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Hausdorff leaf spaces for foliations of codimension one

By Szymon M. Walczak

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Abstract. We discuss the topology of Hausdorff leaf spaces (briefly the HLS) for foliation of codimension one. After examining the connection between HLSs and warped foliations, we describe the HLSs associated with foliations obtained by basic constructions such as transversal and tangential gluing, spinning, turbulization and suspension. We show that the HLS for any foliation of codimension one on a compact Riemannian manifold is isometric to a finite connected metric graph, and any finite connected metric graph is isometric to a certain HLS. In the final part of this paper, we discuss the condition for a sequence of warped foliations to converge the HLS.

1. Introduction.

In 1970s M. Berger has presented the concept of modification of a Riemannian metric of S^3 along the fibers of the Hopf fibration. Following this concept, the author of this paper has introduced the notion of warped foliation [**11**]. Later on, the author has examined the limits of a sequence of warped compact foliations [**10**] and has proposed the notion of the Hausdorff leaf space (briefly the HLS) for a foliation on a compact Riemannian manifold.

This paper is the continuation of the research undertaken in [**10**]. At the beginning, the author shows that the HLS for any foliation $\mathscr{F}$ on a compact Riemannian manifold (M, g) is the Gromov-Hausdorff limit of a sequence of warped foliations with warping functions converging to zero on a dense subset $G \subset M$ (Section 3, Theorem 2). Next, he examines the Hausdorff leaf spaces for all natural constructions of the foliation listed in [**3**]. Namely, the HLS for tangential and transverse gluing, spinning, turbulization, and suspension are studied (Section 4).

The main results of this paper are developed in Section 5 (Theorem 9 and Theorem 10), where the complete description of the Hausdorff leaf space for a codim-1 foliation on a compact Riemannian manifold is presented. It is shown that the HLS for a codim-1 foliation is isometric to a finite connected metric graph, while for every finite connected metric graph G there exists a foliated Riemannian

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manifold $(M, \mathscr{F}, g)$ such that the Hausdorff leaf space for $\mathscr{F}$ is isometric to G. Finally (Theorem 11), the necessary and sufficient condition for the sequence (f_n) of warping function on a compact Riemannian manifold carrying foliation of codim-1 to have a sequence of warped foliations (M_{f_n}) converging to the Hausdorff leaf space for the foliation $\mathscr{F}$ is shown.

For the theory of foliations we refer to [**3**] or [**6**].

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2. Preliminaries.

We assume that all objects in this note are smooth (C^∞).

2.1. Hausdorff leaf spaces.

It often happens that the leaf space of a foliation $\mathscr{F}$ on a Riemannian manifold (M, g) defined as a quotient space of the equivalence relation of belonging to the same leaf of $\mathscr{F}$ is not a Hausdorff space [**3**], [**6**]. To see this, one can consider the Reeb foliation on a solid torus or a number of compact foliations presented by Sullivan [**9**] or Epstein and Vogt [**4**] as an example.

In [**10**], the author of this note, has proposed a definition of a metric space, and hence a Hausdorff space, directly connected with a foliation on a Riemannian manifold. He called it the Hausdorff leaf space (briefly the HLS). Let us now recall the notion of it:

Let $(M, \mathscr{F}, g)$ be a compact foliated manifold. Let us set

$$\rho(L, L') = \inf \left\{ \sum_{i=1}^{n-1} \operatorname{dist}(L_i, L_{i+1}) \right\},$$

where the infimum is taken over all finite sequences of leaves beginning at $L_1 = L$ and ending at $L_n = L'$, and $\operatorname{dist}(F, F') = \inf\{d(x,y) : x \in F, y \in F'\}$ (see Figure 1). Let $\sim$ be an equivalence relation in the space of leaves $\mathscr{L}$ defined by:

$$L \sim L' \Leftrightarrow \rho(L, L') = 0, \quad L, L' \in \mathscr{L}. \tag{1}$$

Let $\tilde{\mathscr{L}} = \mathscr{L}/_{\sim}$. Put

$$\tilde{\rho}([L], [L']) = \rho(L, L'),$$

where $[L], [L'] \in \tilde{\mathscr{L}}$. $(\tilde{\mathscr{L}}, \tilde{\rho})$ is a metric space. We call it *the Hausdorff leaf space* for the foliation $\mathscr{F}$ (briefly the HLS), and we denote it by $\mathrm{HLS}(\mathscr{F})$.

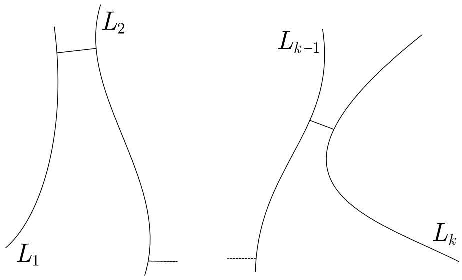

Figure 1. The idea of ρ.

Remark 1. Equivalently, the Hausdorff leaf space can be defined as follows: Following [**1**], one can define in a metric space (X, d) equipped with an equivalence relation R the *quotient pseudo-metric* d_R as

$$d_R(x, y) = \inf \left\{ \sum_{i=1}^{k} d(p_i, q_i) : p_1 = x, q_k = y, k \in \boldsymbol{N} \right\}.$$

where the infimum is taken over all sequences $\{p_i\}_{1 \leq i \leq k}$, $\{q_i\}_{1 \leq i \leq k}$, $k \in \boldsymbol{N}$, such that

$$(p_{i+1}, q_i) \in R.$$

Consider a pseudometric space $(X/R, d_R)$ and identify such points for which d_R is equal to zero. Obtained metric space is called the *quotient metric space.*

Let $(M, \mathscr{F}, g)$ be a compact foliated Riemannian manifold, and let R be the relation of belonging to the same leaf of $\mathscr{F}$. Using R in M we get the alternative definition.

Lemma 2. *For every foliation $\mathscr{F}$ on a compact foliated Riemannian manifold the* $\mathrm{HLS}(\mathscr{F})$ *is a length space.*

Proof. By the definition of the length metric [**5**], for every two points $x, y \in \mathrm{HLS}(\mathscr{F})$ and any curve $c : [0, 1] \to \mathrm{HLS}(\mathscr{F})$ such that $c(0) = x$, $c(1) = y$ we have $\tilde{\rho}(x, y) \leq l(c)$. The opposite inequality follows directly from the definition of the HLS. □

2.2. Gluing metric spaces.

Following [**1**], we now describe how to glue length spaces.

Let (X_α, d_α) be a family of length spaces. Set the length metric d on a disjoint union $\amalg_\alpha X_\alpha$ as follows:

If $x, y \in X_\alpha$, then $d(x, y) = d_\alpha(x, y)$; otherwise, set $d(x, y) = \infty$. The metric d is called the *length metric of disjoint union*.

Now, let (X, d_X) and (Y, d_Y) be two length spaces, while $f : A \to B$ be a bijection between two subsets $A \subset X$ and $B \subset Y$. Equip $Z = X \amalg Y$ with the length metric of disjoint union. Introduce the equivalence relation $\sim$ as the smallest equivalence relation containing relation generated by the relation $x \sim y$ iff $f(x) = y$. The result of gluing X and Y along f is the metric space $(Z/_\sim, d_\sim)$.

2.3. Warped foliations.

As we mentioned in the introduction, in the 70-s M. Berger has presented the concept of modification of a Riemannian metric of S^3 along the fibers of the Hopf fibration. In 1969, Bishop and O'Neil in [**2**] have defined the warped product of two manifolds. Extending these two notions, the author of this paper has defined in [**11**] the notion of warped foliation [**10**], [**11**] as a modification of a Riemannian structure conformally along the leaves of a foliation. The Hausdorff leaf space for warped foliations will be the main topic of our interest in Section 3. The results of Section 3 will be used as a tool in Sections 4 and 5.

Let $(M, \mathscr{F}, g)$ be a smoothly foliated Riemannian manifold and $f : M \to (0, \infty)$ be a smooth basic function on M, i.e. a smooth function constant along the leaves of $\mathscr{F}$. We modify the Riemannian structure g to g_f in the following way: $g_f(v, w) = f^2 g(v, w)$ while both v, w are tangent to the foliation $\mathscr{F}$, but if at least one of vectors v, w is perpendicular to $\mathscr{F}$ then we set $g_f(v, w) = g(v, w)$. The foliated Riemannian manifold $(M, \mathscr{F}, g_f)$ is called here *the warped foliation* and denoted by M_f. The function f is called *the warping function*.

2.4. Gromov-Hausdorff convergence.

Recall the notion of the Gromov-Hausdorff convergence [**5**]. Let (X, d_X) and (Y, d_Y) be arbitrary compact metric spaces. The distance of X and Y can be defined as

$$d_{GH}(X, Y) := \inf\{d_H(X, Y)\},$$

where d ranges over all admissible metrics on the disjoint union $X \amalg Y$, i.e. d is an extension of d_X and d_Y, while d_H denotes the Hausdorff distance. The number $d_{GH}(X, Y)$ is called *the Gromov-Hausdorff distance* of metric spaces X and Y.

THEOREM 1. *$d_{GH}(X, Y) = 0$ iff (X, d_X) is isometric to (Y, d_Y).* $\square$

The Gromov Lemma (below) will be used widely throughout this paper.

LEMMA 3. *Let (X, d_X) and (Y, d_Y) be arbitrary compact metric spaces, and* *let*

$$A = \{x_1, \ldots, x_k\} \subset X,$$
$$B = \{y_1, \ldots, y_k\} \subset Y$$

be ε-nets satisfying for all $1 \leq i,\ j \leq k$ the condition

$$|d_X(x_i, x_j) - d_Y(y_i, y_j)| \leq \varepsilon.$$

Then $d_{GH}(X, Y) \leq 3\varepsilon$. □

3. Convergence theorem.

Consider a sequence $(f_n)_{n\in\boldsymbol{N}}$, $f_n : M \to (0, \infty)$, of warping functions on a compact foliated Riemannian manifold $(M, \mathscr{F}, g)$. One can ask, how does the limit in the Gromov-Hausdorff topology of a sequence of warped foliations $(M_{f_n})_{n\in\boldsymbol{N}}$ look like. Let $G \subset M$ be a dense saturated subset (i.e. a dense subset which is the sum of leaves).

