})$ is endowed with an A_∞-structure quasi-isomorphic to the Sullivan model of X.

Next, denote by $\mathscr{L}_X = (\boldsymbol{L}(U), \partial)$ a Quillen model of a nilpotent space X of finite type. Assume U is of finite type so that $\mathscr{C}^*(\mathscr{L}_X)$ is a Sullivan model of X. In this case denote by I the ideal $\boldsymbol{L}(U)_{\geq 2}\oplus Z_1$ where Z_1 is the vector space of cycles in $\boldsymbol{L}(U)_1$. Applying $\mathscr{C}^*$ to the short exact sequence of DGL's:

$$0 \to I \longrightarrow \mathscr{L}_X \longrightarrow \frac{\mathscr{L}_X}{I} \to 0,$$

we obtain a relative Sullivan algebra

$$\mathscr{C}^*\left(\frac{\mathscr{L}_X}{I}\right) \longrightarrow \mathscr{C}^*(\mathscr{L}_X) \longrightarrow \mathscr{C}^*(I)$$

which is a model for the universal covering of X,

$$\widetilde{X} \longrightarrow X \longrightarrow K(\pi_1 X, 1).$$

This shows that I is a Lie model for $\widetilde{X}$ called the *universal cover of* $\mathscr{L}_X$.

Now consider a pronilpotent space $X = \lim_{\leftarrow_n} X_n$ where each X_n is a rational nilpotent space of finite type. Denote by $p_n \colon \mathscr{L}(n+1) \twoheadrightarrow \mathscr{L}(n)$ a surjective Lie model of the map $X_{n+1} \to X_n$ and define

$$\mathscr{L} = \lim_{\leftarrow_n} \mathscr{L}(n).$$

Lemma 5.

(1) $H_q(\mathscr{L}) = \lim_{\leftarrow_n} \pi_{q+1} X_n = \pi_{q+1} X$, $q \geq 0$.

(2) *Denote by I the ideal $\mathscr{L}_{\geq 2} \oplus Z_1 \subset \mathscr{L}$, where $Z_1 \subset \mathscr{L}$ is the vector space of cycles in degree* 1*. Then I is a Lie model for $\widetilde{X}$.*

Proof.

(1) $H_q(\mathscr{L}) = \lim_{\leftarrow_n} H_q(\mathscr{L}(n)) = \lim_{\leftarrow_n} \pi_{q+1} X_n$. To finish apply [**3**, IX, Theorem 3.1], taking into account that $\lim^1_{\leftarrow}$ vanishes when applied to a tower of vector spaces [**26**], to obtain that $\lim_{\leftarrow_n} \pi_{q+1} X_n = \pi_{q+1} X$.

(2) In particular, from (1) we deduce a weak homotopy equivalence $\widetilde{X} \xrightarrow{\simeq} \lim_{\leftarrow_n} \widetilde{X}_n$.

Consider the ideal in $\mathscr{L}(n)$ defined by

$$I_n = \mathscr{L}(n)_{\geq 2} \oplus Z\big(\mathscr{L}(n)_1\big).$$

We then have a short exact sequence of towers

$$0 \to \lim_{\leftarrow_n} I_n \longrightarrow \mathscr{L} \longrightarrow \lim_{\leftarrow n} \left(\frac{\mathscr{L}(n)}{I_n}\right) \to 0.$$

Consider Quillen models $\mathscr{L}_{\widetilde{X}_n}$ and $\mathscr{L}_{\widetilde{X}}$ of $\widetilde{X}_n$ and $\widetilde{X}$ respectively. Then we have coherent quasi-isomorphisms $\mathscr{L}_{\widetilde{X}_n} \xrightarrow{\simeq} I_n$ which gives, by composition, a quasi-isomorphism $\mathscr{L}_{\widetilde{X}} \xrightarrow{\simeq} \lim_{\leftarrow_n} I_n$. This finishes the proof as $I = \lim_{\leftarrow_n} I_n$. □

4. Lie algebras of derivations modeling mapping spaces $\mathrm{map}_f^*(X, Y)$ with X finite.

Let $\rho\colon L \to L'$ be a DGL morphism and consider the differential graded vector space of Lie ρ-derivations $(Der_\rho(L, L'), \delta)$. Explicitly $Der_\rho(L, L')_n$ is the space of linear maps of degree n, $\theta\colon L_* \longrightarrow L'_{*+n}$, for which $\theta[x, y] = [\theta(x), \rho(y)] + (-1)^{n|x|}[\rho(x), \theta(y)]$, $x, y \in L$. The differential is defined as usual $\delta\theta = \partial\theta + (-1)^n\theta\partial$. Of particular interest is the space $\mathscr{D}er_\rho(L, L')$ of positive ρ-derivations,

$$\mathscr{D}er_\rho(L, L')_i = \begin{cases} Der_\rho(L, L')_i & \text{for } i > 1, \\ ZDer_\rho(L, L')_1 & \text{for } i = 1, \end{cases}$$

in which Z denotes the space of cycles. We shall also denote by δ the differential of this complex.

