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Spectral multipliers for Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates

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Abstract. Let (X, d, μ) be a metric measure space endowed with a distance d and a nonnegative Borel doubling measure μ. Let L be a non-negative self-adjoint operator on $L^2(X)$. Assume that the semigroup e^{-tL} generated by L satisfies the Davies-Gaffney estimates. Let $H^p_L(X)$ be the Hardy space associated with L. We prove a Hörmander-type spectral multiplier theorem for L on $H^p_L(X)$ for $0 < p < \infty$: the operator $m(L)$ is bounded from $H^p_L(X)$ to $H^p_L(X)$ if the function m possesses s derivatives with suitable bounds and $s > n(1/p - 1/2)$ where n is the "dimension" of X. By interpolation, $m(L)$ is bounded on $H^p_L(X)$ for all $0 < p < \infty$ if m is infinitely differentiable with suitable bounds on its derivatives. We also obtain a spectral multiplier theorem on L^p spaces with appropriate weights in the reverse Hölder class.

1. Introduction.

Let (X, d, μ) be a metric measure space endowed with a distance d and a nonnegative Borel doubling measure μ on X. Recall that a metric is doubling provided that there exists a constant $C > 0$ such that for all $x \in X$ and for all $r > 0$,

$$V(x, 2r) \leq CV(x, r) < \infty, \tag{1.1}$$

where $B(x, r) = \{y \in X : d(x, y) < r\}$ and $V(x, r) = \mu(B(x, r))$. In particular, X is a space of homogeneous type. A more general definition and further studies of these spaces can be found in [**CW1**, Chapter 3].

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Note that the doubling property implies the following strong homogeneity property,

$$V(x, \lambda r) \leq C\lambda^n V(x, r) \tag{1.2}$$

for some $C, n > 0$ uniformly for all $\lambda \geq 1$ and $x \in X$. The smallest value of the parameter n is a measure of the dimension of the space. There also exist C and N so that

$$V(y, r) \leq C\bigg(1 + \frac{d(x, y)}{r}\bigg)^N V(x, r) \tag{1.3}$$

uniformly for all $x, y \in X$ and $r > 0$. Indeed, property (1.3) with $N = n$ is a direct consequence of the triangle inequality for the metric d and the strong homogeneity property (1.2). When X is Ahlfors regular, i.e. $V(x, r) \sim r^n$ uniformly in x, the value N can be taken to be 0.

Suppose that L is a non-negative self-adjoint operator on $L^2(X)$. Let $E(\lambda)$ be the spectral resolution of L. For any bounded Borel function $m : [0, \infty) \to \boldsymbol{C}$, by using the spectral theorem we can define the operator

$$m(L) = \int_0^\infty m(\lambda)dE(\lambda). \tag{1.4}$$

It is well known that the operator $m(L)$ is bounded on $L^2(X)$. It is an interesting problem to give sufficient conditions on m and L which imply the boundedness of $m(L)$ on various spaces on X. This has been a very active topic of harmonic analysis and it was studied extensively. The reader is referred, in particular, to [**A**], [**AL**], [**B**], [**BK**], [**C**], [**DeM**], [**DOS**], [**Dz**] and [**FS**] and the references therein.

The following shall be assumed throughout this article unless otherwise specified:

(H1) L is a non-negative self-adjoint operator on $L^2(X)$;
(H2) The operator L generates an analytic semigroup $\{e^{-tL}\}_{t>0}$ on $L^2(X)$ which satisfies the Davies-Gaffney estimate. That is, there exist constants $C, c > 0$ such that for any open subsets $U_1, U_2 \subset X$,

$$\big|\langle e^{-tL} f_1, f_2\rangle\big| \leq C \exp\bigg(-\frac{\mathrm{dist}(U_1, U_2)^2}{ct}\bigg)\|f_1\|_{L^2(X)}\|f_2\|_{L^2(X)}, \quad \forall\, t > 0, \tag{1.5}$$

for every $f_i \in L^2(X)$ with $\operatorname{supp} f_i \subset U_i$, $i = 1, 2$, where $\operatorname{dist}(U_1, U_2) := \inf_{x \in U_1, y \in U_2} d(x, y)$.

Examples of families of operators for which condition (1.5) holds include semigroups generated by second order elliptic self-adjoint operators in divergence form on the Euclidean spaces $\boldsymbol{R}^n$, Schrödinger operators with real potentials and magnetic field (see, for example [**Da1**]). Condition (1.5) is well-known to hold for Laplace-Beltrami operators on all complete Riemannian manifolds (see [**Da1**], [**Da2**], [**Ga**]). In the more general setting of Laplace type operators acting on vector bundles, condition (1.5) is proved in [**Si**]. Condition (1.5) also holds in the setting of local Dirichlet forms (see for instance, [**Stu**]). In this case the metric measure spaces under consideration are possibly not equipped with any differential structure. However, the semigroups associated with these Dirichlet forms satisfy usually Davies-Gaffney estimates with respect to an intrinsic distance.

We shall be working with an auxiliary nontrivial function ϕ with compact support. Let ϕ be a non-negative C_0^∞ function on $\boldsymbol{R}$ such that

$$\operatorname{supp} \phi \subseteq \left(\frac{1}{4}, 1\right) \quad \text{and} \quad \sum_{\ell \in \boldsymbol{Z}} \phi(2^{-\ell}\lambda) = 1 \quad \text{for all } \lambda > 0. \tag{1.6}$$

For $s \geq 0$, let $[s]$ denote the integer part of s. Recall that C^s is the space of functions m on $\boldsymbol{R}$ for which

$$\|m\|_{C^s} = \begin{cases} \sum_{k=0}^{s} \sup_{\lambda \in \boldsymbol{R}} \left|m^{(k)}(\lambda)\right| & \text{if } s \in \boldsymbol{Z}, \\ \left\|m^{([s])}\right\|_{\mathrm{Lip}(s-[s])} + \sum_{k=0}^{[s]} \sup_{\lambda \in \boldsymbol{R}} \left|m^{(k)}(\lambda)\right| & \text{if } s \notin \boldsymbol{Z} \end{cases}$$

is finite.

The aim of this paper is to prove a Hörmander-type spectral multiplier theorem on Hardy spaces $H_L^p(X)$ for $p > 0$, where $H_L^p(X)$ is a new class of Hardy spaces associated to L ([**HLMMY**] and [**DL**], see Section 2 below). The following is the main result of this paper.

Theorem 1.1. *Let L be a non-negative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimate* (1.5). *Let ϕ be a non-negative C_0^∞ function satisfying* (1.6). *If $0 < p \leq 1$ and the bounded measurable function $m : [0, \infty) \to \boldsymbol{C}$ satisfies*

$$C_{\phi,s} = \sup_{t>0} \|\phi(\cdot) m(t\cdot)\|_{C^s} + |m(0)| < \infty \tag{1.7}$$

for some $s > n(1/p - 1/2)$*, then* $m(L)$ *is bounded on* $H_L^p(X)$*, i.e., there exists a constant* $C > 0$ *such that*

$$\|m(L)f\|_{H_L^p(X)} \le C\|f\|_{H_L^p(X)}.$$

If m *satisfies* (1.7) *for all* $s > 0$*, then by interpolation and duality,* $m(L)$ *is bounded on* $H_L^p(X)$ *for all* $0 < p < \infty$.

REMARKS. We would like to list two consequences of Theorem 1.1.

(a) If m satisfies (1.7) for some $s > n/2$, then $m(L)$ is $H_L^p(X)$ bounded for all $p \in [1, +\infty)$.
(b) If m is a bounded analytic function on a sector

$$S_\mu^0 = \{z \in \boldsymbol{C} : z \neq 0 \text{ and } |\arg(z)| < \mu\}$$

for some $\mu > 0$, then m satisfies (1.7) for all $s > 0$. Hence $m(L)$ is bounded on $H_L^p(X)$ for all $0 < p < \infty$. An example is $m(\lambda) = \lambda^{i\gamma}$ for some real value γ. See Corollary 4.3.

