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A smooth family of intertwining operators

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Abstract. Let N be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ and let $\mathscr{W}$ be a submanifold of $\mathfrak{n}^*$ such that the dimension of all polarizations associated to elements of $\mathscr{W}$ is fixed. We choose $(\mathfrak{p}(w))_{w\in\mathscr{W}}$ and $(\mathfrak{p}'(w))_{w\in\mathscr{W}}$ two smooth families of polarizations in $\mathfrak{n}$. Let $\pi_w = \mathrm{ind}_{P(w)}^N \chi_w$ and $\pi'_w = \mathrm{ind}_{P'(w)}^N \chi_w$ be the corresponding induced representations, which are unitary and irreducible. It is well known that π_w and π'_w are unitary equivalent. In this paper, we prove the existence of a smooth family of intertwining operator $(T_w)_w$ for theses representations, where w runs through an appropriate non-empty relatively open subset of $\mathscr{W}$. The intertwining operators are given by an explicit formula.

Introduction.

Let $N = \exp \mathfrak{n}$ be a connected, simply connected nilpotent Lie group. The Kirillov orbit method makes it possible to describe all the irreducible unitary representations of N. Let $l \in \mathfrak{n}^*$, $\mathfrak{p}(l)$ be an arbitrary polarization of l (a subalgebra of $\mathfrak{n}$ with maximal dimension satisfying $\langle l, [\mathfrak{p}(l), \mathfrak{p}(l)]\rangle \equiv 0$) and let $P(l) = \exp \mathfrak{p}(l)$. The induced unitary representation $\pi_l := \mathrm{ind}_{P(l)}^N \chi_l$, where $\chi_l(p) = e^{-i\langle l, \log p\rangle}$ for all $p \in P(l)$, is then irreducible and all the irreducible unitary representations are obtained in this way, up to unitary equivalence. For two distinct linear forms l and l', π_l and $\pi_{l'}$ are unitary equivalent if and only if l and l' belong to the same coadjoint orbit. If $\mathfrak{p}(l)$ and $\mathfrak{p}'(l)$ are two distinct polarizations of the same l, then π_l and $\pi'_l := \mathrm{ind}_{P'(l)}^N \chi_l$, where $P'(l) = \exp \mathfrak{p}'(l)$, are unitary equivalent. This means that there exists a unitary operator $T(l) : \mathscr{H}_{\pi_l} \longrightarrow \mathscr{H}_{\pi'_l}$ (between the respective representation spaces) such that $T(l) \circ \pi_l(x) = \pi'_l(x) \circ T(l)$ for all $x \in N$. In [**11**], G. Lion gave an explicit formula for such an intertwining operator:

$$T(l)\xi(g) = \int_{P'(l)/P(l)\cap P'(l)} \xi(gp)\chi_l(p)dp, \qquad g \in N$$

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for all $\xi \in \mathscr{S}(N/P(l), \chi_l) = \mathscr{H}_{\pi_l}^{\infty}$. This operator is unitary for an appropriate normalization of the measure dp on $P'(l)/P(l) \cap P'(l)$. In this formula, $l \in \mathfrak{n}^*$ is fixed. But one may ask the question of the dependency in l, if $(\mathfrak{p}(l))_l$ and $(\mathfrak{p}'(l))_l$ are two smooth families of polarizations.

As a matter of fact one often comes upon smooth families of polarizations, where l runs through a submanifold of $\mathfrak{n}^*$. Let us give some examples. Let $\mathscr{Z} = \{Z_1, \ldots, Z_n\}$ be a fixed Jordan-Hölder basis of $\mathfrak{n}$ and $\mathscr{W}$ be a submanifold of $\mathfrak{n}^*$ (for instance $\mathscr{W} = \mathfrak{n}^*_{gen}$, the open set of generic elements of $\mathfrak{n}^*$, as in [**12**]). For each $w \in \mathscr{W}$, we denote by $\mathfrak{p}_{\mathscr{Z}}(w)$ the Vergne polarization associated to w with respect to the basis $\mathscr{Z}$ (see [**3**]). Then $(\mathfrak{p}_{\mathscr{Z}}(w))_{w\in\mathscr{W}}$ is a smooth family of Vergne polarizations. Changing to another Jordan-Hölder basis $\mathscr{Z}'$, would yield a second smooth family of Vergne polarizations $(\mathfrak{p}_{\mathscr{Z}'}(w))_{w\in\mathscr{W}}$.

A second important example is obtained in the following way: Let us assume that H is a closed subgroup of the group $\mathrm{Aut}(N)$ of automorphisms of N acting smoothly on N. Let $l \in \mathfrak{n}^*$ be fixed. Then the orbit $H \cdot l$ is a submanifold of $\mathfrak{n}^*$, provided it is locally closed, by [**15**]. For a given fixed Jordan-Hölder basis $\mathscr{Z}$ one may again consider the smooth family of Vergne polarizations $(\mathfrak{p}_{\mathscr{Z}}(w))_{w\in H\cdot l} = (\mathfrak{p}_{\mathscr{Z}}(h \cdot l))_{h\in H}$. But for this example there exists another natural smooth family of polarizations, $(h \cdot \mathfrak{p}_{\mathscr{Z}}(l))_{h\in H}$. Note that here the actions of H on $\mathfrak{n}$ and $\mathfrak{n}^*$ are defined by $h\cdot X = \log(h\cdot \exp X)$ and $\langle h\cdot l, X\rangle = \langle l, h^{-1}\cdot X\rangle$. Particular cases of this example are obtained if either H is the group of conjugations of N or if $H = K$ is any compact subgroup of $\mathrm{Aut}(N)$. In the first case, $H \cdot l = \mathrm{Ad}^*(N) \cdot l = \Omega(l)$ is the co-adjoint orbit of l, which is closed as N is nilpotent. In the second case, the orbit $H \cdot l$ is compact and hence closed. So both cases yield submanifolds of $\mathfrak{n}^*$.

We are hence naturally led to the following question: Given a submanifold $\mathscr{W}$ of $\mathfrak{n}^*$ and given two distinct smooth families of polarizations $(\mathfrak{p}(w))_{w\in\mathscr{W}}$ and $(\mathfrak{p}'(w))_{w\in\mathscr{W}}$, is it possible to define a smooth family of intertwining operators $(T(w))_w$, via the formula of G. Lion? One immediately sees that some restrictions are necessary. So we certainly must assume that all our polarizations, and hence all the co-adjoint orbits $\mathrm{Ad}^*(N) \cdot w$, have the same dimension. But even then, one notices that the formula for $T(w)$ strongly depends on the dimension of $P'(w)/P(w) \cap P'(w)$ and will present singularities each time this dimension changes. In the example of the action of a group H, this is for instance the case at the point $l = e \cdot l$, where e is the identity element of H. So a smooth family of intertwining operators, if it exists, will probably only be defined on a non-empty relatively open subset of the submanifold $\mathscr{W}$.

In this paper, we prove the existence of such a smooth family of intertwining operators for an appropriate non-empty relatively open subset of the manifold $\mathscr{W}$, based on the formula of G. Lion [**11**]. The first section is devoted to introduce some definitions and tools. Then we explain the construction of different smooth bases and we define the generalized Schwartz spaces. In the third section, we use these tools to prove the main result of a generalized Kirillov theory. This result is a crucial tool to show that the family of operators $(T(w))_w$ constructed in the next section, is a smooth family of intertwining operators and defines a linear homeomorphism between $\mathscr{K}\mathscr{S}(\mathscr{V}_0', \mathscr{S}(N/P,\chi))$ and $\mathscr{K}\mathscr{S}(\mathscr{V}_0', \mathscr{S}(N/P',\chi))$ where $\mathscr{V}_0'$ denotes the appropriate non-empty relatively open subset of $\mathscr{W}$.

The existence of a smooth family of intertwining operators is very useful result. In the study of certain harmonic analysis properties, it will make it possible to choose a particularly well suited smooth family of polarizations and then to transpose the obtained result to an arbitrary setting. Hence, for instance in [**12**], the generalized Fourier inversion theorem gives the existence of a Schwartz-retract if the induced representations for $l \in \mathfrak{n}^*_{gen}$ and their operator kernels are realized via specially chosen Vergne polarizations. The existence of smooth families of intertwining operators will make it possible to transpose this result to other realizations of the representations, at least within a certain open subset of $\mathfrak{n}^*_{gen}$. A similar result is obtained for the example of the action of a compact Lie group K on a connected simply connected nilpotent Lie group N. The existence of a local retract for the families of Vergne polarizations $(\mathfrak{p}(k\cdot l))_k$ implies the existence of such retract for the family $(k\cdot\mathfrak{p}(l))_k$ of polarizations and the corresponding representations at least locally. Finally, the existence of singularities in this process may already be studied in the example of the Heisenberg group.

1. On indices and bases.

We start this section by recalling some basic notions and by introducing some useful definitions which are needed in the next sections. As general references, we recommend [**3**], [**2**], [**14**]...

First of all, let $\mathfrak{n}$ be a nilpotent Lie algebra and $N = \exp\mathfrak{n}$ the corresponding connected, simply connected nilpotent Lie group. In this section, $\mathscr{W}$ will denote any non-empty manifold.

1.1. Definitions.

Definition 1. A family of r vectors $\{X_1(w),\dots,X_r(w)\}_{w\in\mathscr{W}}$ is said to be a smooth family of vectors if for all $j\in\{1,\dots,n\}$, the map $w\longmapsto X_j(w)$ is smooth.

Definition 2. A vector space basis $\{Z_1,\dots,Z_n\}$ of the Lie algebra $\mathfrak{n}$ is said to be a Jordan-Hölder basis if $[Z_i,Z_j]\in\langle Z_{r+1},\dots,Z_n\rangle$ where $r=\max(i,j)$, for all $i,j\in\{1,\dots,n\}$ and where $\langle Z_{r+1},\dots,Z_n\rangle$ denotes the subspace generated by $Z_{r+1},\dots,Z_n$. If the family of vectors is smooth, we talk about a smooth Jordan-Hölder basis.

Definition 3. A vector space basis $\{Z_1,\dots,Z_n\}$ of the Lie algebra $\mathfrak{n}$ is said to be a Malcev basis if $\mathfrak{n}_j := \langle Z_j, \dots, Z_n \rangle$ is a subalgebra of $\mathfrak{n}$ for all $j \in \{1, \dots, n\}$. If the family of vectors is smooth, we call it a smooth Malcev basis.

DEFINITION 4. We define a smooth family of subalgebras $\mathfrak{q} = (\mathfrak{q}(w))_{w \in \mathscr{W}}$ in the following way:

- All algebras $\mathfrak{q}(w)$, $w \in \mathscr{W}$, have the same fixed dimension denoted by r.
- There exists $\{X_1(w), \dots, X_r(w)\}_{w \in \mathscr{W}}$ a smooth family of Malcev bases of $(\mathfrak{q}(w))_{w \in \mathscr{W}}$.

  If $\mathscr{W}$ is a submanifold of $\mathfrak{n}^*$ and if $\mathfrak{p} = (\mathfrak{p}(w))_{w \in \mathscr{W}}$ is a smooth family of subalgebras, we say that $(\mathfrak{p}(w))_{w \in \mathscr{W}}$ is a smooth family of polarisations if it satisfies moreover:

- $\mathfrak{p}(w)$ is a polarization of w in $\mathfrak{n}$, for all $w \in \mathscr{W}$.

DEFINITION 5. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{n}$. A family of vectors $\{Z_1, \dots, Z_d\}$ is called a Malcev basis of $\mathfrak{n}$ relative to $\mathfrak{h}$ if $\mathfrak{n} = \bigoplus_{j=1}^{d} \boldsymbol{R} Z_j \oplus \mathfrak{h}$ and if, for every $j = 1, \dots, d$, the subspace $\mathfrak{h}_j := \bigoplus_{i=j}^{d} \boldsymbol{R} Z_i \oplus \mathfrak{h}$ is a subalgebra of $\mathfrak{n}$.

If $\mathfrak{q} = (\mathfrak{q}(w))_{w \in \mathscr{W}}$ is a smooth family of subalgebras, we define similarly a smooth Malcev basis $\{X_1(w), \dots, X_r(w)\}_{w \in \mathscr{W}}$ by requiring that:

- The family $\{X_1(w), \dots, X_r(w)\}_{w \in \mathscr{W}}$ is smooth.
- For any fixed $w \in \mathscr{W}$, $\{X_1(w), \dots, X_r(w)\}$ is a Malcev basis of $\mathfrak{n}$ relative to $\mathfrak{q}(w)$.

DEFINITION 6. Let $\mathfrak{q} = (\mathfrak{q}(w))_{w \in \mathscr{W}}$ be a smooth family of subalgebras. A smooth family of vectors $\{X_1(w), \dots, X_d(w)\}_{w \in \mathscr{W}}$ is said to be a smooth supplementary basis of polynomial character (smooth SP-basis) of $\mathfrak{q} = (\mathfrak{q}(w))_{w \in \mathscr{W}}$, if, given any smooth family of bases $\{Y_{d+1}(w), \dots, Y_n(w)\}_{w \in \mathscr{W}}$ of $(\mathfrak{q}(w))_{w \in \mathscr{W}}$, the maps

$$\begin{aligned} \phi_w : \boldsymbol{R}^d \times \boldsymbol{R}^{n-d} &\longrightarrow N \\ (x_1, \dots, x_d, y_{d+1}, \dots, y_n) &\longmapsto \exp x_1 X_1(w) \cdots \exp x_d X_d(w) \\ &\qquad \cdot \exp(y_{d+1} Y_{d+1}(w) + \cdots + y_n Y_n(w)) \end{aligned}$$

are smooth polynomial diffeomorphisms, which means:

- For any fixed w, ϕ_w is a diffeomorphism.
- $$\begin{aligned} &\exp x_1 X_1(w) \cdots \exp x_d X_d(w) \cdot \exp\big(y_{d+1} Y_{d+1}(w) + \cdots + y_n Y_n(w)\big) \\ &\quad = \exp\big(P_1(w,x,y) X_1(w) + \cdots + P_d(w,x,y) X_d(w) \\ &\qquad + P_{d+1}(w,x,y) Y_{d+1}(w) + \cdots + P_n(w,x,y) Y_n(w)\big) \end{aligned}$$

with $P_j(w,x,y)$ polynomials in x,y with smooth coefficients in w.

- The inverse maps defined by

$$\phi_w^{-1} : N \longrightarrow \boldsymbol{R}^d \times \boldsymbol{R}^{n-d}$$
$$g = \exp\big(x'_1X_1(w) + \cdots + x'_dX_d(w) + y'_{d+1}Y_{d+1}(w) + \cdots + y'_nY_n(w)\big)$$
$$\longmapsto \big(Q_1(w,x',y'),\ldots,Q_d(w,x',y'),Q_{d+1}(w,x',y'),\ldots,Q_n(w,x',y')\big)$$

are such that $Q_j(w,x',y')$ is polynomial in x',y' with smooth coefficients in w for every j, where the Q_j's are defined by

$$\begin{aligned}&\exp\big(x'_1X_1(w) + \cdots + x'_dX_d(w) + y'_{d+1}Y_{d+1}(w) + \cdots + y'_nY_n(w)\big)\\ &\quad= \exp Q_1(w,x',y')X_1(w)\cdots\exp Q_d(w,x',y')X_d(w)\\ &\qquad\cdot\exp\big(Q_{d+1}(w,x',y')Y_{d+1}(w) + \cdots + Q_n(w,x',y')Y_n(w)\big).\end{aligned}$$

It is of course enough to impose these conditions for one arbitrary smooth family of bases $\{Y_{d+1}(w),\ldots,Y_n(w)\}_{w\in\mathscr{W}}$ of $(\mathfrak{q}(w))_{w\in\mathscr{W}}$.

Example 1. A smooth Malcev basis of $\mathfrak{n}$ with respect to $\mathfrak{q} = (\mathfrak{q}(w))_{w\in\mathscr{W}}$ is a smooth SP-basis. The proof of this fact is similar to the one in the fixed case (see [**3**]).

1.2. Indices and a new family of smooth bases of polarizations.

Let $(\mathfrak{q}(w))_{w\in\mathscr{W}}$ be a smooth family of subalgebras in $\mathfrak{n}$ of fixed dimension r and $\mathscr{Z} = \{Z_1,\ldots,Z_n\}$ be a Jordan-Hölder basis of $\mathfrak{n}$. For each w, we define the associated index sets in the following way:

$$I_{\mathscr{Z}}^{\mathfrak{q}(w)} = \{j\in\{1,\ldots,n\} \mid \mathfrak{n}_j+\mathfrak{q}(w) = \mathfrak{n}_{j+1}+\mathfrak{q}(w)\} \quad\text{and}\quad J_{\mathscr{Z}}^{\mathfrak{q}(w)} = \{1,\ldots,n\}\backslash I_{\mathscr{Z}}^{\mathfrak{q}(w)}.$$

By construction, the set $\{Z_j \mid j\in J_{\mathscr{Z}}^{\mathfrak{q}(w)}\}$ is a basis of a supplementary vector subspace to $\mathfrak{q}(w)$ in $\mathfrak{n}$. Hence, the cardinality of this set is $n-r$ and the cardinality of $I_{\mathscr{Z}}^{\mathfrak{q}(w)}$ is r.

Let us write, for any fixed $w\in\mathscr{W}$, $J_{\mathscr{Z}}^{\mathfrak{q}(w)} = \{u_1(w),\ldots,u_{n-r}(w)\} \subset \{1,\ldots,n\}$, ordered such that $u_{n-r}(w) < \cdots < u_1(w)$.

Let

$$u_1 = \max_{w\in\mathscr{W}} u_1(w),$$
$$U_0 = \mathscr{W} \text{ and } U_1 = \{w\in U_0 \mid u_1(w) = u_1\}.$$

By induction, we define for $i = 1,\ldots,n-r$,

$$u_i = \max_{w \in U_{i-1}} u_i(w) \text{ and } U_i = \{w \in U_{i-1} \mid u_i(w) = u_i\}.$$

This inductive process will stop if $i = n - r$. Thus, we get fixed index sets $J = \{u_{n-r} < \cdots < u_1\}$ and $I = \{1, \ldots, n\} \setminus J$.