Next, let $f\colon X \to Y$ be a map of nilpotent complexes with X finite, let L' be a Lie model of Y and choose a Quillen model of X of the form $\mathscr{L}(C)$ for some CDGC, C. This is always possible by taking C, for instance, the dual of a commutative differential graded algebra of the rational homotopy type of X. Finally choose any DGL morphism $\gamma\colon \mathscr{L}(C) \to L'$ modeling the homotopy type of f.

The restriction of γ to $U = s^{-1}\overline{C}$ gives a linear map $\gamma\colon s^{-1}\overline{C} \to L'$ which is also identified to a map $\gamma\colon \overline{C} \to sL'$. Composing with the degree -1 isomorphism $sL' \to L'$ we obtain the map $\overline{\gamma}\colon \overline{C} \longrightarrow L'$.

Next consider the vector space $Hom(\overline{C}, L')$ with the usual bracket, $[f, g] = [\ ,\ ] \circ f \otimes g \circ \Delta$, and the perturbed differential $D_\gamma = D + ad_{\overline{\gamma}}$:

$$D_\gamma f = \partial_L f + (-1)^{|f|} f\delta + [\overline{\gamma}, f].$$

Finally, discard the negative graded part by defining $\mathscr{H}om(\overline{C}, L')$:

$$\mathscr{H}om_i(\overline{C}, L') = \begin{cases} Hom_i(\overline{C}, L') & \text{for } i > 1, \\ Z(Hom_1(\overline{C}, L')) & \text{for } i = 1. \end{cases}$$

Then, the following holds:

Theorem 2 ([**6**, Corollary 15]). *$(\mathscr{H}om(\overline{C}, L'), D_\gamma)$ is a Lie model of $\mathrm{map}_f^*(X, Y)$.*

From this we can prove the following:

THEOREM 3. *The DGL $s^{-1}\mathscr{D}er_\gamma(\mathscr{L}(C),L')$, equipped with the differential and the bracket defined by*

$$\delta\theta = \partial\circ\theta + (-1)^{|\theta|}\theta\circ\partial,$$
$$[f,g](a) = \sum_i (-1)^{1+|a_i|\,|g|}[f(a_i),g(b_i)], \quad \text{where } \overline{\Delta}(a) = \sum_i a_i\otimes b_i,$$

is a Lie model for $\mathrm{map}^*_f(X,Y)$.

PROOF. Observe that the restriction of a given derivation in $\mathscr{D}er_\gamma$ $(\mathscr{L}(C),L')$ to $s^{-1}\overline{C}$ produces an isomorphism of graded vector spaces,

$$\Upsilon\colon s^{-1}\mathscr{D}er_\gamma(\mathscr{L}(C),L') \xrightarrow{\cong} \mathscr{H}om(\overline{C},L'), \quad \Upsilon(s^{-1}\theta)(c) = (-1)^{|\theta|}\theta(s^{-1}c).$$

We now prove that Υ commutes with the differentials $s^{-1}\delta$ and D_γ respectively:

$$D_\gamma\Upsilon(s^{-1}\theta)(c) = \Upsilon(s^{-1}\delta)(s^{-1}\theta)(c),$$

for any derivation $\theta\in\mathscr{D}er_\gamma(\mathscr{L}(C),L')$ and $c\in\overline{C}$.

On the one hand,

$$\begin{aligned} D_\gamma\Upsilon(s^{-1}\theta)(c) &= \partial_{L'}\Upsilon(s^{-1}\theta)(c) + (-1)^{|\theta|+1}\Upsilon(s^{-1}\theta)(dc) + \big[\overline{\gamma},\Upsilon(s^{-1}\theta)\big](c)\\ &= (-1)^{|\theta|}\partial_{L'}\theta(s^{-1}c) - \theta(s^{-1}dc)\\ &\quad + \sum_i(-1)^{|a_i|(|\theta|+1)+|\theta|}\sum_i\big[\gamma(s^{-1}a_i),\theta(s^{-1}b_i)\big], \end{aligned}$$

where $\partial_{L'}$ and d are the differentials in L' and C respectively, and $\overline{\Delta}(c)=\sum_i a_i\otimes b_i$.

On the other hand,

$$\Upsilon(s^{-1}\delta)(s^{-1}\theta)(c) = (-1)^{|\theta|}\partial_{L'}\theta(s^{-1}c) + \theta(\partial s^{-1}c).$$

where $\partial = \partial_1+\partial_2$ is the differential in $L=\boldsymbol{L}(s^{-1}\overline{C})$. Hence,

$$\begin{aligned} \theta(\partial s^{-1}c) &= -\theta(s^{-1}dc) + \theta\bigg(\frac{1}{2}\sum_i(-1)^{|a_i|}\big[s^{-1}a_i,s^{-1}b_i\big]\bigg)\\ &= -\theta(s^{-1}dc) + \sum_i(-1)^{|a_i|+|\theta|(|a_i|+1)}\big[\gamma(s^{-1}a_i),\theta(s^{-1}b_i)\big] \end{aligned}$$

as θ is a γ-derivation and C is cocommutative.