The second main result is Theorem 5.2 (see Section 5) in which L^p-boundedness of spectral multipliers of L is obtained with appropriate weights.

We remark that when the semigroup e^{-tL} generated by L has a kernel $p_t(x, y)$ satisfying a Gaussian upper bound, that is

$$\big|p_t(x,y)\big| \le \frac{C}{V(x,\sqrt{t})} \exp\Big(-\frac{d^2(x,y)}{ct}\Big) \tag{1.8}$$

for all $t > 0$, and $x, y \in X$, then $m(L)$ is of weak type $(1,1)$ and bounded on $L^p(X)$ for p in $(1,\infty)$ (see [**DOS**] for instance). Note that in this case, the Hardy space $H_L^p(X)$ coincides with $L^p(X)$ for every $1 < p < \infty$. However, there are many important operators L which do not satisfy (1.8) but still satisfy (1.5), for example, the Hodge Laplacian on Riemannian manifold with doubling measure ([**AMR**]). The main contribution of this article is to obtain a Hörmander-type spectral multiplier theorem for Hardy spaces using only Davies-Gaffney type estimates (1.5) in place of pointwise kernel bounds (1.8).

The paper is organized as follows. In Section 2, we recall some preliminary results about Hardy space $H_L^p(X)$ associated to an operator L. In Section 3, we give a criterion for boundedness of $m(L)$ on the Hardy spaces $H_L^p(X)$ for $0 < p \le 1$. In Section 4, we prove our main result, Theorem 1.1, whose proof relies on Davies-

Gaffney heat kernel estimates and finite propagation speed of the wave operator to obtain estimates for the kernel of the operator $m(L)$ away from the diagonal. In Section 5, we obtain a spectral multiplier result for L on certain L^p spaces with weights in an appropriate reverse Hölder class.

Throughout, the letter "C" and "c" will denote (possibly different) constants that are independent of the essential variables.

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2. Notation and preliminaries.

Let (X, d, μ) be a metric measure space endowed with a distance d and a nonnegative Borel doubling measure μ. We first have the following time derivative estimate of the semigroup $\{e^{-tL}\}_{t>0}$.

Proposition 2.1. *Assume that the operator L satisfies* (H1)–(H2). *Then for every $K \in \boldsymbol{N}$, the family of operators*

$$\{(tL)^K e^{-tL}\}_{t>0}$$

satisfies the Davies-Gaffney estimate (1.5) *with* $c, C > 0$ *depending on* K, n *and* N *in* (1.2) *and* (1.3) *only.*

Proof. For the proof, see Proposition 2.1, [**HLMMY**]. $\square$

2.1. Hardy spaces $H^p_L(X)$ for $p \geq 1$.

In order to define the Hardy spaces based upon these various operators, we follow [**AMR**] and first define the L^2 adapted Hardy space

$$H^2(X) := \overline{R(L)}, \tag{2.1}$$

that is, the closure of the range of L in $L^2(X)$. Then $L^2(X)$ is the orthogonal sum of $H^2(X)$ and the null space $N(L)$.

Consider the following quadratic operators associated to L

$$S_{h,K}f(x) = \bigg(\int_0^\infty \int_{d(x,y)<t} \big|(t^2L)^K e^{-t^2L} f(y)\big|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \bigg)^{1/2}, \quad x \in X \tag{2.2}$$

where $f \in L^2(X)$. We shall write S_h in place of $S_{h,1}$. For each $K \geq 1$ and $1 \leq p < \infty$, we now define

$$D_{K,p} = \big\{f \in H^2(X) : S_{h,K} f \in L^p(X)\big\}, \quad 1 \le p < \infty.$$

Definition 2.2. Let L be a non-negative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimate (1.5).

(i) For each $1 \le p \le 2$, the Hardy space $H^p_{L,S_h}(X)$ associated to L is the completion of the space $D_{1,p}$ in the norm

$$\|f\|_{H^p_{L,S_h}(X)} = \|S_h f\|_{L^p(X)}.$$

(ii) For each $2 < p < \infty$, the Hardy space $H^p_L(X)$ associated to L is the completion of the space $D_{K_0,p}$ in the norm

$$\|f\|_{H^p_{L,S_h}(X)} = \|S_{h,K_0} f\|_{L^p(X)}, \quad K_0 = \left[\frac{n}{4}\right] + 1.$$

Under the assumption of Gaussian upper bounds (1.8) for the heat kernel of the operator L, it was proved in [**ADM**] that $H^p_{L,S_h}(X) = L^p(X)$ for all $1 < p < \infty$. Note that, in the framework of the present paper, we only assume the Davies-Gaffney estimates on the heat kernel of L, and hence for $1 < p < \infty$, $p \ne 2$, $H^p_{L,S_h}(X)$ may or may not coincide with the space $L^p(X)$. However, it can be verified that $H^2_{L,S_h}(X) = H^2(X)$ and the dual of $H^p_{L,S_h}(X)$ is $H^{p'}_{L,S_h}(X)$, with $1/p + 1/p' = 1$ (See Proposition 9.4 of [**HLMMY**]).

We also recall that the $H^p_L(X)$ spaces $(1 \le p < +\infty)$ are a family of interpolation spaces for the complex interpolation method (See Proposition 9.5 of [**HLMMY**]).

2.2. The atomic Hardy spaces $H^p_{L,at,M}(X)$ for $p \le 1$.

Let us describe the notion of a $(p, 2, M)$-*atom*, $0 < p \le 1$, associated to operators on spaces (X, d, μ). In what follows, assume that

$$M \in \boldsymbol{N} \ \text{ and } \ M > \frac{n(2-p)}{4p}, \tag{2.3}$$

where the parameter n, thought of as a measure of the dimension of the space X, is defined in (1.2). Let us denote by $\mathscr{D}(T)$ the domain of an operator T. We shall often just use B for $B(x_B, r_B)$. Also given $\lambda > 0$, we will write λB for the $\lambda-$dilated ball, which is the ball with the same center as B and with radius $r_{\lambda B} = \lambda r_B$. We set

$$U_0(B) = B, \ \text{ and } \ U_j(B) = 2^j B \backslash 2^{j-1} B \text{ for } j = 1, 2, \ldots. \tag{2.4}$$

DEFINITION 2.3. If $0 < p \le 1$, a function $a(x) \in L^2(X)$ is called a $(p, 2, M)$-atom associated to an operator L if there exist a function $b \in \mathscr{D}(L^M)$ and a ball B of X such that

(i) $a = L^M b$;
(ii) $\operatorname{supp} L^k b \subset B$, $k = 0, 1, \ldots, M$;
(iii) $\|(r_B^2 L)^k b\|_{L^2(X)} \le r_B^{2M} V(B)^{1/2-1/p}$, $k = 0, 1, \ldots, M$.

In the case $\mu(X) < \infty$ the constant function having value $[\mu(X)]^{-1/p}$ is also considered to be an atom.

