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Sublinear elliptic equations with singular coefficients on the boundary

Dedicated to Professor Shinnosuke Oharu on the occasion of his 70th birthday

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Abstract. A sublinear elliptic equation whose coefficient is singular on the boundary is studied in any bounded domain Ω under the zero Dirichlet boundary condition. It is proved that the equation has a unique positive solution and infinitely many sign-changing solutions which belong to $C^1(\overline{\Omega})$ or $C^2(\overline{\Omega})$. Moreover, it is proved that the solutions have the higher order regularity corresponding to the smoothness of the coefficient.

1. Introduction.

We study the existence of positive solutions and infinitely many solutions without positivity for the sublinear elliptic equation under the Dirichlet condition,

$$\begin{cases} -\Delta u = h(x)|u|^p \operatorname{sgn} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain in $\boldsymbol{R}^N$ with smooth boundary $\partial\Omega$ and the nonlinear term is sublinear, i.e., $0 < p < 1$ and $h(x)$ is a measurable function in Ω. We study (1.1) when $h(x)$ has singularity on the boundary. However, we expect the existence of regular solutions of class $C^1(\overline{\Omega})$ or $C^2(\overline{\Omega})$. The coefficient $h(x)$ may diverge to $\pm\infty$ as x tends to the boundary but $u(x)$ converges to zero owing to the Dirichlet boundary condition. Then $h(x)|u|^p \operatorname{sgn} u$ can be bounded or may belong to a suitable $L^q(\Omega)$ or to a Hölder space. Accordingly, a solution lies in $C^1(\overline{\Omega})$ or $C^2(\overline{\Omega})$. A study in this direction has been obtained by Senba, Ebihara and Furusho [**11**] and by Hashimoto and Ôtani [**6**], [**7**]. They studied the problem in the case where $\Omega = B$ is a unit ball and $h(x)$ has a power singularity $h(x) = (1-|x|)^{-a}$.

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Then they proved the existence of positive radial solutions in $C^2(B) \cap C^1(\overline{B})$. In this case, (1.1) is reduced to the two point boundary value problem of the ordinary differential equation. However, in this paper, we consider any bounded domain with a general coefficient $h(x)$. Then we prove the existence of positive solutions and infinitely many solutions without positivity.

We sketch our idea to get results. We observe that a positive solution, if exists, behaves like $\rho(x)$ near the boundary. Here $\rho(x)$ is a distance function defined by,

$$\rho(x) \equiv \mathrm{dist}(x, \partial\Omega) \equiv \inf\{|x-y| : y \in \partial\Omega\}. \tag{1.2}$$

In the right-hand side of the first equation in (1.1), we substitute $\rho(x)$ instead of $u(x)$ and consider the equation

$$-\Delta u = h(x)\rho(x)^p \quad \text{in } \Omega, \tag{1.3}$$

with the zero Dirichlet boundary condition. By the regularity theorem of elliptic equation, if the right-hand side of (1.3) lies in $L^q(\Omega)$ or in $C^\theta(\overline{\Omega})$, then the solution belongs to $W^{2,q}(\Omega)$ or to $C^{2,\theta}(\overline{\Omega})$, respectively. We prove that this assertion is valid for (1.1) also. Indeed, we show that the condition $h\rho^p \in L^q(\Omega)$ or $h\rho^p \in C^\theta(\overline{\Omega})$ is necessary and sufficient for the existence and uniqueness of positive solutions for (1.1) belonging to $W^{2,q}(\Omega)$ or to $C^{2,\theta}(\overline{\Omega})$, respectively. Furthermore, we prove the existence of infinitely many sign-changing solutions and obtain the higher order regularity up to the C^∞-regularity of a positive solution. Our tools are mountain pass lemma and symmetric mountain pass lemma with the elliptic regularity theorem.

This paper is organized into five sections. In Section 2, we state the main results. In Section 3, we prove the existence and uniqueness of positive solutions. In Section 4, we show the existence of infinitely many solutions without positivity. In Section 5, we prove the C^2 or higher order regularity of solutions.

2. Main results.

In this section, we state main results. We assume that $\partial\Omega$ is sufficiently smooth. The exact definition of the smoothness will be stated in Section 5. We first introduce two assumptions below.

(h1) Let $h(x)$ be a measurable function in Ω such that

$$\mathrm{meas}\{x \in \Omega : h(x) > 0\} > 0, \tag{2.1}$$

where $\mathrm{meas}(A)$ denotes the $\boldsymbol{R}^N$-Lebesgue measure of A.

(h2) Suppose that $h(x) \geq 0$ a.e. in Ω.

Under the assumption (h1) without (h2), we allow $h(x)$ to change its sign in Ω. We denote by $W^{m,q}(\Omega)$ the Sobolev space which consists of all $u \in L^q(\Omega)$ such that all the distributional derivatives up to order m lie in $L^q(\Omega)$. Let $W_0^{m,q}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{m,q}(\Omega)$. We call u a $W^{2,q}(\Omega)$-solution if $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ and it satisfies (1.1) in the distribution sense (hence satisfies (1.1) a.e. in Ω). Define $\rho(x)$ by (1.2) and let $0 < p < 1$. We obtain a positive solution and infinitely many solutions in a Sobolev space as below.

THEOREM 2.1.

(i) *Suppose that* (h1) *holds and let* $h\rho^p \in L^q(\Omega)$ *with a certain* $q > N$. *Then* (1.1) *has a non-negative non-trivial* $W^{2,q}(\Omega)$*-solution.*

(ii) *Suppose that* (h1) *and* (h2) *hold and let* $q > N$. *Then* (1.1) *has a unique positive* $W^{2,q}(\Omega)$*-solution if and only if* $h\rho^p \in L^q(\Omega)$.

THEOREM 2.2. *Suppose that* (h1) *holds and let* $h\rho^p \in L^q(\Omega)$ *with a certain* $q > N$. *Then* (1.1) *has a sequence* $\{u_k\}$ *of non-trivial* $W^{2,q}(\Omega)$*-solutions whose* $W^{2,q}(\Omega)$*-norm converges to zero as* $k \to \infty$.

REMARK 2.3. The assumption $h\rho^p \in L^q(\Omega)$ allows $h(x)$ to have singularity on the boundary of Ω because $\rho = 0$ on $\partial\Omega$. In Theorem 2.1(i), the uniqueness of non-negative non-trivial solutions does not hold. Indeed, there is an example of $h(x)$ satisfying (h1) and $h\rho^p \in L^q(\Omega)$ but (1.1) has many non-negative non-trivial solutions.