PROPOSITION 1. *The subsets U_i satisfy the following properties*:

(i) *For each $i = 1, \ldots, n-r$, there exists a continuous function $\mathscr{P}_i$ on $\mathscr{W}$ such that,*

$$U_i = \{w \in U_0 \mid \mathscr{P}_1(w) \neq 0, \mathscr{P}_2(w) \neq 0, \ldots, \mathscr{P}_i(w) \neq 0\}.$$

(ii) *Each U_i is a non-empty relatively open subset of $\mathscr{W}$, as well as the subset $\tilde{U} = \bigcap_{i=1}^{n-r} U_i$, and the index sets $I_{\mathscr{Z}}^{\mathfrak{q}(w)}$ and $J_{\mathscr{Z}}^{\mathfrak{q}(w)}$ are constant if $w \in \tilde{U}$. They are denoted by I and J.*

PROOF. Let $\{X_1(w), \ldots, X_r(w)\}$ denote the given basis of $\mathfrak{q}(w)$. The fixed index set $J = \{u_{n-r} < \cdots < u_1\}$ is obtained by construction. By construction also, the sets $U_i = \{w \in U_{i-1} \mid u_i(w) = u_i\}$ are non-empty. Then, we get

$$\begin{aligned} w \in U_i \Longleftrightarrow\ & w \in U_{i-1} \text{ and } rg(Z_{u_i}, \ldots, Z_n, X_1(w), \ldots, X_r(w)) = rg(Z_{u_i+1}, \ldots, Z_n, \\ & X_1(w), \ldots, X_r(w)) + 1 = r + (i-1) + 1 = r + i \\ \Longleftrightarrow\ & w \in U_{i-1} \text{ and } \textstyle\sum D_{r+i}^2 \neq 0 \text{ where } D_{r+i} \text{ runs through all the sub-} \\ & \text{determinants } (r+i) \times (r+i) \text{ chosen in the coefficient matrix of} \\ & Z_{u_i}, Z_{u_i+1}, \ldots, Z_n, X_1(w), \ldots, X_r(w) \\ \Longleftrightarrow\ & w \in U_{i-1} \text{ and } \mathscr{P}_i(w) \neq 0. \end{aligned}$$

□

PROPOSITION 2. *$\{Z_j \mid j \in J\}$ is a smooth Malcev basis of $\mathfrak{n}$ relative to $\mathfrak{q}(w)$ for all $w \in \tilde{U}$. Similarly if the original basis is smooth.*

PROOF. Same argument as in the fixed case. We refer the reader to [**2**]. □

PROPOSITION 3. *There exists a non-empty relatively open subset V of $\tilde{U}$ such that, for all $j \in I$ and for all $w \in V$, there is a unique vector $\tilde{Z}_j(w) \in \mathfrak{q}(w)$ of the form*

$$\tilde{Z}_j(w) = Z_j - \sum_{i=j+1, i \notin I}^{n} \tilde{\beta}_{ij}(w) Z_i \in \mathfrak{q}(w).$$

Then $\tilde{Z}_j(w) \in \mathfrak{n}_j \backslash \mathfrak{n}_{j+1}$.

Moreover, the map $w \longmapsto \tilde{Z}_j(w)$ *is smooth and* $\{\tilde{Z}_j(w) \mid j \in I\}$ *is a smooth Jordan-Hölder basis of* $\mathfrak{q}(w)$. *Let us put* $\tilde{Z}_j(w) = Z_j$ *if* $j \notin I$. *Then the set*

$$\tilde{\mathscr{Z}} := \{\tilde{Z}_1(w), \ldots, \tilde{Z}_n(w)\}$$

is a smooth Jordan-Hölder basis of all of $\mathfrak{n}$. *For the two bases* $\mathscr{Z} := \{Z_1, \ldots, Z_n\}$ *and* $\tilde{\mathscr{Z}}$, *the different subspaces* $\mathfrak{n}_j$ *are the same. Hence, the indices computed in each one of the two bases are the same for any subalgebra. The same procedure may of course be applied if the original basis* $\mathscr{Z}$ *is already a smooth basis in* w.

Proof. Let us first prove the existence and the uniqueness of $\tilde{Z}_j(w)$ for any fixed $w \in \tilde{U}$:

$$\begin{aligned} j \in I &\Longleftrightarrow Z_j \in \mathfrak{n}_{j+1} + \mathfrak{q}(w) \\ &\Longleftrightarrow Z_j = \sum_{i=j+1}^{n} \alpha_{ij}(w) Z_i + Z'_j(w) \text{ for some } \alpha_{ij}(w) \in \boldsymbol{R} \text{ and } Z'_j(w) \in \mathfrak{q}(w) \\ &\Longrightarrow Z'_j(w) = Z_j - \sum_{i=j+1, i \notin I}^{n} \alpha_{ij}(w) Z_i - \sum_{i=j+1, i \in I}^{n} \alpha_{ij}(w) Z_i. \end{aligned}$$

Then, we repeat the same process for $i \geq j+1$, $i \in I$. We finally see that for each $j \in I$, there exists a vector $\tilde{Z}_j(w) \in \mathfrak{q}(w)$ of the form

$$\tilde{Z}_j(w) = Z_j - \sum_{i=j+1, i \notin I}^{n} \tilde{\beta}_{ij}(w) Z_i \in \mathfrak{q}(w).$$

An easy argument by contradiction shows the uniqueness of $\tilde{Z}_j(w)$. To prove the smoothness of $\tilde{Z}_j(w)$, we use the fact that there exists a non-empty relatively open subset V of $\tilde{U}$ such that for all $w \in V$, $\mathfrak{q}(w)$ is characterized by $(n-r)$ independent Cartesian equations $\sum_{i=1}^{n} b_{ki}(w) x_i = 0$ for all $1 \leq k \leq n-r$, where $w \longmapsto b_{ki}(w)$ are smooth functions and where x_i are the coordinates in the fixed basis of $\mathfrak{n}$. Introducing $\tilde{Z}_j(w)$ in these equations, we obtain the conditions

$$\sum_{i=j+1, i \notin I}^{n} b_{ki}(w) \tilde{\beta}_{ij}(w) = b_{kj}(w) \qquad 1 \leq k \leq n-r. \tag{1}$$

Let $\mathfrak{a}_j(w)$ be the affine subspace defined by the following equations

$$\begin{cases} x_i = 0 & \text{if } 1 \leq i \leq j-1 \\ x_j = 1 & \\ x_k = 0 & \text{if } k \geq j+1 \text{ and } k \in I \end{cases} \tag{2}$$

which means that all vectors of $\mathfrak{a}_j(w)$ have the form $Z_j + \sum_{i=j+1, i \notin I}^{n} x_i Z_i$. In particular $\tilde{Z}_j(w) \in \mathfrak{a}_j(w)$. We know that the combined system (1) and (2) has $\tilde{Z}_j(w)$ as a unique solution. As the coefficients of this system are smooth in w, the maps $w \longmapsto \tilde{Z}_j(w)$ are smooth in a suitable non-empty relatively open subset of $\tilde{U}$. It is easy to prove that $\{\tilde{Z}_j(w) \mid j \in I\}$ is a Jordan-Hölder basis of $\mathfrak{q}(w)$.

The other statements of the proposition are deduced easily from the particular form of the $\tilde{Z}_j(w)$'s. □

2. Smooth bases and generalized Schwartz spaces.

2.1. Construction of smooth SP-bases.

Let $(\mathfrak{q}(w))_{w\in\mathscr{W}}$ and $(\mathfrak{q}'(w))_{w\in\mathscr{W}}$ be two smooth families of subalgebras.

Lemma 1. *There exists a non-empty relatively open subset $\tilde{V}$ of $\mathscr{W}$ such that for all $w \in \tilde{V}$, the dimension of $\mathfrak{q}(w)\cap\mathfrak{q}'(w)$ is constant and minimal. Furthermore we can choose a smooth Jordan-Hölder basis of $\mathfrak{q}(w) \cap \mathfrak{q}'(w)$.*

Proof. Let $\mathfrak{q}^{\perp}(w) = \{\psi \in \mathfrak{n}^* \mid \langle \psi, \mathfrak{q}(w)\rangle = 0\}$ be the orthogonal of $\mathfrak{q}(w)$ in $\mathfrak{n}^*$. The space $\mathfrak{q}^{\perp}(w)$ may be described by a homogenous system of $n-r$ independent equations for the coordinates in the basis $\{Z_1^*, \dots, Z_n^*\}$, whose coefficients are smooth functions of w, for w running through a non-empty relatively open subset U of $\mathscr{W}$. Thus, by solving this system, we may construct a smooth basis of $\mathfrak{q}(w)^{\perp}$ denoted $\{\varphi_{r+1}(w), \dots, \varphi_n(w)\}$ for all $w \in U$. Applying Proposition 3 to $\mathfrak{q}'(w)$, we then conclude that there exists a non-empty relatively open subset V in U such that $\{\tilde{Z}'_{j'_1}(w), \dots, \tilde{Z}'_{j'_r}(w)\}$ is a smooth Jordan-Hölder basis of $\mathfrak{q}'(w)$ where $\tilde{Z}'_{j'_i}(w) = Z_{j'_i} - \sum_{k=j'_i+1, k\notin I'}^{n} \tilde{\beta}'_{kj'_i}(w) Z_k$ for all $i \in \{1, \dots, r\}$ and where I' is the index set associated to all $\mathfrak{q}'(w)$'s.

In order for a vector $\tilde{C}(w) = \sum_{i=1}^{r} c_{j'_i} \tilde{Z}'_{j'_i}(w) \in \mathfrak{q}'(w)$ to belong to $\mathfrak{q}(w)\cap\mathfrak{q}'(w)$, it is necessary and sufficient to have, for $r+1 \leq a \leq n$,

$$\sum_{f=1}^{r} \left\langle \varphi_a(w), \tilde{Z}'_{j'_f}(w) \right\rangle c_{j'_f} = \sum_{f=1}^{r} A_{af}(w) c_{j'_f} = 0$$

where $(A_{af}(w))_{r+1\leq a\leq n; 1\leq f\leq r} = (\langle \varphi_a(w), \tilde{Z}'_{j'_f}(w)\rangle)_{r+1\leq a\leq n; 1\leq f\leq r}$ is a smooth matrix. The solution set of this system is the sub-vector space $\mathfrak{q}'(w) \cap \mathfrak{q}(w)$ of $\mathfrak{q}'(w)$

whose dimension is equal to $r - \operatorname{rank} A_{af}(w)$. To realize the constant and minimal dimension of the previous sub-vector space, we choose one w_0 which realizes the biggest possible rank for the matrix $A_{af}(w)$, which will be denoted by p. In particular there is a sub-matrix $\tilde{A}(w_0)$ of dimension $p \times p$ such that $\det \tilde{A}(w_0) \neq 0$ and the dimension of $\mathfrak{q}(w_0) \cap \mathfrak{q}'(w_0)$ is equal to $r - p$. On the non-empty relatively open subset $V_1 = \{w \in V \mid \det \tilde{A}(w) \neq 0\}$, the dimension of $\mathfrak{q}(w) \cap \mathfrak{q}'(w)$ is constant and minimal. Furthermore, by an argument similar to the one of Proposition 3, there is a method to choose a smooth Jordan-Hölder basis of $\mathfrak{q}(w) \cap \mathfrak{q}'(w)$. To do this, we may have to limit ourselves to another non-empty relatively open subset $\tilde{V}$ of V_1. □

We will now show the existence of a special smooth SP-basis associated to $(\mathfrak{q}(w))_{\mathscr{W}}$ and $(\mathfrak{q}'(w))_{w \in \mathscr{W}}$. In order to do this, we generalize the proof of G. Lion [**11**] to the smooth case.

PROPOSITION 4. *Let $(\mathfrak{q}(w))_{\mathscr{W}}$ and $(\mathfrak{q}'(w))_{w \in \mathscr{W}}$ be two smooth families of subalgebras. Let us assume that the dimension of $\mathfrak{q}(w) \cap \mathfrak{q}'(w)$ is constant and minimal on $\mathscr{W}$ (by restriction of $\mathscr{W}$ if necessary). Then there exists a non-empty relatively open subset $\tilde{V}$ of $\mathscr{W}$ and a family of smooth SP-bases $\{U_1(w), \ldots, U_s(w)\}$ of $\mathfrak{n}$ with respect to $\mathfrak{q}(w)$, such that $\{U_{b+1}(w), \ldots, U_s(w)\}$ is a smooth SP-basis of $\mathfrak{q}'(w)$ with respect to $\mathfrak{q}(w) \cap \mathfrak{q}'(w)$ for all $w \in \tilde{V}$.*

PROOF. By induction. To make this induction rigorous, we may even assume that both $(\mathfrak{q}(w))_w$ and $(\mathfrak{q}'(w))_w$ are contained in a smooth family of subalgebras $(\mathfrak{n}(w))_w$ (spanned by a smooth Jordan-Hölder basis) of the bigger Lie algebra $\mathfrak{n}$.

The induction is then made on $\dim(\mathfrak{n}(w)/\mathfrak{q}'(w))$.

If $\mathfrak{q}'(w) = \mathfrak{n}(w)$ for all w in a non-empty relatively open subset of $\mathscr{W}$, then $\mathfrak{q}(w) \cap \mathfrak{q}'(w) = \mathfrak{q}(w)$ and Proposition 2 gives the SP-basis.

Otherwise, there exists a smooth family $(\mathfrak{n}_0(w))_w$ of ideals of codimension 1 in $\mathfrak{n}(w)$ such that $\mathfrak{q}'(w) \subset \mathfrak{n}_0(w)$ for all w. In fact, by Proposition 2, there exists a smooth Malcev basis $\{Z_1(w), \ldots, Z_a(w)\}$ of $\mathfrak{n}(w)$ relative to $\mathfrak{q}'(w)$. We may then take $\mathfrak{n}_0(w) = \bigoplus_{j=2}^{a} \boldsymbol{R} Z_j(w) \oplus \mathfrak{q}'(w)$. It is an ideal in $\mathfrak{n}(w)$, as it is a subalgebra of codimension 1.

We have to distinguish two cases.

Either $\mathfrak{q}(w) \subset \mathfrak{n}_0(w)$ for all w in a non-empty relatively open subset of $\mathscr{W}$. By the induction hypothesis, there exists a smooth SP-basis $\{R_1(w), \ldots, R_u(w)\}$ of $\mathfrak{n}_0(w)$ with respect to $\mathfrak{q}(w) \cap \mathfrak{n}_0(w)$ such that $\{R_{v+1}(w), \ldots, R_u(w)\}$ is a smooth SP-basis of $\mathfrak{q}'(w)$ with respect to $\mathfrak{q}'(w) \cap \mathfrak{q}(w) \cap \mathfrak{n}_0(w) = \mathfrak{q}'(w) \cap \mathfrak{q}(w)$. By Proposition 2 there exists a smooth vector $S(w)$ which is a Malcev basis of $\mathfrak{n}(w)$ relative to $\mathfrak{n}_0(w)$. Then $\{S(w), R_1(w), \ldots, R_u(w)\}$ is the SP-basis of $\mathfrak{n}(w)$ relative to $\mathfrak{q}(w)$ we are looking for.

In the second case, there exists w such that $\mathfrak{q}(w) \not\subset \mathfrak{n}_0(w)$. By smoothness, this is the case for all w in a non-empty relatively open subset of $\mathscr{W}$. Then the smooth SP-basis $\{R_1(w), \ldots, R_u(w)\}$ of $\mathfrak{n}_0(w)$ relative to $\mathfrak{q}'(w) \cap \mathfrak{q}(w) \cap \mathfrak{n}_0(w) = \mathfrak{q}'(w) \cap \mathfrak{q}(w)$ also works for $\mathfrak{n}(w)$ relative to $\mathfrak{q}(w)$, for all w in that non-empty relatively open subset.

At the end of the induction process, we take $\mathfrak{n}(w) = \mathfrak{n}$ for all w. □

2.2. SP-basis and invariant measure.

Let $\mathfrak{q} = (\mathfrak{q}(w))_{w \in \mathscr{W}}$ be any smooth family of subalgebras. We have the following smooth versions of results of [**11**].

Proposition 5. *Let $\{U_1(w), \ldots, U_s(w)\}_{w \in \mathscr{W}}$ and $\{V_1(w), \ldots, V_s(w)\}_{w \in \mathscr{W}}$ be two smooth SP-bases of $\mathfrak{n}$ relative to $\mathfrak{q} = (\mathfrak{q}(w))_{w \in \mathscr{W}}$. Let us define the maps*

$$\psi_w : \boldsymbol{R}^s \longrightarrow \boldsymbol{R}^s$$

by $(y_1, \ldots, y_s)_w = \psi_w(x_1, \ldots, x_s)_w$ if and only if

$$\exp x_1 U_1(w) \cdots \exp x_s U_s(w) = \exp y_1 V_1(w) \cdots \exp y_s V_s(w) \bmod \exp \mathfrak{q}(w).$$

Then the maps ψ_w are polynomial with smooth coefficients in w. Similarly for ψ_w^{-1}. This is in particular the case for two smooth Malcev bases. One may have to restrict $\mathscr{W}$ to a non-empty relatively open subset.

Proof. Obvious. □

Remark 1. The maps ψ_w and ψ_w^{-1} define a change of coordinates in $N/Q(w)$ where $Q(w) = \exp \mathfrak{q}(w)$. As the Jacobians of this change of coordinates are polynomial functions with smooth coefficients in w as well for ψ_w and ψ_w^{-1}, these Jacobians have to be constants.

This leads to the following result:

Proposition 6. *Let $\{U_1(w), \ldots, U_s(w)\}_{w \in \mathscr{W}}$ be a smooth SP-basis of $\mathfrak{n}$ relative to $\mathfrak{q} = (\mathfrak{q}(w))_{w \in \mathscr{W}}$. Then*

$$f \longmapsto \int_{\boldsymbol{R}^d} f(\exp u_1 U_1(w) \cdots \exp u_s U_s(w)) du_1 \cdots du_s$$

defines a left-invariant measure on $N/Q(w)$, $Q(w) = \exp \mathfrak{q}(w)$, for every w. One may have to restrict $\mathscr{W}$.