THEOREM 2. *For an arbitrary compact foliated manifold $(M, \mathscr{F}, g)$ and any sequence $(f_n)_{n\in\boldsymbol{N}}$, $f_n : M \to (0, 1)$, of warping functions on M converging to zero on G, the Gromov-Hausdorff limit of a sequence of warped foliations is isometric to* HLS$(\mathscr{F})$.

Before we present a proof of the above Theorem, we should show that a conformal modification of a Riemannian structure in the direction tangent to $\mathscr{F}$ (warping) does not change the Hausdorff leaf space. This is true because the distance $\tilde{\rho}$ defined by the formula (1) can be approximated by the length of curves which are "almost perpendicular to the foliation" (see Figure 2). Readers, who are not familiar with the methods of the Gromov-Hausdorff topology can omit following proofs and continue reading of Section 4.

We say that two metric structures g and g' on a compact foliated Riemannian manifold $(M, \mathscr{F})$ *coincide on the orthogonal bundle* $\mathscr{F}^{\perp}$ if every vector v perpendicular to $\mathscr{F}$ in g is perpendicular in g' and vice versa, and $g(v, w) = g'(v, w)$ for any vectors v, w perpendicular to $\mathscr{F}$ either in g or g'.

LEMMA 4. *Let g and g' be any Riemannian structures on M which coincide on the orthogonal bundle $\mathscr{F}^{\perp}$. Then*

$$\tilde{\rho} = \tilde{\rho'},$$

where $\tilde{\rho}$ and $\tilde{\rho'}$ are defined by (1).

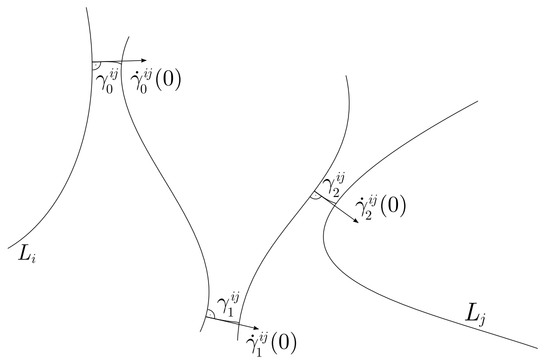

Figure 2. The idea for proofs of Lemma 4 and Theorem 2.

Proof. Since M is compact, we can assume that $g \leq C \cdot g'$ for a certain constant $C \geq 1$. Let ρ and ρ' be pseudometrics given by

$$\rho(L, L') = \inf \left\{ \sum_{i=1}^{n-1} \operatorname{dist}(L_i, L_{i+1}) \right\},$$

$$\rho'(L, L') = \inf \left\{ \sum_{i=1}^{n-1} \operatorname{dist}'(L_i, L_{i+1}) \right\},$$

where dist and dist$'$ denote the distance of the leaves defined by g and g', respectively. Denote by $l(\gamma)$ ($l'(\gamma)$) the g-length (g'-length) of a curve γ. Since the geometry of M is bounded, then for every $\epsilon > 0$ there exists $\delta > 0$ and $A > 0$ such that for every smooth curve $\gamma : [0, l(\gamma)] \to M$ parametrized naturally satisfying

1. $\dot\gamma(0)$ is perpendicular to $\mathscr{F}$,
2. the g'-length $l'(\gamma)$ is smaller than δ,
3. the g'-geodesic curvature $|k_{g'}(\gamma)|$ is smaller than A,

the g-length of the component tangent to $\mathscr{F}$ satisfies

$$\left|\dot\gamma^\top\right| < \epsilon.$$

Let $\epsilon > 0$, $L, L' \in \mathscr{F}$ be such that $d = \operatorname{dist}'(L, L') < \delta$. Let $\gamma : [0, l'(\gamma)] \to M$ be a curve with $\gamma(0) \in L$, $\gamma(l'(\gamma)) \in L'$ such that its length in g' satisfies $d \leq l'(\gamma) \leq \delta$. We have

$$\begin{aligned}\operatorname{dist}(L,L') \le l(\gamma) = \int_{[0,l'(\gamma)]} |\dot\gamma| &\le \int_{[0,l'(\gamma)]} \left|\dot\gamma^{\top}\right| + \int_{[0,l'(\gamma)]} \left|\dot\gamma^{\perp}\right| \\ &\le C\cdot l'(\gamma)\cdot\epsilon + \int_{[0,l'(\gamma)]} \left|\dot\gamma^{\perp}\right|' \le (1+C\epsilon)\cdot l'(\gamma).\end{aligned}$$

Since γ was chosen arbitrarily, we conclude that

$$\operatorname{dist}(L,L') \le (1+C\epsilon)\cdot \operatorname{dist}'(L,L').$$

Now, for every sequence of leaves $L_1,\ldots,L_n$ such that $L_1 = L$, $L_n = L'$ and satisfying

$$\sum_{i=1}^{n-1} \operatorname{dist}'(L_i, L_{i+1}) \le \rho'(L,L') + \epsilon,$$

and such that $\operatorname{dist}'(L_i,L_{i+1}) < \delta$ for all $i \in \{1,\ldots,n-1\}$, we obtain

$$\begin{aligned}\rho(L,L') \le \sum_{i=1}^{n-1} \operatorname{dist}(L_i,L_{i+1}) &\le (1+C\epsilon)\cdot\left(\sum_{i=1}^{n-1}\operatorname{dist}'(L_i,L_{i+1})\right) \\ &\le (1+C\epsilon)\cdot(\rho'(L,L')+\epsilon).\end{aligned}$$

Tending with ϵ to zero we get that $\rho \le \rho'$. Consequently $\tilde\rho \le \tilde\rho'$. Similarly, we can show that $\tilde\rho' \le \tilde\rho$. □

We now turn to a proof of Theorem 2. The idea of a proof is to find, for given $\epsilon > 0$, two ϵ-nets in M_{f_n} and $\mathrm{HLS}(F)$ satisfying the condition of Lemma 3.

Denote by $\pi : M \to \mathrm{HLS}(\mathscr{F})$ the natural projection given by $\pi(x) = [L_x]_\sim$, where $\sim$ is the equivalence relation defined in Section 2.1.

PROOF. Let $\epsilon > 0$ and let $\{x_1,\ldots,x_k\}$ be an ϵ-net on M contained in G. Let $i,j \in \{1,\ldots,k\}$. Choose a family of leaves $F^{ij} = \{F_0^{ij},\ldots,F_{\delta_i^j}^{ij}\}$ such that $L_{x_i} = F_0^{ij}$, $L_{x_j} = F_{\delta_i^j}^{ij}$, $F_\nu^{ij} \subset G$ for any $0 \le \nu \le \delta_i^j$, and

$$\sum_{\nu=0}^{\delta_i^j-1} \operatorname{dist}\left(F_\nu^{ij}, F_{\nu+1}^{ij}\right) \le \tilde\rho\big(\pi(L_{x_i}), \pi(L_{x_j})\big) + \epsilon.$$

Consider a family of curves

$$\gamma_0^{ij}, \ldots, \gamma_{\delta_i^j - 1}^{ij} : [0,1] \to M$$

satisfying $\gamma_\nu^{ij}(0) \in F_\nu^{ij}$, $\gamma_\nu^{ij}(1) \in F_{\nu+1}^{ij}$, and

$$\sum_{\nu=0}^{\delta_i^j - 1} l\big(\gamma_\nu^{ij}\big) \leq \tilde{\rho}\big(\pi(L_{x_i}), \pi(L_{x_j})\big) + 2\epsilon. \tag{2}$$

Let $\delta = \max\{\delta_i^j\}$. Since $f_n \to 0$ on G, and the number of leaves involved in sets F^{ij}, $i, j = 1, \ldots, k$, is finite, there exists $N \in \boldsymbol{N}$ such that for any $n > N$, $i, j \in \{1, \ldots, k\}$ and $\nu \in \{0, \ldots, \delta_i^j - 1\}$ we have

$$\begin{aligned} d_{n, F_\nu^{ij}}\big(\gamma_\nu^{ij}(1), \gamma_{\nu+1}^{ij}(0)\big) &\leq \frac{\epsilon}{\delta}, \\ d_{n, F_0^{ij}}\big(x_i, \gamma_0^{ij}(0)\big) &\leq \frac{\epsilon}{\delta}, \\ d_{n, F_{\delta_i^j - 1}^{ij}}\big(\gamma_{\delta_i^j - 1}^{ij}(1), x_j\big) &\leq \frac{\epsilon}{\delta}, \end{aligned} \tag{3}$$

where $d_{n,F}$ is the distance on the leaf F induced from the Riemannian metric $f_n(F) g\,|_F$.

Let us pick one point in each $\{x_1, \ldots, x_k\} \cap \pi^{-1}(\pi(x_i))$, $i = 1, \ldots, k$. We obtain a set $\{y_1, \ldots, y_m\}$ ($m \leq k$) with the property $\pi(y_i) \neq \pi(y_j)$ iff $i \neq j$.

Let $n > N$. Direct calculation shows that the points $y_1, \ldots, y_m$ form a 4ϵ-net on (M, g_{f_n}). Moreover, by (2) and (3),

$$d_n(y_i, y_j) \leq \tilde{\rho}\big(\pi(L_{y_i}), \pi(L_{y_j})\big) + 3\epsilon,$$

where d_n is a metric on M defined by the Riemannian structure g_{f_n} (see Section 2.3). Denote by $\tilde{\rho}_n$ the metric defined by g_{f_n} and the formula (1). By Lemma 4,

$$\tilde{\rho}\big([L_{y_i}], [L_{y_j}]\big) = \tilde{\rho}_n\big(\pi(L_{y_i}), \pi(L_{y_j})\big) \leq d_n(y_i, y_j).$$

The set $\pi(\{y_1, \ldots, y_m\}) = \pi(\{x_1, \ldots, x_k\})$ provides an ϵ-net on $\mathrm{HLS}(\mathscr{F})$. By Lemma 3, $d_{GH}(M_{f_n}, \mathrm{HLS}(\mathscr{F})) \leq 9\epsilon$. Tending with ϵ to zero we get that

$$d_{GH}(M_{f_n}, \mathrm{HLS}(\mathscr{F})) = 0.$$

The uniqueness of the limit and Theorem 1 completes our proof. □

4. Basic constructions.

Studying foliations one can learn that there are several basic constructions for building foliations [**3**]. In this chapter we examine the HLS for the following constructions: tangential and transverse gluing, two transverse modifications – turbulization and spinning along a transverse boundary component, and for suspension.