Finally, a straightforward computation shows that Υ respects the Lie brackets and therefore, is an isomorphism of DGL's. □

Now let $\varphi\colon L \xrightarrow{\simeq} \mathscr{L}(C)$ be a quasi-isomorphism where L is a cofibrant DGL. Then:

COROLLARY 1. *$H_*(\mathscr{D}er_{\gamma\varphi}(L,L'),\delta)$ is naturally isomorphic (as graded vector space) to $\pi_*\Omega\,\mathrm{map}^*_f(X,Y)$.*

The proof is an immediate consequence of the following:

LEMMA 6. *Let $\gamma\colon L_1 \to L'$ and $\psi\colon L_2 \xrightarrow{\simeq} L_1$ be DGL morphisms, with L_1, L_2 cofibrant and ψ a quasi-isomorphism of graded vector spaces. Then, the induced map*

$$\psi_*\colon \mathscr{D}er_\gamma(L_1,L') \xrightarrow{\simeq} \mathscr{D}er_{\gamma\psi}(L_2,L')$$

is a quasi-isomorphism.

PROOF. Write $L_1 = \boldsymbol{L}(W)$, $L_2 = \boldsymbol{L}(U)$, and filter the spaces $\mathscr{D}er_\gamma(\boldsymbol{L}(U),L')$ and $\mathscr{D}er_{\gamma\psi}(\boldsymbol{L}(W),L')$ respectively by

$$F^p = \big\{f \in \mathscr{D}er_\gamma(\boldsymbol{L}(U),L'),\ f(U) \in L'^{\geq p}\big\}$$

and

$$G^p = \big\{g \in \mathscr{D}er_{\gamma\psi}(\boldsymbol{L}(W),L'),\ g(W) \in L'^{\geq p}\big\},$$

so that ψ_* is a morphism of filtered spaces. At the 0-level, the induced morphism of the resulting spectral sequences has the form $(E_0(\psi_*), d_0)$ where

$$E_0(\psi_*)\colon \mathscr{D}er_\gamma\big((\boldsymbol{L}(U),\partial_0),(L',\overline{\partial})\big) \longrightarrow \mathscr{D}er_{\gamma\psi}\big((\boldsymbol{L}(W),\partial_0),(L',\overline{\partial})\big)$$

is again given by pre-composition; ∂_0 denotes the indecomposable part of the differential in the corresponding free DGL; $\overline{\partial}$ denotes the induced differential on the associated graded space $L' = \oplus_p L'^{\geq p}/L'^{\geq p+1}$; and d_0 is the usual differential. However, in this setting, the map

$$\Upsilon_{\boldsymbol{L}(U)}\colon \mathscr{D}er_\gamma\big((\boldsymbol{L}(U),\partial_0),(L',\overline{\partial})\big) \xrightarrow{\cong} \mathscr{H}om\big((U,\partial_0),(L',\overline{\partial})\big),$$

is an isomorphism of differential vector spaces, this time with the usual differentials (the same for $\Upsilon_{\boldsymbol{L}(W)}$).

Therefore, $E_0(\psi_*)$ can be seen as:

$$Q(\psi)_*\colon \mathscr{H}om\big((U,\partial_0),(L',\overline{\partial})\big) \xrightarrow{\simeq} \mathscr{H}om\big((W,\partial_0),(L',\overline{\partial})\big)$$

where $Q(\psi)\colon (U,\partial_0) \xrightarrow{\simeq} (W,\partial_0)$ is induced by ψ on the indecomposables. The map $Q(\psi)_*$ is clearly a quasi-isomorphism as $Q(\psi)$ is. Hence, $E_1(\psi_*)$ is an isomorphism and, by comparison, ψ_* is a quasi-isomorphism. $\square$

The construction presented in this section is natural in X. Let $g\colon X \to X'$ be a map between finite complexes and $\psi\colon C \to C'$ a CDGC model for g. Let now $f\colon X' \to Y$ be a continuous map, L a Quillen model for Y and $\gamma\colon \mathscr{L}(C') \to L$ a model for f.

Theorem 4. *With the above notations, the induced map*

$$s^{-1}\mathscr{D}er_\gamma(\mathscr{L}(C'),L) \to s^{-1}\mathscr{D}er_{\gamma\psi}(\mathscr{L}(C),L)$$

is a DGL model for $\operatorname{map}^*(g,Y)\colon \operatorname{map}^*_f(X',Y) \to \operatorname{map}^*_{fg}(X,Y)$.

Proof. We show that, applying $\mathscr{C}^*$ to

$$s^{-1}\mathscr{D}er_\gamma(\mathscr{L}(C'),L) \to s^{-1}\mathscr{D}er_{\gamma\psi}(\mathscr{L}(C),L),$$

we obtain a DGA model of $\operatorname{map}^*(g,Y)\colon \operatorname{map}^*_f(X',Y) \to \operatorname{map}^*_{fg}(X,Y)$.