Proof. We know that this is the case if the basis is a smooth Malcev basis. As there always exist smooth Malcev bases and as the transition between a smooth Malcev basis and a smooth SP-basis has a constant Jacobian, the result is true for any SP-bases. □

2.3. Generalized Schwartz spaces.

Let $\mathscr{W}$ be a submanifold of $\mathfrak{n}^*$ and $\mathfrak{p} = (\mathfrak{p}(w))_{w\in\mathscr{W}}$ a smooth family of polarizations, i.e. such that all polarizations $\mathfrak{p}(w)$, $w \in \mathscr{W}$, have a fixed dimension r.

Definition 7. Let W be a non-empty relatively open subset of $\mathscr{W}$. We denote $\mathscr{P}\mathscr{D}(W, \boldsymbol{R}^s)$ the set of differential operators in the coordinates $(v_1, \ldots, v_s) \in \boldsymbol{R}^s$ with polynomial coefficients in the coordinates $(v_1, \ldots, v_s)$, the coefficients of these polynomials being smooth functions on W. Formally, for each $w \in W$, we write

$$D_w(v) = \sum_{|\alpha|\le a} \sum_{|\beta|\le b} c_{\alpha\beta}(w) v^\beta \frac{\partial^\alpha}{\partial v^\alpha}$$

for such a differential operator where $c_{\alpha\beta}(w)$ are smooth coefficients, α, β multi-indices and a, $b \in \boldsymbol{N}$.

For all $A, B, C \in \boldsymbol{N}$ and for all compact subset K contained in a local chart (U, φ) of W, we denote by $\partial^a/\partial w^a$ the partial derivatives with respect to the coordinate system of this chart U. Then, we define a semi-norm on the smooth functions from $W \times \boldsymbol{R}^s$ to $\boldsymbol{C}$ in the following way:

$$\|\tilde{\xi}\|^K_{A,B,C} = \sup_{w\in K,(v_1,\ldots,v_s)\in\boldsymbol{R}^s} \sup_{|a|\le A,|b|\le B,|c|\le C} \left| \frac{\partial^a}{\partial w^a} v^b \frac{\partial^c}{\partial v^c} \tilde{\xi}(w, v_1, \ldots, v_s) \right|. \quad (3)$$

We can now define the generalized Schwartz space as follows:

Definition 8. Let

$$\begin{aligned}
&\mathscr{K}\mathscr{S}\big(W, \mathscr{S}(N/P, \chi)\big) \\
&\quad \equiv \mathscr{K}\mathscr{S}(W, \mathscr{S}(\boldsymbol{R}^s)) \\
&\quad = \big\{\xi = (\xi(w))_{w\in W} \mid \xi : W \longrightarrow \mathscr{S}(N/P, \chi) \text{ smooth}, \ \xi(w; g(w)q(w)) \\
&\qquad := \xi(w)(g(w)\cdot q(w)) = \overline{\chi_w(q(w))}\xi(w; g(w)) \text{ if } q(w) \in P(w) \text{ and} \\
&\qquad \|\tilde{\xi}\|^K_{A,B,C} < \infty \text{ for all } K \text{ and all } A, B, C\big\}
\end{aligned}$$

where $\tilde{\xi}(w, v_1, \ldots, v_s) := \xi(w, \exp v_1 V_1(w) \cdots \exp v_s V_s(w))$ and where $\mathscr{S}(N/P, \chi)$ $:= (\mathscr{S}(N/P(w), \chi_w))_{w \in W}$.

The space $\mathscr{K}\mathscr{S}(W, \mathscr{S}(N/P, \chi))$ will be called generalized Schwartz space. Here $\{V_1(w), \ldots, V_s(w)\}$ denotes an arbitrary smooth Malcev basis of $\mathfrak{n}$ relative to $\mathfrak{p}(w)$ (see Proposition 7 below).

Then, the semi-norms given in (3) define the same topology on our function space as the following family of semi-norms:

$$\|\tilde{\xi}\|_{A,D}^{K} = \sup_{w \in K, (v_1, \ldots, v_s) \in \boldsymbol{R}^s} \sup_{|a| < A} \left| \frac{\partial^a}{\partial w^a} D_w(v) \tilde{\xi}(w, v_1, \ldots, v_s) \right| < \infty \tag{4}$$

for all $A \in \boldsymbol{N}$, for all $D = (D_w)_{w \in W} \in \mathscr{P}\mathscr{D}(W, \boldsymbol{R}^s)$ and for all compact subset K contained in a chart of W.

PROPOSITION 7. *The generalized Schwartz space is independent of the choice of the smooth Malcev basis of* $\mathfrak{n}$ *relative to* $\mathfrak{p}(w)$*, provided we are willing to restrict the domain of* w *appropriately, if necessary.*

PROOF. Let $\{V_1'(w), \ldots, V_s'(w)\}$ be another smooth Malcev basis of $\mathfrak{n}$ relative to $\mathfrak{p}(w)$. Let $\{X_1(w), \ldots, X_r(w)\}$ be a smooth Malcev basis of $\mathfrak{p}(w)$. We may have to restrict $\mathscr{W}$ for the existence of $\{X_1(w), \ldots, X_r(w)\}$. Then, by Proposition 5, for all $\xi \in \mathscr{K}\mathscr{S}(W_1, \mathscr{S}(N/P, \chi))$, for some open subset W_1 of $\mathscr{W}$, we have

$$\begin{aligned}
&\tilde{\tilde{\xi}}\big(w; v_1', \ldots, v_s'\big) \\
&\quad := \xi\big(w; \exp v_1' V_1'(w) \cdots \exp v_s' V_s'(w)\big) \\
&\quad = \xi\big(w; \exp\big(P_1(w; v_1', \ldots, v_s') V_1(w)\big) \cdots \exp\big(P_s(w; v_1', \ldots, v's) V_s(w)\big) \\
&\qquad \cdot \exp\big(Q_1(w; v_1', \ldots, v_s') X_1(w)\big) \cdots \exp\big(Q_r(w; v_1', \ldots, v_s') X_r(w)\big)\big) \\
&\quad = \prod_{j=1}^{r} e^{i Q_j(w; v_1', \ldots, v_s') \langle w, X_j(w) \rangle} \tilde{\xi}\big(w; P_1(w; v_1', \ldots, v_s'), \ldots, P_s(w; v_1', \ldots, v_s')\big)
\end{aligned}$$

where for all $j \in \{1, \ldots, r\}$, $Q_j(w; v_1', \ldots, v_s')$ is a polynomial function in the variables $v_1', \ldots, v_s'$ with smooth coefficients in w and the map $(v_1', \ldots, v_s') \longmapsto (P_1(w; v_1', \ldots, v_s'), \ldots, P_s(w; v_1', \ldots, v_s'))$ is a bi-polynomial diffeomorphism with smooth coefficients in w. We have similar relations for the passage from $\{V_1'(w), \ldots, V_s'(w)\}$ to $\{V_1(w), \ldots, V_s(w)\}$. Since our differential operator satisfies the smooth version Lemma A.2.1 in [**3**], we get the result. □

3. The generalized Kirillov theorem.

3.1. The statement of the theorem.

We denote by $\mathscr{U}(\mathfrak{n})$ the universal enveloping algebra of $\mathfrak{n}$. Let $\mathscr{Y} = \mathscr{Y}(w) = \{V_1(w), \ldots, V_s(w)\}$ be a smooth family of Malcev bases of $\mathfrak{n}$ relative to $\mathfrak{p}(w)$. We identify the Hilbert space $L^2(N/P(w), \chi_w)$ with $L^2(\boldsymbol{R}^s)$ via the unitary operator U defined by $U\xi(w)(v_1, \ldots, v_s) := \xi(w; \exp v_1 V_1(w) \cdots \exp v_s V_s(w))$. We denote by $\rho_w = U \circ \pi_w \circ U^*$ the corresponding realization of the representation π_w of N on the space $L^2(\boldsymbol{R}^s)$. This realization ρ_w of π_w on the space $L^2(\boldsymbol{R}^s)$ depends of course on the choice of the basis of $\mathfrak{n}$ relative to $\mathfrak{p}(w)$.

We may now present one of the main results of this paper, the generalized Kirillov theorem.

THEOREM 1. *There exists a non-empty relatively open subset $\mathscr{D}$ of $\mathscr{W}$ such that for every $w_0 \in \mathscr{D}$ and for every relative neighborhood $\mathscr{V}_0$ of w_0 contained in $\mathscr{D}$, there are $w_0' \in \mathscr{V}_0$ and a relative neighborhood $\mathscr{V}_0'$ of w_0' contained in $\mathscr{V}_0$ which satisfy: For every $D = (D_w)_{w \in \mathscr{V}_0'} \in \mathscr{P}\mathscr{D}(\mathscr{V}_0', \boldsymbol{R}^s)$, there exists $(X_w)_{w \in \mathscr{V}_0'}$ with*

$$X_w = \sum_{|\zeta| < C} c_\zeta(w) Z^\zeta \in \mathscr{U}(\mathfrak{n}), \qquad \forall w \in \mathscr{V}_0'$$

where Z^ζ is defined by $Z^\zeta = Z_1^{\zeta_1} \cdots Z_n^{\zeta_n}$ if $\zeta = (\zeta_1, \ldots, \zeta_n)$ and where the maps $w \longmapsto c_\zeta(w)$ are smooth, such that $d\rho_w(X_w) = D_w$ for all $w \in \mathscr{V}_0'$. This result is independent of the choice of the family of smooth Malcev bases used to define the realization ρ_w of π_w.

Here $\{Z_1, \ldots, Z_n\}$ denotes the fixed Jordan-Hölder basis of $\mathfrak{n}$.

3.2. The theorem in variable Lie groups.

To prove Theorem 1, we will proceed by induction on the dimension of $\mathfrak{n}$. However, new parameters and new variations will appear in the proof. In order to control those, we shall carry out the proof not only for one single Lie algebra, but for so-called variable Lie algebras. We refer the reader to the references [**13**], [**12**], [**10**] for more details about variable Lie structures and we present only a few aspects which are useful in our proof.

DEFINITION 9. Let $\mathfrak{n}$ be a real vector space of finite dimension n and $\mathfrak{b}$ be a submanifold of $\boldsymbol{R}^d$. We say $\mathfrak{n}_b := (\mathfrak{n}, [\cdot,\cdot]_b)$ is a variable nilpotent Lie algebra if it satisfies the following conditions:

1. For every $b \in \mathfrak{b}$, there exists a Lie bracket $[\cdot,\cdot]_b$ defined on $\mathfrak{n}$ such that $(\mathfrak{n}, [\cdot,\cdot]_b)$ is a nilpotent Lie algebra.
2. There exists a fixed basis $\{Z_1, \ldots, Z_n\}$ of $\mathfrak{n}$ such that the structure constants

$c_{ij}^k(b)$ defined by $[Z_i, Z_j]_b = \sum_{k=1}^n c_{ij}^k(b) Z_k$ are smooth and verify the following property: For all $b \in \mathfrak{b}$, for $k \le \max(i,j), c_{ij}^k(b) = 0$. This means that $\{Z_1, \dots, Z_n\}$ is a Jordan-Hölder basis for every $\mathfrak{n}_b$.

Let $N_b = \exp_b \mathfrak{n}_b$ be the associated connected, simply connected variable nilpotent Lie group. We denote by $\mathscr{W}'$ a smooth submanifold in $\mathfrak{b} \times \mathfrak{n}^*$.

For every $(b,w) \in \mathscr{W}'$, we consider the largest ideal $\mathfrak{a}(b,w)$ in $\mathfrak{n}_b$ contained in the stabilizer $\mathfrak{n}_b(w) := \{X \in \mathfrak{n}, \mid \langle w, [X, \mathfrak{n}]_b \rangle = 0\}$. We now define indices $j_1(b,w)$ and $k_1(b,w)$ by

$$\begin{aligned} j_1(b,w) &= \max\{j \in \{1,\dots,n\} \mid \langle w, [Z_j, \mathfrak{n}]_b \rangle \neq 0\} \\ &= \max\{j \in \{1,\dots,n\} \mid Z_j \notin \mathfrak{a}(b,w)\} \\ k_1(b,w) &= \max\{k \in \{1,\dots,n\} \mid \langle w, [Z_{j_1(b,w)}, Z_k]_b \rangle \neq 0\}. \end{aligned}$$

The equality in the definition of $j_1(b,w)$ is due to the fact that $\{Z_1, \dots, Z_n\}$ is a Jordan-Hölder basis. We consider also

$$\begin{aligned} j_1 &= \max\{j_1(b,w) \mid (b,w) \in \mathscr{W}'\} \\ k_1 &= \max\{k_1(b,w) \mid (b,w) \in \mathscr{W}' \text{ and } j_1(b,w) = j_1\}. \end{aligned}$$

Set $\mathscr{D}_1 := \{(b,w) \in \mathscr{W}' \mid j_1(b,w) = j_1 \text{ and } k_1(b,w) = k_1\}$. Then

$$\mathscr{D}_1 = \{(b,w) \in \mathscr{W}' \mid \langle w, [Z_{k_1}, Z_{j_1}]_b \rangle \neq 0\}.$$

and it is a non-empty relatively open subset of $\mathscr{W}'$. For further details, we refer the reader to [**12**], even if our definition of the indices j_1 and k_1 is slightly different. It is easy to see that, for all $(b,w) \in \mathscr{D}_1$, $\mathfrak{n}_1(b,w) := \{U \in \mathfrak{n}_b \mid \langle w, [U, Z_{j_1}]_b \rangle = 0\}$ is an ideal of codimension 1 in $\mathfrak{n}_b$ and $\mathscr{B}_1(b,w) = \{Z_i^1(b,w) \mid i \in I^1\}$ is a smooth Jordan-Hölder basis of $\mathfrak{n}_1(b,w)$ for all $(b,w) \in \mathscr{D}_1$, where $I^1 := \{1,\dots,n\} \backslash \{k_1\}$ and

$$Z_i^1(b,w) := Z_i - \frac{\langle w, [Z_{j_1}, Z_i]_b \rangle}{\langle w, [Z_{j_1}, Z_{k_1}]_b \rangle} Z_{k_1}$$

(see [**14**]).

We define a new variable Lie algebra $\mathfrak{n}_1$ in the following way: As a vector space, $\mathfrak{n}_1$ is spanned by a basis denoted $\{Z_u^1, u \in I^1\}$, where, for fixed (b,w), Z_u^1 is identified with $Z_u^1(b,w)$. The variable Lie brackets on $\mathfrak{n}_1$ are defined by $[Z_u^1, Z_v^1]_{(b,w)} := [Z_u^1(b,w), Z_v^1(b,w)]_b$ for all $(b,w) \in \mathscr{D}_1$. Then, $(\mathfrak{n}_1, [\cdot,\cdot]_{(b,w)})$ may be identified with $(\mathfrak{n}_1(b,w), [\cdot,\cdot]_b)$. Since we use this algebra in the induction, we will define a new variation on $\mathfrak{n}_1$, i.e. a new submanifold $\mathscr{W}_1$ of $(\mathfrak{b} \times \mathfrak{n}^*) \times \mathfrak{n}_1^*$ as follows:

$$(b,w,\tilde{w}) \in \mathscr{W}_1 \text{ with } (b,w) \in \mathscr{W}' \text{ and } \tilde{w} \in \mathfrak{n}_1^* \equiv \mathfrak{n}_1(b,w)^*$$
$$\Longleftrightarrow (b,w) \in \mathscr{W}' \text{ and } \tilde{w} \text{ is defined by } \langle \tilde{w}, Z_j^1 \rangle := \langle w, Z_j^1(b,w) \rangle \forall j \in I^1.$$

To prove Theorem 1, it suffices to show the corresponding result in the variable case and then reduce $\mathfrak{b}$ to a single point. As in the fixed case, we choose a family of smooth Malcev bases to define the realization $\rho_{(b,w)}$ of $\pi_{(b,w)}$.

Let's notice that the results of Sections 1 and 2 remain valid if we replace the variation in w by a variation in (b,w) and we denote $\rho_{(b,w)}$ the realization of the representation $\pi_{(b,w)} := \operatorname{ind}_{P(b,w)}^{N} \chi_{(b,w)}$ of N_b on the space $L^2(\boldsymbol{R}^s)$.

Theorem 2. *There exists a non-empty relatively open subset $\mathscr{D}' \subset \mathscr{W}'$ such that for all $(b_0,w_0) \in \mathscr{D}'$ and for every relative neighborhood $\mathscr{V}_0$ of (b_0,w_0) contained in $\mathscr{D}'$, there exist $(b_0',w_0') \in \mathscr{V}_0$ and a relative neighborhood $\mathscr{V}_0'$ of (b_0',w_0') contained in $\mathscr{V}_0$ such that: For every $D = (D_{(b,w)})_{(b,w)} \in \mathscr{PD}(\mathscr{V}_0', \boldsymbol{R}^s)$, there is a smooth family of vectors $X = (X_{(b,w)})_{(b,w)\in\mathscr{V}_0'}$ in the enveloping algebras, i.e. $X_{(b,w)} = \sum_{|\zeta|<C} c_\zeta(b,w) Z^\zeta \in \mathscr{U}(\mathfrak{n}_b)$, satisfying $d\rho_{(b,w)}(X_{(b,w)}) = D_{(b,w)}$. In the previous formula, Z^ζ is defined by $Z^\zeta = Z_1^{\zeta_1} \cdots Z_n^{\zeta_n}$ if $\zeta = (\zeta_1, \dots, \zeta_n)$ and the map $(b,w) \longmapsto c_\zeta(b,w)$ is smooth. This result is independent of the choice of the family of smooth Malcev bases used to define the realizations $\rho_{(b,w)}$.*

3.3. Proof of the variable generalized Kirillov theorem.

The Kirillov theorem in the fixed case is known. But the proof has to be rewritten, in order to study the dependency on (b,w).

It is easy to see that Theorem 2 is independent of the choice of the smooth Malcev basis of $\mathfrak{n}_b$ relative to $\mathfrak{p}(b,w)$ since the transition between two such smooth Malcev bases is given, modulo $\mathfrak{p}(b,w)$, by a smooth bi-polynomial function in the coordinates, with coefficients which are smooth functions in (b,w) for all $(b,w) \in W' \subset \mathscr{W}'$, where W' is a non-empty relatively open subset of $\mathscr{W}'$. In particular, our differential operators are transformed into differential operators of the same type via this transition. So we may choose appropriate bases to make the proof.