We only provide here a detailed proof for tangential gluing and turbulization, which are used in Section 5. For transverse gluing and spinning we give the characterization of their HLS's and we only provide an outline of a proof. Details, analogical as for tangential gluing and turbulization, are left to the reader.

4.1. Tangential gluing.

Let us assume that $(M_i, \mathscr{F}_i, g_i)$ $(i = 1, 2)$ are compact foliated Riemannian manifolds with boundary, while $\mathscr{F}_i$ is a foliation tangent to the boundary. Let $S_i \subset \partial M_i$ $(i = 1, 2)$ be a union of boundary components, and let $h : S_1 \to S_2$ be an isometry mapping leaves onto leaves. According to [**3**], identify S_1 with S_2 using $x \equiv h(x)$, and form the quotient foliated manifold $M = M_1 \cup_h M_2$ with foliation $\mathscr{F} = \mathscr{F}_1 \cup_h \mathscr{F}_2$ defined by the leaves of $\mathscr{F}_i$ (Figure 3).

Let us assume that one can obtain a smooth Riemannian structure g on M with the property $g \mid M_i = g_i$ $(i = 1, 2)$.

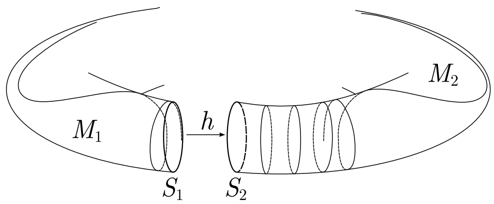

Figure 3. Tangential gluing.

Denote by $\pi : M \to \mathrm{HLS}(\mathscr{F})$, $\pi_i : M_i \to \mathrm{HLS}(\mathscr{F}_i)$ $(i = 1, 2)$ natural projections. Consider the smallest equivalence relation $\sim$ in disjoint union

$$\mathrm{HLS}(\mathscr{F}_1) \amalg \mathrm{HLS}(\mathscr{F}_2)$$

containing the relation defined as follows:

$$\pi_1(L) \sim \pi_2(L') \Leftrightarrow \exists_{x \in \pi_1^{-1}(\pi_1(L))} \quad \pi_2(h(x)) = \pi_2(L').$$

Let $X = \mathrm{HLS}(\mathscr{F}_1) \amalg \mathrm{HLS}(\mathscr{F}_2)/_{\sim}$ endowed with quotient metric d_X.

Denote by $\Phi : \mathrm{HLS}(\mathscr{F}_1) \amalg \mathrm{HLS}(\mathscr{F}_2) \to X$, $\tilde{\pi} : M_1 \amalg M_2 \to \mathrm{HLS}(\mathscr{F}_1) \amalg \mathrm{HLS}(\mathscr{F}_2)$, and $p : M_1 \amalg M_2 \to M$ natural projections (see Figure 4).

$$\begin{tikzcd}
M_1 \amalg M_2 \arrow[r, "p"] \arrow[d, "\tilde{\pi}"] & M \arrow[r, "\pi"] & \mathrm{HLS}(\mathscr{F}) \\
\mathrm{HLS}(\mathscr{F}_1) \amalg \mathrm{HLS}(\mathscr{F}_2) \arrow[r, "\Phi"'] & X &
\end{tikzcd}$$

Figure 4. The projections for Theorem 3.

Theorem 3. *The space* $\mathrm{HLS}(\mathscr{F})$ *is isometric to* (X, d_X).

We begin the proof by the following:

Lemma 5. *For any* $x, y \in M_1 \amalg M_2$

$$d_X\big(\Phi(\tilde{\pi}(x)), \Phi(\tilde{\pi}(y))\big) = \tilde{\rho}\big(\pi(p(x)), \pi(p(y))\big).$$

Proof. Let $\epsilon > 0$. Consider points $\pi(p(x))$ and $\pi(p(y))$. By the definition of $\mathrm{HLS}(\mathscr{F})$, there exist points $r_1, q_1, \dots, r_k, q_k$ in the disjoint union $M_1 \amalg M_2$ such that r_ν, q_ν belong to the same component, and $r_1 \in L_x$, $q_k \in L_y$ (L_x and L_y are here the leaves of the appropriate foliation $\mathscr{F}_1$ or $\mathscr{F}_2$). Moreover, $p(q_\nu)$ and $p(r_{\nu+1})$ lie in the same leaf of $\mathscr{F}$, and

$$\sum_{\nu=1}^{k} \bar{d}\big(r_\nu, q_\nu\big) \le \tilde{\rho}\big(\pi(p(x)), \pi(p(y))\big) + \epsilon,$$

where $\bar{d}$ is the length metric of disjoint union in $M_1 \amalg M_2$ (see Section 2.2). Denote by $\tilde{d}$ the length metric of disjoint union in $\mathrm{HLS}(\mathscr{F}_1) \amalg \mathrm{HLS}(\mathscr{F}_2)$. By the definition of X, we have

$$\begin{aligned}\sum_{\nu=1}^{k} \bar{d}(r_\nu, q_\nu) &\ge \sum_{\nu=1}^{k} \tilde{d}(\tilde{\pi}(r_\nu), \tilde{\pi}(q_\nu)) \\ &\ge d_X\big(\Phi(\tilde{\pi}(x)), \Phi(\tilde{\pi}(y))\big).\end{aligned}$$

Finally,

$$d_X\big(\Phi(\tilde{\pi}(x)), \Phi(\tilde{\pi}(y))\big) \le \tilde{\rho}\big(\pi(p(x)), \pi(p(y))\big) + \epsilon. \tag{4}$$

Next, consider points $\Phi(\tilde{\pi}(x))$ and $\Phi(\tilde{\pi}(y))$. There exist points $r_1, q_1, \dots, r_k, q_k$ in the disjoint union $\mathrm{HLS}(\mathscr{F}_1) \amalg \mathrm{HLS}(\mathscr{F}_2)$ such that $\Phi(q_\nu) = \Phi(r_{\nu+1})$ ($\nu = 1, \dots, k$), $\Phi(r_1) = \Phi(\tilde{\pi}(x))$, $\Phi(q_k) = \Phi(\tilde{\pi}(y))$, and

$$\sum_{\nu=1}^{k} \tilde{d}(r_\nu, q_\nu) \le d_X\big(\Phi(\tilde{\pi}(x)), \Phi(\tilde{\pi}(y))\big) + \epsilon,$$

where $\tilde{d}$ denotes again the length metric of disjoint union in $\mathrm{HLS}(\mathscr{F}_1) \amalg \mathrm{HLS}(\mathscr{F}_2)$.

Now, for every $\nu = 1, \dots, k$, one can find a sequence of leaves $L_1^\nu, \dots, L_{l_\nu}^\nu$ of the appropriate foliation ($\mathscr{F}_1$ if $r_\nu, q_\nu \in \mathrm{HLS}(\mathscr{F}_1)$ or $\mathscr{F}_2$ if $r_\nu, q_\nu \in \mathrm{HLS}(\mathscr{F}_2)$) satisfying $r_\nu \in L_1^\nu$, $q_\nu \in L_{l_\nu}^\nu$, and

$$\sum_{\mu=1}^{l_\nu - 1} \operatorname{dist}\left(L_\mu^\nu, L_{\mu+1}^\nu\right) \le \tilde{d}(r_\nu, q_\nu) + \frac{\epsilon}{k}.$$

Since h maps leaves onto leaves, one can consider the leaves described above as leaves of a foliation $\mathscr{F}$. Moreover, $\tilde{\pi}(x) \in \tilde{\pi}(L_1^1)$, and $\tilde{\pi}(y) \in \tilde{\pi}(L_{l_k}^k)$. Hence we have

$$\tilde{\rho}\big(\pi(p(x)), \pi(p(y))\big) \le d_X\big(\Phi(\tilde{\pi}(x)), \Phi(\tilde{\pi}(y))\big) + 2\epsilon. \tag{5}$$

Passing with ϵ to zero in inequalities (4) and (5), we get that

$$\tilde{\rho}\big(\pi(p(x)), \pi(p(y))\big) = d_X\big(\Phi(\tilde{\pi}(x)), \Phi(\tilde{\pi}(y))\big).$$

This completes our proof. □

Now, we turn to the proof of Theorem 3.

Proof. Let us consider a sequence $(f_n : M \to (0,1])$ of constant functions on M converging to zero. Obviously, it can be used as a sequence of warping functions. By Theorem 2, the limit $\lim_{GH} M_{f_n} = \mathrm{HLS}(\mathscr{F})$. We will show, that $\lim_{GH} M_{f_n} = (X, d_X)$.

Let $\{x_1^1, \dots, x_{k_1}^1\} \subset M_1$ and $\{x_1^2, \dots, x_{k_2}^2\} \subset M_2$ be $\epsilon/2$-nets. We can assume that there exist $N \in \boldsymbol{N}$, $K_1 \le k_1$, and $K_2 \le k_2$ such that

- $\tilde{\pi}(x_i^1) \ne \tilde{\pi}(x_j^1)$ while $i \ne j$ ($i, j \le K_1$), $\tilde{\pi}(x_l^2) \ne \tilde{\pi}(x_m^2)$ while $l \ne m$ ($l, m \le K_2$);
- $\{x_1^1, \dots, x_{K_1}^1\}$ is an ϵ-net in $(M_1)_{f_n}$, while $\{x_1^2, \dots, x_{K_2}^2\}$ provides an ϵ-net in $(M_2)_{f_n}$, $n > N$.

Denote

- $y_j = \Phi\big(\tilde{\pi}(x_j^1)\big)$, $j = 1, \dots, K_1$;
- $y_{K_1+j} = \Phi\big(\tilde{\pi}(x_j^2)\big)$, $j = 1, \dots, K_2$;
- $x_j = \pi\big(p(x_j^1)\big)$, $j = 1, \dots, K_1$;
- $x_{K_1+j} = \pi\big(p(x_j^2)\big)$, $j = 1, \dots, K_2$.