For it consider the following diagram of DGL's

$$\begin{array}{ccc}
s^{-1}\mathscr{D}er_\gamma(\mathscr{L}(C'),L) & \longrightarrow & s^{-1}\mathscr{D}er_{\gamma\psi}(\mathscr{L}(C),L) \\
\Upsilon \big\downarrow \cong & & \cong \big\downarrow \Upsilon \\
\mathscr{H}om(\overline{C'},L) & \longrightarrow & \mathscr{H}om(\overline{C},L) \\
\cong \big\downarrow & & \big\downarrow \cong \\
s^{-1}\mathscr{D}er(\Lambda V,\overline{C'}^\sharp) & \longrightarrow & s^{-1}\mathscr{D}er(\Lambda V,\overline{C}^\sharp)
\end{array}$$

in which: $V=(sL)^\sharp$, Υ is the isomorphism defined in the proof of Theorem 3, and the lower square commutes by [**6**, Theorem 13]. The DGL structure in the last arrow is the one given in [**5**, Theorem 1] or [**6**, Definition 5]. Hence,

$$\mathscr{C}^*\big(s^{-1}\mathscr{D}er_\gamma(\mathscr{L}(C'),L)\big) \longleftarrow \mathscr{C}^*\big(s^{-1}\mathscr{D}er_{\gamma\psi}(\mathscr{L}(C),L)\big)$$

is identified to

$$\mathscr{C}^*\big(s^{-1}\mathscr{D}er(\Lambda V,\overline{C'}^\sharp)\big) \longleftarrow \mathscr{C}^*\big(s^{-1}\mathscr{D}er(\Lambda V,\overline{C}^\sharp)\big).$$

However, by [**6**, Theorem 9], and using the same notation than in this reference, this last morphism equals to

$$\Lambda(1\otimes\overline{\psi})\colon \Lambda\big(\overline{V\otimes A}^1\oplus(V\otimes A)^{\geq 2}\big)\longrightarrow\Lambda\big(\overline{V\otimes A'}^1\oplus(V\otimes A')^{\geq 2}\big),$$

which is a DGA model of $\mathrm{map}^*(g,Y)\colon\ \mathrm{map}^*_f(X',Y)\to\mathrm{map}^*_{fg}(X,Y)$. Here, $A=\overline{C}^\sharp$, $A'=\overline{C'}^\sharp$ and $\overline{\psi}\colon\overline{C}\to\overline{C'}$. □

As a final remark, consider the particular case of the space of self-equivalences of X homotopic to the identity map $\mathrm{map}_1(X,X)=aut_1(X)$, and choose $L=\mathscr{L}(C)$ a model of X as in Theorem 3. Then, by this result, $s^{-1}\mathscr{D}er(L,L)$ is a Lie model of $aut_1(X)$. Note that the Lie bracket is not related with the usual commutator bracket of derivations of $\mathscr{D}er(L,L)$ which is known to model the classifying space $Baut_1(X)$ [**22**], [**23**], [**24**].

5. L_∞ algebras of derivations modeling mapping spaces $\mathrm{map}^*_f(X,Y)$ with X finite.

As before, let $f\colon X\to Y$ be a map of nilpotent complexes with X finite, let L be the minimal Quillen model of X and let L' be any Lie model of Y. Choose also a DGL morphism $\gamma\colon L\to L'$ modeling the homotopy type of f. Then we have:

Theorem 5. *There is a structure of L_∞ algebra on $s^{-1}\mathscr{D}er_\gamma(L,L')$ for which it becomes an L_∞ model of* $\mathrm{map}^*_f(X,Y)$.

For its proof we need the following results, which are of a classical type in deformation theory. However, we have added here a complete proof in order to have the precise hypothesis and statements we want in our applications.

Proposition 4. *Let $\varphi\colon(L,\partial)\xrightarrow{\simeq}(E,\partial)$ be a surjective quasi-isomorphism of finite type complexes where (L,∂) is an L_∞ algebra for which $\ell_1=\partial$. Then, E has an structure of L_∞ algebra for which $\ell_1=\partial$ and there is a quasi-isomorphism of L_∞ algebras $\{f^{(k)}\}$ from L to E with $f^{(1)}=\varphi$.*

Moreover, this construction is natural with respect to surjective morphisms: consider the following commutative diagram of finite type complexes

$$
\begin{array}{ccccc}
\ker\varphi_1 & \hookrightarrow & L_1 & \overset{\varphi_1}{\underset{\simeq}{\twoheadrightarrow}} & E_1 \\
\downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\
\ker\varphi_2 & \hookrightarrow & L_2 & \overset{\varphi_2}{\underset{\simeq}{\twoheadrightarrow}} & E_2
\end{array}
$$

where α is a morphism of L_∞ algebras, φ_i are surjective quasi-isomorphsms, $i = 1, 2$, and all the vertical arrows are surjections. Then, β is also a morphism of L_∞ algebras.