In order to get $\mathscr{PD}(\mathscr{V}_0', \boldsymbol{R}^s) \subset (d\rho_{(b,w)}(\mathscr{U}(\mathfrak{n}_b)))_{(b,w)}$, we proceed by induction on the dimension of $\mathfrak{n}_b$.

If N_b is abelian for all b, then $s = 0$ and the result is trivial.

Assume now that the theorem is true for all variable nilpotent Lie groups and algebras of dimension less than or equal to $n-1$. We will show that the theorem also holds for $(N_b)_{b\in\mathfrak{b}}$. Put

$$X(b,w) = \frac{Z_{k_1}}{\langle w, [Z_{k_1}, Z_{j_1}]_b\rangle};\ Y = Z_{j_1};\ Z(b,w) = [X(b,w), Y]_b \in \mathfrak{a}(b,w)$$

for all $(b,w) \in \mathscr{D}_1$. If we replace Y by $Y(b,w) = Y - \langle w, Y\rangle Z(b,w)$, we obtain

$$\langle w, Z(b,w)\rangle = 1;\ \langle w, Y(b,w)\rangle = 0 \text{ and}$$
$$[X(b,w), Y(b,w)]_b = Z(b,w) - \langle w, Y\rangle[X(b,w), Z(b,w)]_b =: \tilde{Z}(b,w)$$

where $X(b,w)$, $Y(b,w)$, $Z(b,w)$ and $\tilde{Z}(b,w)$ are smooth functions.

Since $Z(b,w) \in \mathfrak{a}(b,w) \subset \mathfrak{n}(b,w)$, $\langle w, \tilde{Z}(b,w)\rangle = 1$. So we may replace $Z(b,w)$ by $\tilde{Z}(b,w)$ and call it again $Z(b,w)$.

Given $(b_0, w_0) \in \mathscr{D}_1$, we distinguish three cases:

First Case: $\langle w, [\mathfrak{p}(b,w), Z_{j_1}]_b\rangle = 0$, i.e. $\mathfrak{p}(b,w) \subset \mathfrak{n}_1(b,w)$ for all $(b,w) \in \mathscr{V}_0 \subset \mathscr{D}_1$ where $\mathscr{V}_0$ is a relative neighborhood of (b_0, w_0) contained in $\mathscr{D}_1$.

Let $N_1(b,w) = \exp_b \mathfrak{n}_1(b,w)$ be the closed normal subgroup of N_b of Lie algebra $\mathfrak{n}_1(b,w)$ and $N^1_{(b,w)} = \exp_{(b,w)} \mathfrak{n}_1$ be the associated variable group of dimension $n-1$. It's easy to see that $\mathscr{W}_1 \subset \mathscr{W}' \times \mathfrak{n}_1^*$ is a submanifold of $\mathfrak{b} \times \mathfrak{n}^* \times \mathfrak{n}_1^*$ such that

$$\begin{aligned}
&(b,w,\tilde{w}) \in \mathscr{W}_1 \text{ with } (b,w) \in \mathscr{V}_0\\
&\quad\Longrightarrow (b,w) \in \mathscr{W}'\\
&\quad\Longrightarrow \dim \mathfrak{p}(b,w) \text{ is constant and by hypothesis } \mathfrak{p}(b,w) \subset \mathfrak{n}_1(b,w)\\
&\quad\Longrightarrow \text{the dimension of a polarization at } \tilde{w} \text{ in } \mathfrak{n}_1(b,w) \text{ is constant.}
\end{aligned}$$

Let $\pi^1_{(b,w,\tilde{w})}$ be the representation of $N^1_{(b,w)}$ identified with the representation $\pi^1_{(b,w)} = \operatorname{ind}_{P(b,w)}^{N_1(b,w)} \chi_{(b,w)}$ of $N_1(b,w)$. By stage induction, we have $\pi_{(b,w)} \simeq \operatorname{ind}_{N_1(b,w)}^{N_b} \pi^1_{(b,w)}$. The intertwining operator between $\pi_{(b,w)}$ and $\operatorname{ind}_{N_1(b,w)}^{N_b} \pi^1_{(b,w)}$ is given by $(V_{(b,w)}\xi(b,w))(t)(g_1) := \xi(b,w)(\exp_b(tX(b,w)) \cdot g_1)$ for all $\xi(b,w) \in L^2(N_b/P(b,w), \chi_{(b,w)})$. We can hence identify the Hilbert space $L^2(N_b/P(b,w), \chi_{(b,w)})$ with $L^2(\boldsymbol{R}, L^2(N_1(b,w)/P(b,w), \chi_{(b,w,\tilde{w})})) = L^2(\boldsymbol{R}, \mathscr{H}_{\pi^1_{(b,w)}})$. According to Proposition 1 and Proposition 2, there exists a non-empty relatively open subset $\mathscr{V}_0'$ of $\mathscr{V}_0$ such that $I_1 := I^{\mathfrak{p}(b,w)}_{\mathscr{B}_1(b,w)}$ is a fixed set and $\{Z^1_j(b,w), j \notin I_1\}$ is a smooth Malcev basis of $\mathfrak{n}_1(b,w)$ relative to $\mathfrak{p}(b,w)$ for all $(b,w) \in \mathscr{V}_0'$. We denote this basis by $\mathscr{Y}^1_{(b,w)} := \{Y^1_2(b,w), \ldots, Y^1_s(b,w)\}$. If we put $Y^1_1(b,w) = X(b,w)$, then $\mathscr{Y}'_{(b,w)} := \{Y^1_1(b,w), \ldots, Y^1_s(b,w)\}$ is a smooth Malcev basis of $\mathfrak{n}_b$ relative to $\mathfrak{p}(b,w)$.

Let $\pi'_{(b,w)} := \operatorname{ind}_{N_1(b,w)}^{N_b} \pi^1_{(b,w)}$ be the induced representation of N_b on the space $L^2(\boldsymbol{R}, \mathscr{H}_{\pi^1_{(b,w)}})$ and $E_{\mathscr{Y}^1_{(b,w)}}(t_2, \ldots, t_s) := \exp_b t_2 Y^1_2(b,w) \cdots \exp_b t_s Y^1_s(b,w)$.

Then $\pi'_{(b,w)}$ and $\rho_{(b,w)}$ are equivalent and the corresponding intertwining operator is given by $(V'_{(b,w)}\eta(b,w))(t,t_2,\ldots,t_s) := \eta(b,w)(t)(E_{\mathscr{Y}^1_{(b,w)}}(t_2,\ldots,t_s))$ for all $\eta(b,w) \in L^2(\boldsymbol{R},\mathscr{H}_{\pi^1_{(b,w)}})$. A simple computation shows that we have for all $\xi(b,w) \in L^2(\boldsymbol{R},\mathscr{H}_{\pi^1_{(b,w)}})$ and all $t \in \boldsymbol{R}$,

$$\begin{aligned}\big(\pi'_{(b,w)}(\exp_b xX(b,w))\xi(b,w)\big)(t) &= \xi(b,w)(t-x)\\ \big(\pi'_{(b,w)}(\exp_b yY(b,w))\xi(b,w)\big)(t) &= e^{ity}\xi(b,w)(t)\\ \big(\pi'_{(b,w)}(u^1_{(b,w)})\xi(b,w)\big)(t) &= \pi^1_{(b,w)}\big((u^1_{(b,w)})^{(-t)}\big)(\xi(b,w)(t))\end{aligned}$$

where $u^1_{(b,w)} \in N^1_{(b,w)}, x, y \in \boldsymbol{R}$ and where $(u^1_{(b,w)})^{-t} := \exp_b(-tX(b,w)) \cdot u^1_{(b,w)} \cdot \exp_b(tX(b,w))$. From now on we will work with $\mathscr{C}^\infty$ vectors, i.e. with Schwartz functions only. It follows that for $X(b,w), Y(b,w) \in \mathfrak{n}_b$, $d^1_{(b,w)} \in \mathscr{U}(\mathfrak{n}_1(b,w))$ and for a smooth vector $\xi(b,w)$, we have

$$\begin{aligned}\big(d\pi'_{(b,w)}(X(b,w))\xi(b,w)\big)(t) &= -\frac{d}{dt}\xi(b,w)(t)\\ \big(d\pi'_{(b,w)}(Y(b,w))\xi(b,w)\big)(t) &= it\xi(b,w)(t)\\ \big(d\pi'_{(b,w)}(d^1_{(b,w)})\xi(b,w)\big)(t) &= d\pi^1_{(b,w)}\big((d^1_{(b,w)})^{-t}\big)(\xi(b,w)(t))\end{aligned}$$

where $(d^1_{(b,w)})^{-t} = \sum_{j=0}^{\infty}((-t)^j/j!)(\mathrm{ad}_b(X(b,w)))^j d^1_{(b,w)}$. Since $\mathfrak{n}_b$ is nilpotent, the previous sum is finite. We denote it by $\sum_{j=0}^{m}$.

Now we consider $d_{(b,w)} \in \mathscr{U}(\mathfrak{n}_b)$ and we define

$$\tilde{d}_{(b,w)} = \sum_{j=0}^{m} \frac{1}{j!i^j} Y^j(b,w)\mathrm{ad}_b^j X(b,w) d_{(b,w)} \in \mathscr{U}(\mathfrak{n}_b).$$

Let us compute $d\pi'_{(b,w)}(\tilde{d}^1_{(b,w)})$ where $d^1_{(b,w)} \in \mathscr{U}(\mathfrak{n}_1(b,w))$. For $\xi(b,w) \in L^2(\boldsymbol{R},\mathscr{H}_{\pi^1_{(b,w)}})$ and $t \in \boldsymbol{R}$, we have

$$\begin{aligned}&\big(d\pi'_{(b,w)}(\tilde{d}^1_{(b,w)})\xi(b,w)\big)(t)\\ &= d\pi^1_{(b,w)}\big((\tilde{d}^1_{(b,w)})^{-t}\big)(\xi(b,w)(t))\\ &= d\pi^1_{(b,w)}\left(\left(\sum_{j=0}^{m}\frac{1}{j!i^j}Y^j(b,w)\mathrm{ad}_b^j X(b,w) d^1_{(b,w)}\right)^{-t}\right)(\xi(b,w)(t))\end{aligned}$$

$$= \left(d\pi^1_{(b,w)}\left(\sum_{j=0}^{m}\frac{1}{j!i^j}(Y^j(b,w))^{-t}\big(\mathrm{ad}_b^j X(b,w)\big)d^1_{(b,w)}\right)^{-t}\right)(\xi(b,w)(t)).$$

One shows easily that $d\pi^1_{(b,w)}(Y^{-t}(b,w)) = it\boldsymbol{I}_{\mathscr{H}_{\pi^1_{(b,w)}}}$. Then we get

$$\begin{aligned}\big(d\pi'_{(b,w)}(\tilde{d}^1_{(b,w)})\xi(b,w)\big)(t) &= d\pi^1_{(b,w)}\left(\sum_{j=0}^{m}\frac{1}{j!}t^j\big(\mathrm{ad}_b^j X(b,w)\big)\big(d^1_{(b,w)}\big)^{-t}\right)(\xi(b,w)(t))\\ &= d\pi^1_{(b,w)}\big(d^1_{(b,w)}\big)(\xi(b,w)(t)).\end{aligned}$$

Let $\rho'_{(b,w)}$ be the realization of the representation $\pi'_{(b,w)}$ on $L^2(\boldsymbol{R}, L^2(\boldsymbol{R}^{s-1}))$ with an appropriate choice of smooth Malcev bases of $\mathfrak{n}$ relative to $\mathfrak{p}(b,w)$. Since $\pi'_{(b,w)} \equiv \pi_{(b,w)}$, then $\rho_{(b,w)}$ and $\rho'_{(b,w)}$ are equivalent and we denote $S_{(b,w)}$ the corresponding intertwining operator. It is defined by $(S_{(b,w)}\tilde{\xi}(b,w))(t_1)(t_2,\dots,t_s) = \tilde{\xi}(b,w)(t_1,t_2,\dots,t_s)$.

For all $\eta(b,w) \in \mathscr{S}(\boldsymbol{R}^s)$, we have

$$\begin{aligned}\big(d\rho_{(b,w)}(X(b,w))\eta(b,w)\big)(t_1,\dots,t_s) &= -\frac{d}{dt_1}\eta(b,w)(t_1,\dots,t_s)\\ \big(d\rho_{(b,w)}(Y(b,w))\eta(b,w)\big)(t_1,\dots,t_s) &= it_1\eta(b,w)(t_1,\dots,t_s)\\ \big(d\rho_{(b,w)}(\tilde{d}^1_{(b,w)})\eta(b,w)\big)&(t_1,t_2,\dots,t_s)\\ &= \big(d\rho^1_{(b,w)}\big(d^1_{(b,w)}\big)((S_{(b,w)}\eta(b,w))(t_1))\big)(t_2,\dots,t_s)\end{aligned}$$

where $\rho^1_{(b,w)}$ is the induced representation $\pi^1_{(b,w)}$ realized on $L^2(\boldsymbol{R}^{s-1})$.

Let now $D_{(b,w)}$ be a smooth differential operator in the coordinates $(t_2,\dots,t_s)$ with polynomial coefficients in the coordinates $(t_2,\dots,t_s)$. We may of course identify it with a differential operator in the coordinates $(t_1,\dots,t_s)$, which is constant in the direction of t_1 and write

$$(D_{(b,w)}\eta(b,w))(t_1,\dots,t_s) = D_{(b,w)}((S_{(b,w)}\eta(b,w))(t_1))(t_2,\dots,t_s)$$

for all $\eta(b,w) \in \mathscr{S}(\boldsymbol{R}^s)$. The induction hypothesis applied to $d\rho^1_{(b,w)}$ gives us the existence of a smooth family $(U^1_{(b,w)})_{(b,w)}$ in the enveloping algebras such that

$$\begin{aligned}(D_{(b,w)}\eta(b,w))(t_1,\ldots,t_s) &= D_{(b,w)}((S_{(b,w)}\eta(b,w))(t_1))(t_2,\ldots,t_s)\\ &= d\rho^1_{(b,w)}\big(U^1_{(b,w)}\big)((S_{(b,w)}\eta(b,w))(t_1))(t_2,\ldots,t_s)\\ &= d\rho_{(b,w)}\big(\tilde{U}^1_{(b,w)}\big)\eta(b,w)(t_1,t_2,\ldots,t_s).\end{aligned}$$

This implies that $(d\rho_{(b,w)}(\mathscr{U}(\mathfrak{n}_b)))_{(b,w)\in\mathscr{V}'_0}$ contains all the (b,w)-smooth differential operators with polynomial coefficients in the variables $(t_2,\ldots,t_s)$. As it also contains the operator $\partial/\partial t_1$ and the multiplication operator by t_1, it contains all of $\mathscr{P}\mathscr{D}(\mathscr{V}'_0,\boldsymbol{R}^s)$.

Second case: $\langle w,[\mathfrak{p}(b,w),Z_{j_1}]_b\rangle\neq 0$ for all $(b,w)\in\mathscr{V}_0\subset\mathscr{D}_1$, i.e. for all $(b,w)\in\mathscr{V}_0$, a relative neighborhood of (b_0,w_0), we have $\mathfrak{p}(b,w)\not\subset\mathfrak{n}_1(b,w)$. Let us now show that there exists a smooth family of vectors $\tilde{X}(b,w)$ in $\mathfrak{n}$ satisfying the following conditions

$$\tilde{X}(b,w)\in\mathfrak{p}(b,w);\ \ \tilde{X}(b,w)-X(b,w)\in\mathfrak{n}_1(b,w);\ \ \langle w,\tilde{X}(b,w)\rangle=0.$$

In fact, there exists $\tilde{X}(b,w)\in\mathfrak{p}(b,w)$ such that $\langle w,[\tilde{X}(b,w),Z_{j_1}]\rangle\neq 0$. As, $\mathfrak{n}=\boldsymbol{R}X(b,w)\oplus\mathfrak{n}_1(b,w)$, we may choose this vector of the form $\tilde{X}(b,w)=X(b,w)+U(b,w)$ with $U(b,w)\in\mathfrak{n}_1(b,w)$. By adding an appropriate multiple of $Z(b,w)$, we may even assume that $\langle w,\tilde{X}(b,w)\rangle=0$.

Let us now show that the previous choice implies that the family $(\tilde{X}(b,w))_{(b,w)}$ can be chosen smoothly in a certain non-empty relatively open subset $\mathscr{V}'$ of $\mathscr{V}_0$. In fact, if we characterize $\mathfrak{p}(b,w)$ by a system of Cartesian equations with smooth coefficients in (b,w), then the three defining conditions of $\tilde{X}(b,w)$ can be expressed as a system of linear equations in the coordinates of $\tilde{X}(b,w)$ with smooth coefficients in (b,w). By construction, this system has solutions for every (b,w). We may then construct a smooth solution of the system in a certain non-empty relatively open subset $\mathscr{V}'$ in $\mathscr{V}_0$.

One has $\mathfrak{p}(b,w)=\mathfrak{p}(b,w)\cap\mathfrak{n}_1(b,w)+\boldsymbol{R}\tilde{X}(b,w)$. Let us now consider $\mathfrak{p}'(b,w)=\mathfrak{p}(b,w)\cap\mathfrak{n}_1(b,w)+\boldsymbol{R}Y(b,w)$. It is easy to check that this is again a smooth family of polarizations.

According to the first case, we know that the generalized Kirillov result is true for the family of polarizations $\mathfrak{p}'(b,w)\subset\mathfrak{n}_1(b,w)$. We are going to transfer this result to the family of polarizations $(\mathfrak{p}(b,w))_{(b,w)}$ by studying explicitly the intertwining operators between the families of associated representations, as well as their dependency in (b,w). Since $\mathfrak{p}'(b,w)$ is a polarization of w in $\mathfrak{n}_b$, it implies that $\pi_{(b,w)}\simeq\pi'_{(b,w)}$ where $\pi'_{(b,w)}=\operatorname{ind}_{P'(b,w)}^{N_b}\chi_{(b,w)}$ and $P'(b,w)=\exp_b\mathfrak{p}'(b,w)$. Hence there exists a family of intertwining operators $T_{(b,w)}$. For a fixed (b,w), this intertwining operator may be given by

$$T_{(b,w)}\xi((b,w))\big(g_1 \cdot \exp_b x\tilde{X}(b,w)\big)$$
$$:= \int_{\boldsymbol{R}} \xi(b,w)\big(g_1 \cdot \exp_b x\tilde{X}(b,w) \cdot \exp_b yY(b,w)\big) \underbrace{\chi_{(b,w)}(\exp_b yY(b,w))}_{=1} dy$$
$$= \int_{\boldsymbol{R}} e^{ixy}\xi(b,w)(g_1 \cdot \exp_b yY(b,w))dy$$

according to [**11**], for all $g_1 \in N_1(b,w)$ and $x \in \boldsymbol{R}$.