One can easily check that $\{y_1, \dots, y_{K_1+K_2}\}$ and $\{x_1, \dots, x_{K_1+K_2}\}$ have the same number of elements, and $y_i = y_j$ iff $x_i = x_j$. Thus, putting $K = K_1 + K_2$ we get two ϵ-nets $\{y_1, \dots, y_K\}$ and $\{x_1, \dots, x_K\}$ in X and $\mathrm{HLS}(\mathscr{F})$, respectively. By Lemma 5

$$d_X(x_i, x_j) = \tilde{\rho}(y_i, y_j)$$

for all $i, j \leq K$. By Lemma 3, $d_{GH}((X, d_X), \mathrm{HLS}(\mathscr{F})) = 0$. Finally, by Theorem 1, we get the statement. $\square$

REMARK 6. Note that it can be impossible to construct a smooth Riemannian structure g on M such that $g \mid M_i = g_i$ $(i = 1, 2)$. But all Riemannian structures on a compact manifold are equivalent. We slightly modify the Riemannian structures g_i to obtain structures with desired properties. In this case we can only prove that $\mathrm{HLS}(\mathscr{F})$ of the glued foliation is homeomorphic to (X, d_X).

4.2. Transverse gluing.

Following [**3**], let $(M_1, \mathscr{F}_1, g_1)$, $(M_2, \mathscr{F}_2, g_2)$ be smooth compact foliated Riemannian manifolds of dimension n with nonempty boundary and codimension q foliations. Suppose that $S_i \subset \partial M_i$ is a union of boundary components $(i = 1, 2)$ and $\phi : S_1 \to S_2$ is an isometry mapping leaves to leaves. Suppose further that $\mathscr{F}_i$ is g_i-orthogonal to S_i. Form a manifold $M = M_1 \cup_\phi M_2$ from the disjoint union $M_1 \amalg M_2$ by identifying x with $\phi(x)$. Endow M with an induced foliation.

Let us denote the natural projections as shown on the Figure 5.

$$\begin{tikzcd}
M_1 \amalg M_2 \arrow[r, "p"'] \arrow[d, "\tilde{\pi}"] & M \arrow[r, "\pi"] & \mathrm{HLS}(\mathscr{F}) \\
\mathrm{HLS}(\mathscr{F}_1) \amalg \mathrm{HLS}(\mathscr{F}_2) \arrow[r, "\Phi"] & X &
\end{tikzcd}$$

Figure 5. The projections for Theorem 4.

Consider the smallest equivalence relation $\sim$ in disjoint union

$$\mathrm{HLS}(\mathscr{F}_1) \amalg \mathrm{HLS}(\mathscr{F}_2)$$

containing the relation defined by

$$\tilde{\pi}(x) \sim \tilde{\pi}(\phi(x)).$$

Next, glue HLS($\mathscr{F}_1$) with HLS($\mathscr{F}_2$) along $\sim$ and denote the result endowed with quotient metric by (X, d_X).

LEMMA 7. *For any two points $x, y \in M_1 \amalg M_2$*

$$d_X\big(\Phi(\tilde{\pi}(x)), \Phi(\tilde{\pi}(y))\big) = \tilde{\rho}\big(\pi(p(x)), \pi(p(y))\big).$$

PROOF. Analogical to the proof of Lemma 5. Left to the reader. □

THEOREM 4. HLS($\mathscr{F}$) *coincides with* (X, d_X).

PROOF. Denote by $A_1 = \{x_1, \ldots, x_k\} \subset M_1 \setminus \partial M_1$ and $A_2 = \{y_1, \ldots, y_m\} \subset M_2 \setminus \partial M_2$ two ϵ-nets. One can easily check that $p(A_1 \cup A_2)$ is an ϵ-net in M, $\pi(p(A_1 \cup A_2))$ is an ϵ-net in HLS($\mathscr{F}$) and $\Phi(\tilde{\pi}(A_1 \cup A_2))$ is an ϵ-net in X. Moreover, by the construction of X we have that

$$\sharp\big(\Phi(\tilde{\pi}(A_1 \cup A_2))\big) = \sharp\big(\pi(p(A_1 \cup A_2))\big).$$

Lemma 7 and Lemma 3 yield the statement. □

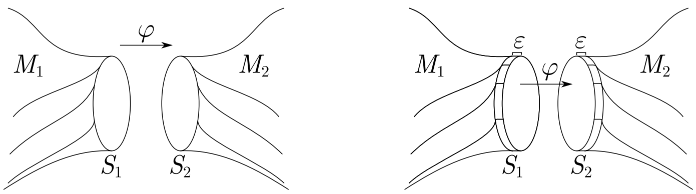

Figure 6. Transverse gluing.

REMARK 8. Of course, not every foliation transverse to the boundary component is orthogonal to it. But one can easily modify (see [**3**]) any transverse foliation to obtain a foliation orthogonal to the boundary component with the same space as the Hausdorff leaf space (see Figure 6). Hence, Theorem 4 is true for any foliations transverse to the boundary.

4.3. Turbulization.

Let now $(M, \mathscr{F}, g)$ be a foliated Riemannian manifold of dimension $n+1$

endowed with a codimension one foliation which is leaf-wise and transversely orientable. Let $\gamma : [0,1] \to M$ be a closed transversal curve and let $N(\gamma)$ be a fixed foliated tubular neighbourhood of γ. Let us equip $N(\gamma) = D^n \times S^1$ with cylindrical coordinates (r, z, t) (we take t *modulo* 1, while the leaves of $\mathscr{F}|N(\gamma)$ are the sets $D^n \times \{t\}$). Let

$$\omega = \cos\lambda(r)\mathrm{d}r + \sin\lambda(r)\mathrm{d}t,$$

where $\lambda : [0,1] \to [-\pi/2, \pi/2]$ is a smooth, strictly increasing on $[0, 3/4]$ function satisfying $\lambda(0) = -\pi/2$, $\lambda(2/3) = 0$, $\lambda(t) = \pi/2$ for all $t \geq 3/4$, and with derivatives of all orders at zero vanishing. Since ω is integrable, it defines a foliation $\mathscr{F}_\gamma$ of $N(\gamma)$, which agrees with $\mathscr{F}$ near $\partial N(\gamma)$ and has a Reeb component R inside $N(\gamma)$. Modified foliation $\mathscr{F}_\gamma$ of M is called a *turbulized* foliation, while this deformation is called *the turbulization along the curve* γ [**3**] (Figure 7).

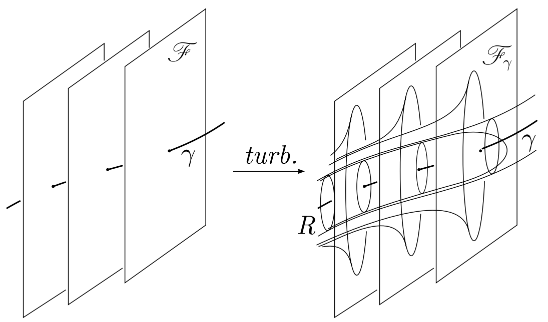

Figure 7. Turbulization.

Denote by L_x (L_x^γ) the leaf of $\mathscr{F}$ ($\mathscr{F}_\gamma$) passing through $x \in M$. Next, let $\pi : M \to \mathrm{HLS}(\mathscr{F})$ be the natural projection, and let X be a metric space obtained from $\mathrm{HLS}(\mathscr{F})$ by identification $\pi(\gamma([0,1]))$ to a point (Figure 9). Equip X with the quotient metric d_X.

Theorem 5. $\mathrm{HLS}(\mathscr{F}_\gamma)$ *is isometric with* (X, d_X).

Before we start a proof, we shall prove technical lemmas.

Lemma 9. *For every two leaves $L_1, L_2 \in \mathscr{F}$ and every $\epsilon > 0$ there exists a sequence of leaves $F_1, \ldots, F_k \in \mathscr{F}_\gamma$, $k \leq 3$, satisfying*

1. $F_1 \setminus N(\gamma) = L_1 \setminus N(\gamma)$ *and* $F_k \setminus N(\gamma) = L_2 \setminus N(\gamma)$,
2. $\sum_{\nu=1}^{k-1} \mathrm{dist}(F_\nu, F_{\nu+1}) \leq \mathrm{dist}(L_1, L_2) + \epsilon$.

$$\begin{tikzcd}
{[0,1]} \arrow[r, "\gamma"] & (M, \mathscr{F}, g) \arrow[r, "turb."] \arrow[d, "\pi"'] & (M, \mathscr{F}_\gamma, g) \arrow[d, "\pi_\gamma"] \\
& \mathrm{HLS}(\mathscr{F}) \arrow[d, "p"'] & \mathrm{HLS}(\mathscr{F}_\gamma) \\
& (X, d_X) \arrow[ur, "f"'] &
\end{tikzcd}$$

Figure 8. The projections for Theorem 5.

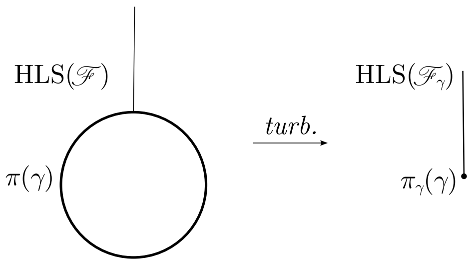

Figure 9. HLS for turbulized foliation.