Recall that the spaces $H^p(X)$ when $p < 1$ (see for example, [**FS**] and [**CW2**]) are not spaces of functions on X but spaces of distributions. In the present setting we need to use an appropriate space of linear functionals to define the Hardy spaces. In order to do this we follow the approach in [**HLMMY**] and [**HM**] to introduce adapted Lipschitz spaces $\Lambda_L^{\alpha,s}(X), \alpha > 0$ and $s \in \boldsymbol{N}$, associated to an operator L on the space X. Let $\phi = L^M \nu$ be a function in $L^2(X)$, where $\nu \in \mathscr{D}(L^M)$. For $\epsilon > 0$ and $M \in \boldsymbol{N}$, we introduce the norm

$$\|\phi\|_{\mathscr{M}_0^{p,2,M,\epsilon}(L)} = \sup_{j \ge 0} \left[2^{j\epsilon} V(x_0, 2^j)^{1/p-1/2} \sum_{k=0}^{M} \left\| L^k \nu \right\|_{L^2(U_j(B_0))} \right]$$

where B_0 is the ball centered at some $x_0 \in X$ with radius 1. We set

$$\mathscr{M}_0^{p,2,M,\epsilon}(L) = \left\{ \phi = L^M \nu \in L^2(X) : \|\phi\|_{\mathscr{M}_0^{p,2,M,\epsilon}(L)} < \infty \right\}.$$

Let $\left(\mathscr{M}_0^{p,2,M,\epsilon}(L)\right)^*$ be the dual of $\mathscr{M}_0^{p,2,M,\epsilon}(L)$, and let A_t denote either $(I+t^2L)^{-1}$ or e^{-t^2L}. We claim that if $f \in \left(\mathscr{M}_0^{p,2,M,\epsilon}(L)\right)^*$, then the distribution $(I - A_t)^M f$ belongs to $L^2_{\mathrm{loc}}(X)$. Indeed, if $\varphi \in L^2(B)$ for some ball B, it follows from the Davies-Gaffney estimate (1.5) that $(I - A_t)^M \varphi \in \mathscr{M}_0^{p,2,M,\epsilon}(L)$ for every $\epsilon > 0$. Thus,

$$\begin{aligned} \left|\langle (I - A_t)^M f, \varphi \rangle\right| &= \left|\langle f, (I - A_t)^M \varphi \rangle\right| \\ &\le C_{t,\, r_B \operatorname{dist}(B, x_0)} \|f\|_{(\mathscr{M}_0^{p,2,M,\epsilon}(L))^*} \|\varphi\|_{L^2(B)} V(B)^{1/2-1/p}. \end{aligned} \tag{2.5}$$

Since B was arbitrary, the claim follows. Similarly, $(t^2L)^M e^{-t^2L} f \in L^2_{\mathrm{loc}}(X)$.

In order to define the adapted $\Lambda_L^{\alpha,s}(X)$ spaces, we need one more space. For any $0 < p \le 1$ and $M \in \boldsymbol{N}$, we set

$$\mathscr{E}_{M,p} = \bigcap_{\epsilon>0} \big(\mathscr{M}_0^{p,2,M,\epsilon}(L)\big)^*. \tag{2.6}$$

DEFINITION 2.4. Let L be an operator satisfying (H1)–(H2). For $\alpha \geq 0$ and an integer $s \geq [n\alpha/2]$, an element $\ell \in \mathscr{E}_{M,(\alpha+1)^{-1}}$ is said to belong to $\Lambda_L^{\alpha,s}(X)$ if

$$\left[\frac{1}{\mu(B)^{1+2\alpha}} \int_B \big|(I-(I+r_B^2L)^{-1})^s\ell(x)\big|^2 d\mu(x)\right]^{1/2} \leq C \tag{2.7}$$

where B is any ball in X and C depends only on ℓ. Let $\Re^{\alpha,s}(\ell)$ be the infimum of all C for which (2.7) holds. The norm of ℓ in this space is denoted by

$$\|\ell\|_{\Lambda_L^{\alpha,s}(X)} = \begin{cases} \Re^{\alpha,s}(\ell), & \text{if } \mu(X)=\infty, \\ \Re^{\alpha,s}(\ell) + \big|\int_X |\ell(x)|^2 d\mu(x)\big|, & \text{if } \mu(X)<\infty. \end{cases}$$

In the sequel, we will often write $\mathrm{BMO}_L(X)$ in place of $\Lambda_L^{0,1}(X)$, the adapted space of functions with bounded mean oscillations on X.

In this case the mapping $\ell \to \|\ell\|_{\Lambda_L^{\alpha,s}(X)}$ is a norm. We shall see that this definition is independent of the choice of $M > [n(2-p)/4p]$ (up to "modding out" elements in the null space of the operator L^M, as these are annihilated by $(I-(I+r_B^2L)^{-1})^s$). Compared to the classical definition (see, for example, [**CW2**]), in (2.7) the resolvent $(I+r_B^2L)^{-1}$ plays the role of averaging over the ball, and the power $M > [n(2-p)/4p]$ provides the necessary "L-cancellation".

DEFINITION 2.5. Given $0 < p \leq 1$ and $M > n(2-p)/4p$. Let $f \in (\Lambda_L^{1/p-1,M}(X))^*$. An atomic $(p,2,M)$-representation of f is a series $f = \sum_j \lambda_j a_j$ where $\{\lambda_j\}_{j=0}^\infty \in \ell^p$, each a_j is a $(p,2,M)$-atom, and the sum converges in $L^2(X)$. Set

$$\boldsymbol{H}^p_{L,at,M}(X) := \{f : f \text{ has an atomic } (p,2,M)\text{-representation}\},$$

with the norm given by

$$\|f\|_{\boldsymbol{H}^p_{L,at,M}(X)} = \inf\bigg\{\Big(\sum_{j=0}^\infty |\lambda_j|^p\Big)^{1/p} : f = \sum_{j=0}^\infty \lambda_j a_j \text{ is an atomic } (p,2,M)\text{-representation}\bigg\}.$$

The atomic Hardy space $H^p_{L,at,M}(X)$ is then defined as the completion of $\boldsymbol{H}^p_{L,at,M}(X)$ in $(\Lambda_L^{1/p-1,M}(X))^*$ with respect to this norm.

In this case the mapping $h \to \|h\|_{H^p_{L,at,M}(X)}, 0 < p < 1$, is not a norm and $d(h,g) = \|h-g\|_{H^p_{L,at,M}(X)}$ is a quasi-metric. For $p = 1$, the mapping $h \to \|h\|_{H^p_{L,at,M}(X)}$ is a norm and $H^1_{L,at,M}(X)$ is a Banach space. It also follows easily from the above definitions that

$$H^p_{L,at,M_2}(X) \subseteq H^p_{L,at,M_1}(X) \tag{2.8}$$

whenever $0 < p \leq 1$ and two integers $M_i \in \boldsymbol{N}, i = 1,2$, with $[n(2-p)/4p] \leq M_1 \leq M_2 < \infty$. A basic result concerning these spaces is the following proposition.

Proposition 2.6. *If an operator L satisfies conditions* (H1) *and* (H2)*, then for every $0 < p \leq 1$ and every integer $M \in \boldsymbol{N}$ with $M > [n(2-p)/4p]$,*

$$H^p_{L,at,M}(X) = H^p_{L,at,M_0}(X),$$

where $M_0 = \min\{M \in \boldsymbol{N} : M > [n(2-p)/4p]\}$.

We next describe the notion of a $(p,2,M,\epsilon)$-*molecule* associated to an operator L which satisfies (H1)–(H2).

Definition 2.7. Let $0 < p \leq 1$, $0 < \epsilon$ and $M \in \boldsymbol{N}$. A function $m(x) \in L^2(X)$ is called a $(p,2,M,\epsilon)$-molecule associated to L if there exist a function $b \in \mathscr{D}(L^M)$ and a ball B such that

(i) $m = L^M b$;
(ii) For every $k = 0,1,2,\ldots,M$ and $j = 0,1,2,\ldots$, there holds

$$\big\|(r_B^2 L)^k b\big\|_{L^2(U_j(B))} \leq r_B^{2M} 2^{-j\epsilon} V(2^j B)^{1/2-1/p},$$

where the annuli $U_j(B)$ were defined in (2.4).

Given any $\epsilon > 0$ and $M \geq 1$, it is clear that every $(p,2,M)$-atom is a $(p,2,M,\epsilon)$ molecule. We note that if $\phi \in \mathscr{M}_0^{p,2,M,\epsilon}(L)$ with norm 1, then ϕ is a $(p,2,M,\epsilon)$-molecule adapted to B_0. Conversely, if m is a $(p,2,M,\epsilon)$-molecule adapted to any ball, then $m \in \mathscr{M}_0^{p,2,M,\epsilon}(L)$. Moreover, we have the following result. For its proof, we refer to Theorem 5.1 of [**HLMMY**] for $p = 1$, and Section 3 of [**DL**] for $p < 1$.