We now have to study the realizations of $\pi_{(b,w)}$ and $\pi'_{(b,w)}$ on the space $L^2(\boldsymbol{R}^s)$. As the result of the theorem will be independent of the choice of the corresponding smooth families of Malcev bases, let's proceed as follows: It is easy to check that $\mathfrak{p}(b,w) + \mathfrak{p}'(b,w)$ is a subalgebra of $\mathfrak{n}_b$ for all (b,w). Hence we may pick an arbitrary smooth family $\{V_1(b,w), \ldots, V_{s-1}(b,w)\}$ of Malcev bases of $\mathfrak{n}_b$ relative to $\mathfrak{p}(b,w) + \mathfrak{p}'(b,w)$. This may for instance be done by the index method in Proposition 2, which remains valid in the variable case. It is even possible to choose these vectors $V_1(b,w), \ldots, V_{s-1}(b,w)$ in $\mathfrak{n}_1(b,w)$, what we will do. In the process of choosing this family of bases, we might have to restrict once more the relatively open set $\mathscr{V}'$ to a non-empty relatively open subset $\mathscr{V}'_0$ of $\mathscr{V}_0$.

Then one checks that $\{V_1(b,w), \ldots, V_{s-1}(b,w), Y(b,w)\}$ is a smooth family of Malcev bases of $\mathfrak{n}_b$ relative to $\mathfrak{p}(b,w)$ and that $\{V_1(b,w), \ldots, V_{s-1}(b,w), \tilde{X}(b,w)\}$ is a smooth family of Malcev bases of $\mathfrak{n}_b$ relative to $\mathfrak{p}'(b,w)$. These two families of bases are then used to define $\rho_{(b,w)}$ and $\rho'_{(b,w)}$. For any $\xi(b,w) \in \mathscr{H}_{\pi_{(b,w)}}$, we define $\tilde{\xi}(b,w) \in L^2(\boldsymbol{R}^s)$ by

$$\tilde{\xi}(b,w)(v_1, \ldots, v_{s-1}, y)$$
$$:= \xi(b,w)\big(\exp v_1V_1(b,w) \cdots \exp v_{s-1}V_{s-1}(b,w) \cdot \exp yY(b,w)\big)$$

and we identify thus $\mathscr{H}_{\pi_{(b,w)}}$ with $L^2(\boldsymbol{R}^s)$, and $\pi_{(b,w)}$ with $\rho_{(b,w)}$. Similarly, for $\xi'(b,w) \in \mathscr{H}_{\pi'_{(b,w)}}$, we define $\tilde{\xi}'(b,w) \in L^2(\boldsymbol{R}^s)$ by

$$\tilde{\xi}'(b,w)(v_1, \ldots, v_{s-1}, x)$$
$$:= \xi'(b,w)\big(\exp v_1V_1(b,w) \cdots \exp v_{s-1}V_{s-1}(b,w) \cdot \exp x\tilde{X}(b,w)\big)$$

and we identify thus $\mathscr{H}_{\pi'_{(b,w)}}$ with $L^2(\boldsymbol{R}^s)$, and $\pi'_{(b,w)}$ with $\rho'_{(b,w)}$ acting on $L^2(\boldsymbol{R}^s)$. The intertwining operator $\tilde{T}_{(b,w)}$ between $\rho_{(b,w)}$ and $\rho'_{(b,w)}$ is then deduced from $T_{(b,w)}$ and is given by

$$\big(\tilde{T}_{(b,w)}\tilde{\xi}(b,w)\big)(v_1,\ldots,v_{s-1},x) = \int_{\boldsymbol{R}} e^{ixy}\tilde{\xi}(b,w)(v_1,\ldots,v_{s-1},y)dy.$$

It is nothing but a partial Fourier transform which sends $\mathscr{S}(\boldsymbol{R}^s)$ onto itself and $\mathscr{P}\mathscr{D}(\mathscr{V}_0',\boldsymbol{R}^s)$ onto itself. As the Kirillov result is true for the family $(\rho'_{(b,w)})_{(b,w)}$ by the first case, it remains hence true for the family $(\rho_{(b,w)})_{(b,w)}$, thanks to this smooth family of intertwining operators, i.e. $\mathscr{P}\mathscr{D}(\mathscr{V}_0',\boldsymbol{R}^s) \subset (d\rho_{(b,w)}(\mathscr{U}(\mathfrak{n}_b)))_{(b,w)\in\mathscr{V}_0'}$.

Third case: $\langle w_0, [\mathfrak{p}(b_0,w_0), Z_{j_1}]_{b_0}\rangle = 0$ and for every neighborhood $\mathscr{V}_0$ of (b_0,w_0) there exists $(b',w')\in\mathscr{V}_0$ such that $\langle w', [\mathfrak{p}(b',w'), Z_{j_1}]_{b'}\rangle \neq 0$. Then there exists a neighborhood $\mathscr{V}'$ of (b',w') contained in $\mathscr{V}_0$ such that for all $(b'',w'')\in\mathscr{V}'$, we have $\langle w'', [\mathfrak{p}(b'',w''), Z_{j_1}]_{b''}\rangle \neq 0$ and we apply the second case to all $(b'',w'')\in\mathscr{V}'$. This proves that there exists a non-empty relatively open subset $\mathscr{V}_0'$ of $\mathscr{V}'$ such that $\mathscr{P}\mathscr{D}(\mathscr{V}_0',\boldsymbol{R}^s) \subset (d\rho_{(b,w)}(\mathscr{U}(\mathfrak{n}_b)))_{(b,w)\in\mathscr{V}_0'}$.

The other inclusion, $(d\rho_{(b,w)}(\mathscr{U}(\mathfrak{n}_b)))_{(b,w)\in\mathscr{V}'} \subset \mathscr{P}\mathscr{D}(\mathscr{V}',\boldsymbol{R}^s)$, for any non-empty relatively open subset $\mathscr{V}'$ in $\mathscr{W}'$, may be easily checked. □

REMARK 2. If $\mathfrak{b}$ is reduced to a single point, i.e. if there is no variable structure but a fixed Lie group, then we have Theorem 1.

REMARK 3. In the particular case where $\mathscr{W} = \mathfrak{n}^*_{gen}$, the set of generic linear forms in the sense of Ludwig-Zahir [**14**], there exists a more precise result: For every differential operator D on $\boldsymbol{R}^s$, there exists a rational mapping

$$\begin{aligned} \mathfrak{n}^*_{gen} &\longrightarrow \mathscr{U}(n) \\ w &\longmapsto A(w) = \sum_{|I|\le n_0} a_I(w)Z^I \end{aligned}$$

such that $d\rho_w(A(w)) = D$ for all $w\in\mathfrak{n}^*_{gen}$ (see [**13**]).

4. Smooth families of intertwining operators.

Let $(\mathfrak{p}(w))_{w\in\mathscr{W}}$ and $(\mathfrak{p}'(w))_{w\in\mathscr{W}}$ be two smooth families of polarisations in $\mathfrak{n}$. Let us first recall that there exist a non-empty relatively open subset W of $\mathscr{W}$ and a smooth SP-basis $\{U_1(w),\ldots,U_s(w)\}$ of $\mathfrak{n}$ relative to $\mathfrak{p}(w)$ such that $\{U_{b+1}(w),\ldots,U_s(w)\}$ is a smooth SP-basis of $\mathfrak{p}'(w)$ relative to $\mathfrak{p}(w)\cap\mathfrak{p}'(w)$ for all $w\in W$ (Proposition 4). Similarly, there exist a non-empty relatively open subset $\tilde{W}$ of W and a smooth SP-basis $\{U_1'(w),\ldots,U_s'(w)\}$ of $\mathfrak{n}$ relative to $\mathfrak{p}'(w)$ such that $\{U_{b+1}'(w),\ldots,U_s'(w)\}$ is a smooth SP-basis of $\mathfrak{p}(w)$ relative to $\mathfrak{p}(w)\cap\mathfrak{p}'(w)$ for all $w\in\tilde{W}$. Let us also recall that we may use SP-bases instead of Malcev bases to define invariant measures on the quotients.

We may then apply the results of G. Lion [**11**] pointwise. Let $w \in W$ be fixed. For every $\xi(w) \in \mathscr{H}^\infty_{\pi_w} = \mathscr{S}(N/P(w), \chi_w)$, the integral

$$\begin{aligned} T_w\xi(w; g) &:= T_w\xi(w)(g) \\ &= \int_{P'(w)/P(w)\cap P'(w)} \xi(w; gu)\chi_w(u)du \\ &= \int_{\boldsymbol{R}^{s-b}} \xi(w; g\cdot \exp u_{b+1}U_{b+1}(w)\cdots \exp u_sU_s(w)) \\ &\qquad \cdot e^{-i\langle w, \log(\exp u_{b+1}U_{b+1}(w)\cdots \exp u_sU_s(w))\rangle} du_{b+1}\cdots du_s \end{aligned}$$

converges for every $g \in N$. The operator T_w defines a homeomorphism from $\mathscr{H}^\infty_{\pi_w} = \mathscr{S}(N/P(w), \chi_w)$ onto $\mathscr{H}^\infty_{\pi'_w} = \mathscr{S}(N/P'(w), \chi_w)$. It may be extended to an intertwining operator for π_w and π'_w, between the spaces $\mathscr{H}_{\pi_w} = L^2(N/P(w), \chi_w)$ and $\mathscr{H}_{\pi'_w} = L^2(N/P'(w), \chi_w)$. If the measure on $P'(w)/P(w)\cap P'(w)$ is correctly normalized (which may for instance be obtained by multiplying one of the vectors of the basis $\mathscr{Z}_w$ by a constant), then T_w is an isometry.

In this section, our aim is to establish similar results for the family $(T_w)_{w\in\mathscr{U}}$ as a whole, for some non-empty relatively open subset $\mathscr{U}$ of W and to do it in a smooth way. This has of course to be specified. We need some preliminary results.

Let us first introduce some definitions and notations (see also Definition 8). For every $\xi \in \mathscr{K}\mathscr{S}(W, \mathscr{S}(N/P, \chi))$ we denote, as previously, the corresponding element of $\mathscr{K}\mathscr{S}(W, \mathscr{S}(\boldsymbol{R}^s))$ by $\tilde{\xi}$, the identification of $\mathscr{S}(N/P, \chi)$ with $\mathscr{S}(\boldsymbol{R}^s)$ being made via any smooth Malcev basis. For $A \in \boldsymbol{N}$, $D = (D_w)_{w\in W} \in \mathscr{P}\mathscr{D}(W, \boldsymbol{R}^s)$, we put

$$\|\tilde{\xi}(w)\|_{A,D} = \sup_{|a|\le A}\ \sup_{(u_1,\dots,u_s)\in\boldsymbol{R}^s} \left| \frac{\partial^a}{\partial w^a} D_w(u)\tilde{\xi}(w; u_1, \dots, u_s)\right| < \infty,$$

if w is in a fixed chart of W, so that $\partial^a/\partial w^a$ makes sense.

For any compact subset K contained in a chart of W, we have

$$\|\tilde{\xi}\|^K_{A,D} = \sup_{w\in K} \|\tilde{\xi}(w)\|_{A,D}.$$

Let now $g := (g_w)_{w\in W}$ a family of elements of N such that the map $w \longmapsto g_w$ is smooth and let $\xi \in \mathscr{K}\mathscr{S}(W, \mathscr{S}(N/P, \chi))$. We write ${}_{g^{-1}}\xi$ for the function defined by

$$({}_{g^{-1}}\xi)(w)(x) = {}_{g_w^{-1}}(\xi(w))(x) = \xi(w)(g_w \cdot x) = \xi(w; g_w \cdot x).$$

It is easy to check that ${}_{g^{-1}}\xi \in \mathscr{K}\mathscr{S}(W, \mathscr{S}(N/P, \chi))$ too.

Let $\mathscr{B}_w = \{V_1(w), \dots, V_n(w)\}$ be any smooth Malcev basis of $\mathfrak{n}$. If $g_w = \prod_{i=1}^n \exp(g_i(w)V_i(w))$, we write $\|g_w\|^2_{\mathscr{B}_w} = \sum_{i=1}^n g_i(w)^2$. If $\mathscr{B}'_w$ is any other such smooth Malcev basis, then there exist a strictly positive continuous function $w \longmapsto C(w)$ and $R \in \boldsymbol{N}^*$ such that

$$\|g_w\|^2_{\mathscr{B}'_w} \le C(w)\big(1 + \|g_w\|^2_{\mathscr{B}_w}\big)^R \tag{5}$$

for all w. The exponent R may be chosen independently of w. As a matter of fact it only depends on the structure of the Campbell-Baker-Hausdorff formula, i.e. on the degree of nilpotency of $\mathfrak{n}$. This inequality is obtained by passing from coordinates of the second kind to coordinates of the first kind, making the change of basis and going back to coordinates of the second kind.

Now, we introduce some technical lemmas.

LEMMA 2. *Let* $\tilde{\mathscr{B}}_w = \{V_1(w), \dots, V_s(w), X_1(w), \dots, X_r(w)\}$ *be a any smooth Malcev basis of* $\mathfrak{n}$ *such that* $\{V_1(w), \dots, V_s(w)\}$ *is a smooth Malcev basis of* $\mathfrak{n}$ *relative to* $\mathfrak{p}(w)$ *and* $\{X_1(w), \dots, X_r(w)\}$ *is a smooth Malcev basis of* $\mathfrak{p}(w)$*. So* $s + r = n$*. Let* $g = (g_w)_w$ *be smooth and let* $g_1(w), \dots, g_n(w)$ *denote the coordinates of the second kind of* g_w*. If one defines* $u : w \longmapsto u(w) = \exp u_1V_1(w) \cdots \exp u_sV_s(w)$ *where* $u_1, \dots, u_s \in \boldsymbol{R}$ *are independent of* w*, then*

$$\begin{aligned} g_w \cdot u(w) = \exp\big(&Q_1(w; g_1(w), \dots, g_n(w), u_1, \dots, u_s)V_1(w)\big) \cdots \\ &\exp\big(Q_s(w; g_1(w), \dots, g_n(w), u_1, \dots, u_s)V_s(w)\big) \\ &\cdot R(w; g_1(w), \dots, g_n(w), u_1, \dots, u_s) \end{aligned}$$

where $R(w; g_1(w), \dots, g_n(w), u_1, \dots, u_s) \in P(w)$*.*

The map $(u_1, \dots, u_s) \longmapsto (Q_1(w; g_1(w), \dots, g_n(w), u_1, \dots, u_s), \dots, Q_s(w; g_1(w), \dots, g_n(w), u_1, \dots, u_s))$ *is bi-polynomial with respect to the variables* $(u_1, \dots, u_s)$*. The coefficients of these polynomial functions may be expressed as polynomial functions in* $g_1(w), \dots, g_n(w)$ *with coefficients that are* $\mathscr{C}^\infty$ *in* w*. Similarly for the inverse map.*

PROOF. Similar arguments as in the fixed case. See for instance [**3**]. □

LEMMA 3. *Let us take the same hypotheses as in Lemma* 2 *and let's assume that* w *runs through a fixed chart of* W*. Given* $A \in \boldsymbol{N}$ *and* $D \in \mathscr{P}\mathscr{D}(W, \boldsymbol{R}^s)$, *there exists a finite collection* $N_j, A_j \in \boldsymbol{N}$, $D_j \in \mathscr{P}\mathscr{D}(W, \boldsymbol{R}^s)$ *and smooth positive functions* $C_j(w)$, *for* $j \in \{1, \dots, d\}$ *such that for all* $\xi \in \mathscr{K}\mathscr{S}(W, \mathscr{S}(N/P, \chi))$ *and every smooth family* $g = (g_w)_{w \in W}$ *of elements of* N *we have,*

$$\begin{aligned}\big\|[({}_{g^{-1}}\xi)(w)]\tilde{}\,\big\|_{A,D} &= \big\|[{}_{g_w^{-1}}(\xi(w))]\tilde{}\,\big\|_{A,D} \\ &\le \sum_{j=1}^{d} C_j(w)\big(1 + \|g_w\|^2_{\mathscr{B}_w}\big)^{N_j} \big\|\tilde{\xi}(w)\big\|_{A_j, D_j} < \infty.\end{aligned}$$

If K *is any compact subset of* W *contained in the fixed chart of* W, *we have for some positive constant* C *depending on* g

$$\big\|({}_{g^{-1}}\xi)\tilde{}\,\big\|^K_{A,D} \le C \sum_{j=1}^{d} \big\|\tilde{\xi}\big\|^K_{A_j, D_j} < \infty.$$

If $A = 0$, *i.e. if there is no derivative in* w, *then all the* A_j *'s may be taken to be* 0 *too.*

PROOF. Obvious by Lemma 2 and by the facts that all the functions in w are bounded on the compact set K. □

REMARK 4. In case the coordinates $g_i = g_i(w)$ are all fixed, the second bound may be written as

$$\big\|({}_{g^{-1}}\xi)\tilde{}\,\big\|^K_{A,D} \le C \bigg(1 + \sum_{i=1}^{n} g_i^2\bigg)^N \sum_{j=1}^{d} \big\|\tilde{\xi}\big\|^K_{A_j, D_j} < \infty$$

where $N = \max(N_1, \dots, N_d)$.