REMARK 2.4. The $W^{2,q}(\Omega)$-solutions obtained by Theorems 2.1 and 2.2 belong to $C^1(\overline{\Omega})$ because $q > N$. Hashimoto and Ôtani [**7**] studied (1.1) when Ω is a unit ball B and $h(x) = (1 - |x|)^{-a}$. They proved that (1.1) has a radially symmetric positive solution in $C^2(B) \cap C^1(\overline{B})$ if $0 < a < p+1$ and in $C^2(B) \cap C^\theta(\overline{B})$ with all $\theta \in (0, (2-a)/(1-p))$ if $p + 1 \leq a < (p+1)/2 + 1$. We see that Theorem 2.1 gives the same result as Hashimoto and Ôtani's one for $0 < a < p+1$. Indeed, since $\Omega = B$, $\rho(x)$ is equal to $1 - |x|$. Then the condition $h\rho^p \in L^q(B)$ with a certain $q > N$ is equivalent to $a < p + 1$. Therefore Theorem 2.1(ii) provides a unique positive radial solution $u(r)$ in $W^{2,q}(B)$, and hence in $C^1(\overline{B})$. Moreover, $u(r)$ belongs to $C^2(B)$ because $h(r) \in C[0, 1)$ and $u(r)$ is a solution of the ordinary differential equation

$$u'' + \frac{n-1}{r}u' + hu^p = 0, \quad \text{in } (0, 1).$$

We emphasize that our theorem is applicable to any bounded domain and to any coefficient $h(x)$. Moreover, Theorem 2.1(ii) guarantees that our condition $h\rho^p \in L^q(\Omega)$ is necessary and sufficient for a positive solution to exist uniquely in $W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$.

In Theorems 2.1 and 2.2, we have already obtained the $W^{2,q}(\Omega)$-solutions, which are in $C^1(\overline{\Omega})$. Now we consider the $C^2(\overline{\Omega})$ regularity of solutions. Let $C^\theta(\overline{\Omega})$ denote the set of the Hölder continuous functions on $\overline{\Omega}$ with exponent θ. We define $C^{m,\theta}(\overline{\Omega})$ by the set of m times continuously differentiable functions whose m-th order derivatives belong to $C^\theta(\overline{\Omega})$. Although $h(x)$ is not defined on $\partial\Omega$, we use the assumption $h\rho^p \in C^\theta(\overline{\Omega})$ in the next theorem. This means that $h\rho^p$ is continuous in Ω and can be extended on $\overline{\Omega}$ as a Hölder continuous function with exponent θ.

Theorem 2.5. *Suppose that $h\rho^p \in C^\theta(\overline{\Omega})$ with a certain $\theta \in (0,1)$. Then any $W^{2,q}(\Omega)$-solution with $q > N$ belongs to $C^{2,\alpha}(\overline{\Omega})$ with $\alpha = \min(\theta, p)$.*

Note that if $\partial\Omega$ is smooth, then $\rho(x)$ is also smooth near the boundary but it is not differentiable at some points in Ω. Indeed, $\rho(x)$ is not differentiable at the center of the maximal ball that is included in Ω. To get the $C^\infty(\overline{\Omega})$-regularity of a positive solution, instead of $\rho(x)$, we employ an auxiliary function $\sigma(x)$ such that

$$\sigma \in C^{m+1,\theta}(\overline{\Omega}), \quad \sigma > 0 \text{ in } \Omega, \quad \sigma = 0, \ \frac{\partial \sigma}{\partial \nu} < 0 \text{ on } \partial\Omega. \tag{2.2}$$

Here $\partial/\partial\nu$ denotes the outward normal derivative. It is well known that if $\partial\Omega$ is smooth enough, (2.2) is fulfilled by the solution $e(x)$ of the equation,

$$-\Delta e = 1 \ \text{ in } \Omega, \quad e = 0 \ \text{ on } \partial\Omega. \tag{2.3}$$

Theorem 2.6. *Suppose that* (h1) *and* (h2) *hold and $\sigma(x)$ satisfies* (2.2). *Then there exists a unique positive solution in $C^{m+2,\theta}(\overline{\Omega})$ if and only if $h\sigma(x)^p$ is in $C^{m,\theta}(\overline{\Omega})$. Especially, if $\partial\Omega$ is of C^∞ and $he(x)^p \in C^\infty(\overline{\Omega})$, then there is a unique positive solution in $C^\infty(\overline{\Omega})$.*

Remark 2.7. The condition $h\sigma(x)^p \in C^{m,\theta}(\overline{\Omega})$ depends only on $h(x)$ and does not on the choice of $\sigma(x)$. Indeed, in Lemma 5.9, it will be proved that if σ_1 and σ_2 satisfy (2.2), then $h\sigma_1^p \in C^{m,\theta}(\overline{\Omega})$ is equivalent to $h\sigma_2^p \in C^{m,\theta}(\overline{\Omega})$.

Remark 2.8. Although Theorem 2.5 is valid for all solutions, Theorem 2.6 holds for a positive solution only and is not valid for sign-changing solutions. Indeed, the assumption $h\sigma^p \in C^{1,\theta}(\overline{\Omega})$ does not guarantee the conclusion that a sign-changing solution belongs to $C^{3,\theta}(\overline{\Omega})$. To see it, let $N = 1$ and $\Omega = (0,1)$.

Then (1.1) is rewritten as

$$-u'' = h(x)|u|^p \operatorname{sgn} u \quad \text{in } (0,1), \tag{2.4}$$

with $u(0) = u(1) = 0$. We solve (2.3) with $N = 1$ to get $e(x) = (x - x^2)/2$. We define

$$h(x) = e(x)^{-p} = \left(\frac{x - x^2}{2}\right)^{-p}.$$

Then $h(x)e(x)^p \equiv 1 \in C^\infty[0,1]$. Because of Theorem 2.2 with Theorem 2.1(ii), Problem (2.4) has a sign-changing solution $u(x)$, which has an interior zero z. Then $u'(z) \neq 0$. Indeed, if $u'(z) = 0$, then $u \equiv 0$ on $[0,1]$. This assertion, which will be proved in Lemma 2.9, is not evident because the nonlinear term of (2.4) is not Lipschitz continuous at $u = 0$. Since $u(z) = 0$, $u'(z) \neq 0$ and $0 < p < 1$, the right-hand side of (2.4) is not differentiable at z. Hence u does not belong to $C^3[0,1]$.

In the next lemma, we show that $u(z) = u'(z) = 0$ implies $u \equiv 0$.

Lemma 2.9. *Let $h(x) = ((x-x^2)/2)^{-p}$ and u be a solution of* (2.4) *in* $(0,1)$. *If $u(z) = u'(z) = 0$ at some $z \in (0,1)$, then u identically vanishes.*

Proof. We define the energy,

$$E(x) \equiv \frac{1}{2}u'(x)^2 + \frac{1}{p+1}h(x)|u(x)|^{p+1},$$

which has a derivative

$$E'(x) = \frac{1}{p+1}h'(x)|u(x)|^{p+1}. \tag{2.5}$$

Let a and b satisfy $0 < a < z < b < 1$. Then there is a $C > 0$ such that $|h'(x)| \le Ch(x)$ in $[a,b]$. Integrating (2.5) over (z,x) and using $E(z) = 0$, we get

$$\begin{aligned} E(x) &= \int_z^x (p+1)^{-1}h'(t)|u|^{p+1}dt \\ &\le C\int_z^x (p+1)^{-1}h(t)|u|^{p+1}dt \le C\int_z^x E(t)dt. \end{aligned}$$

The Gronwall inequality shows $E(x) \leq 0$ on $[z, b]$, and hence $u(x) \equiv 0$ on $[z, b]$. Similarly, we see $u(x) \equiv 0$ on $[a, z]$. Since a and b are arbitrary, $u \equiv 0$ on $(0, 1)$. $\square$

Example 2.10. We give an example of $h(x)$ satisfying the assumptions of theorems. Let $h(x) = e(x)^{-\alpha} g(x)$, where $0 < \alpha \leq p$ and $g(x)$ satisfies (2.1). If $g \not\equiv 0$ on $\partial\Omega$, then $h(x)$ has a singularity on $\partial\Omega$. If $g \in L^q(\Omega)$, $g \in C^\theta(\overline{\Omega})$ or $g(x) \in C^\infty(\overline{\Omega})$ with $\alpha = p$, then $h(x)$ satisfies the assumption of Theorem 2.1, 2.5 or 2.6, respectively.