Proof. Let $\epsilon > 0$, and $x \in L_1$, $y \in L_2$ be such that $d(x,y) \leq \mathrm{dist}(L_1, L_2) + \epsilon$. We shall consider three cases:

1. $x, y \notin M \setminus N(\gamma)$. Put $F_1 = L_x^\gamma$ and $F_2 = L_y^\gamma$. Then $\mathrm{dist}(F_1, F_2) \leq d(x,y) \leq \mathrm{dist}(L_1, L_2) + \epsilon$.
2. $x, y \in N(\gamma)$. Choose points $x' \in L_1 \setminus N(\gamma)$ and $y' \in L_2 \setminus N(\gamma)$. Put $F_1 = L_{x'}^\gamma$, $F_2 = L_{y'}^\gamma$. By the definition of the turbulization we have that

$$\mathrm{dist}(F_1, F_2) = 0 \leq \mathrm{dist}(L_1, L_2).$$

3. $x \in N(\gamma)$ and $y \in M \setminus N(\gamma)$. Let $z \in L_1 \setminus N(\gamma)$. Put $F_1 = L_z^\gamma$, $F_2 = L_x^\gamma$, $F_3 = L_y^\gamma$. We have

$$\mathrm{dist}(F_1, F_2) + \mathrm{dist}(F_2, F_3) \leq 0 + d(x,y) \leq \mathrm{dist}(L_1, L_2) + \epsilon.$$

This completes the proof. $\square$

Lemma 10. *For any $x, y \in M$ the following inequality holds*:

$$d_X\big(p(\pi(x)), p(\pi(y))\big) \leq \mathrm{dist}\left(L_x^\gamma, L_y^\gamma\right).$$

PROOF. Let $x, y \in M$. We shall consider few cases.
Case 1: $L_x^\gamma \cap N(\gamma) = L_y^\gamma \cap N(\gamma) = \emptyset$. Then $L_x = L_x^\gamma$, $L_y = L_y^\gamma$, and

$$d_X\big(p(\pi(x)), p(\pi(y))\big) \leq \operatorname{dist}\big(L_x^\gamma, L_y^\gamma\big).$$

Case 2: $L_x^\gamma \cap N(\gamma) \neq \emptyset$, $L_y^\gamma \cap N(\gamma) \neq \emptyset$. Then, by the construction of $\mathscr{F}_\gamma$,

$$L_x \cap N(\gamma) \neq \emptyset, \quad \text{and} \quad L_y \cap N(\gamma) \neq \emptyset.$$

Finally,

$$d_X\big(p(\pi(x)), p(\pi(y))\big) = 0 = \operatorname{dist}\big(L_x^\gamma, L_y^\gamma\big).$$

Case 3: $L_x^\gamma \cap N(\gamma) \neq \emptyset$, $L_y^\gamma \cap N(\gamma) = \emptyset$. Let $\epsilon > 0$, and $r \in L_x^\gamma$, $q \in L_y^\gamma$ be such points that $d(r,q) \leq \operatorname{dist}(L_x^\gamma, L_y^\gamma) + \epsilon$.

Suppose first that $x \notin N(\gamma)$, $r \notin N(\gamma)$. Then $L_x = L_r$, $L_y = L_q$, and

$$\begin{aligned} d_X\big(p(\pi(x)), p(\pi(y))\big) &\leq \operatorname{dist}(L_x, L_y) \leq d(r,q) \\ &\leq \operatorname{dist}\big(L_x^\gamma, L_y^\gamma\big) + \epsilon. \end{aligned}$$

Next, suppose that $x \notin N(\gamma)$, $r \in N(\gamma)$. Set $L_1 = L_x$, $L_2 = L_r$, $L_3 = L_y$. Recall that $L_q = L_y$. Hence, by Case 1,

$$\begin{aligned} d_X\big(p(\pi(x)), p(\pi(y))\big) &\leq d_X\big(p(\pi(x)), p(\pi(r))\big) + d_X\big(p(\pi(r)), p(\pi(q))\big) \\ &\leq 0 + d(r,q) \leq \operatorname{dist}\big(L_x^\gamma, L_y^\gamma\big) + \epsilon. \end{aligned}$$

Analogically we show that

$$d_X\big(p(\pi(x)), p(\pi(y))\big) \leq \operatorname{dist}\big(L_x^\gamma, L_y^\gamma\big) + \epsilon,$$

for $x \in N(\gamma)$, $r \notin N(\gamma)$, and $x \in N(\gamma)$, $r \in N(\gamma)$.

Passing with ϵ to zero gives the desired inequality. $\square$

Let us denote by π_γ the natural projection from $(M, \mathscr{F}_\gamma, g)$ to $\mathrm{HLS}(\mathscr{F}_\gamma)$ (see Figure 8), and by $\tilde{\rho}_\gamma$ the metric in $\mathrm{HLS}(\mathscr{F}_\gamma)$.

Let $f : X \to \mathrm{HLS}(\mathscr{F}_\gamma)$ be defined as follows:

$$f\big(p(\pi(x))\big) = \pi_\gamma\big(L_x^\gamma\big). \tag{6}$$

Lemma 11. *For any $x, y \in M$ we have*

$$d_X\big(p(\pi(x)), p(\pi(y))\big) \leq \tilde{\rho}_\gamma\big(\pi_\gamma(L_x^\gamma), \pi_\gamma(L_y^\gamma)\big).$$

Proof. Let $\epsilon > 0$, $x, y \in M$. There exists a sequence of leaves $L_1^\gamma, \dots, L_k^\gamma$ such that $L_1^\gamma = L_x^\gamma$, $L_k^\gamma = L_y^\gamma$, and

$$\sum_{\nu=1}^{k-1} \operatorname{dist}\big(L_\nu^\gamma, L_{\nu+1}^\gamma\big) \leq \tilde{\rho}_\gamma\big(\pi_\gamma(L_x^\gamma), \pi_\gamma(L_y^\gamma)\big) + \epsilon.$$

Let $r_1, q_1, \dots, r_{k-1}, q_{k-1} \in M$ be such that $r_\nu \in L_\nu^\gamma$, $q_\nu \in L_{\nu+1}^\gamma$, and

$$d(r_\nu, q_\nu) \leq \operatorname{dist}\big(L_\nu^\gamma, L_{\nu+1}^\gamma\big) + \frac{\epsilon}{k}.$$

Note that $L_x^\gamma = L_{r_1}^\gamma$, $L_y^\gamma = L_{q_{k-1}}^\gamma$, and $L_{r_{\nu+1}}^\gamma = L_{q_\nu}^\gamma$. By Lemma 10,

$$\begin{aligned} d_X\big(p(\pi(x)), p(\pi(r_1))\big) &= 0,\\ d_X\big(p(\pi(y)), p(\pi(q_{k-1}))\big) &= 0,\\ d_X\big(p(\pi(r_{\nu+1})), p(\pi(q_\nu))\big) &= 0 \end{aligned}$$

for all $\nu \in \{1, \dots, k-1\}$. By the construction of X,

$$\begin{aligned} &d_X\big(p(\pi(x)), p(\pi(y))\big)\\ &\quad\leq \sum_{\nu=1}^{k-1} d_X\big(p(\pi(r_\nu)), p(\pi(q_\nu))\big) + \sum_{\nu=1}^{k-2} d_X\big(p(\pi(r_{\nu+1})), p(\pi(q_\nu))\big)\\ &\quad\quad + d_X\big(p(\pi(x)), p(\pi(r_1))\big) + d_X\big(p(\pi(y)), p(\pi(q_{k-1}))\big)\\ &\quad\leq \sum_{\nu=1}^{k-1} d(r_\nu, q_\nu) \leq \tilde{\rho}_\gamma\big(\pi_\gamma(L_x^\gamma), \pi_\gamma(L_y^\gamma)\big) + 2\epsilon. \end{aligned}$$

Passing with ϵ to zero gives us the statement. □

Lemma 12. *For any $x, y \in M$ we have*

$$\tilde{\rho}_\gamma\big(\pi_\gamma(L_x^\gamma), \pi_\gamma(L_y^\gamma)\big) \leq d_X\big(p(\pi(x)), p(\pi(y))\big).$$

Proof. Let $x, y \in M$, $\epsilon > 0$. There exist points $r_1, q_1, \ldots, r_k, q_k \in \mathrm{HLS}(\mathscr{F})$ such that $p(q_i) = p(r_{i+1})$, $\pi(x) = r_1$, $\pi(y) = q_k$, and

$$\sum_{i=1}^{k-1} \tilde{\rho}(r_i, q_i) \leq d_X\big(p(\pi(x)), p(\pi(y))\big) + \epsilon. \tag{7}$$

For any $i \in \{1, \ldots, k\}$ one can find a family of leaves $L_{i,1}, \ldots, L_{i,\mu_i}$ satisfying $L_{i+1,1} = L_{i,\mu_i}$, $x \in L_{1,1}$, $y \in L_{k,\mu_k}$, and

$$\sum_{\nu=1}^{\mu_i - 1} \mathrm{dist}(L_{i,\nu}, L_{i,\nu+1}) \leq \tilde{\rho}(r_i, q_i) + \frac{\epsilon}{k}. \tag{8}$$

By (7), (8), and Lemma 9, one can find a finite sequence $L_1^\gamma, \ldots, L_m^\gamma$ of leaves of $\mathscr{F}_\gamma$ such that $L_1^\gamma = L_x^\gamma$, $L_m^\gamma = L_y^\gamma$ and

$$\begin{aligned}\tilde{\rho}_\gamma\big(\pi_\gamma(L_x^\gamma), \pi_\gamma(L_y^\gamma)\big) &\leq \sum_{\nu=1}^{m-1} \mathrm{dist}\left(L_\nu^\gamma, L_{\nu+1}^\gamma\right) \\ &\leq d_X\big(p(\pi(x)), p(\pi(y))\big) + 3\epsilon.\end{aligned}$$

Passing with ϵ to zero gives us the statement. □

Lemma 13. *f is bijective.*

Proof. Suppose that $p(\pi(x)) = p(\pi(y))$, $x, y \in M$. Consider two cases.
Case 1: $\pi(x) = \pi(y)$. If $\pi^{-1}(\pi(x)) \cap \gamma([0,1]) = \emptyset$ then $\pi^{-1}(\pi(y)) \cap \gamma([0,1]) = \emptyset$, and $\pi_\gamma^{-1}(\pi_\gamma(x)) = \pi_\gamma^{-1}(\pi_\gamma(y))$. Hence $\pi_\gamma(L_x^\gamma) = \pi_\gamma(L_y^\gamma)$, and $f(p(\pi(x))) = f(p(\pi(y)))$.
Case 2: If $\pi(x) \neq \pi(y)$, then there exist $\xi_x \in \pi^{-1}(\pi(x)) \cap \gamma([0,1])$ and $\xi_y \in \pi^{-1}(\pi(y)) \cap \gamma([0,1])$. Hence, $\pi_\gamma(\xi_x) = \pi_\gamma(\xi_y) = \pi_\gamma(R)$, where R denotes the Reeb component. But $\pi_\gamma(L_x^\gamma) = \pi_\gamma(L_{\xi_x}^\gamma)$. Thus $f(p(\pi(x))) = f(p(\pi(y)))$.

Finally, f is well defined. By the definition, f is "onto" $\mathrm{HLS}(\mathscr{F}_\gamma)$. Checking that f is one-to-one we leave to the reader. □

Now, we can turn to the proof of Theorem 5.