We denote $\tilde{\xi}$ and $\tilde{\xi}(w; \cdot)$ for the function ξ written in the coordinates of the second kind in any smooth Malcev basis $\{V_1(w), \dots, V_s(w)\}$ of $\mathfrak{n}$ relative to $\mathfrak{p}(w)$. On the other hand, we will write $\tilde{\xi}_{SP}$ for the corresponding function on $\boldsymbol{R}^{s-b} \equiv P'(w)/P(w) \cap P'(w)$, resp. on $\boldsymbol{R}^s \equiv N/P(w)$, defined by

$$\begin{aligned}\tilde{\xi}_{SP}(w; 0, \dots, 0, u_{b+1}, \dots, u_s) &= \xi\big(w; \exp u_{b+1}U_{b+1}(w) \cdots \exp u_s U_s(w)\big) \\ \tilde{\xi}_{SP}(w; u_1, \dots, u_s) &= \xi\big(w; \exp u_1 U_1(w) \cdots \exp u_s U_s(w)\big)\end{aligned}$$

where the $U_j(w)$'s are the vectors of the SP-basis constructed in Proposition 4.

This will in particular be used to compute integrals over $P'(w)/P(w) \cap P'(w)$.

We remark that for fixed $w \in W$, we have the intertwining operator T_w defined by

$$T_w \xi(w; g) = \int_{P'(w)/P(w) \cap P'(w)} \xi(w; gu) \chi_w(u) du.$$

This defines $T = (T_w)_w$ on $\mathscr{K}\mathscr{S}(W, \mathscr{S}(N/P, \chi))$ by $(T\xi)(w) := T_w(\xi(w))$. We may also view it as acting on $\mathscr{K}\mathscr{S}(W, \mathscr{S}(\boldsymbol{R}^s))$ by $\tilde{T} = (\tilde{T}_w)_w$ where $(\tilde{T}\tilde{\xi})(w) = \tilde{T}_w(\tilde{\xi}(w)) := (T_w(\xi(w)))\tilde{}$. For fixed w, $\tilde{T}_w$ is then an intertwining operator between ρ_w and ρ'_w. We will use the smooth Malcev basis $\{V_1(w), \dots, V_s(w)\}$ to compute the Schwartz semi-norms on $\mathscr{S}(N/P(w), \chi_w)$, as explained in the beginning of this section, by identifying $\xi(w)$ and $\tilde{\xi}(w)$.

We have the following result:

Lemma 4. *For every $\tilde{M} \in \boldsymbol{N}$, there exists $M \in \boldsymbol{N}$ and a continuous family of positive constants $(C(w))_w$ such that*

$$\begin{aligned} \left\|\tilde{\xi}_{SP}(w)\right\|_{0, D_{\tilde{M}}} &:= \sup_{(u_1, \dots, u_s) \in \boldsymbol{R}^s} \left| \left(1 + \sum_{j=1}^s u_j^2\right)^{\tilde{M}} \tilde{\xi}_{SP}(w)(u_1, \dots, u_s) \right| \\ &\leq C(w) \sup_{(v_1, \dots, v_s) \in \boldsymbol{R}^s} \left| \left(1 + \sum_{j=1}^s v_j^2\right)^{M} \tilde{\xi}(w)(v_1, \dots, v_s) \right| \\ &=: C(w) \left\|\tilde{\xi}(w)\right\|_{0, D_M}. \end{aligned}$$

Proof.

$$\begin{aligned} \left\|\tilde{\xi}_{SP}(w)\right\|_{0, D_{\tilde{M}}} &:= \sup_{(u_1, \dots, u_s) \in \boldsymbol{R}^s} \left| \left(1 + \sum_{j=1}^s u_j^2\right)^{\tilde{M}} \tilde{\xi}_{SP}(w)(u_1, \dots, u_s) \right| \\ &= \sup_{(u_1, \dots, u_s) \in \boldsymbol{R}^s} \left| \left(1 + \sum_{j=1}^s u_j^2\right)^{\tilde{M}} \xi(w)(\exp u_1 U_1(w) \cdots \exp u_s U_s(w)) \right| \\ &= \sup_{(u_1, \dots, u_s) \in \boldsymbol{R}^s} \left| \left(1 + \sum_{j=1}^s u_j^2\right)^{\tilde{M}} \xi(w)(\exp P_1(w, u_1, \dots, u_s) V_1(w) \cdots \right. \\ &\qquad \left. \exp P_s(w, u_1, \dots, u_s) V_s(w)) \right| \end{aligned}$$

for polynomial functions $P_j(w, u_1, \dots, u_s)$ with smooth coefficients in w. In fact, as a smooth Malcev basis is also a smooth SP-basis, we know by Proposition 5, that the change of coordinates

$$(u_1, \dots, u_s) \mapsto (v_1, \dots, v_s)$$

is bipolynomial with smooth coefficients in both directions, if $(v_1, \dots, v_s)$ denote the coordinates of the second kind in the Malcev basis $\{V_1(w), \dots, V_s(w)\}$ of $\mathfrak{n}$ relative to $\mathfrak{p}(w)$. Hence, there are polynomial functions Q_j such that

$$\begin{aligned}
&\big\|\tilde{\xi}_{SP}(w)\big\|_{0,D_{\tilde{M}}} \\
&\quad := \sup_{(v_1,\dots,v_s)\in \boldsymbol{R}^s} \left| \left(1+\sum_{j=1}^{s} Q_j(w, v_1, \dots, v_s)^2\right)^{\tilde{M}} \xi(w)(\exp v_1 V_1(w) \cdots \exp v_s V_s(w)) \right| \\
&\quad \leq C(w) \sup_{(v_1,\dots,v_s)\in \boldsymbol{R}^s} \left| \left(1+\sum_{j=1}^{s} v_j^2\right)^{M} \tilde{\xi}(w)(v_1, \dots, v_s) \right| \\
&\quad =: C(w)\big\|\tilde{\xi}(w)\big\|_{0,D_M}
\end{aligned}$$

as any polynomial $P(w, x_1, \dots, x_s)$ is bounded by some $C(w)(1+\sum_{j=1}^{s} x_j^2)^N$. $\square$

We then get the following bounds for these intertwining operators T_w and $\tilde{T}_w$.

Lemma 5. *There exist a continuous family of positive constants $(C(w))_w$ and an integer $M \in \boldsymbol{N}^*$ such that, for all w,*

$$|T_w\xi(w;e)| = \big|\tilde{T}_w\tilde{\xi}(w;0)\big| \leq C(w)\big\|\tilde{\xi}(w)\big\|_{0,D_M} < \infty$$

where e is the unit element of the group N and D_M is the element of $\mathscr{P}\mathscr{D}(W, \boldsymbol{R}^s)$ defined by multiplication by $(1+\sum_{j=1}^{s} v_j^2)^M$ if the coordinates of $\tilde{\xi}(w)$ are denoted by $(v_1, \dots, v_s)$.

If K is a compact subset of a fixed chart and if $w \in K$ we have of course

$$|T_w\xi(w;e)| = \big|\tilde{T}_w\tilde{\xi}(w;0)\big| \leq C\big\|\tilde{\xi}\big\|_{0,D_M}^{K} < \infty$$

where $C = \sup_{w\in K} C(w)$.

Proof. Let $\tilde{M} \in \boldsymbol{N}^*$ such that

$$A := \int_{\boldsymbol{R}^{s-b}} \frac{1}{(1+\sum_{j=b+1}^{s} u_j^2)^{\tilde{M}}} du_{b+1} \cdots du_s < \infty.$$

Then

$$\begin{aligned}
\big|\tilde{T}_w \tilde{\xi}(w;0)\big| &= \big|T_w \xi(w;e)\big| \\
&\le \int_{\boldsymbol{R}^{s-b}} \big|\tilde{\xi}_{SP}(w;0,\dots,0,u_{b+1},\dots,u_s)\big| du_{b+1}\cdots du_s \\
&\le A \cdot \sup_{(u_{b+1},\dots,u_s)\in \boldsymbol{R}^{s-b}} \left|\left(1+\sum_{j=b+1}^{s} u_j^2\right)^{\tilde{M}} \tilde{\xi}_{SP}(w;0,\dots,0,u_{b+1},\dots,u_s)\right| \\
&\le A \cdot \big\|\tilde{\xi}_{SP}(w)\big\|_{0,D_{\tilde{M}}} \\
&\le C(w)\big\|\tilde{\xi}(w)\big\|_{0,D_M} < \infty
\end{aligned}$$

for some M and some continuous $C(w)$, by Lemma 4. □

Let $\tilde{\mathscr{B}}_w$ be the smooth Malcev basis as in Lemma 2. Similarly, let $\tilde{\mathscr{B}}'_w = \{V'_1(w),\dots,V'_s(w),X'_1(w),\dots,X'_r(w)\}$ be any similar smooth Malcev basis of $\mathfrak{n}$, replacing $\mathfrak{p}(w)$ by $\mathfrak{p}'(w)$. We have:

Lemma 6. *There exist a finite family of smooth positive functions $\tilde{C}_j(w)$, integers $A_j, M_j, \tilde{M}_j \in \boldsymbol{N}$, partial differential operators $D_{M_j} \in \mathscr{P}\mathscr{D}(W,\boldsymbol{R}^s)$, $j \in \{1,\dots,d\}$ such that*

$$|T_w\xi(w;g_w)| \le \sum_{j=1}^{d} \tilde{C}_j(w)\big(1+\|g_w\|^2_{\tilde{\mathscr{B}}'_w}\big)^{\tilde{M}_j} \big\|\tilde{\xi}(w)\big\|_{0,D_{M_j}} < \infty$$

for all $\xi \in \mathscr{K}\mathscr{S}(W,\mathscr{S}(N/P,\chi))$, for every smooth family $g=(g_w)_w$, of elements N and for all $w \in W$.

Proof.

$$\begin{aligned}
|T_w\xi(w;g_w)| &= \big|T_w\big({}_{g_w^{-1}}\xi(w;e)\big)\big| \\
&\le C(w)\big\|\big({}_{g^{-1}}\xi\big)\tilde{}(w)\big\|_{0,D_M} \quad \text{by Lemma 5} \\
&\le C(w)\sum_{j=1}^{d} C_j(w)\big(1+\|g_w\|^2_{\tilde{\mathscr{B}}_w}\big)^{N_j}\big\|\tilde{\xi}(w)\big\|_{0,D_{M_j}} \quad \text{by Lemma 3} \\
&\le \sum_{j=1}^{d} \tilde{C}_j(w)\big(1+\|g_w\|^2_{\tilde{\mathscr{B}}'_w}\big)^{\tilde{M}_j}\big\|\tilde{\xi}(w)\big\|_{0,D_{M_j}} < \infty \quad \text{by (5).}
\end{aligned}$$

□

REMARK 5. If g_w is of the form $g_w = \exp v_1 V_1'(w) \cdots \exp v_s V_s'(w)$ for fixed $v_1, \dots, v_s \in \boldsymbol{R}$, we have

$$\left|\tilde{T}_w \tilde{\xi}(w; v_1, \dots, v_s)\right| \leq \left(1 + \sum_{i=1}^{s} v_i^2\right)^M \sum_{j=1}^{d} \tilde{C}_j(w) \left\|\tilde{\xi}(w)\right\|_{0, D_{M_j}}$$

where $M = \sup_{j=1,\dots,d} \tilde{M}_j$. For $w \in K$, we even have

$$\left|\tilde{T}_w \tilde{\xi}(w; v_1, \dots, v_s)\right| \leq C \left(1 + \sum_{i=1}^{s} v_i^2\right)^M \sum_{j=1}^{d} \left\|\tilde{\xi}\right\|_{0, D_{M_j}}^K .$$

REMARK 6. Let now K be a compact subset contained in a chart so that $\partial^a / \partial w^a$ makes sense. For a multi-index a and $w \in K$, we have

$$\begin{aligned}
&\frac{\partial^a}{\partial w^a} (\tilde{T}_w \tilde{\xi})(w; 0) \\
&\quad = \frac{\partial^a}{\partial w^a} \int_{\boldsymbol{R}^{s-b}} \tilde{\xi}_{SP}(w; 0, \dots, 0, u_{b+1}, \dots, u_s) \prod_{j=b+1}^{s} e^{-iu_j \langle w, U_j(w) \rangle} du_{b+1} \cdots du_s \\
&\quad = \int_{\boldsymbol{R}^{s-b}} \frac{\partial^a}{\partial w^a} \left[\tilde{\xi}_{SP}(w; 0, \dots, 0, u_{b+1}, \dots, u_s) \prod_{j=b+1}^{s} e^{-iu_j \langle w, U_j(w) \rangle} \right] du_{b+1} \cdots du_s .
\end{aligned}$$

In fact, $\tilde{\xi}_{SP} \in \mathscr{K}\mathscr{S}(W, \mathscr{S}(\boldsymbol{R}^s))$, as this is the case for $\tilde{\xi}$ and as the change of coordinates is bipolynomial with smooth coefficients. It is then easy to check the hypothesis of the theorem which allows to interchange derivation and integration [**5**]. If we apply successively the formula of derivation of a product of functions, we see that there exist two finite families of polynomial functions $P_k(w, 0, \dots, 0, u_{b+1}, \dots, u_s)$ and $Q_i(w, 0, \dots, 0, u_{b+1}, \dots, u_s)$, indexed by multi-indices k and i which are $\mathscr{C}^\infty$ in w such that

$$\begin{aligned}
&\frac{\partial^a}{\partial w^a} \big(\tilde{T}_w \tilde{\xi}\big)(w; 0) \\
&\quad = \int_{\boldsymbol{R}^{s-b}} \sum_{|k| \leq |a|} \left[\left(\frac{\partial^k}{\partial w^k} \tilde{\xi}_{SP} \right)(w; 0, \dots, 0, u_{b+1}, \dots, u_s) \right] \\
&\qquad\qquad \cdot P_k(w, 0, \dots, 0, u_{b+1}, \dots, u_s) \prod_{j=b+1}^{s} e^{-iu_j \langle w, U_j(w) \rangle} du_{b+1} \cdots du_s
\end{aligned}$$

$$= \int_{\boldsymbol{R}^{s-b}} \sum_{|i|\le|a|} \frac{\partial^i}{\partial w^i}\big[Q_i(w,0,\dots,0,u_{b+1},\dots,u_s)\tilde{\xi}_{SP}(w;0,\dots,0,u_{b+1},\dots,u_s)\big]$$

$$\cdot \prod_{j=b+1}^{s} e^{-iu_j\langle w,U_j(w)\rangle} du_{b+1}\cdots du_s.$$

Let us consider $Q_i(w,0,\dots,0,u_{b+1},\dots,u_s)$ as a polynomial in $u_1,\dots,u_s$ which is constant in $u_1,\dots,u_b$ and which we will also denote by Q_i. Let $(v_1,\dots,v_s)$ denote the coordinates in the Malcev basis $(V_1(w),\dots,V_s(w))$. We know that the change of coordinates between the u's and the v's is bipolynomial with smooth coefficients in w. In this change of coordinates the polynomial Q_i is hence changed into a new polynomial R_i in the variables $v_1,\dots,v_s$, such that $Q_i=(R_i)_{SP}$. Hence $Q_i(w,\cdot)\tilde{\xi}_{SP}(w,\cdot)$ becomes $R_i(w,\cdot)\tilde{\xi}(w,\cdot)$ if we change from coordinates u_j to v_j and

$$\begin{aligned}\left|\frac{\partial^a}{\partial w^a}\big(\tilde{T}_w\tilde{\xi}\big)(w;0)\right| &\le \sum_{|i|\le|a|}\left|\tilde{T}_w\bigg(\frac{\partial^i}{\partial w^i}\big(Q_i(w;\cdot)\tilde{\xi}_{SP}(w;\cdot)\big)(0)\bigg)\right| \\ &\le C(w)\sum_{|i|\le|a|}\left\|\frac{\partial^i}{\partial w^i}\big(R_i(w;\cdot)\tilde{\xi}(w;\cdot)\big)\right\|_{0,D_{M_i}} \quad \text{by lemma 5} \\ &\le C(w)\sum_{|i|\le|a|}\big\|\tilde{\xi}(w)\big\|_{|a|,D_{M_i}D_i} \\ &= C(w)\sum_{|i|\le|a|}\big\|\tilde{\xi}(w)\big\|_{|a|,\tilde{D}_i}<\infty\end{aligned}$$

where D_i is the element of $\mathscr{P}\mathscr{D}(W,\boldsymbol{R}^s)$ obtained by multiplication by R_i and $\tilde{D}_i=D_{M_i}D_i$.

Finally, we get:

Lemma 7. *For any compact subset K contained in a fixed chart of W and for any multi-index a, there exist $C>0$, $M\in\boldsymbol{N}$ and finite families of multi-indices A_j and of differential operators $D_j\in\mathscr{P}\mathscr{D}(W,\boldsymbol{R}^s)$, $j\in\{1,\dots,d\}$ such that*

$$\left|\frac{\partial^a}{\partial w^a}\big(\tilde{T}_w\tilde{\xi}\big)(w;v_1,\dots,v_s)\right|\le C\bigg(1+\sum_{i=1}^{s}v_i^2\bigg)^M\sum_{j=1}^{d}\big\|\tilde{\xi}\big\|^K_{A_j,D_j}<\infty$$

for all $\tilde{\xi}\in\mathscr{K}\mathscr{S}(W,\mathscr{S}(\boldsymbol{R}^s))$ and for all $w\in K$.

Proof. By arguments similar to the ones in Lemma 6 and Remark 5. □

Definition 10. A family $T = (T_w)_w$ of linear operators from $\mathscr{K}\mathscr{S}(W, \mathscr{S}(N/P, \chi))$ to $\mathscr{K}\mathscr{S}(W, \mathscr{S}(N/P', \chi))$ is said to be smooth if, for every $\xi \in \mathscr{K}\mathscr{S}(W, \mathscr{S}(N/P, \chi))$ and $\xi' \in \mathscr{K}\mathscr{S}(W, \mathscr{S}(N/P', \chi))$, the map $w \longmapsto \langle T_w(\xi(w)), \xi'(w)\rangle_{L^2(N/P'(w), \chi_w)}$ is smooth. In this definition, we assume of course that we have chosen a smooth family of bases of $\mathfrak{n}$ relative to $\mathfrak{p}'(w)$ and that we endow $N/P'(w)$ with the corresponding Lebesgue measure.

We are now prepared for our final result.