3. Positive solution.

In this section, we prove Theorem 2.1. Denote the $L^q(\Omega)$-norm by $\|\cdot\|_q$ and the $W^{2,q}(\Omega)$-norm by $\|\cdot\|_{2,q}$ and the $C^1(\overline{\Omega})$-norm by $\|\cdot\|_{C^1(\overline{\Omega})}$. We begin with elementary but important inequalities.

Lemma 3.1.

(i) *If $u \in C^1(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$, then*

$$|u(x)| \leq \|\nabla u\|_\infty \rho(x) \quad \text{in } \Omega. \tag{3.1}$$

(ii) *For any $q > N$, there is a positive constant $a(q)$ such that*

$$|u(x)| \leq a(q)\|u\|_{2,q}\, \rho(x), \tag{3.2}$$

for $x \in \Omega$ and $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$.

(iii) *For $1 < q < \infty$, there is a constant $b(q) > 0$ such that*

$$\|u\|_{2,q} \leq b(q)\|\Delta u\|_q, \tag{3.3}$$

for $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$.

Proof. Fix $x \in \Omega$ arbitrarily. Let r be the radius of the maximal ball centered at x that is included in Ω. Then $B(x, r) \subset \Omega$ and $\rho(x) = r$. Hereafter, $B(x, r)$ denotes the ball centered at x with radius r. Choose a point ξ on $\partial B(x, r) \cap \partial\Omega$. Put $v(t) = u(tx + (1 - t)\xi)$. Then $v(0) = u(\xi) = 0$, $v(1) = u(x)$ and $|x - \xi| = \rho(x)$. Therefore we have

$$u(x) = \int_0^1 v'(t)dt = \int_0^1 \nabla u(tx + (1 - t)\xi) \cdot (x - \xi)dt,$$

which is estimated as

$$|u(x)| \leq \|\nabla u\|_\infty |x - \xi| = \|\nabla u\|_\infty \rho(x).$$

Thus we have (3.1). Let $q > N$. By the Sobolev imbedding, we have a constant $a(q) > 0$ such that $\|u\|_{C^1(\overline{\Omega})} \leq a(q)\|u\|_{2,q}$. Then (3.1) leads to (3.2). The assertion (iii) is a well known elliptic regularity theorem. □

To prove Theorems 2.1 and 2.2, we need the next proposition.

PROPOSITION 3.2. *Let $h \in L^1_{loc}(\Omega)$ satisfy* (2.1) *and let $q > 0$. Then there exist $\delta > 0$ and a sequence $\{\phi_n\}_{n=1}^\infty \subset C_0^\infty(\Omega)$ such that $\phi_n \geq 0$, each support of ϕ_n is compact and included in $\Omega \setminus \Omega_\delta$, the supports of ϕ_n and ϕ_m are disjoint for $n \neq m$ and*

$$\int_\Omega h\phi_n^q dx > 0 \quad \textit{for } n \in \boldsymbol{N}.$$

Here Ω_δ is defined by

$$\Omega_\delta \equiv \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}. \tag{3.4}$$

The proof of the proposition above is based on the lemma below.

LEMMA 3.3. *Let D be an open subset of $\boldsymbol{R}^N$ and let $h \in L^1(D)$ satisfy* (2.1) *with Ω replaced by D. Then for any $q > 0$, there exists a non-negative function $\phi \in C_0^\infty(D)$ satisfying*

$$\int_D h\phi^q dx > 0 \tag{3.5}$$

PROOF. Note that $h(x)$ may change its sign in D. Put

$$A \equiv \left\{ x \in D : \delta \leq h(x) \leq \frac{1}{\delta} \right\} \quad \text{with } \delta > 0.$$

We fix $\delta > 0$ so small that $\operatorname{meas}(A) > 0$. Define $u(x) = h(x)$ on A and $u(x) = 0$ in $D \setminus A$. Since $u \in L^\infty(D)$, we have a sequence $\{u_n\} \subset C_0^\infty(D)$ such that $0 \leq u_n \leq 1/\delta$ in D, u_n converges to $u(x)$ a.e. in D and strongly in $L^p(D)$ for any $p \in [1, \infty)$. Then the integral of hu_n^q over $D \setminus A$ converges to zero because of the Lebesgue convergence theorem. Hence

$$\int_D h u_n^q dx \longrightarrow \int_A h(x)^{q+1} dx > 0.$$

Thus we can fix n so large that u_n satisfies (3.5). The proof is complete. $\square$

PROOF OF PROPOSITION 3.2. Let $h \in L^1_{loc}(\Omega)$ satisfy (2.1). Then we choose a $\delta > 0$ so small that

$$\operatorname{meas}\{x \in \Omega \setminus \Omega_\delta : h(x) > 0\} > 0, \tag{3.6}$$

where Ω_δ is defined by (3.4). We claim the existence of $\{D_n\}_{n=1}^\infty$ such that each D_n is an open subset of $\Omega \setminus \Omega_\delta$ and $D_n \cap D_m = \emptyset$ if $n \neq m$ and the set of $x \in D_n$ satisfying $h(x) > 0$ has a positive Lebesgue measure. This claim seems to be known, but for the sake of completeness, we give a proof. By (3.6), there exists a small ball $B(x_0, \varepsilon)$ in $\Omega \setminus \Omega_\delta$ such that the set of $x \in B(x_0, \varepsilon)$ satisfying $h(x) > 0$ has a positive Lebesgue measure. We put, for $0 \le r \le \varepsilon$

$$g(r) \equiv \int_{B(x_0,r)} h^+(x) dx, \quad h^+(x) = \max(h(x), 0).$$

Then $g(\varepsilon) > 0$. Choose a strictly increasing sequence $\{a_n\}_{n=1}^\infty$ such that

$$0 < a_1 < a_2 < \cdots < \lim_{n\to\infty} a_n \le g(\varepsilon).$$

Since $g(r)$ is continuous, there exists an r_n satisfying $g(r_n) = a_n$. Then $\{r_n\}$ is strictly increasing and bounded from above by ε. Put $D_n = B(x_0, r_{n+1}) \setminus \overline{B(x_0, r_n)}$. Then it is easy to verify that $\{D_n\}_{n=1}^\infty$ satisfy our claim. Choose $\phi_n \in C_0^\infty(D_n)$ by Lemma 3.3. This completes the proof. $\square$

The uniqueness of positive solutions for a sublinear elliptic equation has already been proved by Brezis and Oswald [**3**]. However, since $h(x)$ has a singularity on $\partial\Omega$, their result is not applicable to our problem. To get the uniqueness, we show the comparison theorem for positive solutions. To this end, we define a supersolution and a subsolution of the equation

$$-\Delta u = h|u|^p \operatorname{sgn} u \quad \text{in } D, \tag{3.7}$$

where D is a bounded open subset of $\boldsymbol{R}^N$. We put

$$C_0^\infty(D)^+ \equiv \{\phi \in C_0^\infty(D) : \phi \ge 0\}.$$

For $u \in L^1_{loc}(D)$ satisfying $h|u|^p \in L^1_{loc}(D)$, we call u a *weak subsolution* of (3.7) if

$$\int_D \big(u\Delta\phi + h|u|^p(\operatorname{sgn} u)\phi\big)dx \geq 0 \quad \text{for any } \phi \in C_0^\infty(D)^+. \tag{3.8}$$

We define a *weak supersolution* by the reverse inequality.