PROPOSITION 2.8. *Suppose* $0 < p \leq 1$ *and* $M \geq [n(2-p)/4p]$. *If* m *is* $(p, 2, M, \epsilon)$*-molecule associated to* L, *then* $m \in H^p_{L,at,M}(X)$. *Moreover,* $\|m\|_{H^p_{L,at,M}(X)}$ *is independent of* m.

We recall the following natural analogue of the Fefferman-Stein duality result ([**FS**]).

PROPOSITION 2.9. *The dual of* $H^1_{L,at,M}$ *is the space* $\mathrm{BMO}_L(X)$. *If* $0 < p < 1$ *and* $\alpha = 1/p - 1$, *then for every integer* $M > [n(2-p)/4p]$, $\Lambda^{\alpha,M}_L(X)$ *is the dual of* $H^p_{L,at,M}(X)$.

For the proof of Proposition 2.9 when $p = 1$, we refer the reader to Theorem 2.7 of [**HLMMY**] where it is proved that the dual of $H^1_{L,at,M}$ is the space $\mathrm{BMO}_L(X)$. For $p < 1$, we refer to Section 3, [**DL**].

Consequently, from Proposition 2.9 one may write $\Lambda^\alpha_L(X)$ in place of $\Lambda^{\alpha,\widetilde{M}}_L(X)$ when $\widetilde{M} \in \boldsymbol{N}$ with $\widetilde{M} > [n(2\alpha+1)/2]$ as these spaces are all equivalent, and hence define

$$\Lambda^\alpha_L(X) := \Lambda^{\alpha,\widetilde{M}}_L(X), \quad \widetilde{M} > \left[\frac{n(2\alpha+1)}{4}\right].$$

2.3. A characterization of $H^p_{L,at,M}(X)$ in terms of square function. In Section 2.1, the Hardy spaces $H^p_{L,S_h}(X)$ were defined for $p \geq 1$. Now consider the case $0 < p < 1$. The space $H^p_{L,S_h}(X)$ is defined as the completion of

$$\left\{ f \in H^2(X) : \|S_h f\|_{L^p(X)} < \infty \right\}$$

in the norms given by the L^p norm of the square function; i.e.,

$$\|f\|_{H^p_{L,S_h}(X)} = \|S_h f\|_{L^p(X)}, \quad 0 < p < 1.$$

Then the "square function" and "atomic" H^p spaces are equivalent if the parameter $M > [n(2-p)/4p]$. In fact, we have the following result.

PROPOSITION 2.10. *Suppose* $0 < p \leq 1$ *and* $M > [n(2-p)/4p]$. *Then we have* $H^p_{L,at,M}(X) = H^p_{L,S_h}(X)$ *for* $0 < p \leq 1$. *Moreover,*

$$\|f\|_{H^p_{L,at,M}} \approx \|f\|_{H^p_{L,S_h}},$$

where the implicit constants depend only on M, n *and* N *in* (1.2) *and* (1.3) *only.*

PROOF. For the proof, see Section 3, [**DL**]. □

It follows from Proposition 2.6 that one may write $H^p_{L,at}(X)$ in place of $H^p_{L,at,M}(X)$, when $M > [n(2-p)/4p]$, as these spaces are all equivalent.

DEFINITION 2.11. The Hardy space $H^p_L(X), 0 < p \leq 1$, is the space

$$H^p_L(X) := H^p_{L,S_h}(X) = H^p_{L,at}(X) := H^p_{L,at,M}(X), \quad M > \left[\frac{n(2-p)}{4p}\right].$$

3. A criterion for boundedness of spectral multipliers on $H^p_L(X)$.

As mentioned before, we shall give a criterion that allows us to derive estimates on Hardy spaces $H^p_L(X)$. This generalizes the classical Calderón-Zygmund theory and we would like to emphasize that the conditions imposed involve the operator and its action on some functions but not its kernel.

The main result of this section is the following theorem.

THEOREM 3.1. *Let L be a non-negative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimate* (1.5). *Let m be a bounded Borel function. Suppose that $0 < p \leq 1$ and $M > n(2-p)/4p$. Assume that there exist constants $D > n(1/p - 1/2)$ and $C > 0$ such that for every $j = 2, 3 \ldots,$*

$$\left\|m(L)(1-e^{-r_B^2 L})^M f\right\|_{L^2(U_j(B))} \leq C 2^{-jD} \|f\|_{L^2(B)} \tag{3.1}$$

for any ball B with radius r_B and for all $f \in L^2(X)$ with supp $f \subset B$. Then the operator $m(L)$ extends to a bounded operator on $H^p_L(X)$. More precisely, there exists a constant $C > 0$ such that for all $f \in H^p_L(X)$

$$\|m(L)f\|_{H^p_L(X)} \leq C\|f\|_{H^p_L(X)}. \tag{3.2}$$

PROOF. We first note that since $H^p_L(X) \cap L^2(X)$ is dense in $H^p_L(X)$, we can define $m(L)$ on $H^p_L(X) \cap L^2(X)$. Once we can show $H^p_L(X)$ boundedness of $m(L)$ on this dense set, the operator $m(L)$ can be extended on $H^p_L(X)$.

To prove Theorem 3.1, we claim that there exists a constant $\epsilon = D - n(1/p - 1/2) > 0$ such that, for every $(p, 2, 2M)$-atom a associated to a ball B of X, $m(L)a$ is a constant multiple of a $(p, 2, M, \epsilon)$-molecule associated to the ball B. The conclusion of the theorem is then an immediate consequence of Proposition 2.9 and L^2-boundedness of $m(L)$.

Let us prove the claim. By the definition of $(p, 2, 2M)$-atom, there exists a function $b \in \mathscr{D}(L^{2M})$ such that $a = L^{2M}b$ satisfies (ii) and (iii) in Definition 2.3.

By spectral theory, one may write

$$m(L)a = L^M\big[m(L)L^M b\big]. \tag{3.3}$$

From the definition of $(p, 2, M, \epsilon)$-molecule, it remains to show that for $k = 0, 1, \dots, M$ and for all $j = 1, 2, \dots,$

$$\big\|\big(r_B^2 L\big)^k\big[m(L)L^M b\big]\big\|_{L^2(U_j(B))} \le Cr_B^{2M} 2^{-j\epsilon} V(2^j B)^{1/2-1/p}. \tag{3.4}$$

We now prove estimate (3.4). Note that $m(L)$ is bounded on $L^2(X)$. For every $k = 0, 1, \dots, M$, it follows from (iii) in Definition 2.3 that $\|L^{M+k}b\|_{L^2(B)} \le Cr_B^{2(M-k)} V(B)^{1/2-1/p}$. We can write for $j = 0, 1, 2,$

$$\begin{aligned}\big\|\big(r_B^2 L\big)^k\big[m(L)L^M b\big]\big\|_{L^2(U_j(B))} &\le r_B^{2k}\big\|m(L)L^{M+k}b\big\|_{L^2(X)}\\ &\le Cr_B^{2k}\big\|L^{M+k}b\big\|_{L^2(B)}\\ &\le Cr_B^{2M} V(B)^{1/2-1/p}.\end{aligned}$$