Theorem 3. *Let $\mathscr{W}$ be a submanifold of $\mathfrak{n}^*$ such that the dimension of polarizations associated to elements of $\mathscr{W}$ is fixed. We consider $(\mathfrak{p}(w))_{w\in\mathscr{W}}$ and $(\mathfrak{p}'(w))_{w\in\mathscr{W}}$ two smooth families of polarizations of the elements of $\mathscr{W}$ in $\mathfrak{n}$. Then, there exists a non-empty relative open subset $\mathscr{U} \subset \mathscr{W}$ such that for every $w_0 \in \mathscr{U}$ and for every relative open neighborhood $\mathscr{V}_0$ of w_0 in $\mathscr{U}$ there exist $w_0' \in \mathscr{V}_0$ and a relative open neighborhood $\mathscr{V}_0'$ of w_0' in $\mathscr{V}_0$ such that*

$$T := (T_w)_w : \mathscr{K}\mathscr{S}\big(\mathscr{V}_0', \mathscr{S}(N/P, \chi)\big) \longrightarrow \mathscr{K}\mathscr{S}\big(\mathscr{V}_0', \mathscr{S}(N/P', \chi)\big)$$

defines a continuous linear operator. Moreover, $(T_w)_w$ is a smooth family of intertwining operators between (π_w) and (π_w').

Proof. We know by [**11**], that $T_w(\xi(w))$ converges for every w and may be extended to all $\xi(w) \in L^2(N/P(w), \chi_w)$, that T_w intertwines π_w and π_w', and that $\tilde{T}_w$ intertwines ρ_w and ρ_w'. The smoothness of $w \longmapsto \langle T_w(\xi(w)), \xi'(w)\rangle_{L^2(N/P'(w), \chi_w)}$ is due to the fact that it is possible interchange derivation and integration (see [**5**]).

In the proof of the continuity of $T = (T_w)_w$, we will use the bounds established for $|\partial^a/\partial w^a \tilde{T}_w(\tilde{\xi}(w))|$ in Lemma 7.

By Theorem 1, there exists a non-empty relatively open subset $\mathscr{U}$ of $\mathscr{W}$ such that for every $w_0 \in \mathscr{U}$ and for every relative neighborhood $\mathscr{V}_0$ of w_0 in $\mathscr{U}$ there exists $w_0' \in \mathscr{V}_0$ and a relative neighborhood $\mathscr{V}_0'$ of w_0' in $\mathscr{V}_0$ with the following properties: For every $D = (D_w)_w \in \mathscr{P}\mathscr{D}(\mathscr{V}_0', \boldsymbol{R}^s)$, there exists $U_w = \sum_{|\zeta|<C} c_\zeta(w) Z^\zeta \in \mathscr{U}(\mathfrak{n})$ where Z^ζ is defined by $Z^\zeta = Z_1^{\zeta_1} \cdots Z_n^{\zeta_n}$ if $\zeta = (\zeta_1, \ldots, \zeta_n)$ and $c_\zeta(w)$ are smooth such that $d\rho_w'(U_w) = D_w$ for all $w \in \mathscr{V}_0'$. Let M is the integer obtained by Lemma 7 (taken large enough in order to be valid for all a such that $|a| \leq A$). Let us choose an integer $\tilde{M} \geq M$ and denote by $D_{\tilde{M}} \in \mathscr{P}\mathscr{D}(\mathscr{V}_0', \boldsymbol{R}^s)$ the operator obtained by multiplication by $(1 + \sum_{i=1}^s v_i^2)^{\tilde{M}}$. For any $D' = (D_w')_w \in \mathscr{P}\mathscr{D}(\mathscr{V}_0', \boldsymbol{R}^s)$, we define $D = (D_w)_w$ by $D_w := D_{\tilde{M}} D_w'$. We then obtain, for every $\tilde{\xi} \in \mathscr{K}\mathscr{S}(\mathscr{V}_0', \mathscr{S}(\boldsymbol{R}^s))$,

$$\begin{aligned}\left|D_{\tilde M}\frac{\partial^a}{\partial w^a}D'_w(v)\big(\tilde T\tilde\xi\big)(w;v_1,\dots,v_s)\right| &= \left|\frac{\partial^a}{\partial w^a}D_w(v)\big(\tilde T\tilde\xi\big)(w;v_1,\dots,v_s)\right|\\ &= \left|\frac{\partial^a}{\partial w^a}d\rho'_w(U_w)\big(\tilde T\tilde\xi\big)(w;v_1,\dots,v_s)\right|\\ &= \left|\frac{\partial^a}{\partial w^a}\tilde T\big(d\rho_w(U_w)\tilde\xi\big)(w;v_1,\dots,v_s)\right|\\ &= \left|\frac{\partial^a}{\partial w^a}\tilde T\big(D_1\tilde\xi\big)(w;v_1,\dots,v_s)\right|\\ &\le C\Big(1+\sum_{i=1}^s v_i^2\Big)^M\sum_{j=1}^d \big\|D_1\tilde\xi\big\|^K_{A_j,D_{M_j}}\\ &\le C\Big(1+\sum_{i=1}^s v_i^2\Big)^M\sum_{j=1}^d \big\|\tilde\xi\big\|^K_{A_j,D_{M_j}D_1}\end{aligned}$$

for some smooth families $(U_w)_w$ in $\mathscr{U}(\mathfrak{n})$ and $D_1=(D_{1,w})_w\in\mathscr{P}\mathscr{D}(\mathscr{V}_0',\boldsymbol{R}^s)$ such that $d\rho_w(U_w)=D_{1,w}$. Hence

$$\begin{aligned}\left|\frac{\partial^a}{\partial w^a}D'_w(v)\big(\tilde T_w\tilde\xi\big)(w;v_1,\dots,v_s)\right| &\le C\frac{1}{\big(1+\sum_{i=1}^s v_i^2\big)^{\tilde M-M}}\sum_{j=1}^d\big\|\tilde\xi\big\|^K_{A_j,D_{M_j}D_1}\\ &< \infty.\end{aligned}$$

As $\tilde M\ge M$, $\|\tilde T\tilde\xi\|^K_{A,D'}<\infty$ and $T\xi\in\mathscr{K}\mathscr{S}(\mathscr{V}_0',\mathscr{S}(N/P',\chi))$ for all $\xi\in\mathscr{K}\mathscr{S}(\mathscr{V}_0',\mathscr{S}(N/P,\chi))$. The continuity of T for the topology of the generalized Schwartz spaces is deduced from the inequality

$$\big\|(\tilde T\tilde\xi)\big\|^K_{A,D'}\le C\sum_{k=1}^r\big\|\tilde\xi\big\|^K_{B_k,D_k}$$

(for some possibly new choice of r, B_k, D_k). □

Proposition 8. *There exist a smooth function $|\alpha(w)|$ and a smooth normalization of the bases and measures of $(N/P(w))_w$ and $(N/P'(w))_w$ such that $R=(R_w)_w$ is a smooth family of unitary intertwining operators between $(\pi_w)_w$ and $(\pi'_w)_w$ where $R_w=(1/|\alpha(w)|)T_w$.*

Proof. It is well known that for a fixed w, π_w acts algebraically irreducibly on the space of the $\mathscr{C}^\infty$-vectors and there exists a unitary intertwining operator S_w between π_w and π'_w. Since T_w is also an intertwining operator between π_w and π'_w, $S_w^{-1} \circ T_w$ intertwines π_w. By Schur's Lemma, $S_w^{-1} \circ T_w = \alpha(w) \boldsymbol{I}_{\mathscr{S}(N/P(w),\chi_w)}$ where $\alpha(w) \in \boldsymbol{C}^*$ and $R_w := (1/|\alpha(w)|)T_w$ is a unitary operator. In addition, the map $w \longmapsto |\alpha(w)|$ is smooth. This is due to smoothness of the map

$$w \longrightarrow \langle T_w \xi(w), T_w \xi(w) \rangle_{L^2(N/P'(w),\chi_w)}$$

by Theorem 3. In fact, let's choose a smooth family $\xi = (\xi(w))_w$ with $\xi(w) \in L^2(N/P(w), \chi_w)$ and $\|\xi(w)\|_2^2 = 1$ for all w. Then

$$\begin{aligned} d(w) &:= \langle T_w \xi(w), T_w \xi(w) \rangle_{L^2(N/P'(w),\chi_w)} \\ &= |\alpha(w)|^2 \langle S_w \xi(w), S_w \xi(w) \rangle_{L^2(N/P'(w),\chi_w)} \\ &= |\alpha(w)|^2 \|\xi(w)\|_2^2 \quad \text{since } S_w \text{ is a unitary operator} \\ &= |\alpha(w)|^2 \end{aligned}$$

and $w \longmapsto |\alpha(w)| = \sqrt{d(w)}$ is smooth. Hence $R = (R_w)$ is a smooth family of unitary intertwining operators. □

We may now make a similar construction by interchanging the roles of $\mathfrak{p}(w)$ and $\mathfrak{p}'(w)$. We get a smooth family of intertwining operators

$$T' := (T'_w)_w : \mathscr{K}\mathscr{S}\big(\tilde{\mathscr{V}}'_0, \mathscr{S}(N/P', \chi)\big) \longrightarrow \mathscr{K}\mathscr{S}\big(\tilde{\mathscr{V}}'_0, \mathscr{S}(N/P, \chi)\big)$$

by

$$T'_w \xi'(w; g) = \int_{P(w)/P(w)\cap P'(w)} \xi'(w; gu) \chi_w(u) du.$$

Note that the non-empty relatively open subset of $\mathscr{W}$ had perhaps to be restricted once more and is now denoted by $\tilde{\mathscr{V}}'_0$. By the same arguments as previously, there exist $\alpha'(w) \in \boldsymbol{C}^*$ and a unitary intertwining operator S'_w such that $(S'_w)^{-1} \circ T'_w = \alpha'(w) \boldsymbol{I}_{\mathscr{S}(N/P'(w),\chi_w)}$ and also $R'_w := (1/|\alpha'(w)|)T'_w$ defines a smooth family of unitary intertwining operators and the map $w \longmapsto |\alpha'(w)|$ is smooth.

On the other hand, $T'_w \circ T_w$ intertwines π_w. But such an operator must be a multiple $c(w)$ of the identity by Schur's lemma where $c(w) \in \boldsymbol{C}^*$ and the map $w \longmapsto c(w)$ is smooth by the same argument as previously. By [**6**] and [**11**], we know that $c(w) > 0$. Since

$$R'_w \circ R_w = \frac{1}{|\alpha'(w)||\alpha(w)|} T'_w \circ T_w = \frac{1}{|\alpha'(w)||\alpha(w)|} c(w) \boldsymbol{I}_{\mathscr{S}(N/P(w),\chi_w)}$$

is unitary and since $c(w) > 0$, we obtain $|\alpha'(w)||\alpha(w)| = c(w)$ and $R'_w = R_w^{-1}$. This implies that

$$R = (R_w)_w : \mathscr{K}S\big(\tilde{\mathscr{V}}'_0, \mathscr{S}(N/P,\chi)\big) \to \mathscr{K}S\big(\tilde{\mathscr{V}}'_0, \mathscr{S}(N/P',\chi)\big)$$

is a linear homeomorphism.

Let us now define for all K compact subset of $\mathscr{V}'_0$ and for every $\xi(w) \in L^2(N/P(w),\chi_w)$ such that $w \longmapsto \|\xi(w)\|_{L^2(N/P(w),\chi_w)}$ is measurable, the following semi-norm

$$\|\xi\|_{2,K} = \left(\int_K \|\xi(w)\|^2_{L^2(N/P(w),\chi_w)} d\mu(w) \right)^{1/2}$$

where $d\mu(w)$ denotes the usual measure on the submanifold $\mathscr{V}'_0$ obtained locally from the Lebesgue measure on the charts. Then we consider

$$\begin{aligned} L^2_c\big(\mathscr{V}'_0, L^2(N/P,\chi)\big) = \big\{ \xi = (\xi(w))_{w\in\mathscr{V}'_0} \mid \xi(w) \in L^2(N/P(w),\chi_w)\ \forall w, \text{ such that} \\ w \mapsto \|\xi(w)\|_{L^2(N/P(w),\chi_w)} \text{ is measurable and } \|\xi\|_{2,K} < \infty, \\ \forall K \subset \mathscr{V}'_0 \text{ compact} \big\}. \end{aligned}$$

Note that $L^2_c(\mathscr{V}'_0, L^2(N/P,\chi))$ is a Fréchet space and that

$$L^2_c\big(\mathscr{V}'_0, L^2(N/P,\chi)\big) = \overline{\mathscr{K}\mathscr{S}\big(\mathscr{V}'_0, \mathscr{S}(N/P,\chi)\big)}^{L^2_c}.$$

Proposition 9. *The families of intertwining operators $T = (T_w)_w$ and $R = (R_w)_w$ may be extended continuously to families of intertwining operators from $L^2_c(\mathscr{V}'_0, L^2(N/P,\chi))$ to $L^2_c(\mathscr{V}'_0, L^2(N/P',\chi))$.*

Proof. For $\xi \in \mathscr{K}\mathscr{S}(\mathscr{V}'_0, \mathscr{S}(N/P,\chi))$, we have

$$\begin{aligned} \|T\xi\|^2_{2,K} &= \int_K \|T_w\xi(w)\|^2_{L^2(N/P'(w),\chi_w)} d\mu(w) \\ &= \int_K |\alpha(w)|^2 \|R_w\xi(w)\|^2_{L^2(N/P'(w),\chi_w)} d\mu(w) \le C\|\xi\|^2_{2,K} \end{aligned}$$

where $0 \leq C := \sup_{w \in K} |\alpha(w)|^2 < \infty$. So the extension of T to the Fréchet space $L^2_c(\mathscr{V}'_0, L^2(N/P, \chi))$ is possible. For $R = (R_w)_w$ the argument is similar, except that $|\alpha(w)|^2$ is replaced by 1. So T and R can be extend to operators from $L^2_c(\mathscr{V}'_0, L^2(N/P, \chi))$ to $L^2_c(\mathscr{V}'_0, L^2(N/P', \chi))$. □

5. Kernel functions and intertwining operators.

5.1.

As before, let $(\mathfrak{p}(w))_{w\in\mathscr{W}}$ and $(\mathfrak{p}'(w))_{w\in\mathscr{W}}$ be two smooth families of polarizations of the elements of $\mathscr{W}$. Let π_w and π'_w be the corresponding unitary irreducible representations and $R = (R_w)_w$ and $R' = (R'_w)_w$ be the smooth families of unitary intertwining operators obtained in Proposition 8, where w runs through a certain non-empty open subset of $\mathscr{W}$. Let now $f \in \mathscr{S}(N)$. Then the operator $\pi_w(f)$ (resp. $\pi'_w(f)$) is a kernel operator whose kernel function F (resp. F') is given by the formula

$$F(w, x, y) = \int_{P(w)} f(xpy^{-1})\chi_w(p)dp$$

(resp. $F'(w, x, y) = \dots$ with $P'(w)$ instead of $P(w)$ in the formula). It is easy to check that in this case the kernel functions F and F' are related by the formula

$$\begin{aligned} F'(w, x, y) &= (R_w \otimes \overline{R_w})F(w, x, y) \\ &= \frac{1}{|\alpha(w)|^2} \int_{P'(w)/P(w)\cap P'(w)} \int_{P'(w)/P(w)\cap P'(w)} \\ &\quad \cdot F(w, xu, yv)\chi_w(u)\overline{\chi_w(v)}dudv. \end{aligned}$$

5.2.

Conversely, one may ask the question whether, given an appropriate kernel function F, there exists $f \in \mathscr{S}(N)$ such that $\pi_w(f)$ is a kernel operator and has $F(w, \cdot, \cdot)$ as a kernel function. In this case, $R(F) := f$ is called a retract for F.

For fixed w, this question has been solved by R. Howe [**7**]. If w runs through $\mathfrak{n}^*_{gen}$, set of generic linear forms in the sense of Ludwig-Zahir [**14**] and if the polarizations and bases are particularly well chosen, a solution to this problem is given by the generalized Fourier inversion theorem, found in [**12**]. In particular, the function F has to satisfy the following covariance condition

$$F(w, xp, yp') = \overline{\chi_w(p)}\chi_w(p')F(w, x, y) \quad \forall p, p' \in P(w), \forall x, y \in N.$$

If moreover, the polarizations $(\mathfrak{p}(w))_w$ are Vergne polarizations with respect to a fixed Jordan-Hölder basis, we need the following compatibility relation for kernel functions corresponding to equivalent representations:

$$w' = \mathrm{Ad}^*(m)w \text{ for some } m \in N \Longrightarrow F(w', x, y) = F(w, xm, ym), \forall x, y \in N.$$

This condition just translates the equivalence relation between the corresponding representations into a relation between the kernel functions. Furthermore, these conditions are necessary (but not sufficient) for the existence of a retract.

The compatibility condition may of course be transferred to any other smooth choice of polarizations $(\mathfrak{p}'(w))_w$, at least locally, when Theorem 3 and Proposition 8 apply. A corresponding family of operator kernels $F'(w, \cdot, \cdot)$ will have to satisfy

$$\begin{aligned} w' = \mathrm{Ad}^*(m)w \text{ for some } m \in N \Longrightarrow &\big((R'_{w'} \otimes \overline{R'}_{w'})F'\big)(w', x, y) \\ &= \big((R'_w \otimes \overline{R'}_w)F'\big)(w, xm, ym) \quad \forall x, y \in N. \end{aligned}$$

5.3.

The smoothness of the family of intertwining operators is necessary for the following reason: the theorems for the existence of a retract, when w runs through an appropriate non-empty relatively open set in our submanifold $\mathscr{W}$, require the potential kernel functions to be generalized Schwartz functions and smooth families of intertwining operators send such kernel functions into functions of the same type. This allows, under appropriate conditions, to transfer results on the existence of a retract for representations built by using a particular type of polarizations to a more general or more suitable family of polarizations and representations, at least locally, where we have the existence of the smooth family of intertwining operators. The next paragraph will give a more precise example.

5.4.