Proof. By Lemma 13, f defined in (6) is a bijection from X onto $\mathrm{HLS}(\mathscr{F}_\gamma)$. By Lemmas 11 and 12, f is an isometry. □

4.4. Spinning.

Following the definition given in [**3**] we recall the notion of *spinning* a foliation along a transverse boundary component (see Figure 10).

Let $(M, \mathscr{F}, g)$ be a compact Riemannian manifold carrying codim-1 foliation transverse to the boundary $\partial M \neq \emptyset$. Let S be a connected component of ∂M with $\partial S = \emptyset$. Assume that $\mathscr{F}|S$ can be defined by a closed non-singular 1-form $\omega \in A^1(S)$.

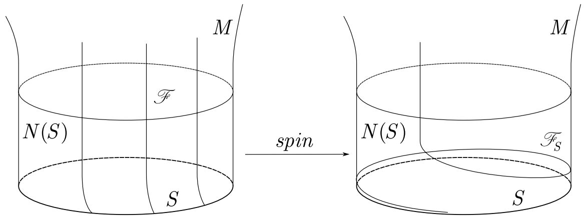

Figure 10. Spinning along the boundary component.

Let $N(S) = S \times [0,1)$ be a foliated collar, i.e. the leaves of $\mathscr{F} \mid N(S)$ are of the form $L \times [0,1)$, where L is a leaf of $\mathscr{F} \mid S$.

Decompose $T_{(x,t)}(N(S)) = T_x(S) \oplus T_t([0,1))$. One can write a vector field $\zeta \in X(N(S))$ as

$$\zeta = v_t + g\partial_t,$$

where $g \in C^\infty(N(S))$, $\partial_t = \partial/\partial t$, and v_t denotes the component of ζ tangent to S. Thus, ω extends to a closed non-singular form $\omega_{N(S)}$ by

$$\omega_{N(S)}(v_t + g\partial_t) = \omega(v_t).$$

Let $h : [0,1) \to [0,1]$ be a C^∞-function such that $h(t) = 0$ for $t \in [1/2,1)$, $h(0) = 1$, and h is decreasing strictly monotonically on $[0,1/2]$. Moreover, let the derivatives of all orders of h vanish at $t = 0$. Set

$$\theta = (1 - h(t))\omega_{N(S)} + h(t)\mathrm{d}t.$$

θ agrees with $\omega_{N(S)}$ on $S \times [1/2,1)$ and with $\mathrm{d}t$ on $S \times \{0\}$. Moreover, θ is integrable and S becomes a leaf of a new foliation $\mathscr{F}_S$ on $S \times [0,1)$. But $\mathscr{F}$ coincides with $\mathscr{F}_S$ outside the collar $S \times [0,1/2)$. Thus, we extend $\mathscr{F}_S$ to a foliation $\mathscr{F}_S$ on M which is tangent to the boundary component S.

Now, identify in $\mathrm{HLS}(\mathscr{F})$ the points of $\pi(S)$ and denote the result by X. Endow X with the quotient metric denoted by d_X.

Before we examine the Hausdorff leaf space for a spinned foliation we formulate technical lemmas. Easy proofs are omitted and left to the reader.

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[d, "\pi_S"] & \mathrm{HLS}(\mathscr{F}) \arrow[d, "\phi"'] \\ \mathrm{HLS}(\mathscr{F}_S) & X \end{tikzcd}$$

Figure 11. The projections for Theorem 6.

Let $\pi : M \to \mathrm{HLS}(\mathscr{F})$, $\pi_S : M \to \mathrm{HLS}(\mathscr{F}_S)$, and $\phi : \mathrm{HLS}(\mathscr{F}) \to X$ denote the natural projections (Figure 11). Denote by L_z (L^S_z) a leaf of $\mathscr{F}$ ($\mathscr{F}_S$) passing through $z \in M$.

LEMMA 14. *For every two points $p, q \in M$ such that $L^S_p = L^S_q$ we have*

$$d_X\big(\phi(\pi(L_p)), \phi(\pi(L_q))\big) = 0. \qquad \square$$

LEMMA 15. *For any two points $p, q \in M$ such that $L_p = L_q$ we have*

$$\tilde{\rho}_S\big(\pi_S(L^S_p), \pi_S(L^S_q)\big) = 0. \qquad \square$$

LEMMA 16. *For any two points $x, y \in M$ we have*

$$d_X\big(\phi(\pi(L_x)), \phi(\pi(L_y))\big) = \tilde{\rho}_S\big(\pi_S(x), \pi_S(y)\big). \qquad \square$$

THEOREM 6. $\mathrm{HLS}(\mathscr{F})$ *coincides with* (X, d_X).

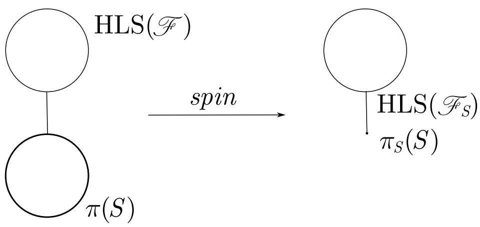

Figure 12. HLS of a foliation spinned along the boundary component S.

PROOF. Let $f_n = 1/n$ be a constant function on M. Let $A' = \{x_1, \dots, x_{k'}\} \subset M$ be an $\epsilon/2$-net on M. One can select a subset $A = \{x_1, \dots, x_k\} \subset A'$ and $N \in \boldsymbol{N}$ such that $\pi(x_i) \neq \pi(x_j)$ $(i \neq j)$ and A an ϵ-net on $M_{1/n} = (M, \mathscr{F}, g_{1/n})$ for all $n > N$. We may assume that the points $x_{k-l}, \dots, x_k$ are the only ones that belong to $\pi^{-1}(\pi(S))$. Now, pick from the points $x_{k-l}, \dots, x_k$ exactly one, let say

x_{k-l}.

Observe that $\pi_S(\{x_{k-l}, \dots, x_k\})$ is a single point in $\mathrm{HLS}(\mathscr{F}_S)$. Hence, there exists N' such that $\{x_1, \dots, x_{k-l}\}$ is an ϵ-net on $M^S_{1/n} = (M, \mathscr{F}_S, g_{1/n})$ for all $n > N'$. Moreover, since $x_{k-l}, \dots, x_k \in \pi^{-1}(\pi(S))$,

$$\phi(\pi(x_\mu)) = \phi(\pi(x_\nu)), \quad \mu, \nu \in \{k-l, \dots, k\}.$$

Set $\zeta_i = \phi(\pi(x_i))$, $\xi_j = \pi_S(x_j)$ $(i, j = 1, \dots, k-l)$. By the construction and Lemma 4, the sets $\{\zeta_i\}$ and $\{\xi_j\}$ are 2ϵ-nets on X and $\mathrm{HLS}(\mathscr{F}_S)$, respectively. By Lemma 16,

$$d_X(\zeta_i, \zeta_j) = \tilde{\rho}_S(\xi_i, \xi_j), \quad \text{for all } i, j \in \{1, \dots, k\}.$$

By Lemma 3, $d_{GH}(X, \mathrm{HLS}(\mathscr{F}_S)) = 0$, and by Theorem 1, X is isometric to $\mathrm{HLS}(\mathscr{F}_{\mathscr{S}})$. □

4.5. Suspension.

Denote by B a smooth connected manifold, and by $p : \tilde{B} \to B$ the universal covering of B. Let $x_0 \in B$. Recall that the covering transformation group Γ acts from the right on $\tilde{B}$ and hence $\Gamma \subset \mathrm{Diff}(\tilde{B})$. Let F be a q-dimensional manifold. Consider a group homomorphism $h : \Gamma \to \mathrm{Diff}(F)$. Then Γ acts on $\tilde{B} \times F$ by

$$\gamma(x, z) = (x \cdot \gamma, h(\gamma)(z)), \quad (x \in \tilde{B},\ z \in F).$$

Consider a foliation $\tilde{F} = \{\tilde{B} \times \{z\}, z \in F\}$. Using canonical projection one can project $\tilde{F}$ onto a foliation $\mathscr{F}$ of $M = (\tilde{B} \times F)/\Gamma$. The foliation $\mathscr{F}$ is called the *suspension* of the homomorphism h. One can check that M is a fibre bundle over B, and F coincides with its fibre.

Analogically as in Section 2.1, one can define the *Hausdorff orbit space*:

Let G be a group acting on a metric space (X, d_X). Denote by $\mathscr{O}$ the space of orbits of G-action. Set

$$\rho(G(x), G(y)) = \inf\left\{\sum_{i=1}^{n-1} d_X(G_i, G_{i+1})\right\},$$

where the infimum is taken over all finite sequences of orbits beginning at $G_1 = G(x)$ and ending at $G_n = G(y)$, and $G(z)$ denotes the orbit of $z \in X$. Define an equivalence relation $\sim$ in $\mathscr{O}$ by:

$$G(x) \sim G(y) \Leftrightarrow \rho(G(x), G(y)) = 0, \quad x, y \in X.$$

Let $\tilde{\mathscr{O}} = \mathscr{O}/_\sim$. Put

$$\tilde{\rho}\big([G(x)], [G(y)]\big) = \rho(G(x), G(y)),$$

where $[G(x)], [G(y)] \in \tilde{\mathscr{O}}$. $(\tilde{\mathscr{O}}, \tilde{\rho})$ is a metric space. We call it the *Hausdorff orbit space* of the G-action, and we denote it by $\mathrm{HOS}(X/G)$.

THEOREM 7. $\mathrm{HLS}(\mathscr{F})$ *is homeomorphic to* $\mathrm{HOS}(F/h(\Gamma))$.

PROOF. By the construction of suspension, there exists a homeomorphism between the space of leaves of $\mathscr{F}$ and the space of orbits of $h(\Gamma)$. It induces a homeomorphism between $\mathrm{HLS}(\mathscr{F})$ and $\mathrm{HOS}(F/h(\Gamma))$. □

5. Main results – HLS for codim-1 foliations.

5.1. HLS for compact *I*-bundles.

Let $(M, \mathscr{F}, \mathrm{pr})$ be a foliated I-bundle, $I = [0, 1]$. Note that there are at most two boundary leaves. Let us denote by L_0 the boundary leaf passing through the points $0 \in I$ of every fiber. Consider the function $d : \mathscr{L} \to [0, 1]$ ($\mathscr{L}$ denotes here the space of leaves of the foliation $\mathscr{F}$) defined by $d(L) = \tilde{\rho}(L_0, L)$, where $\tilde{\rho}$ denotes the metric in $\mathrm{HLS}(\mathscr{F})$. Let $\pi : M \to \mathrm{HLS}(\mathscr{F})$ again be the natural projection.