Assume now that for $j \ge 3$. Following (8.7) and (8.8) in [**HM**], we write

$$\begin{aligned} I &= 2\bigg(r_B^{-2}\int_{r_B}^{\sqrt{2}r_B} s\,ds\bigg)\cdot I\\ &= 2r_B^{-2}\int_{r_B}^{\sqrt{2}r_B} s(I - e^{-s^2L})^M ds + \sum_{j=1}^{M} C_{j,M} r_B^{-2}\int_{r_B}^{\sqrt{2}r_B} se^{-js^2L}ds, \end{aligned} \tag{3.5}$$

where $C_{j,M} \in \boldsymbol{R}$ are some constants depending on j, M only. However, $\partial_s e^{-js^2L} = -2jsLe^{-js^2L}$ and therefore,

$$\begin{aligned} 2jL\int_{r_B}^{\sqrt{2}r_B} se^{-js^2L}ds &= e^{-jr_B^2L} - e^{-2jr_B^2L} = e^{-jr_B^2L}(I - e^{-jr_B^2L})\\ &= e^{-jr_B^2L}(I - e^{-r_B^2L})\sum_{i=0}^{j-1} e^{-ir_B^2L}. \end{aligned} \tag{3.6}$$

Applying the procedure outline in (3.5)–(3.6) M times, we have for every $x \in X$,

$$
\begin{aligned}
&\big(r_B^2 L\big)^k \big[m(L)L^M b(x)\big] \\
&\quad= 2^M r_B^{2k}\bigg(r_B^{-2}\int_{r_B}^{\sqrt{2}r_B} s(I-e^{-s^2L})^M ds\bigg)^M m(L)(L^{M+k}b)(x) \\
&\qquad+\sum_{\ell=1}^{M} r_B^{2(k-\ell)}L^{-\ell}(I-e^{-r_B^2L})^\ell\bigg(r_B^{-2}\int_{r_B}^{\sqrt{2}r_B} s(I-e^{-s^2L})^M ds\bigg)^{M-\ell} \\
&\qquad\times\sum_{j=1}^{(2M-1)\ell} C(\ell,j,M)e^{-jr_B^2L}m(L)(L^{M+k}b)(x) \\
&\quad=\sum_{\ell=0}^{M} G^{(k)}_{\ell,M,r_B}(x)
\end{aligned}
\tag{3.7}
$$

for some constants $C(\ell,j,M)\in \boldsymbol{R}$.

Now, let us estimate $\{G^{(k)}_{\ell,M,r_B}\}_{\ell=0}^M$ by examining ℓ in three cases.

Subcase (1.1). $\ell=0$ and $k=0,1,2,\dots,M$.

First, we set $P_{M,r_B}(L)=r_B^{-2}\int_{r_B}^{\sqrt{2}r_B} s(I-e^{-s^2L})^M ds$. From (3.7), we have

$$
\begin{aligned}
G^{(k)}_{0,M,r_B}(x) &= \sum_{i=0}^{\infty} 2^M r_B^{2k}\int_{r_B}^{\sqrt{2}r_B}\Big(\frac{s}{r_B}\Big)^2 P^{M-1}_{M,r_B}(L) \\
&\qquad\times\Big(\big[m(L)(1-e^{-s^2L})^M(L^{M+k}b)\big]\chi_{U_i(B)}\Big)(x)\frac{ds}{s}.
\end{aligned}
$$

It follows from condition (1.5) that the operator $P^{M-1}_{M,r_B}(L)$ satisfies L^2 off-diagonal estimates, and there exist some constants c, $C>0$ such that for every $i,j=0,1,2,\dots$

$$
\begin{aligned}
\big\|P^{M-1}_{M,r_B}(L)f\big\|_{L^2(U_j(B))} &\le Ce^{-\operatorname{dist}(U_j(B),U_i(B))^2/cr_B^2}\|f\|_{L^2(U_i(B))} \\
&\le Ce^{-c2^{|j-i|}}\|f\|_{L^2(U_i(B))},
\end{aligned}
$$

where in the last inequality we have used the fact that $\operatorname{dist}(U_j(B),U_i(B))\ge C2^{|j-i|}r_B$ for every $j,i\ge 0$. Hence,

$$
\begin{aligned}
&\big\|G^{(k)}_{0,M,r_B}\big\|_{L^2(U_j(B))}\\
&\quad\le 2^M r_B^{2k}\sum_{i=0}^{\infty}\int_{r_B}^{\sqrt{2}r_B}\Big(\frac{s}{r_B}\Big)^2\\
&\qquad\times\big\|P^{M-1}_{M,r_B}(L)\big(\big[m(L)(1-e^{-s^2L})^M(L^{M+k}b)\big]\chi_{U_i(B)}\big)\big\|_{L^2(U_j(B))}\frac{ds}{s}\\
&\quad\le Cr_B^{2k}\sum_{i=0}^{\infty}e^{-c2^{|j-i|}}\int_{r_B}^{\sqrt{2}r_B}\big\|m(L)(1-e^{-s^2L})^M(L^{M+k}b)\big\|_{L^2(U_i(B))}\frac{ds}{s}.
\end{aligned}
$$

Note that for every $s\in[r_B,\sqrt{2}r_B]$, we have that $U_0(B)=B\subset B(x_B,s)$ and $U_i(B)\subset U_{i-1}(B(x_B,s))\cup U_i(B(x_B,s))$ for $i\ge 1$. Those, in combination with the condition (3.1), give for every $s\in[r_B,\sqrt{2}r_B]$,

$$
\begin{aligned}
\big\|m(L)(1-e^{-s^2L})^M(L^{M+k}b)\big\|_{L^2(U_i(B))}&\le C2^{-iD}\big\|L^{M+k}b\big\|_{L^2(B)}\\
&\le C2^{-iD}r_B^{2(M-k)}V(B)^{1/2-1/p}. \qquad (3.8)
\end{aligned}
$$

From the doubling property (1.2) again, we have that $V(2^jB)\le C2^{jn}V(B)$. This, together with (3.8), shows

$$
\begin{aligned}
\big\|G^{(k)}_{0,M,r_B}\big\|_{L^2(U_j(B))}&\le Cr_B^{2k}\sum_{i=0}^{\infty}e^{-c2^{|j-i|}}2^{-iD}r_B^{2(M-k)}V(B)^{1/2-1/p}\\
&\le Cr_B^{2k}2^{-jD}r_B^{2(M-k)}V(B)^{1/2-1/p}\\
&\le Cr_B^{2M}2^{-j(D-n(1/p-1/2))}V(2^jB)^{1/2-1/p}. \qquad (3.9)
\end{aligned}
$$

Subcase (1.2). $\ell=M$ and $k=0,1,2,\dots,M$. In those cases, one has

$$
\begin{aligned}
&\big|G^{(k)}_{M,M,r_B}(x)\big|\\
&\quad\le C\sum_{u=1}^{(2M-1)M}r_B^{2(k-M)}\sum_{i=0}^{\infty}\big|e^{-ur_B^2L}\big(\big[m(L)(I-e^{-r_B^2L})^M(L^kb)\big]\chi_{U_i(B)+}\big)(x)\big|.
\end{aligned}
$$

It follows from the condition (1.5) that the operators $\{e^{-ur_B^2L}\}_{u=1}^{(2M-1)M}$ satisfy L^2 off-diagonal estimate, and then

$$
\begin{aligned}
\big\|G^{(k)}_{M,M,r_B}\big\|_{L^2(U_j(B))} &\leq Cr_B^{2(k-M)}\sum_{i=0}^{\infty} e^{-c2^{|j-i|}}\big\|m(L)(I-e^{-r_B^2L})^M(L^kb)\big\|_{L^2(U_i(B))}\\
&\leq Cr_B^{2(k-M)}\sum_{i=0}^{\infty} e^{-c2^{|j-i|}}2^{-iD}\|L^kb\|_{L^2(B)}\\
&\leq Cr_B^{2(k-M)}2^{-jD}r_B^{2(2M-k)}V(B)^{1/2-1/p}\\
&\leq Cr_B^{2M}2^{-j(D-n(1/p-1/2))}V(2^jB)^{1/2-1/p}. \qquad (3.10)
\end{aligned}
$$