LEMMA 3.4. *Let D be a bounded open subset of $\boldsymbol{R}^N$ and $h(x)$ be a nonnegative measurable function in D. Suppose that $u, v \in C(\overline{D})$, $u, v > 0$ in D, $hu^p, hv^p \in L^1(D)$, u and v are a weak subsolution and a weak supersolution of (3.7), respectively. If $u \leq v$ on ∂D, then $u \leq v$ in D.*

PROOF. Suppose that $u \leq v$ on ∂D but $u > v$ at some points in D. Then

$$E \equiv \{x \in D : u(x) > v(x)\} \neq \emptyset. \tag{3.9}$$

We use the same argument as in the proof of our result [**9**, Theorem 2.2]. Choose a function $J \in C_0^\infty(\boldsymbol{R}^N)$ such that $J(x) \geq 0$ and

$$\operatorname{supp} J \subset \{x : |x| < 1\}, \quad \int_{\boldsymbol{R}^N} J(x)dx = 1.$$

Hereafter $\operatorname{supp} J$ denotes the support of J. To use a mollifier, we put $J_\varepsilon(x) = \varepsilon^{-N}J(x/\varepsilon)$ for $\varepsilon > 0$. Denote the convolution of u and v by $u * v$. Put $u_\varepsilon = J_\varepsilon * u$ and $v_\varepsilon = J_\varepsilon * v$. Since u and v are a weak subsolution and a weak supersolution, we have

$$-\Delta u_\varepsilon \leq J_\varepsilon * f(\cdot, u), \quad -\Delta v_\varepsilon \geq J_\varepsilon * f(\cdot, v), \quad \text{in } D(\varepsilon), \tag{3.10}$$

where

$$f(x, s) \equiv h(x)s^p, \quad D(\varepsilon) \equiv \{x \in D : \operatorname{dist}(x, \partial D) > \varepsilon\}.$$

For $\delta, \varepsilon > 0$, we define

$$E(\delta, \varepsilon) \equiv \{x \in D(\varepsilon) : u_\varepsilon(x) > v_\varepsilon(x) + \delta\}.$$

By (3.9), there is a $\delta_0 > 0$ such that $E(\delta, \varepsilon) \neq \emptyset$ for $\delta, \varepsilon \in (0, \delta_0)$. If $\partial E(\delta, \varepsilon)$ is not smooth, by the Sard theorem we construct an approximate sequence $E_n(\delta, \varepsilon)$ with smooth boundary. For the rigorous proof, we refer the readers to [**9**]. We suppose that $\partial E(\delta, \varepsilon)$ is sufficiently smooth. Fix $\delta \in (0, \delta_0)$ arbitrarily. Since $u \leq v$ on

∂D, we choose $\varepsilon_0 \in (0, \delta_0)$ so small that $u_\varepsilon - v_\varepsilon < \delta/2$ on $\partial D(\varepsilon)$ for $\varepsilon \in (0, \varepsilon_0)$. Hence ε_0 depends on δ. Fix $\varepsilon \in (0, \varepsilon_0)$ arbitrarily. Then $\partial E(\delta, \varepsilon) \cap \partial D(\varepsilon) = \emptyset$. This means

$$v_\varepsilon = u_\varepsilon - \delta, \quad \frac{\partial}{\partial \nu}(v_\varepsilon - u_\varepsilon) \geq 0 \quad \text{on } \partial E(\delta, \varepsilon). \tag{3.11}$$

Here $\partial/\partial\nu$ denotes the outward normal derivative. By (3.10), we get

$$\int_{E(\delta,\varepsilon)} (u_\varepsilon \Delta v_\varepsilon - v_\varepsilon \Delta u_\varepsilon) dx \leq \int_{E(\delta,\varepsilon)} \big((J_\varepsilon * f(\cdot, u)) v_\varepsilon - (J_\varepsilon * f(\cdot, v)) u_\varepsilon\big) dx. \tag{3.12}$$

Since $\partial E(\delta, \varepsilon)$ is smooth, we use the Green formula with (3.11) to get

$$\begin{aligned}
&\int_{E(\delta,\varepsilon)} (u_\varepsilon \Delta v_\varepsilon - v_\varepsilon \Delta u_\varepsilon) dx \\
&= \int_{\partial E(\delta,\varepsilon)} u_\varepsilon \left(\frac{\partial v_\varepsilon}{\partial \nu} - \frac{\partial u_\varepsilon}{\partial \nu} \right) ds + \delta \int_{\partial E(\delta,\varepsilon)} \frac{\partial u_\varepsilon}{\partial \nu} ds \\
&\geq \delta \int_{E(\delta,\varepsilon)} \Delta u_\varepsilon dx \geq -\delta \int_{E(\delta,\varepsilon)} J_\varepsilon * f(\cdot, u) dx \\
&\geq -\delta \|f(\cdot, u)\|_{L^1(D)}.
\end{aligned}$$

This inequality with (3.12) yields

$$-\delta \|f(\cdot, u)\|_{L^1(D)} \leq \int_{E(\delta,\varepsilon)} \big(J_\varepsilon * f(\cdot, u) v_\varepsilon - J_\varepsilon * f(\cdot, v) u_\varepsilon\big) dx.$$

Letting $\varepsilon \to 0+$ and then $\delta \to 0+$, we have

$$\int_E h(x)(u^p v - u v^p) dx \geq 0.$$

Since $u > v > 0$, $h \geq 0$ in E and $0 < p < 1$, it follows that $h \equiv 0$ in E. Then (3.8) is reduced to

$$\int_E u \Delta \phi dx \geq 0, \quad \int_E v \Delta \phi dx \leq 0,$$

for $\phi \in C_0^\infty(E)^+$. Then u and v are called weakly subharmonic and weakly superharmonic in E, respectively. For such functions, the comparison theorem holds (see [**5**, p. 23, p. 29]). Since $u = v$ on ∂E, we get $u \leq v$ in E. This contradicts the definition of E. Consequently, we conclude that $u \leq v$ in D if $u \leq v$ on ∂D. $\square$

PROOF OF THEOREM 2.1. For each $n \in \boldsymbol{N}$, we define

$$h_n(x) \equiv \begin{cases} n & \text{if } \ h(x) > n, \\ h(x) & \text{if } \ -n \leq h(x) \leq n, \\ -n & \text{if } \ h(x) < -n. \end{cases} \tag{3.13}$$