LEMMA 17. *For any two leaves $L \neq L'$ such that $\pi(L) \neq \pi(L')$ we have $d(L) \neq d(L')$.*

PROOF. Since $\pi(L) \neq \pi(L')$ then $\tilde{\rho}(\pi(L), \pi(L')) > 0$. Let $\epsilon > 0$, and let $L_1, \ldots, L_k$ be a family of leaves such that $L_k = L'$, and

$$\sum_{\nu=0}^{k-1} \mathrm{dist}(L_\nu, L_{\nu+1}) < \tilde{\rho}(L_0, L') + \epsilon.$$

Without loss of generality we can assume that there exists $j \in \{0, \ldots, k-1\}$ satisfying $L_j = L$ (if not then rename the leaf L to L' and L' to L). Then

$$\tilde{\rho}(L_0, L) + \tilde{\rho}(L, L') \leq \sum_{\nu=0}^{j-1} \mathrm{dist}(L_\nu, L_{\nu+1}) + \sum_{\nu=j}^{k-1} \mathrm{dist}(L_\nu, L_{\nu+1}) < \tilde{\rho}(L_0, L') + \epsilon.$$

Hence,

$$d(L) + \tilde{\rho}(L, L') \leq d(L').$$

By the triangle inequality and the above, we obtain

$$d(L) + \tilde{\rho}(L, L') = d(L').$$

But $\tilde{\rho}(L, L') > 0$. Hence, $d(L) < d(L')$. This completes the proof. $\square$

THEOREM 8. *Let* $(M, \mathscr{F}, \mathrm{pr})$ *be a foliated* I*-bundle.* $\mathrm{HLS}(\mathscr{F})$ *is isometric to a metric segment or a singleton.*

PROOF. Let L_0 denote the same leaf as in Lemma 17, d be a function on the space of leaves of $\mathscr{F}$ defined by $d(L) = \tilde{\rho}(L_0, L)$, and let $\delta = \max_{L \in \mathscr{F}} d(L)$. Let $\pi : M \to \mathrm{HLS}(\mathscr{F})$ be a natural projection, while $p : M \to [0, \delta]$ be the mapping defined by $p(x) = d(L_x)$. By Lemma 17, for any two leaves such that $\pi(L) \neq \pi(L')$ we have

$$d(L) \neq d(L').$$

Let $\epsilon > 0$, and $L, L' \in \mathscr{F}$ be two arbitrary leaves such that $d(L) < d(L')$. Let $L_1, \ldots, L_k,\ L_{k+1}, \ldots, L_{k+l}$ be a family of leaves satisfying $L_k = L$, $L_{k+l} = L'$, and

$$\sum_{\nu=0}^{k+l-1} \mathrm{dist}(L_\nu, L_{\nu+1}) \leq \tilde{\rho}(\pi(L_0), \pi(L')) + \epsilon.$$

Since $\tilde{\rho}(\pi(L_0), \pi(L)) \leq \sum_{\nu=0}^{k-1} \mathrm{dist}(L_\nu, L_{\nu+1})$, we have

$$\begin{aligned} \tilde{\rho}(\pi(L), \pi(L')) &\leq \sum_{\nu=k}^{k+l-1} \mathrm{dist}(L_\nu, L_{\nu+1}) \\ &\leq \tilde{\rho}\big(\pi(L_0), \pi(L')\big) + \epsilon - \tilde{\rho}\big(\pi(L_0), \pi(L)\big) = d(L') - d(L) + \epsilon. \end{aligned} \tag{9}$$

Now, let $L_1, \ldots, L_k$, $L_{k+1}, \ldots, L_{k+l}$ be a family of leaves such that $L_k = L$, $L_{k+l} = L'$

$$\sum_{\nu=0}^{k-1} \mathrm{dist}(L_\nu, L_{\nu+1}) \leq \tilde{\rho}(\pi(L_0), \pi(L)) + \frac{\epsilon}{2},$$

and

$$\sum_{\nu=k}^{k+l-1} \mathrm{dist}(L_\nu, L_{\nu+1}) \leq \tilde{\rho}(\pi(L), \pi(L')) + \frac{\epsilon}{2}.$$

Then

$$d(L') \leq \sum_{\nu=0}^{k+l-1} \mathrm{dist}(L_\nu, L_{\nu+1}) \leq \tilde{\rho}(\pi(L_0), \pi(L)) + \frac{\epsilon}{2} + \tilde{\rho}(\pi(L), \pi(L')) + \frac{\epsilon}{2}$$
$$\leq d(L) + \tilde{\rho}(\pi(L), \pi(L')) + \epsilon.$$

We get

$$d(L') - d(L) \leq \tilde{\rho}(\pi(L), \pi(L')) + \epsilon. \tag{10}$$

We finally get, by (9) and (10),

$$\big| |d(L) - d(L')| - \tilde{\rho}(\pi(L), \pi(L')) \big| \leq \epsilon. \tag{11}$$

Let $A = \{x_1, \dots, x_k\}$ be an ϵ-net on M. Then $\pi(A)$ and $p(A)$ are ϵ-nets on $\mathrm{HLS}(\mathscr{F})$ and $([0,d], |\cdot|)$, respectively. Moreover, $\sharp\pi(A) = \sharp p(A)$. By (11), we have

$$\big| |p(L_i) - p(L_j)| - \tilde{\rho}(\pi(L_i), \pi(L_j)) \big| \leq \epsilon,$$

where $L_\nu = L_{x_\nu}$. By Lemma 3, $d_{GH}(\mathrm{HLS}(\mathscr{F}), [0,d]) \leq 3\epsilon$. Finally,

$$d_{GH}(\mathrm{HLS}(\mathscr{F}), [0,d]) = 0,$$

and, by Theorem 1, $\mathrm{HLS}(\mathscr{F})$ is isometric to the metric segment $I = ([0,d], |\cdot|)$. □

5.2. HLS for codim-1 foliations.

Recall now [**1**] that the metric graph G is the result of gluing of a set of a disjoint metric segments $E = \{E_i\}$ and points $V = \{v_i\}$ along an equivalence relation defined in the union of V and the set of the endpoints of the segments equipped with the length metric. A graph G is called *finite* if V and E are finite.

THEOREM 9. $\mathrm{HLS}(\mathscr{F})$ *of any codimension one foliation on a compact Riemannian manifold is isometric to a finite connected metric graph.*

PROOF. Following the proof of the main theorem of [**7**], we can cover M by a finite number of mutually disjoint saturated neighbourhoods N_i ($i = 1, \dots, k$) such that the HLS of the foliation restricted to N_i is a singleton, and a finite number of mutually disjoint foliated I-bundles (denoted by $C_1, \dots, C_m$) with their HLS's, by Lemma 8, isometric to $[0, d_j]$, $d_j > 0$, $1 \leq j \leq m$. We can assume that $N_i \cap C_j \subset \partial N_i \cap \partial C_j$, $1 \leq i \leq k$, $1 \leq j \leq m$ (Figure 13), and that the sets N_i

($i = 1, 2, \ldots, k$) are maximal, i.e. $\pi^{-1}(\pi(N_i)) = N_i$, where $\pi : M \to \mathrm{HLS}(\mathscr{F})$ denotes the natural projection.

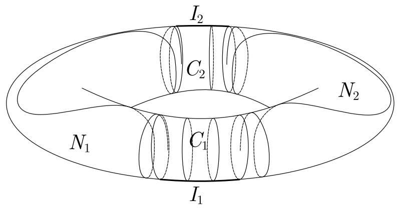

Figure 13. The sets N_i and C_j.

Let $v_i = \mathrm{HLS}(\mathscr{F}|N_i)$, and $V = \{v_1, \ldots, v_k\}$. Next, let

$$E = \{I_1, \ldots, I_m\}, \quad I_j = \mathrm{HLS}(\mathscr{F} \mid C_j) = [0, d_j].$$

Denote by $\pi_j : C_j \to [0, d_j]$ natural projections.

Introduce in V and in the set of the endpoints of the segments I_j, $1 \le j \le m$, the smallest equivalence relation $\sim$ generated by the following relation:

A point v_i is in the relation with an endpoint a (a can be equal to 0 or d_j) of the segment I_j iff $N_i \cap \pi_j^{-1}(a) \neq \emptyset$.

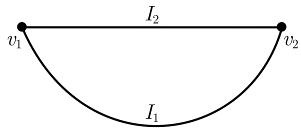

Figure 14. Construction of a graph.

Now, let glue points from V and segments from E along $\sim$. We now endow obtained space with the length metric. In this way we create a metric graph G (Figure 14). By the construction of G and Theorem 3, $\mathrm{HLS}(\mathscr{F})$ is isometric to G. □

Remark 18. One can easily check that, for a given foliation $\mathscr{F}$ of codimension one, it is possible to construct a number of metric graphs, not necessarily finite, isometric to $\mathrm{HLS}(\mathscr{F})$, but all of them are isometric as metric spaces with length metric. For example, consider a foliation of T^2 by a infinite number of Kronecker components separated by circles foliation (Figure 15). Then every Kronecker component can define itself a node of a graph, and every circle foliation can define an edge. One also can select only one Kronecker component to be a node, and the rest of foliation to be an edge. One can check that any metric graph

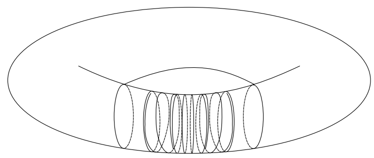

Figure 15. A part of a foliation by Kronecker components and circles.

constructed this way is isometric to a circle.

Example 19. Recall that any compact connected manifold of dimension 1 is either an interval I or a 1-dimensional sphere S^1. Hence, a foliated bundle of codim-1 is either I-bundle or S^1-bundle. One can see that the Hausdorff leaf space for a codim-1 foliated bundle is a singleton, a metric segment or a circle S^1.