Subcase (1.3). $\ell = 1,2,\dots,M-1$ and $k=0,1,2,\dots,M$. In those cases, one has

$$
\begin{aligned}
\big|G^{(k)}_{\ell,M,r_B}(x)\big| \leq Cr_B^{2(k-M)} &\sum_{j=1}^{(2M-1)\ell}\sum_{i=0}^{\infty}\int_{r_B}^{\sqrt{2}r_B} \big|\big(r_B^2L\big)^{M-\ell}e^{-jr_B^2L}\big(I-e^{-r_B^2L}\big)^{\ell}\\
&\times P^{M-\ell-1}_{M,r_B}(L)\big(\big[m(L)(1-e^{-s^2L})^M(L^kb)\big]\chi_{U_i(B)}\big)(x)\big|\frac{ds}{s}.
\end{aligned}
$$

Then we can use an argument similar to Subcase (1.1) again to obtain

$$
\big\|G^{(k)}_{\ell,M,r_B}\big\|_{L^2(U_j(B))} \leq Cr_B^{2M}2^{-j(D-n(1/p-1/2))}V(2^jB)^{1/2-1/p}.
$$

This, together with estimates (3.9) and (3.10), gives the desired estimate (3.4). The proof of Theorem 3.1 is complete. $\square$

A natural question about Theorem 3.1 is how strong the assumption (3.1) is, and its relation with the regularity condition on the kernel.

Proposition 3.2. *Let L be a non-negative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimate* (1.5). *Suppose $0<p\leq 1$ and $M>n(2-p)/4p$. Assume that the operator $m(L)(I-e^{-r_B^2L})^M$ has an associated kernel $K_{m(L)(I-e^{-r_B^2L})^M}(x,y)$ which satisfies for every $j=2,3,\dots$,*

$$
\int_{2^jr_B\leq d(x,y)<2^{j+1}r_B}\big|K_{m(L)(I-e^{-r_B^2L})^M}(x,y)\big|^2d\mu(x)\leq C\frac{2^{-2jD}}{V(y,r_B)},\qquad y\in X, \quad (3.11)
$$

where $D>n(1/p-1/2)$. Then $m(L)(I-e^{-r_B^2L})^M$ satisfies condition (3.1) *of*

Theorem 3.1. *More precisely, there exists a constant $C > 0$ such that for every $j = 2, 3 \dots,$*

$$\big\|m(L)(1 - e^{-r_B^2 L})^M f\big\|_{L^2(U_j(B))} \leq C 2^{-jD} \|f\|_{L^2(B)}$$

for any ball B with radius r_B and for all $f \in L^2(X)$ with supp $f \subset B$.

Proof. Let $f \in L^2(X)$ with supp $f \subset B$. Note that for every $y \in B$, we have that $V(y, r_B) \sim V(x_B, r_B)$. The Cauchy-Schwarz inequality, together with the condition (3.11), gives for every $j = 2, 3, \dots,$

$$\begin{aligned}
&\big\|m(L)(I - e^{-r_B^2 L})^M f\big\|_{L^2(U_j(B))} \\
&\quad = \bigg\{ \int_{U_j(B)} \bigg| \int_B K_{m(L)(I - e^{-r_B^2 L})^M}(x, y) f(y) d\mu(y) \bigg|^2 d\mu(x) \bigg\}^{1/2} \\
&\quad \leq \|f\|_{L^2(B)} \bigg\{ \int_B \int_{2^{j-2} r_B \leq d(x,y) < 2^{j+1} r_B} \big|K_{m(L)(I - e^{-r_B^2 L})^M}(x, y)\big|^2 d\mu(x) d\mu(y) \bigg\}^{1/2} \\
&\quad \leq C \|f\|_{L^2(B)} \sum_{i=-2}^{0} 2^{-(j-i)D} \bigg\{ \int_B V(y, r_B)^{-1} d\mu(y) \bigg\}^{1/2} \\
&\quad \leq C 2^{-jD} \|f\|_{L^2(B)}.
\end{aligned}$$

This proves Proposition 3.2. $\square$

4. Proof of Theorem 1.1.

Let L be an operator satisfying (H1)–(H2). Recall that, if L is a non-negative, self-adjoint operator on $L^2(X)$, and $E_L(\lambda)$ denotes its spectral decomposition, then for every bounded Borel function $F : [0, \infty) \to \boldsymbol{C}$, one defines the operator $F(L) : L^2(X) \to L^2(X)$ by the formula

$$F(L) := \int_0^\infty F(\lambda) dE_L(\lambda). \tag{4.1}$$

In particular, the operator $\cos(t\sqrt{L})$ is then well-defined on $L^2(X)$. Moreover, it follows from Theorem 3.4 of [**CS**] that there exists a constant c_0 such that the Schwartz kernel $K_{\cos(t\sqrt{L})}$ of $\cos(t\sqrt{L})$ satisfies

$$\operatorname{supp} K_{\cos(t\sqrt{L})}(x,y) \subset \big\{(x,y) \in X \times X : d(x,y) \leq c_0 t\big\}. \tag{4.2}$$

More precisely, we have the following result.

PROPOSITION 4.1. *Let L be a non-negative self-adjoint operator acting on $L^2(X)$. Then the finite speed propagation property* (4.2) *and Davies-Gaffney estimate* (1.5) *are equivalent.*

PROOF. For the proof, we refer the reader to Theorem 2 in [**S2**] and Theorem 3.4 in [**CS**]. See also [**CGT**] and [**T**]. □

By the Fourier inversion formula, whenever F is an even bounded Borel function with $\hat{F} \in L^1(\boldsymbol{R})$, we can represent $F(\sqrt{L})$ in terms of $\cos(t\sqrt{L})$. More specifically, by recalling (4.1) we have

$$F(\sqrt{L}) = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{F}(t) \cos(t\sqrt{L})\, dt,$$

which, when combined with (4.2), gives

$$K_{F(\sqrt{L})}(x,y) = (2\pi)^{-1} \int_{|t| \geq c_0^{-1} d(x,y)} \hat{F}(t) K_{\cos(t\sqrt{L})}(x,y)\, dt, \tag{4.3}$$

which will be often used in the sequel.

PROOF OF THEOREM 1.1. We begin to prove Theorem 1.1 by using Theorem 3.1. Observe that m satisfies the condition (1.7) if and only if the function $\lambda \to m(\lambda^2)$ satisfies the same property. For this reason, we shall consider $m(\sqrt{L})$ rather than $m(L)$. Notice that $m(\lambda) = m(\lambda) - m(0) + m(0)$ and hence

$$m\big(\sqrt{L}\big) = (m(\cdot) - m(0))\big(\sqrt{L}\big) + m(0)I.$$

Replacing m by $m - m(0)$, we may assume in the sequel that $m(0) = 0$. Let ϕ be a function as in (1.6). We have for all $\lambda > 0$,

$$m(\lambda) = \sum_{\ell=-\infty}^{\infty} \phi(2^{-\ell}\lambda) m(\lambda) = \sum_{\ell=-\infty}^{\infty} m_\ell(\lambda). \tag{4.4}$$

This decomposition implies that the sequence $\sum_{\ell=-N}^{N} m_\ell(\sqrt{L})$ converges strongly in $L^2(X)$ to $m(\sqrt{L})$. We shall prove that $\sum_{\ell=-N}^{N} m_\ell(\sqrt{L})$ is bounded on $H_L^p(X)$

with its bound independent of N. This together with the strong convergence in $L^2(X)$ and the fact that $H^p_L(X)\cap L^2(X)$ is dense in $H^p_L(X)$ imply the theorem.