Then $h_n \in L^\infty(\Omega)$. For $u \in H_0^1(\Omega)$, we define

$$I_n(u) \equiv \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} h_n(x) |u|^{p+1} \right) dx. \tag{3.14}$$

By the standard argument, we verify that $I_n(\cdot)$ is a C^1-functional on $H_0^1(\Omega)$ and satisfies the Palais-Smale condition. Since $|h_n(x)| \leq n$, we have

$$I_n(u) \geq \frac{1}{2} \|\nabla u\|_2^2 - \frac{n}{p+1} C^{p+1} \|\nabla u\|_2^{p+1}, \tag{3.15}$$

where we have used the imbedding, $\|u\|_{p+1} \leq C \|\nabla u\|_2$. By (3.15) with $0 < p < 1$, $I_n(u)$ is bounded from below. Therefore $I_n(u)$ has a minimizer $u_n \in H_0^1(\Omega)$ and it becomes a critical point of I_n, i.e.,

$$I_n'(u_n) = 0, \quad I_n(u_n) = \min_{u \in H_0^1(\Omega)} I_n(u).$$

Since $|u_n| \in H_0^1(\Omega)$ is also a minimizer, we rewrite $|u_n|$ as u_n. Then u_n is a non-negative critical point, i.e., it is a non-negative weak solution of (1.1) with $h = h_n$. Since $h_n \in L^\infty(\Omega)$, by the bootstrap argument with the elliptic regularity theorem, we verify that u_n belongs to $W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ for any $r \in [1, \infty)$. In particular, $u_n \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. We shall show *a priori* upper estimate for $I_n(u_n)$. Note that $h \in L_{loc}^q(\Omega)$ because $h\rho^p \in L^q(\Omega)$. By Proposition 3.2 with $q = p + 1$, we take a non-negative function $\phi_1 \in C_0^\infty(\Omega)$ such that the integral of $h\phi_1^{p+1}$ over Ω is positive. By the Lebesgue convergence theorem, we choose an $n_0 \in \boldsymbol{N}$ such that

$$\int_\Omega h_n \phi_1^{p+1} dx \geq \frac{1}{2} \int_\Omega h \phi_1^{p+1} dx > 0 \quad \text{for } n \geq n_0.$$

Then for $t > 0$ and $n \geq n_0$, we have

$$I_n(t\phi_1) \leq \frac{t^2}{2}\|\nabla\phi_1\|_2^2 - \frac{t^{p+1}}{2(p+1)}\int_\Omega h\phi_1^{p+1}dx.$$

We fix $t > 0$ so small that the right-hand side is negative, and then denote the right-hand side by $-c$, which is independent of n. Therefore

$$I_n(u_n) = \inf_{H_0^1(\Omega)} I_n(u) \leq -c < 0 \quad \text{for } n \geq n_0. \tag{3.16}$$

We show that $\|u_n\|_{2,q}$ is bounded. Applying (3.2) to the right-hand side of (1.1) with $h = h_n$, we have

$$|\Delta u_n| \leq a(q)^p\|u_n\|_{2,q}^p|h_n|\rho(x)^p \leq a(q)^p\|u_n\|_{2,q}^p|h(x)|\rho(x)^p.$$

Taking the L^q-norm of both sides and using (3.3), we get

$$\|u_n\|_{2,q} \leq b(q)\|\Delta u_n\|_q \leq a(q)^p b(q)\|h\rho^p\|_q\|u_n\|_{2,q}^p,$$

or equivalently

$$\|u_n\|_{2,q}^{1-p} \leq a(q)^p b(q)\|h\rho^p\|_q.$$

Since $0 < p < 1$, $\|u_n\|_{2,q}$ is bounded as $n \to \infty$. We extract a subsequence (again denoted by $\{u_n\}$) from $\{u_n\}$ which weakly converges to a certain limit u_∞ in $W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. Since $W^{2,q}(\Omega)$ is compactly imbedded in $C^1(\overline{\Omega})$, u_n converges to u_∞ strongly in $C^1(\overline{\Omega})$. We show that u_∞ is a $W^{2,q}(\Omega)$-solution of (1.1). Using Lemma 3.1, we get

$$\left|h_n u_n^p\right| \leq \|\nabla u_n\|_\infty^p|h(x)|\rho^p \leq C|h(x)|\rho(x)^p \in L^q(\Omega),$$

with some $C > 0$. Let $\phi \in C_0^\infty(\Omega)$ be any test function. By the Lebesgue convergence theorem, we see

$$\int_\Omega h_n u_n^p\phi dx \longrightarrow \int_\Omega h u_\infty^p\phi dx \quad \text{as } n \to \infty. \tag{3.17}$$

Since u_n is a weak solution of (1.1), we have

$$\int_\Omega \nabla u_n \cdot \nabla \phi dx = \int_\Omega h_n u_n^p \phi dx.$$

Letting $n \to \infty$, we get

$$\int_\Omega \bigl(\nabla u_\infty \cdot \nabla \phi - h u_\infty^p \phi\bigr) dx = 0.$$

Thus u_∞ is a $W^{2,q}(\Omega)$-solution of (1.1). Since $u_n \ge 0$, u_∞ is also non-negative. Letting $n \to \infty$ in (3.16), we get $I(u_\infty) \le -c < 0$, where $I(\cdot)$ is defined by (3.14) with h_n replaced by h. Accordingly, u_∞ is a non-trivial and non-negative solution. Thus we get the assertion (i).

We show (ii). Assume that (h1) and (h2) hold. Let $h\rho^p \in L^q(\Omega)$. In the proof of (i), we have already obtained a $W^{2,q}(\Omega)$-solution u_∞ such that $u_\infty \ge 0$ and $u_\infty \not\equiv 0$. Since $h \ge 0$ in Ω by (h2), we have

$$-\Delta u_\infty = h(x) u_\infty^p \ge 0 \quad \text{in } \Omega.$$

From the strong maximum principle, it follows that $u_\infty > 0$ in Ω. The uniqueness of positive solutions follows directly from Lemma 3.4.

Conversely, assume that (1.1) has a positive $W^{2,q}(\Omega)$-solution $u(x)$. Then $u \in C^1(\overline{\Omega})$ and we have

$$-\Delta u = h(x) u^p \ge 0 \quad \text{in } \Omega.$$

By Hopf's maximum principle, $\partial u/\partial\nu < 0$ on $\partial\Omega$. Hence there are constants $d, c_0 > 0$ such that $\partial u/\partial\nu \le -c_0$ in $\overline{\Omega}_d$, where Ω_d is defined by (3.4). Since $\partial\rho/\partial\nu = -1$ on $\partial\Omega$ and $u > 0$ in Ω, there is a $\delta > 0$ such that $u(x) \ge \delta\rho(x)$ in Ω. Since $u \in W^{2,q}(\Omega)$, we have

$$0 \le \delta^p h\rho(x)^p \le h(x)u(x)^p = -\Delta u \in L^q(\Omega).$$

Consequently, $h\rho(x)^p \in L^q(\Omega)$ and the proof is complete. □

4. Infinitely many solutions.

In this section, we prove Theorem 2.2. Throughout this section, we assume that (h1) holds and $h\rho^p \in L^q(\Omega)$. Our method is based on the symmetric mountain pass lemma, which needs a notion of Krasnoselskii's genus. We define it as below.