The proof for this result requires its dual counterpart:

Lemma 7. *Let $(\Lambda W, D)$ be a Sullivan algebra with $D = \sum_{i\geq 1} D_i$, $D_i(W) \subset \Lambda^i W$, and let $j\colon (V, d) \overset{\simeq}{\hookrightarrow} (W, D_1)$ be an injective quasi-isomorphism. Then, there exists a differential D on ΛV for which $D_1 = d$ and a quasi-isomorphism $\psi\colon (\Lambda V, D) \xrightarrow{\simeq} (\Lambda W, D)$ extending j, i.e., such that $(\psi - j)(V) \in \Lambda^{\geq 2} V$.*

Moreover, this construction is natural: consider the following commutative diagram of complexes

$$
\begin{array}{ccccc}
(V_2, d) & \overset{j_2}{\underset{\simeq}{\hookrightarrow}} & (W_2, d) & \longrightarrow & (W_2/V_2, \overline{d}) \\
\phi\big\downarrow & & \theta\big\downarrow & & \big\downarrow \\
(V_1, d) & \overset{j_1}{\underset{\simeq}{\hookrightarrow}} & (W_1, d) & \longrightarrow & (W_1/V_1, \overline{d})
\end{array}
$$

where j_1, j_2 are injective quasi-isomorphisms and all the vertical arrows are injections. Suppose there are differentials D in ΛW_1 and ΛW_2, with $D_1 = d$, for which $\Lambda\theta\colon (\Lambda W_2, D) \to (\Lambda W_1, D)$ is a CDGA morphism. Then, the following is a commutative diagram of CDGA's:

$$
\begin{array}{ccc}
(\Lambda V_2, D) & \overset{\psi_2}{\underset{\simeq}{\longrightarrow}} & (\Lambda W_2, D) \\
\Lambda\phi\big\downarrow & & \big\downarrow\Lambda\theta \\
(\Lambda V_1, D) & \overset{\psi_1}{\underset{\simeq}{\longrightarrow}} & (\Lambda W_1, D)
\end{array}
$$

Proof. For the first statement, choose a basis of the acyclic complex $(W/V, \overline{D}_1)$ of the form $\{x_i, y_i\}$, where $\overline{D}_1 x_i = y_i$, and take elements $w_i \in W$ such that $[w_i] = x_i$. Then, the section $\sigma\colon (W/V, \overline{D}_1) \to W$, $\sigma(x_i) = w_i$, $\sigma(y_i) = D_1 w_i$ induces an isomorphism of differential vector spaces $j \oplus \sigma\colon (V, d) \oplus (W/V, \overline{D}_1) \overset{\cong}{\to} (W, D_1)$. Endow $\Lambda(V \oplus \langle x_i, y_i\rangle)$ with a differential D so that

$$\Lambda(j\oplus\sigma)\colon \big(\Lambda(V\oplus\langle x_i,y_i\rangle),D\big)\xrightarrow{\cong}(\Lambda W,D)$$

becomes a CDGA isomorphism. As $Dx_i=y_i+\Omega_i$ with Ω_i decomposable,

$$\big(\Lambda(V\oplus\langle x_i,y_i\rangle),D\big)=\big(\Lambda(V\oplus\langle x_i,y_i'\rangle),D\big)$$

with $Dx_i=y_i'=y_i+\Omega_i$. Note that, in the quotient quasi-isomorphism $p\colon(\Lambda V\oplus\langle x_i,y_i'\rangle,D)\overset{\simeq}{\twoheadrightarrow}(\Lambda V,\overline{D})$, $\overline{D}_1(v)=dv$. We now show that we may choose a section ρ of p such that $\rho(v)=v+\Phi$, with Φ decomposable:

Let $(v_j)_{j\geq1}$ be a basis of V such that $\overline{D}(v_j)\in\Lambda V_{<j}$ and assume ρ is a section of p for which $\rho(v_k)-v_k$ is decomposable for $k<j$. Then, the linear part of $\rho(v_j)-v_j$ is a D_1-cycle in $\langle x_i,y_i'\rangle$, and so is a D_1-boundary, $\rho(v_j)=v_j+D_1(u)+\omega$, with ω decomposable. Define $\rho'=\rho$ on $\Lambda V_{<j}$, $\rho'(v_j)=\rho(v_j)-D(u)$ and extend ρ' into a map on all of ΛV homotopic to ρ. Proceeding inductively in this way we obtain the required section.

To finish, define $\psi=\Lambda(j\oplus\sigma)\circ\rho$.

For the second statement, observe that the quotient map $(W_2/V_2,\overline{d})\hookrightarrow(W_1/V_1,\overline{d})$ is injective, Hence, as above, we may decompose W_2 and W_1 as follows:

$$(W_2,d)\cong(V_2,d)\oplus(S,d),\qquad (W_1,d)\cong(V_1,d)\oplus(S,d)\oplus(T,d),$$

where both (S,d) and (T,d) are acyclic complexes. With this decomposition the lemma easily follows. □

Proof of Proposition 4. For the first assertion, the dual map $\varphi^\vee\colon(E^\vee,\partial^\vee)\hookrightarrow(L^\vee,\partial^\vee)$ is an injective quasi-isomorphism. Moreover (see Proposition 3) $\mathscr{C}^\infty(L)=(\Lambda sL^\vee,D)$ where $D_1=s\partial^\vee$. By the lemma aboe there is a differential D on $\Lambda sE^\vee$ where $D_1=s\partial^\vee$ and a CDGA quasi-isomorphism $\psi\colon(\Lambda sE^\vee,D)\xrightarrow{\simeq}(\Lambda sL^\vee,D)$ extending $s\varphi^\vee$.