We now fix $s > n(1/p-1/2)$ in Theorem 1.1 and let $M\in \boldsymbol{N}$ such that $M > s/2$. For every $\ell\in \boldsymbol{Z}$ and $r>0$, we set for $\lambda>0$

$$F_{r,M}(\lambda) = m(\lambda)(1-e^{-(r\lambda)^2})^M, \tag{4.5}$$

$$F^{\ell}_{r,M}(\lambda) = m_\ell(\lambda)(1-e^{-(r\lambda)^2})^M. \tag{4.6}$$

We may write

$$m\big(\sqrt{L}\big)(1-e^{-r^2L})^M = F_{r,M}\big(\sqrt{L}\big) = \lim_{N\to\infty}\sum_{\ell=-N}^{N} F^{\ell}_{r,M}\big(\sqrt{L}\big). \tag{4.7}$$

Fix a ball $B\subset X$. For every $b\in L^2(X)$ with supp $b\in B$, we claim that for every $\ell\in\boldsymbol{Z}$ and every $j\geq 3$,

$$\big\|F^{\ell}_{r_B,M}\big(\sqrt{L}\big)b\big\|_{L^2(U_j(B))} \leq CC_{\phi,s}2^{-sj}\big(2^\ell r_B\big)^{-s}\min\big\{1,(2^\ell r_B)^{2M}\big\}\|b\|_{L^2(B)}. \tag{4.8}$$

This, in combination with (4.7), shows that for every $j\geq 3$,

$$\begin{aligned}
&\big\|m\big(\sqrt{L}\big)(1-e^{-r_B^2L})^Mb\big\|_{L^2(U_j(B))}\\
&\quad\leq CC_{\phi,s}2^{-sj}\lim_{N\to\infty}\sum_{\ell=-N}^{N}\big(2^\ell r_B\big)^{-s}\min\big\{1,(2^\ell r_B)^{2M}\big\}\|b\|_{L^2(B)}\\
&\quad\leq CC_{\phi,s}2^{-sj}\Bigg(\sum_{\ell:\,2^\ell r_B>1}\big(2^\ell r_B\big)^{-s}+\sum_{\ell:\,2^\ell r_B\leq 1}\big(2^\ell r_B\big)^{2M-s}\Bigg)\|b\|_{L^2(B)}\\
&\quad\leq CC_{\phi,s}2^{-sj}\|b\|_{L^2(B)}
\end{aligned}\tag{4.9}$$

since $s>n(1/p-1/2)$ and $M>s/2$. We note that the last inequality follows from the convergence of power series with common ratio $1/2$.

Thus, the assumptions of Theorem 3.1 are satisfied, and the conclusion of Theorem 1.1 is obtained.

It remains to prove our claim (4.8). First, we record a useful auxiliary result. For a proof, see pp. 237–238 in [**H**] (see also [**A**]).

LEMMA 4.2. *Assume that the function $f\in C_k(\boldsymbol{R})$ with compact support.*

Let $A = k + \epsilon$ with $\epsilon \in (0,1)$ and

$$\mathscr{D}_A(f) = \sup\left\{ \frac{|f^{(k)}(t+h) - f^{(k)}(t)|}{h^{\epsilon}} : h > 0,\ t \in \boldsymbol{R} \right\}.$$

Then for every $\lambda > 0$ there is an even bounded integrable function $\psi_\lambda \in C(\boldsymbol{R})$ such that for all $t \in \boldsymbol{R}$

$$\operatorname{supp}(\widehat{\psi_\lambda}) \subseteq [-\lambda, \lambda] \quad \text{and} \quad \big|f(t) - (f * \psi_\lambda)(t)\big| \le C \mathscr{D}_A(f) \lambda^{-A},$$

where C is a constant that depends only on k.

Back to the proof of estimate (4.8). In the sequel, without loss of generality we assume that the constant c_0 in (4.2) equals 1, i.e., $c_0 = 1$. From the compact support property of ϕ, it follows that for every $\ell \in \boldsymbol{Z}$, $\operatorname{supp} F^{\ell}_{r_B,M}(\lambda) \subseteq (2^{\ell-2}, 2^{\ell})$. Fix $\ell \in \boldsymbol{Z}$. For each j, let $\lambda_j = 2^{j-1} r_B$ and use Lemma 4.2 with $f = F^{\ell}_{r_B,M}$ and $k = [n(1/p - 1/2)]$. This insures the existence of an even, bounded, integrable function $\psi_j \in C(\boldsymbol{R})$ such that $\operatorname{supp}(\widehat{\psi_j}) \subseteq [-2^{j-1} r_B, 2^{j-1} r_B]$, and for all $\lambda \in \boldsymbol{R}$,

$$F^{\ell,j}_{r_B,M}(\lambda) = F^{\ell}_{r_B,M}(\lambda) - \big(F^{\ell}_{r_B,M} * \psi_j\big)(\lambda)$$

satisfying

$$\begin{aligned} \big|F^{\ell,j}_{r_B,M}(\lambda)\big| &\le C \mathscr{D}_s\big(F^{\ell}_{r_B,M}(t)\big) \cdot \big(2^j r_B\big)^{-s} \\ &\le C C_{\phi,s} 2^{-sj} \min\big\{1, (2^{\ell} r_B)^{2M}\big\} \big(2^{\ell} r_B\big)^{-s}. \end{aligned} \tag{4.10}$$

Observe that the Fourier transforms of $F^{\ell,j}_{r_B,M}(\lambda)$ and $F^{\ell}_{r_B,M}(\lambda)$ agree on $\{\xi \in \boldsymbol{R} : |\xi| \ge 2^{j-1} r_B\}$. From this and (4.3), it follows that the kernels of $F^{\ell}_{r_B,M}(\sqrt{L})$ and $F^{\ell,j}_{r_B,M}(\sqrt{L})$ agree on the set $\{(x,y) : d(x,y) \ge 2^{j-1} r_B\}$. Hence, for each j, we have

$$\begin{aligned} \big\|F^{\ell}_{r_B,M}(\sqrt{L}) b\big\|_{L^2(U_j(B))} &= \big\|F^{\ell,j}_{r_B,M}(\sqrt{L}) b\big\|_{L^2(U_j(B))} \\ &\le \big\|F^{\ell,j}_{r_B,M}\big\|_{L^\infty(\boldsymbol{R})} \|b\|_{L^2(B)} \\ &\le C C_{\phi,s} 2^{-sj} \big(2^{\ell} r_B\big)^{-s} \min\big\{1, (2^{\ell} r_B)^{2M}\big\} \|b\|_{L^2(B)}. \end{aligned}$$

This proves our claim (4.8). Hence, the proof of Theorem 1.1 is complete. □

In the next corollary, we give endpoint estimates for imaginary powers of self-adjoint operators on Hardy spaces $H^p_L(X)$.

Corollary 4.3. *Let L be a non-negative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimate* (1.5). *Then for every $\gamma \in \boldsymbol{R}$, the operator $L^{i\gamma}$ is bounded on $H^p_L(X)$ with the norm*

$$\|L^{i\gamma}\|_{H^p_L(X)\to H^p_L(X)} \le C_\epsilon(1+|\gamma|)^{n(1/p-1/2)+\epsilon} \tag{4.11}$$

for every $\epsilon > 0$, and C_ϵ is a constant independent of γ.

Hence by interpolation and duality, $L^{i\gamma}$ extends to a bounded operator on $H^p_L(X)$ for all $0 < p < \infty$.