Definition 4.1. Let H be a Banach space and A a subset of H. A is said to be *symmetric* if $x \in A$ implies $-x \in A$. For a closed symmetric set A not containing the origin, we define a *genus* $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\boldsymbol{R}^k \setminus \{0\}$. If there does not exist such a finite k, we define $\gamma(A) = \infty$. For an empty set $\emptyset$, we put $\gamma(\emptyset) = 0$. Let Γ_k denote the family of closed symmetric subsets A such that $0 \notin A$ and $\gamma(A) \geq k$.

Assumption 4.2. Let H be an infinite dimensional Banach space and let $I \in C^1(H, \boldsymbol{R})$ satisfy (I1) and (I2) below.

(I1) $I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition (PS),
(PS) any sequence $\{u_k\}$ in H such that $I(u_k)$ is bounded and $I'(u_k) \to 0$ in H^* as $k \to \infty$ has a convergent subsequence.
(I2) For each $k \in \boldsymbol{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} I(u) < 0$.

Under the assumption above, we define c_k by

$$c_k \equiv \inf_{A \in \Gamma_k} \sup_{u \in A} I(u). \tag{4.1}$$

We state the symmetric mountain pass lemma due to Ambrosetti and Rabinowitz [**2**] and Clark [**4**]. We refer the readers to [**8**], [**10**] and [**12**] also.

Lemma 4.3 ([**2**], [**4**], [**10**]). *Suppose that Assumption* 4.2 *holds. Then each* c_k *is a critical value of* $I(u)$ *and* $c_k \leq c_{k+1} < 0$ *for* $k \in \boldsymbol{N}$ *and* $\{c_k\}$ *converges to zero. Moreover, if* $c_k = c_{k+1} = \cdots = c_{k+p} \equiv c$*, then* $\gamma(K_c) \geq p + 1$*. Here* K_c *is defined by*

$$K_c \equiv \{u \in H : I'(u) = 0,\ I(u) = c\}.$$

If we would define $d_{n,k}$ for $I_n(u)$ such as in (4.1) by

$$d_{n,k} = \inf_{A \in \Gamma_k} \sup_{u \in A} I_n(u),$$

then we need a uniform estimate $\underline{d}_k \leq d_{n,k} \leq \overline{d}_k < 0$ with two sequences $\underline{d}_k$, $\overline{d}_k$ which are independent of n and converge to zero as $k \to \infty$. If we could prove the existence of $\underline{d}_k$ and $\overline{d}_k$, then a critical point $u_{n,k}$ of $I_n(u)$ corresponding to $d_{n,k}$ converges to a limit $u_{\infty,k}$ as $n \to \infty$ along a subsequence. Furthermore, $u_{\infty,k}$ becomes a critical point of $I(u)$ which satisfies $\underline{d}_k \leq I(u_{\infty,k}) \leq \overline{d}_k$. Here $I(u)$ is defined by (3.14) with h_n replaced by h. Thus we can get infinitely many critical points if there exist $\underline{d}_k$ and $\overline{d}_k$. However, it is hard to prove the existence of $\underline{d}_k$.

Therefore, instead of $I_n(u)$, we introduce a new functional $J_n(u)$, for which we shall get a uniform estimate in Lemma 4.6. Moreover, we shall show that a critical point of $J_n(u)$ becomes that of $I_n(u)$. Recall $a(q)$ and $b(q)$ defined by (3.2) and (3.3), respectively. We define $R, R_0 > 0$ by

$$R_0 \equiv \left(4a(q)^p b(q) \|h\rho^p\|_q\right)^{1/(1-p)}, \quad R \equiv a(q)R_0. \tag{4.2}$$

Choose $G_0(t) \in C_0^1(\boldsymbol{R})$ such that $G_0(t)$ is even and

$$\begin{gathered} G_0(t) = 1 \quad \text{for } |t| \le 1, \qquad G_0(t) = 0 \quad \text{for } |t| \ge 2, \\ -2 \le G_0'(t) \le 0 \quad \text{for } 1 \le t \le 2. \end{gathered} \tag{4.3}$$

We define $G(t) \equiv G_0(R^{-1}t)$, where R is defined by (4.2). Then $G \in C_0^1(\boldsymbol{R})$, G is even and

$$G(t) = 1 \quad \text{if } |t| \le R, \qquad G(t) = 0 \quad \text{if } |t| \ge 2R, \tag{4.4}$$

$$0 \le G(t) \le 1, \quad -4 \le tG'(t) \le 0 \quad \text{for } t \in \boldsymbol{R}. \tag{4.5}$$

We define

$$\begin{gathered} H \equiv H_0^1(\Omega), \quad W \equiv W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), \\ J_n(u) \equiv \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1} h_n(x) G\left(\frac{u(x)}{\rho(x)}\right)|u|^{p+1}\right)dx, \end{gathered} \tag{4.6}$$

where $h_n(x)$ is defined by (3.13). Then H is a Hilbert space and W a Banach space, which are equipped with the norms,

$$\|u\|_H \equiv \|\nabla u\|_2, \quad \|u\|_W \equiv \|u\|_{2,q}.$$

Note that $J_n(u)$ are well defined on both H and W since h_n and G are bounded. We use a notation $J_n(u, H)$ or $J_n(u, W)$ when we consider it as a functional on H or W, respectively.

LEMMA 4.4.

(i) *If $\|u\|_{2,q} \le R_0$, then $I_n(u, W) = J_n(u, W)$.*
(ii) *$J_n(\cdot)$ is a C^1-functional on H and W.*
(iii) *$J_n(\cdot, H)$ satisfies the Palais-Smale condition.*
(iv) *For each $n \in \boldsymbol{N}$ fixed, $J_n(\cdot, H)$ is bounded from below.*

(v) *For any $k \in \boldsymbol{N}$, there exists an $A_k \in \Gamma_k$ such that the supremum of $J_n(u, H)$ on A_k is negative. Here Γ_k is defined by Definition* 4.1 *with $H = H_0^1(\Omega)$.*

Proof. We show (i). Let $\|u\|_{2,q} \le R_0$. Then $|u(x)| \le R\rho(x)$ in Ω by (3.2) with (4.2). Hence $G(u(x)/\rho(x)) = 1$ and $I_n(u, W) = J_n(u, W)$.