For the second assertion, take the dual of the diagram and apply Lemma 7. □

Proof of Theorem 5. Recall that, given a coalgebra model of X, C, there is an injective DGL quasi-isomorphism $\psi\colon L\overset{\simeq}{\hookrightarrow}\mathscr{L}(C)$ [**9**, Section 22]. By Lemma 6, this produces a surjective quasi-isomorphism of differential vector spaces

$$\psi_*\colon\mathscr{D}er_{\gamma'}(\mathscr{L}(C),L')\xrightarrow{\simeq}\mathscr{D}er_\gamma(L,L').$$

Here, γ' is a factorization of γ through ψ. To finish apply Proposition 4 taking into account that, by Theorem 3, $s^{-1}\mathscr{D}er_{\gamma'}(\mathscr{L}(C),L')$ is a Lie model of $\mathrm{map}^*_f(X,Y)$. □

Theorem 4.1 is natural in X. Let $i\colon X \hookrightarrow X'$ be an inclusion of finite complexes and $j\colon L_X \hookrightarrow L_{X'}$ a minimal Quillen model for i. Let now $f\colon X' \to Y$ be a continuous map, L a Quillen model for Y and $\gamma\colon L_{X'} \to L'$ a DGL model for f.

THEOREM 6. *With the above notations, the induced map*

$$s^{-1}\mathscr{D}er_\gamma(L_{X'}, L) \to s^{-1}\mathscr{D}er_{\gamma j}(L_X, L')$$

is a surjective L_∞*-model for* $map^*(i,Y)\colon map^*_f(X',Y) \to map^*_{fi}(X,Y)$.

PROOF. Apply Theorem 4 and Proposition 4. □

6. Lie and L_∞ models for mapping spaces without finite dimension hypothesis on X.

Until this point we have dealt with mapping spaces $\mathrm{map}^*_f(X,Y)$ in which X is a finite complex. We now consider the general situation and assume $f\colon X \to Y$ to be a pointed map between nilpotent rational CW-complexes of finite type. Denote by $f_n\colon X_n \to Y$ the restriction of f to the $n-$skeleton of X and let $i_n\colon C_n \hookrightarrow C_{n+1}$ be an injective coalgebra model of the inclusion $j_n\colon X_n \hookrightarrow X_{n+1}$. Then, if we denote by $U_n = H_*(C_n)$, there is a differential on $\boldsymbol{L}(U_n)$ and an injective quasi-isomorphism [**9**, Section 22]

$$\varphi_n\colon \boldsymbol{L}(U_n) \overset{\simeq}{\hookrightarrow} \mathscr{L}(C_n).$$

Moreover, we may assume $U_n \hookrightarrow U_{n+1}$ and we may choose φ_n so that the following commutes:

$$\begin{array}{ccc} \boldsymbol{L}(U_n) & \xrightarrow[\simeq]{\varphi_n} & \mathscr{L}(C_n) \\ \big\downarrow & & \big\downarrow \mathscr{L}(i_n) \\ \boldsymbol{L}(U_{n+1}) & \xrightarrow[\simeq]{\varphi_{n+1}} & \mathscr{L}(C_{n+1}). \end{array}$$

The linear part $(\varphi_n)_1$ of φ_n is injective. We denote by S_n a supplement of the image of $(\varphi_n)_1$ in C_n. This can be done in a functorial way, $\mathscr{L}(i_n)(S_n) \subset S_{n+1}$.

Next, choose L' a Lie model of Y and DGL morphisms $\gamma_n\colon \mathscr{L}(C_n) \to L'$ modeling f_n. Then, the diagram above produces another commutative diagram in which the vertical arrows are surjections and the sequences are exact:

$$\begin{array}{ccccc} Hom(S_{n+1},L') & \hookrightarrow & \mathscr{D}er_{\gamma_{n+1}}(\mathscr{L}(C_{n+1}),L') & \xrightarrow{\simeq} & \mathscr{D}er_{\gamma_{n+1}\varphi_{n+1}}(\boldsymbol{L}(U_{n+1}),L') \\ \downarrow & & \downarrow & & \downarrow \\ Hom(S_n,L') & \hookrightarrow & \mathscr{D}er_{\gamma_n}(\mathscr{L}(C_n),L') & \xrightarrow{\simeq} & \mathscr{D}er_{\gamma_n\varphi_n}(\boldsymbol{L}(U_n),L'). \end{array}$$

Finally, desuspend this diagram and apply Theorem 6 to obtain:

Proposition 5. *$s^{-1}\mathscr{D}er_{\gamma_{n+1}\varphi_{n+1}}(\boldsymbol{L}(U_{n+1}),L') \twoheadrightarrow s^{-1}\mathscr{D}er_{\gamma_n\varphi_n}(\boldsymbol{L}(U_n),L')$ is a surjective L_∞ model of $j_n^*\colon \operatorname{map}_{f_{n+1}}(X_{n+1},Y) \to \operatorname{map}_{f_n}(X_n,Y)$.*

Observe that $C = \lim_{\to_n} C_n$ is a coalgebra model for X,

$$\gamma = \lim_{\to_n} \gamma_n \colon \mathscr{L}(C) \to L'$$

is a Lie model of f and $L = \lim_{\leftarrow_n} \boldsymbol{L}(U_n)$ is the Quillen minimal model of X. Moreover, $\varphi = \lim_{\to_n} \varphi_n \colon L \xrightarrow{\simeq} \mathscr{L}(C)$ is still an injective quasi-isomorphism. Then we have:

Theorem 7.

(1) *$H_*\big(s^{-1}\mathscr{D}er_{\gamma\varphi}(L,L')\big) \cong \pi_*\Omega\big(\operatorname{map}_f^*(X,Y)\big) \otimes \boldsymbol{Q}$ as graded Lie algebras.*
(2) *The universal cover of $s^{-1}\mathscr{D}er_\gamma(\mathscr{L}(C),L')$ is a Lie model for the universal cover of $\operatorname{map}_f^*(X,Y)$.*
(3) *The universal cover of $s^{-1}\mathscr{D}er_{\gamma\varphi}(L,L')$ is a L_∞ model for the universal cover of $\operatorname{map}_f^*(X,Y)$.*

Proof. Observe that,

$$\mathscr{D}er_{\gamma\varphi}(L,L') = \lim_{\leftarrow_n} \mathscr{D}er_{\gamma_n\varphi_n}(\boldsymbol{L}(U_n),L'),$$
$$\mathscr{D}er_\gamma(\mathscr{L}(C),L') = \lim_{\leftarrow_n} \mathscr{D}er_{\gamma_n}(\mathscr{L}(C_n),L'),$$

as DGL and L_∞ algebras respectively. Moreover, in view of diagram above, and by Proposition 4, we have a quasi-isomorphism of L_∞ algebras

$$\mathscr{D}er_\gamma(\mathscr{L}(C),L') \xrightarrow{\simeq} \mathscr{D}er_{\gamma\varphi}(L,L').$$

On the other hand, $\operatorname{map}_f^*(X,Y) = \lim_{\leftarrow_n} \operatorname{map}_{f_n}^*(X_n,Y)$. Apply Theorem 3 and Lemma 5 to get (1) and (2). Finally, Proposition 5 and again Lemma 5 imply (3). □

As an immediate consequence we obtain:

Corollary 2. *For any formal space X, and any coformal space Y, both finite type 1-connected CW-complexes,* $\mathrm{map}_c^*(X,Y)$ *is a coformal space whose homotopy Lie algebra is* $Hom(H_*(X;\boldsymbol{Q}),\pi_*\Omega Y\otimes\boldsymbol{Q})$. *Here c denotes the constant map.*

Example 8. We compute the Lie algebra $\pi_*\Omega(\mathrm{map}_c^*(\boldsymbol{C}P^\infty_{\boldsymbol{Q}},Y_{\boldsymbol{Q}}))$. First, recall that, for a 1-connected finite complex X,

$$\pi_*\Omega\big(\mathrm{map}_c^*(X,Y)\big)_{\boldsymbol{Q}}\cong\oplus_{j-i=*}Hom\big(H_i(X;Q),\pi_j\Omega Y\otimes\boldsymbol{Q}\big)$$

as graded Lie algebras with the bracket in the latter term given by the coalgebra structure Δ on $H_*(X;\boldsymbol{Q})$ and the bracket in $\pi_*\Omega Y\otimes\boldsymbol{Q}$ given by

$$[f,g](x)=\sum_i(-1)^{|g||x_i'|}[f(x_i),g(x_i')],\quad \Delta(x)=\sum_i x_i\otimes x_i'.$$

Therefore, by Theorem 7, for $n\geq 2$,

$$\pi_n\Omega\big(\mathrm{map}_c^*(\boldsymbol{C}P^\infty_{\boldsymbol{Q}},Y_{\boldsymbol{Q}})\big)\cong\lim_{\leftarrow r}\pi_n\Omega\big(\mathrm{map}_c^*(\boldsymbol{C}P^r_{\boldsymbol{Q}},Y_{\boldsymbol{Q}})\big)=\prod_{\substack{r\geq n+1,\\ r-n-1\,\mathrm{even}}}\pi_rY\otimes\boldsymbol{Q}.$$