Let $N = \exp \mathfrak{n}$ be a connected, simply connected, nilpotent Lie group, let $\mathscr{B}$ be a fixed Jordan-Hölder basis and let K be a compact Lie subgroup of $\mathrm{Aut}(N)$, acting smoothly on N. Of course, this action induces a corresponding action of K on $\mathfrak{n}$, $\mathfrak{n}^*$, $\widehat{N}, \ldots$ Let us assume that for any $l' \in \mathfrak{n}^*$, $\mathfrak{p}(l')$ denotes the Vergne polarization of l' with respect to the basis $\mathscr{B}$. The action of K on $\widehat{N}$ is given by

$${}^k\pi_l(x) := \pi_l(k^{-1} \cdot x) \quad \forall x \in N,\ \forall k \in K.$$

Hence the question: Does there exist a local retract theorem for π_l^k, where k runs through an appropriate section of K/K_{π_l}? Here K_{π_l} denotes the stabilizer of π_l

and is equal to the set of all k's in K such that ${}^k\pi_l$ is equivalent to π_l. Let us assume l to be generic in the sense of Ludwig-Zahir. Let us notice that $k \cdot \mathfrak{p}(l)$ is a polarization for $k \cdot l$ and let us define $\tilde{\pi}_k = \mathrm{ind}_{k \cdot P(l)}^N \chi_{k \cdot l}$. It is easy to see that ${}^k\pi_l$ and $\tilde{\pi}_k$ are unitary equivalent through the intertwining operator U_k defined by $U_k\xi(t) := \xi(k^{-1} \cdot t)$ for all $\xi \in \mathscr{H}_{{}^k\pi_l}$. So, instead of proving the existence of a local retract for ${}^k\pi_l$, which turns out to be very complicated, it is sufficient to prove such an existence for the representation $\tilde{\pi}_k$. But even this is not straightforward. Let us therefore define $\pi_{k \cdot l} := \mathrm{ind}_{P(k \cdot l)}^N \chi_{k \cdot l}$ where $P(k \cdot l) = \exp \mathfrak{p}(k \cdot l)$ and where $\mathfrak{p}(k \cdot l)$ is the Vergne polarization for $k \cdot l$ with respect to the fixed basis $\mathscr{B}$. We do this for $k \in \tilde{K} := \{k \in K | k \cdot l \in \mathfrak{n}^*_{gen}\}$. As we are now dealing with generic linear forms $k \cdot l$ and Vergne polarizations $\mathfrak{p}(k \cdot l)$, we may use the Fourier inversion Theorem [**12**], to prove the existence of a local retract for suitable kernel functions (having compact k-support for k in a local section of K/K_{π_l}, generalized Schwartz conditions, covariance and compatibility conditions). Finally, the smooth family of intertwining operators gives the retract result for the representations $(\tilde{\pi}_k)_k$, and hence for the representations $({}^k\pi_l)_k$, locally where Theorem 3 and Proposition 8 hold. The details of the argument require a very careful definition of the potential kernel functions and the proof of the fact that the smooth family of intertwining operators respects these kernel functions.

6. Example of the Heisenberg group.

In the case of the Heisenberg group $H = \exp \mathfrak{h}$ where $\mathfrak{h} = \langle X, Y, Z \rangle$ with $[X, Y] = Z$, we consider the submanifold of $\mathfrak{h}^*$ given by

$$\mathscr{W} = \{l_\epsilon := (1 + \epsilon)Z^* \mid \epsilon \in] - 1, 1[\}.$$

Then $\mathfrak{p}(l_\epsilon) = \boldsymbol{R}Y + \boldsymbol{R}Z$ and $\mathfrak{p}'(l_\epsilon) = \boldsymbol{R}(Y + \epsilon X) + \boldsymbol{R}Z$ define two smooth families of polarizations of the collection of l_ϵ's, for $\epsilon \in] - 1, 1[$, and we denote by $\pi_\epsilon := \mathrm{ind}_{P(l_\epsilon)}^N \chi_{l_\epsilon}$ and $\pi'_\epsilon := \mathrm{ind}_{P'(l_\epsilon)}^N \chi_{l_\epsilon}$ the corresponding induced representations of H where $P(l_\epsilon) = \exp \mathfrak{p}(l_\epsilon)$ and $P'(l_\epsilon) = \exp \mathfrak{p}'(l_\epsilon)$. We will compute explicitly the families of intertwining operators $(T_\epsilon)_\epsilon$ and $(R_\epsilon)_\epsilon$ and we will show how the singularities appear and behave.

6.1. First method.

We construct a basis of $\mathfrak{h}$ relative to $\mathfrak{p}(l_\epsilon)$ which respects $\mathfrak{p}'(l_\epsilon)$ by the index method of Section 2.1. Similarly, we construct a basis of $\mathfrak{h}$ relative to $\mathfrak{p}'(l_\epsilon)$ which respects $\mathfrak{p}(l_\epsilon)$. To do this, we write the basis $\{X, Y, Z\}$ as follows

$$Y_1 = X; Y_2 = Y; Y_3 = Z.$$

Then, we obtain the following index sets $I := I^{\mathfrak{p}(l_\epsilon)} = \{2,3\}$ and $I' := I^{\mathfrak{p}'(l_\epsilon)} = \{1,3\}$, which are fixed for $\epsilon \in]-1,1[\backslash\{0\}$. Moreover, by Proposition 3, we construct $\{Y_1 + (1/\epsilon)Y_2, Y_3\} = \{X + (1/\epsilon)Y, Z\}$ as a smooth Jordan-Hölder basis of $\mathfrak{p}'(l_\epsilon)$ and $\{Y_2, Y_3\} = \{Y, Z\}$ as a smooth Jordan-Hölder basis of $\mathfrak{p}(l_\epsilon)$. Finally, we obtain that $\{X + (1/\epsilon)Y\}$ is a smooth Malcev basis of $\mathfrak{h}$ relative to $\mathfrak{p}(l_\epsilon)$ and that $\{Y\}$ is a smooth Malcev basis of $\mathfrak{h}$ relative to $\mathfrak{p}'(l_\epsilon)$.

Let $\tilde{\xi} \in \mathscr{S}(\boldsymbol{R})$ be arbitrary and let us identify it with a function of $\mathscr{K}\mathscr{S}(\mathscr{W}, \mathscr{S}(\boldsymbol{R}))$ which is constant in ϵ. We then define the function $\xi \in \mathscr{K}\mathscr{S}(\mathscr{W}, \mathscr{S}(H/P(l), \chi))$ by

$$\begin{aligned}\xi\Big(\epsilon, \exp\Big(x\Big(X + \frac{1}{\epsilon}Y\Big)\Big)\exp yY \exp zZ\Big) \\ = e^{i(1+\epsilon)z}\xi\Big(\epsilon, \exp\Big(x\Big(X + \frac{1}{\epsilon}Y\Big)\Big)\Big) := e^{i(1+\epsilon)z}\tilde{\xi}(x),\end{aligned}$$

thanks to the covariance relation. We then have

$$\begin{aligned}(\tilde{T}_\epsilon\tilde{\xi})(y) &:= (T_\epsilon\xi)(\epsilon; \exp yY) \\ &= \int_{\boldsymbol{R}} \xi\Big(\epsilon; \exp yY \cdot \exp x\Big(X + \frac{1}{\epsilon}Y\Big)\Big)\underbrace{\chi_{l_\epsilon}\Big(\exp x\Big(X + \frac{1}{\epsilon}Y\Big)\Big)}_{=1} dx \\ &= \int_{\boldsymbol{R}} \xi\Big(\epsilon; \exp x\Big(X + \frac{1}{\epsilon}Y\Big)\Big)e^{-i(1+\epsilon)xy}dx \\ &= \int_{\boldsymbol{R}} \tilde{\xi}(x)e^{-i(1+\epsilon)xy}dx \\ &= \widehat{\tilde{\xi}}((1+\epsilon)y).\end{aligned}$$

Formally, $(\tilde{T}_\epsilon\tilde{\xi})(y)$ converges to $\widehat{\tilde{\xi}}(y)$ when ϵ tends to 0. But, this makes no sense for our problem, as the norm of the vector $X + (1/\epsilon)Y$ of the Malcev basis of $\mathfrak{h}$ relative to $\mathfrak{p}(l_\epsilon)$ tends to infinity. Even, if we would replace this vector by $V(\epsilon) := \epsilon(X + (1/\epsilon)Y) = Y + \epsilon X$, then $V(\epsilon)$ would converge to $Y \in \mathfrak{p}(l)$, where $l_0 = l$, for $\epsilon \longrightarrow 0$, i.e. the limit vector would no longer be a Malcev basis of $\mathfrak{h}$ relative to $\mathfrak{p}(l)$. This illustrates the fact that the singularities are inherent to this problem and method.

6.2. Second method.

In this particular example of the Heisenberg group, one may find another, "natural" Malcev basis of $\mathfrak{h}$ relative to $\mathfrak{p}(l_\epsilon)$, respectively $\mathfrak{p}'(l_\epsilon)$, i.e. $\{X\}$, which is a good choice for both families of polarizations.

As we will see, the singularity at $\epsilon = 0$ will, in this approach, lead to a nice interpretation: As a matter of fact, for $\epsilon = 0$, $l_\epsilon = l$ and $\mathfrak{p}(l_\epsilon) = \mathfrak{p}'(l_\epsilon) = \mathfrak{p}(l) = \boldsymbol{R}Y + \boldsymbol{R}Z$. So, for $\epsilon = 0$, the polarizations and representations coincide. The intertwining operator may be taken to be the identity. We will show that, for an appropriate choice of the unitary intertwining operators R_ϵ (these are only defined up to a constant of module 1), the family $(R_\epsilon)_\epsilon$ will converge weakly and strongly towards the identity operator. Unfortunately, this cannot be done smoothly, nor in the generalized Schwartz topologies. Moreover, in the case of a general nilpotent Lie group, such an approach is not possible.

Again, we start with an arbitrary function $\tilde{\xi} \in \mathscr{S}(\boldsymbol{R})$ and we define $\xi \in \mathscr{K}\mathscr{S}(\mathscr{W}, \mathscr{S}(H/P, \chi))$ by

$$\xi(\epsilon, \exp xX \cdot \exp yY \cdot \exp zZ) = e^{i(1+\epsilon)z}\xi(\epsilon, \exp xX) =: e^{i(1+\epsilon)z}\tilde{\xi}(x),$$

thanks to the covariance relation. As $\{Y + \epsilon X\}$ is a Malcev basis of $\mathfrak{p}'(l_\epsilon)$ relative to $\mathfrak{p}(l_\epsilon) \cap \mathfrak{p}'(l_\epsilon)$, we have

$$\begin{aligned}
\big(\tilde{T}_\epsilon\tilde{\xi}\big)(x) &:= (T_\epsilon\xi)(\epsilon, \exp xX) \\
&= \int_{\boldsymbol{R}} \xi(\epsilon, \exp xX \cdot \exp y(Y + \epsilon X))\, \underbrace{\chi_{l_\epsilon}(\exp y(Y + \epsilon X))}_{=1}\, dy \\
&= \int_{\boldsymbol{R}} \xi(\epsilon, \exp(x + y\epsilon)X) e^{-i\frac{1}{2}y^2\epsilon(1+\epsilon)} dy \\
&= \frac{1}{\epsilon}\int_{\boldsymbol{R}} \xi(\epsilon; \exp(x + y)X) e^{-i\frac{1}{2}\frac{1+\epsilon}{\epsilon}y^2} dy \\
&= \frac{1}{\epsilon}\int_{\boldsymbol{R}} \tilde{\xi}(x + y) e^{-i\frac{1}{2}\frac{1+\epsilon}{\epsilon}y^2} dy.
\end{aligned}$$

By elementary computation and by the use of integration by parts, the integral

$$\int_{\boldsymbol{R}} e^{\pm ix^2} dx = \frac{(1 \pm i)\sqrt{\pi}}{\sqrt{2}}$$

(see [**8**]) implies that for every Schwartz function $\tilde{\xi}$ on $\boldsymbol{R}$ and every $x \in \boldsymbol{R}$,

$$\lim_{u \longrightarrow +\infty} \sqrt{u} \int_{\boldsymbol{R}} \tilde{\xi}(x + s) e^{\pm ius^2} ds = \frac{(1 \pm i)\sqrt{\pi}}{\sqrt{2}}\tilde{\xi}(x).$$

In order to use this result, let's put $u := ((1+\epsilon)/|\epsilon|)/2$ and $\operatorname{sgn}(\epsilon) = \left\{ \begin{smallmatrix} +1 \text{ if } \epsilon>0 \\ -1 \text{ if } \epsilon<0 \end{smallmatrix} \right.$.

Hence

$$\big(\tilde{T}_\epsilon \tilde{\xi}\big)(x) = \frac{\sqrt{2}}{\operatorname{sgn}(\epsilon)\sqrt{|\epsilon|(1+\epsilon)}} \cdot \left[\sqrt{u}\int_{\boldsymbol{R}} \tilde{\xi}(x+y) e^{-i\operatorname{sgn}(\epsilon)uy^2} dy\right]$$

which implies that $\lim_{\epsilon\longrightarrow 0} |(\tilde{T}_\epsilon\tilde{\xi})(x)| = +\infty$ for all x. Let's define

$$\tilde{R}_\epsilon := \frac{\operatorname{sgn}(\epsilon)\sqrt{|\epsilon|(1+\epsilon)}}{\sqrt{2\pi}} \cdot \frac{1+\operatorname{sgn}(\epsilon)i}{\sqrt{2}} \cdot \tilde{T}_\epsilon.$$

We then have the following pointwise convergence

$$\lim_{\epsilon\longrightarrow 0} \big(\tilde{R}_\epsilon\tilde{\xi}\big)(x) = \tilde{\xi}(x), \quad \forall x \in \boldsymbol{R},$$

if $\tilde{\xi} \in \mathscr{S}(\boldsymbol{R})$. Of course, the $\tilde{R}_\epsilon$'s are intertwining operators. Moreover they are isometries and hence unitary, by the arguments of Proposition 8. To show the isometry property, we make the following computations:

$$\begin{aligned}
\big(\tilde{T}_\epsilon\tilde{\xi}\big)(x) &= \frac{1}{\epsilon}\int_{\boldsymbol{R}} \tilde{\xi}(x+y)e^{-i(1/2)((1+\epsilon)/\epsilon)y^2} dy \\
&= \frac{1}{\epsilon}e^{-i(1/2)((1+\epsilon)/\epsilon)x^2}\int_{\boldsymbol{R}} \underbrace{\tilde{\xi}(u)e^{-i(1/2)((1+\epsilon)/\epsilon)u^2}}_{=:\eta(\epsilon,u)} e^{i((1+\epsilon)/\epsilon)ux} du \\
&= \frac{1}{\epsilon}e^{-i(1/2)((1+\epsilon)/\epsilon)x^2}\widehat{\eta}\left(\epsilon, -\frac{1+\epsilon}{\epsilon}x\right)
\end{aligned}$$

and

$$\begin{aligned}
\big\|\tilde{T}_\epsilon\tilde{\xi}\big\|_2^2 &= \int_{\boldsymbol{R}} \frac{1}{\epsilon^2}\left|\widehat{\eta}\left(\epsilon, -\frac{1+\epsilon}{\epsilon}x\right)\right|^2 dx \\
&= \frac{1}{\epsilon^2}\frac{|\epsilon|}{1+\epsilon}\int_{\boldsymbol{R}} |\widehat{\eta}(\epsilon,u)|^2 du \\
&= \frac{2\pi}{|\epsilon|(1+\epsilon)}\|\eta(\epsilon,\cdot)\|_2^2 \\
&= \frac{2\pi}{|\epsilon|(1+\epsilon)}\|\tilde{\xi}\|_2^2.
\end{aligned}$$

Finally,

$$\|\tilde{R}_\epsilon\tilde{\xi}\|_2 = \frac{\sqrt{|\epsilon|(1+\epsilon)}}{\sqrt{2\pi}}\|\tilde{T}_\epsilon\tilde{\xi}\|_2 = \|\tilde{\xi}\|_2 \qquad \forall\tilde{\xi} \in \mathscr{S}(\boldsymbol{R})$$

and $\|\tilde{R}_\epsilon\|_{op} = 1$. The operators R_ϵ may hence be extended to all of $L^2(\boldsymbol{R})$.

We have seen that $\tilde{R}_\epsilon\tilde{\xi} \longrightarrow \tilde{\xi}$ pointwise, when $\epsilon \longrightarrow 0$. This implies that $\tilde{R}_\epsilon \longrightarrow \boldsymbol{I}_{\mathscr{S}(\boldsymbol{R})}$ weakly and strongly, where $\boldsymbol{I}_{\mathscr{S}(\boldsymbol{R})}$ is the identity operator on $\mathscr{S}(\boldsymbol{R})$, and hence also on $L^2(\boldsymbol{R})$ similarly. As a matter of fact, for all $\tilde{\xi}$, $\tilde{\eta} \in \mathscr{S}(\boldsymbol{R})$,

$$\langle\tilde{R}_\epsilon\tilde{\xi}, \tilde{\eta}\rangle = \tilde{R}_\epsilon\big(\tilde{\xi} * \bar{\check{\tilde{\eta}}}\big)(0) \longrightarrow \tilde{\xi} * \bar{\check{\tilde{\eta}}}(0) = \langle\tilde{\xi}, \tilde{\eta}\rangle, \quad \text{if } \epsilon \to 0,$$

by the pointwise convergence, where $\check{\tilde{\eta}}(x) := \tilde{\eta}(-x)$. So, as the $\tilde{R}_\epsilon$'s are isometries,

$$\|\tilde{R}_\epsilon - \tilde{\xi}\|_2^2 = \|\tilde{\xi}\|_2^2 - \langle\tilde{R}_\epsilon\tilde{\xi}, \tilde{\xi}\rangle - \langle\tilde{\xi}, \tilde{R}_\epsilon\tilde{\xi}\rangle + \|\tilde{\xi}\|_2^2 \longrightarrow 0, \quad \text{if } \epsilon \to 0$$

for all $\tilde{\xi} \in \mathscr{S}(\boldsymbol{R})$. We have weak and strong convergence of $(\tilde{R}_\epsilon)_\epsilon$ towards $\boldsymbol{I}_{\mathscr{S}(\boldsymbol{R})}$ in $\mathscr{S}(\boldsymbol{R})$, and hence in $L^2(\boldsymbol{R})$ similarly, by density of $\mathscr{S}(\boldsymbol{R})$ in $L^2(\boldsymbol{R})$. Unfortunately the family $(\tilde{R}_\epsilon)_{\epsilon\in]-1,1[}$, where $\tilde{R}_0 = \boldsymbol{I}_{\mathscr{S}(\boldsymbol{R})}$, is not smooth at the origin, nor do we have convergence in the generalized Schwartz topologies.

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