We prove (ii). Put

$$F_n(x, s) \equiv \frac{1}{p+1} h_n(x) G\left(\frac{s}{\rho(x)}\right) |s|^{p+1}. \tag{4.7}$$

Then $J_n(u)$ is rewritten as

$$J_n(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - F_n(x, u)\right) dx. \tag{4.8}$$

Denote the partial derivative of $F_n(x, s)$ with respect to s by $f_n(x, s)$. Then

$$f_n(x, s) = h_n(x) G\left(\frac{s}{\rho(x)}\right) |s|^p \operatorname{sgn} s + \frac{1}{p+1} h_n(x) G'\left(\frac{s}{\rho(x)}\right) \frac{|s|^{p+1}}{\rho(x)}. \tag{4.9}$$

We show that $F_n(x, s)$ and $f_n(x, s)$ are bounded on $\overline{\Omega} \times \boldsymbol{R}$. Let $\overline{\rho}$ be the maximum of $\rho(x)$ on $\overline{\Omega}$. If $|s| > 2R\overline{\rho}$, then $|s|/\rho(x) > 2R$, and hence $G(s/\rho(x)) = 0$. If $|s| \le 2R\overline{\rho}$, then we use $|G| \le 1$ to get

$$|F_n(x, s)| \le n(p+1)^{-1}(2R\overline{\rho})^{p+1}.$$

Thus $F_n(x, s)$ is bounded on $\overline{\Omega} \times \boldsymbol{R}$. In the same way, we see that the first term on the right-hand side in (4.9) is bounded. Let us show that the second term is bounded. If $|s| > 2R\overline{\rho}$, then $G'(s/\rho(x))$ vanishes. Let $|s| \le 2R\overline{\rho}$. Since $|tG'(t)| \le 4$ by (4.5), we have

$$\frac{|h_n|}{p+1} \left| G'\left(\frac{s}{\rho}\right) \right| \frac{|s|^{p+1}}{\rho(x)} \le \frac{4n}{p+1} |s|^p \le \frac{4n}{p+1} (2R\overline{\rho})^p.$$

Thus $f_n(x, s)$ is bounded. Then $J_n(u)$ is a C^1-functional on H and has a derivative

$$J_n'(u)v = \int_\Omega \big(\nabla u \cdot \nabla v - f_n(x, u)v\big) dx. \tag{4.10}$$

In the same way, we see that $J_n \in C^1(W, \boldsymbol{R})$. By the standard argument, we verify that $J_n(u,H)$ satisfies the Palais-Smale condition on H. Since $F_n(x,s)$ is bounded, $J_n(u)$ is bounded from below.

We show (v). Fix $n \in \boldsymbol{N}$ arbitrarily. By Proposition 3.2 with $q = p+1$, we have a $\delta > 0$ and functions $\{\phi_i\}_{i=1}^{\infty}$ in $C_0^{\infty}(\Omega)$ such that $\phi_i \geq 0$, ϕ_i has a compact support included in $\Omega \setminus \Omega_\delta$, the supports of ϕ_i and ϕ_j are disjoint for $i \neq j$ and

$$\int_{\Omega} h_n \phi_i^{p+1} dx > 0. \tag{4.11}$$

After replacing ϕ_i by $\phi_i / \|\nabla \phi_i\|_2$, we can assume that $\|\nabla \phi_i\|_2 = 1$. Since the supports of ϕ_i are disjoint to each others, we have

$$(\phi_i, \phi_j)_H \equiv (\nabla \phi_i, \nabla \phi_j)_2 = \delta_{ij},$$

where $(\cdot,\cdot)_2$ denotes the L^2-inner product and δ_{ij} stands for Kronecker's symbol, i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Accordingly, $\{\phi_i\}$ forms an orthonormal system in H (but not a complete system). Let $k \in \boldsymbol{N}$. We define

$$A_k \equiv \left\{ \sum_{i=1}^{k} t_i \phi_i : \sum_{i=1}^{k} t_i^2 = \alpha^2 \right\}, \tag{4.12}$$

where $\alpha > 0$ will be determined later on. Since A_k is a $(k-1)$-dimensional sphere, the genus $\gamma(A_k)$ is equal to k because of Borsuk-Ulam's theorem. Thus $A_k \in \Gamma_k$. Put

$$\alpha_k = R\delta \Big(\max_{1 \leq i \leq k} \|\phi_i\|_\infty \Big)^{-1}.$$

We claim

$$|u(x)| \leq R\rho(x) \quad \text{for } x \in \Omega,\ u \in A_k \text{ and } \alpha \in (0, \alpha_k). \tag{4.13}$$

Let $u = \sum_{i=1}^{k} t_i \phi_i \in A_k$. Since $\operatorname{supp} \phi_i$ is in $\Omega \setminus \Omega_\delta$, $u(x)$ vanishes in Ω_δ and (4.13) holds in Ω_δ. For $x \in \Omega \setminus \Omega_\delta$ and $\alpha \in (0, \alpha_k)$, we have

$$\|u\|_\infty = \max_{1 \leq i \leq k} |t_i| \|\phi_i\|_\infty \leq \alpha \max_{1 \leq i \leq k} \|\phi_i\|_\infty \leq R\rho(x).$$

Hence we obtain (4.13). This gives us

$$G\left(\frac{u(x)}{\rho(x)}\right) = 1 \quad \text{for } x \in \Omega,\ u \in A_k \text{ and } \alpha \in (0, \alpha_k). \tag{4.14}$$

Since any norm is equivalent to each other in a finite dimensional Banach space, there is a $C_k > 0$ independent of α such that

$$\left(\sum_{i=1}^{k} |t_i|^{p+1}\right)^{1/(p+1)} \geq C_k \|\nabla u\|_2 \quad \text{for } u = \sum_{i=1}^{k} t_i \phi_i. \tag{4.15}$$

Put

$$\beta_{n,k} = \min_{1 \leq i \leq k} \int_\Omega h_n \phi_i^{p+1} dx > 0.$$

Note that $\|\nabla u\|_2 = \alpha$ for $u \in A_k$ and recall that the supports of ϕ_i are disjoint to each others. Using (4.14) and (4.15), we obtain, for $u \in A_k$

$$\begin{aligned} J_n(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\sum_{i=1}^{k} \int_\Omega h_n |t_i|^{p+1} \phi_i^{p+1} dx \\ &\leq \frac{1}{2}\|\nabla u\|_2^2 - \frac{\beta_{n,k}}{p+1}\sum_{i=1}^{k} |t_i|^{p+1} \\ &\leq \frac{1}{2}\|\nabla u\|_2^2 - \frac{\beta_{n,k}}{p+1} C_k^{p+1} \|\nabla u\|_2^{p+1} \\ &= \frac{1}{2}\alpha^2 - \frac{\beta_{n,k}}{p+1} C_k^{p+1} \alpha^{p+1} < 0, \end{aligned} \tag{4.16}$$

provided that $\alpha > 0$ is small enough. Thus we have (v). $\square$

By Lemmas 4.3 and 4.4, we can define

$$c_{n,k} \equiv \inf_{A \in \Gamma_k} \sup_{u \in A} J_n(u, H). \tag{4.17}$$

This is a critical value of J_n in H. $J_n(u)$ satisfies the Palais-Smale condition not in W but in H. Therefore we applied Lemma 4.3 to J_n in H and obtained critical values $c_{n,k}$ in (4.17). We explain our method to prove Theorem 2.2. First, we give a lower estimate and an upper estimate of $c_{n,k}$ independent of n. Next, we prove that any critical point of J_n in H belongs to W and becomes a critical point of $I_n(u, W)$. Last, letting $n \to \infty$, we obtain infinitely many solutions of (1.1). To get the uniform estimates of $c_{n,k}$, we need the next lemma. For the proof, we refer the readers to [**10**, Proposition 7.8].