The bracket of two sequences $(a_r)_{r\geq n+1}$ and $(b_s)_{s\geq m+1}$ is the sequence $(c_\ell)_{\ell\geq m+n+2}$ with,

$$c_\ell=\sum_{r+s=\ell}[a_r,b_s].$$

As a final application of all of the above, consider a map $f\colon X\to Y$ between CW-complexes of finite type (non necessarily finite) and let $\alpha\in\pi_*(X)\otimes\boldsymbol{Q}$ be a Whitehead bracket of length $n\geq\mathrm{nil}\,\pi_*(Y)\otimes\boldsymbol{Q}$. Then, f extends to a map $\overline{f}\colon X\cup_\alpha e^{k+1}\to Y$ and we get a fibration

$$\mathrm{map}_c^*(S^{k+1},Y)\to\mathrm{map}_{\overline{f}}^*(X\cup_\alpha e^{k+1},Y)\xrightarrow{q}\mathrm{map}_f^*(X,Y).$$

Here, c is the constant map and $\mathrm{map}_c^*(S^{k+1},Y)=\Omega^{k+1}Y$ acts on $\mathrm{map}_{\overline{f}}^*(X\cup_\alpha e^{k+1},Y)$ via the pinching coaction $\nabla\colon X\cup_\alpha e^{k+1}\to X\cup_\alpha e^{k+1}\vee S^{k+1}$. Then, we have the following generalization of the main result in [**15**]:

Proposition 6. *Let $\alpha \in \pi_*(X)\otimes \boldsymbol{Q}$ be a Whitehead bracket of length $n \geq nil\,\pi_*(Y)\otimes \boldsymbol{Q}$. Then*:

(1) *Rationally, and at the level of universal covers, the above fibration is trivial, i.e.,*

$$\widetilde{\operatorname{map}}^*_{\bar{f}}\big(X\cup_\alpha e^{k+1},Y\big)_{\boldsymbol{Q}} \simeq \widetilde{\operatorname{map}}^*_f(X,Y)_{\boldsymbol{Q}} \times \Omega^{k+1}Y_{\boldsymbol{Q}}.$$

(2) *Moreover, for any $q\geq 1$,*

$$\pi_q \operatorname{map}^*_{\bar{f}}\big(X\cup_\alpha e^{k+1},Y\big)\otimes \boldsymbol{Q} \cong \big(\pi_q \operatorname{map}^*_f(X,Y)\otimes \boldsymbol{Q}\big)\oplus\big(\pi_q\Omega^{k+1}Y\otimes \boldsymbol{Q}\big).$$

Proof. Under our hypothesis, in [**15**, Theorem 1.2] it is proved that, for a finite complex X of dimension bounded by the connectivity of Y, and for the constant map $c\colon X\to Y$, $\operatorname{map}^*_c(X\cup_\alpha e^{k+1},Y)_{\boldsymbol{Q}}\simeq \operatorname{map}^*_c(X,Y)_{\boldsymbol{Q}}\times\Omega^{k+1}Y_{\boldsymbol{Q}}$. However, assuming again X finite, but for any $f\colon X\to Y$, essentially the same proof can be carried out to show that the fibration above splits rationally:

$$\operatorname{map}^*_{\bar{f}}\big(X\cup_\alpha e^{k+1},Y\big)_{\boldsymbol{Q}} \simeq \operatorname{map}^*_f(X,Y)_{\boldsymbol{Q}}\times\Omega^{k+1}Y_{\boldsymbol{Q}}.$$

Now, if X is a (non necessarily finite) CW-complex of finite type, observe that

$$\operatorname{map}^*_{\bar{f}}\big(X\cup_\alpha e^{k+1},Y\big) = \varprojlim_n \operatorname{map}^*_{\bar{f}_n}\big(X_n\cup_\alpha e^{k+1},Y\big),$$

is a pronilpotent complex. Here, X_n denotes the n-skeleton of X and $\bar{f}_n$ is the appropriate restriction.

As at the beginning of this section consider for each n, a coalgebra model $C_n \hookrightarrow C_{n+1}$ of the inclusion $X_n\hookrightarrow X_{n+1}$, so that $C=\lim_{\to_n} C_n$ is a coalgebra model for X. Moreover, choose L' a Lie model of Y and DGL morphisms $\gamma_n\colon \mathscr{L}(C_n)\to L'$ modeling f_n, the restriction of f to X_n, so that $\gamma=\lim_{\to_n}\gamma_n\colon \mathscr{L}(C)\to L'$ is a Lie model of f.

Next, as for each $n\geq k$,

$$\operatorname{map}^*_{\bar{f}}\big(X_n\cup_\alpha e^{k+1},Y\big)\simeq_{\boldsymbol{Q}} \operatorname{map}^*_f(X_n,Y)\times\Omega^{k+1}Y_{\boldsymbol{Q}},$$

apply Theorem 3 to get a Lie model for this space of the form

$$s^{-1}\mathscr{D}er_{\gamma_n}(\mathscr{L}(C_n),L')\times s^{-1}\mathscr{D}er(\boldsymbol{L}(a_k),L').$$

Now observe that the limit of these DGL's is

$$s^{-1}\mathscr{D}er_\gamma(\mathscr{L}(C),L')\times s^{-1}\mathscr{D}er(\boldsymbol{L}(a_k),L').$$

To finish apply Lemma 5 and Theorem 7(2). □

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