Lemma 20. *For every $k \in \boldsymbol{N}$ there exists a compact foliated manifold $(M, \mathscr{F})$ such that M has exactly k boundary components and* $\mathrm{HLS}(\mathscr{F})$ *is a singleton, and the holonomy mappings h of the boundary leaves satisfy $h(0) = 0$, $h'(0) = 1$, $h^{(n)}(0) = 0$ for all $n \geq 2$.*

Proof. Let $\hat{M} = S^1 \times \Sigma$, where Σ is a compact surface of dimension 2, and let $\hat{\mathscr{F}}$ be the product foliation by $\{z\} \times \Sigma$, $z \in S^1$. Let $x_1, \ldots, x_k \in \Sigma$. Let N_i ($i = 1, \ldots, k$) be disjoint tubular neighbourhoods of $\gamma_i = S^1 \times \{x_i\}$. Turbulize $\hat{\mathscr{F}}$ simultaneously along γ_i. One can check [**3**] that it is possible to turbulize in such way that the holonomy mappings h of the compact leaves of the Reeb components satisfy $h(0) = 0$, $h'(0) = 1$, $h^{(n)}(0) = 0$ for all $n \geq 2$.

Next, let M be a foliated manifold obtained from $(\hat{M}, \hat{\mathscr{F}})$ by removing the interior of the Reeb components of the turbulized foliation. It follows that M is compact, and its boundary has exactly k components homeomorphic with the torus T^2. Moreover, every leaf different from boundary leaves accumulate on every boundary component. Thus $\mathrm{HLS}(\mathscr{F})$ is a singleton, and $\mathscr{F}$ is a foliation with desired properties. □

Remark 21. One can see that all leaves of the foliation constructed in Lemma 20 are proper.

Lemma 22. *For any metric segment $I = [0, d]$ there exists a compact foliated Riemannian manifold $(M, \mathscr{F}, g)$ carrying codim-1 foliation such that* $\mathrm{HLS}(\mathscr{F})$ *is isometric to I.*

Proof. Taking $M = [0, d] \times \Sigma$, where again Σ is a compact surface, with product foliation $\{t\} \times \Sigma$ and the product metric we get the statement. □

THEOREM 10. *For every finite connected metric graph G there exists a compact foliated Riemannian manifold $(M, \mathscr{F}, g)$ such that* $\mathrm{HLS}(\mathscr{F})$ *is isometric to G. Moreover, every leaf of $\mathscr{F}$ is proper.*

PROOF. Let $G = (V, E)$ be a finite connected metric graph with k nodes. "Cutting" every edge in the middle we obtain k connected metric graphs G_i (Figure 16).

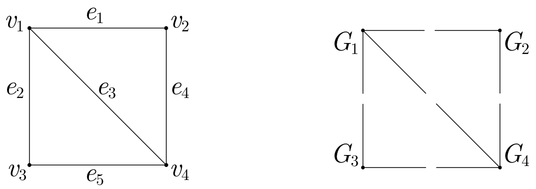

Figure 16. Star graphs G_i.

Consider a graph G_i. If all nodes of G_i have only one edge, then assign for G_i a foliated manifold indicated in Lemma 22.

Let v be a node having more than one edge, let say m. One can assign for v a 3-dimensional foliated Riemannian manifold $(V_i, \mathscr{F}_i, g_i)$ indicated in Lemma 20 with exactly m boundary components homeomorphic to the torus T^2, and such that HLS for V_i is a singleton, and the holonomy mappings h of the boundary leaves satisfy $h(0) = 0$, $h'(0) = 1$, $h^{(n)}(0) = 0$ for all $n \geq 2$.

Next, for every edge assign a manifold $E^i_\nu = [0, d_i] \times T^2$ (as described in Lemma 22), $1 \leq \nu \leq m$. Note that both $\mathscr{F}_i$ and foliations of E^i_ν are tangent to the boundary components.

Since the holonomy mappings h of the boundary leaves satisfy $h(0) = 0$, $h'(0) = 1$, $h^{(n)}(0) = 0$ for all $n \geq 2$, then by Theorem 3, one can glue manifolds V_i and E^i_ν to obtain a compact foliated Riemannian manifold $(M_i, \mathscr{F}_i, g_i)$ with $\mathrm{HLS}(\mathscr{F}_i)$ isometric to G_i (Figure 17). Moreover, the boundary components of M_i $(i = 1, \dots, m)$ are homeomorphic to T^2, the foliation $\mathscr{F}_i$ on each M_i is tangent

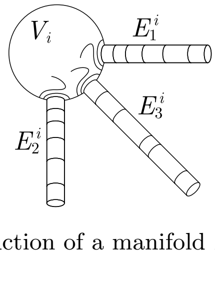

Figure 17. Construction of a manifold M_i for the graph G_i.

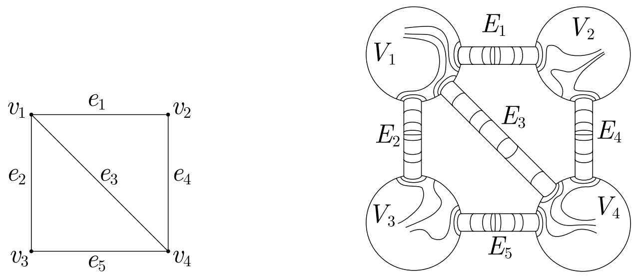

Figure 18. The graph G and the manifold $(M, \mathscr{F}, g)$.

to the boundary, and holonomy mappings h of boundary leaves satisfy $h'(0) = 1$, $h^{(n)}(0) = 0$ for all $n \geq 2$.

Again, by Theorem 3, one can glue manifolds M_i to get a compact foliated manifold $(M, \mathscr{F}, g)$ such that $\mathrm{HLS}(\mathscr{F})$ is isometric to G (Figure 18).

By Remark 21, all leaves of $\mathscr{F}$ are proper. This ends our proof. $\square$

5.3. Warped foliations in codim-1.

Let $(M, \mathscr{F}, g)$ be an arbitrary compact foliated Riemannian manifold with codim-1 foliation. Let $(f_n)_{n\in\boldsymbol{N}}$, $f_n : M \to (0, 1]$, be a sequence of warping functions (see Section 2.3). We will now provide the necessary and sufficient condition for a sequence of warped foliations $(M_{f_n})_{n\in\boldsymbol{N}}$ to converge to the Hausdorff leaf space for the foliation $\mathscr{F}$.

First, note that on any connected finite metric graph G with at least two nodes there exist a measure μ and constants $\beta \geq 1$, $\eta_0 > 0$ such that for all $0 < \eta < \eta_0$ and $x \in G$

$$\frac{1}{\beta}\eta \leq \mu(B_d(x, \eta)) \leq \beta\eta, \tag{12}$$

where $B_d(x, \eta) = \{y \in G : d(x, y) < \eta\}$. Indeed, denote by $V = \{e_1, \dots, e_k\}$ the set of vertices, and by $E = \{I_1, \dots, I_m\}$ the set of all edges of the graph G. Let μ be a measure induced by the Lebesgue measure on edges I_j of G and let $\eta_0 = (1/2) \min l(I_j)$ and $\beta = \max\{2, \max_{i=1,\dots,k} n(e_i)\}$, where $l(I)$ denotes the length of an edge I, and $n(e)$ denotes the number of edges in a vertex e. Such μ satisfies (12).

Let $(f_n)_{n\in\boldsymbol{N}}$, $f_n : M \to (0, 1]$, be a sequence of smooth warping functions on $(M, \mathscr{F}, g)$, where $\mathscr{F}$ is a foliation of codimension one.

THEOREM 11. $d_{GH}((M, g_{f_n}), \mathrm{HLS}(\mathscr{F})) \to 0$ *if and only if for every* $\varepsilon > 0$ *there exists* $N \in \boldsymbol{N}$ *such that for any* $n > N$ *the following is satisfied*:

There exists a finite family of leaves $F^n = \{F_1^n, \dots, F_k^n\}$ *such that*

1. $\bigcup F^n$ *is* ε*-dense in* M,
2. $f_n|_{\bigcup F^n} < \varepsilon$.

The proof of the sufficient condition is analogical to the proof of Theorem 2 in Section 3. The proof of the necessary condition is the same as the proof of Theorem 6.5 in [**10**]. We don't repeat them here and we leave them to the reader.

6. Final remarks.

One can ask, what is the classification of HLS for foliations of codimension greater than one. This question still is open. We only present some results for an arbitrary codimension.

Let $(M, \mathscr{F}, g)$ be a compact connected foliated Riemannian manifold, and again let $\pi : M \to \mathrm{HLS}(\mathscr{F})$ be the natural projection. One can easily check that π is continuous. Moreover, for any leaf $L \in \mathscr{F}$ the set $\pi^{-1}(\pi(L))$ is a closed, nonempty, saturated subset of M.

Let us recall that a subset $A \subseteq M$ is called minimal if it is nonempty, closed and saturated and there is no proper subset of A with these properties [**3**]. From the construction of $\mathrm{HLS}(\mathscr{F})$ it follows that for any leaf $L \in \mathscr{F}$ the set $\pi^{-1}(\pi(L))$ contains a minimal set.

As a simple consequence of the above observations we have:

Theorem 12. *If the number of minimal sets of* $\mathscr{F}$ *is countable then the* $\mathrm{HLS}(\mathscr{F})$ *is a singleton.*

Proof. Since the number of minimal sets is countable, then $\mathrm{HLS}(\mathscr{F})$ is a countable set. The projection $\pi : M \to \mathrm{HLS}(\mathscr{F})$ is continuous, hence $\mathrm{HLS}(\mathscr{F})$ is compact and connected. This ends our proof. □

Theorem 13. *If* $\mathscr{F}$ *contains a compact leaf with finite holonomy then* $\mathrm{HLS}(\mathscr{F})$ *contains an open subset* U *homeomorphic to an open set of* $\boldsymbol{R}^q$, *where* q *is a codimension of* $\mathscr{F}$.

Proof. The statement is a direct consequence of the Reeb Stability Theorem (see [**3**] or [**6**]). □

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Szymon M. WALCZAK
University of Łódź
Faculty of Mathematics and Computer Science
Banacha 22
90-238 Łódź, Poland
E-mail: sajmonw@math.uni.lodz.pl