Proof. We apply Theorem 1.1 with $m(\lambda) = \lambda^{i\gamma}$. It can be verified that for $s > n(1/p-1/2)$,

$$\sup_{t>0} \|\phi(\cdot)m(t\cdot)\|_{C^s} \le C(1+|\gamma|)^s.$$

Then the operator $L^{i\gamma}$ is bounded on $H^p_L(X)$ with

$$\|L^{i\gamma}\|_{H^p_L(X)\to H^p_L(X)} \le C_\epsilon(1+|\gamma|)^{n(1/p-1/2)+\epsilon} \tag{4.12}$$

for all $\epsilon > 0$. This and fact that $\|L^{i\gamma}\|_{L^2(X)\to L^2(X)} \le 1$ imply by the Marcinkiewicz interpolation theorem that $L^{i\gamma}$ extends to a bounded operator on $H^p_L(X)$ for all $0 < p < \infty$ with norm bounded by $C_{p,\epsilon}$. This proves Corollary 4.3. □

5. Spectral multipliers and weights.

In this section we shall study weighted norm inequalities for spectral multipliers associated to non-negative self-adjoint operators, under off-diagonal estimates on the semigroup $\{e^{-tL}\}_{t>0}$. First, let us review some classical classes of weights. A weight w is a non-negative locally integrable function. We say that $w \in A_p$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset X$

$$\left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w^{1-p'}(x)dx\right)^{p-1} \le C.$$

The reverse Hölder classes are defined in the following way: $w \in RH_q$, $1 < q < \infty$, if there is a constant C such that for every ball $B \subset X$

$$\Big(\frac{1}{|B|}\int_B w^q(x)dx\Big)^{1/q} \le C\Big(\frac{1}{|B|}\int_B w(x)dx\Big).$$

5.1. Singular integrals and weights.

The following theorem is Theorem 3.7, [**AM**].

THEOREM 5.1. *Let $1 \le p_0 < q_0 \le \infty$. Let T be a sublinear operator acting on $L^{p_0}(X)$, $\{A_r\}_{r>0}$ a family of operators acting from a subspace $\mathscr{D}$ of $L^{p_0}(X)$ into $L^{p_0}(X)$. Assume that*

$$\Big(\frac{1}{V(B)}\int_B \big|T(I - A_{r_B})f\big|^{p_0} d\mu\Big)^{1/p_0} \le \sum_{j\ge 0}\alpha_j\Big(\frac{1}{V(2^jB)}\int_{2^jB} |f|^{p_0} d\mu\Big)^{1/p_0} \tag{5.1}$$

and

$$\Big(\frac{1}{V(B)}\int_B \big|TA_{r_B}f\big|^{q_0} d\mu\Big)^{1/q_0} \le \sum_{j\ge 0}\alpha_j\Big(\frac{1}{V(2^jB)}\int_{2^jB} |Tf|^{p_0} d\mu\Big)^{1/p_0} \tag{5.2}$$

for all $f \in \mathscr{D}$, and all ball B with radius r_B, for some α_j with $\sum_{j\ge 0}\alpha_j < \infty$. Then for all $p_0 < p < q_0$ and $w \in A_{p/p_0} \cap RH_{(q_0/p)'}$, there exists a constant C such that

$$\|Tf\|_{L^p(w)} \le C\|f\|_{L^p(w)}. \tag{5.3}$$

Note that if w is any given weight so that w, $w^{1-p'} \in L^1_{\mathrm{loc}}(X)$, then a given linear operator T is bounded on $L^p(w)$, $1 < p < \infty$, if and only if its adjoint T^* (with respect to $d\mu$) is bounded on $L^p(w^{1-p'})$. Therefore,

$$T : L^p(w) \to L^p(w), \quad \text{for all } w \in A_{p/p_0} \cap RH_{(q_0/p)'}$$

if and only if

$$T^* : L^{p'}(w) \to L^{p'}(w), \quad \text{for all } w \in A_{p'/q_0'} \cap RH_{(p_0'/p')'}.$$

5.2. Spectral multipliers, off-diagonal estimates and weights.
In this section, we assume the following condition.

(H3) Suppose $2 < q_0 \leq \infty$. Assume that the analytic semigroup e^{-tL} generated by L satisfies L^2-L^{q_0} off-diagonal estimates: there exist coefficients $\{\alpha_j\}_{j\geq 0}$ satisfying $\sum_{j\geq 0} \alpha_j < \infty$ such that, for all balls B and for all function $f \in L^2(X)$,

$$\left(\frac{1}{V(B)} \int_B \left|e^{-r_B^2 L}(f)\right|^{q_0} d\mu\right)^{1/q_0} \leq \sum_{j\geq 0} \alpha_j \left(\frac{1}{V(2^j B)} \int_{2^j B} \left|f\right|^2 d\mu\right)^{1/2}. \tag{5.4}$$

Then the following result holds.

Theorem 5.2. *Let L be a non-negative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimate* (1.5) *and assumption* (H3)*. Let ϕ be a non-negative C_0^∞ function satisfying* (1.6)*, and assume that the bounded measurable function $m : [0,\infty] \to \boldsymbol{C}$ satisfies*

$$C_{\phi,s} = \sup_{t>0} \|\phi(\cdot)m(t\cdot)\|_{C^s} + |m(0)| < \infty \tag{5.5}$$

for some $s > n/2$.

(i) *If $q_0' < p < 2$ and $w \in A_{p/q_0'} \cap RH_{(2/p)'}$; or*
(ii) *$2 < p < q_0$ and $w \in A_{p/2} \cap RH_{(q_0/p)'}$,*

then there exists a constant C such that

$$\|m(L)f\|_{L^p(w)} \leq C\|f\|_{L^p(w)}. \tag{5.6}$$

Proof. We first note that (ii) can be obtained from (i) by a standard duality argument. To prove (i), let us fix a p such that $2 < p < q_0$ and $w \in A_{p/2} \cap RH_{(q_0/p)'}$. Estimate (5.6) follows from Theorem 5.1, applied to $Tf = m(L)f$ and $A_r = I - (I - e^{-r^2 L})^M$ with $M \in \boldsymbol{N}$ and $M > s/2$. Note first that assumption (H3) implies condition (5.2). To verify (5.1), it suffices to show that there exist coefficients $\{\alpha_j\}_{j\geq 0}$ satisfying $\sum_{j\geq 0} \alpha_j < \infty$ such that for all balls B,

$$\left(\frac{1}{V(B)} \int_B \left|m(\sqrt{L})(I - e^{-r_B^2 L})^M f(y)\right|^2 dy\right)^{1/2} \leq \sum_{j\geq 0} \alpha_j \left(\frac{1}{V(2^j B)} \int_{2^j B} |f|^2 d\mu\right)^{1/2} \tag{5.7}$$

for all $f \in L_c^\infty(X)$ (*i.e.* bounded with compact support).

The proof of (5.7) is almost identical to the one of Theorem 1.1. For any ball B, we shall use the following decomposition

$$f = \sum_{j\geq 0} f_j, \qquad f_j = f\chi_{U_j(B)},$$

where $U_j(B)$ were defined in (2.4). Note that $s > n/2$ and $M > s/2$. We use an argument in estimates (4.6), (4.7), (4.8) and (4.9) in the proof of Theorem 1.1 to obtain

$$\begin{aligned}
&\left(\frac{1}{V(B)}\int_B \big|m(\sqrt{L})(I-e^{-r_B^2 L})^M f(y)\big|^2 dy\right)^{1/2} \\
&\quad \leq V(B)^{-1/2}\sum_{j\geq 0}\big\|m(\sqrt{L})(I-e^{-r_B^2 L})^M f_j\big\|_{L^2(B)} \\
&\quad \leq CC_{\phi,s}V(B)^{-1/2}\sum_{j\geq 0}2^{-sj}\left(\sum_\ell \left(2^\ell r_B\right)^{-s}\min\left\{1,(2^\ell r_B)^{2M}\right\}\right)\|f_j\|_{L^2(X)} \\
&\quad \leq C\sum_{j\geq 0}2^{-sj}\left(\frac{V(2^{j+1}B)}{V(B)}\right)^{1/2}\left(\frac{1}{V(2^{j+1}B)}\int_{2^{j+1}B}|f|^2 d\mu\right)^{1/2} \\
&\quad \leq C\sum_{j\geq 0}2^{-(s-n/2)j}\left(\frac{1}{V(2^j B)}\int_{2^j B}|f|^2 d\mu\right)^{1/2}.
\end{aligned}$$

The desired estimate (5.7) is obtained for $\alpha_j = 2^{-(s-n/2)j}$ with $\sum_{j\geq 0} 2^{-(s-n/2)j} < \infty$. This concludes the proof of Theorem 5.1. □

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