LEMMA 4.5. *Let X be a closed linear subspace of H whose codimension is $k-1$. Then $A \cap X \neq \emptyset$ for $A \in \Gamma_k$.*

LEMMA 4.6. *Let $c_{n,k}$ be as in* (4.17). *Then there exist two sequences $\{\underline{c}_k\}$ and $\{\overline{c}_k\}$ such that both of them converge to zero and*

$$\underline{c}_k \leq c_{n,k} \leq \overline{c}_k < 0 \quad \textit{for all } n \in \boldsymbol{N}. \tag{4.18}$$

PROOF. First, we show the existence of $\overline{c}_k$. We use the same argument as in (4.16). Let δ, $\{\phi_i\}$ and A_k be the same as in the proof of Lemma 4.4, where we replace (4.11) by

$$\int_\Omega h\phi_i^{p+1}dx > 0 \quad \text{for } i \in \boldsymbol{N}.$$

Let $k \in \boldsymbol{N}$. By the Lebesgue convergence theorem, there is an $N(k) \in \boldsymbol{N}$ such that

$$\int_\Omega h_n\phi_i^{p+1}dx > \frac{1}{2}\int_\Omega h\phi_i^{p+1}dx > 0 \quad \text{for } 1 \leq i \leq k,\ n \geq N(k).$$

Put

$$\beta_k = \min_{1\leq i\leq k} \frac{1}{2}\int_\Omega h\phi_i^{p+1}dx.$$

Then in the same way as in (4.16), we have

$$J_n(u) \leq \frac{1}{2}\alpha^2 - \frac{\beta_k}{p+1}C_k^{p+1}\alpha^{p+1} \quad \text{for } u \in A_k,\ n \geq N(k).$$

Note that the right-hand side is independent of n. We fix $\alpha > 0$ so small that the right-hand side is negative, which is denoted by $-a_k < 0$. Therefore,

$$\sup_{u\in A_k} J_n(u) \leq -a_k < 0,$$

which shows

$$c_{n,k} \leq \sup_{A_k} J_n(u) \leq -a_k \quad \text{for } n \geq N(k).$$

Since each $c_{n,k}$ with $1 \le n \le N(k)$ is negative by Lemma 4.4 (v), we define $\overline{c}_k \equiv \sup_{n\in \boldsymbol{N}} c_{n,k} < 0$.

We shall show the existence of $\underline{c}_k$ and its convergence to zero. Define r by $1/q + 1/r = 1$. Then we claim that

$$J_n(u) \ge \frac{1}{2}\|\nabla u\|_2^2 - C_0\|u\|_r \quad \text{for } u \in H, \tag{4.19}$$

where $C_0 = (2R)^p(p+1)^{-1}\|h\rho^p\|_q$. Note that u belongs to $L^r(\Omega)$ if $u \in H$ because $q > N$. Let $u \in H$ and put

$$D \equiv \left\{x \in \Omega : \frac{|u(x)|}{\rho(x)} \le 2R\right\}.$$

Since $G(u/\rho) = 0$ in $\Omega \setminus D$, we use the Hölder inequality with $|h_n| \le |h|$ to get

$$\begin{aligned}\int_\Omega |h_n||u|^{p+1}G\left(\frac{u}{\rho}\right)dx &\le \left\||h||u|^pG\left(\frac{u}{\rho}\right)\right\|_{L^q(D)}\|u\|_{L^r(D)} \\ &\le \|h(2R\rho)^p\|_{L^q(D)}\|u\|_{L^r(D)} \\ &= (2R)^p\|h\rho^p\|_q\|u\|_r. \end{aligned}\tag{4.20}$$

Substituting this inequality into (4.6), we get (4.19). Let λ_k and ψ_k be the k-th eigenvalue and the eigenfunction of the problem,

$$-\Delta\psi = \lambda\psi \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega.$$

Let X be a closed linear subspace of H which is spanned by ψ_i with $k \le i < \infty$. Then the codimension of X is $k-1$. By Lemma 4.5, $X \cap A \ne \emptyset$ for $A \in \Gamma_k$. Hence by (4.19) we obtain

$$\begin{aligned} c_{n,k} = \inf_{A\in\Gamma_k}\sup_{u\in A} J_n(u) &\ge \inf_{u\in X} J_n(u) \\ &\ge \inf_{u\in X}\left\{\frac{1}{2}\|\nabla u\|_2^2 - C_0\|u\|_r\right\}. \end{aligned}\tag{4.21}$$

We shall show that there are positive constants a and C independent of k such that

$$\|u\|_r \le C\lambda_k^{-a}\|\nabla u\|_2 \quad \text{for } u \in X. \tag{4.22}$$

Recall that $1/q + 1/r = 1$ and $q > N$, and note

$$\lambda_k \|u\|_2^2 \le \|\nabla u\|_2^2 \quad \text{for } u \in X. \tag{4.23}$$

If $N \ge 2$, then $r < 2$, and hence we have a $C > 0$ such that

$$\|u\|_r \le C\|u\|_2 \le C\lambda_k^{-1/2}\|\nabla u\|_2. \tag{4.24}$$

Thus (4.22) holds. Let $N = 1$. If $r \le 2$, then (4.24) is still valid. Let $r > 2$. Then we have

$$\|u\|_r^r = \int_\Omega |u|^r dx \le \|u\|_\infty^{r-2}\|u\|_2^2. \tag{4.25}$$

Since $N = 1$, $H_0^1(\Omega)$ is imbedded in $L^\infty(\Omega)$, i.e., $\|u\|_\infty \le C\|\nabla u\|_2$ with some $C > 0$. Then (4.25) is rewritten as

$$\|u\|_r^r \le C^{r-2}\|\nabla u\|_2^{r-2}\lambda_k^{-1}\|\nabla u\|_2^2 = C^{r-2}\lambda_k^{-1}\|\nabla u\|_2^r,$$

which means (4.22). Consequently, (4.22) is valid for all N and r. By (4.21) with (4.22), we have

$$\begin{aligned} c_{n,k} &\ge \inf_{u\in X}\left\{\frac{1}{2}\|\nabla u\|_2^2 - C_0 C\lambda_k^{-a}\|\nabla u\|_2\right\} \\ &= \inf_{t\ge 0}\left\{\frac{1}{2}t^2 - C_0 C\lambda_k^{-a}t\right\} = -\frac{1}{2}\big(C_0 C\lambda_k^{-a}\big)^2. \end{aligned}$$

We define $\underline{c}_k$ by the last term. Then it converges to zero as $k \to \infty$. □

In the next lemma, we show that a critical point of $J_n(\cdot, H)$ in H belongs to W and becomes a critical point of $I_n(\cdot, W)$.

Lemma 4.7. *Let $u_0 \in H$ be a critical point of $J_n(\cdot, H)$. Then u_0 belongs to W, $\|u_0\|_{2,q} \le R_0$, $I_n(u_0, W) = J_n(u_0, W)$ and $I_n'(u_0, W) = 0$.*

Proof. Let $J_n'(u_0, H) = 0$. By (4.10), u_0 is a $H_0^1(\Omega)$-weak solution of

$$-\Delta u_0 = f_n(x, u_0). \tag{4.26}$$

Since $f_n(x, s)$ is bounded on $\overline{\Omega} \times \boldsymbol{R}$, u_0 belongs to $W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ for any

$r \in [1, \infty)$. In particular, $u_0 \in W$. We rewrite $f_n(x, u_0)$ in (4.9) as

$$f_n(x, u_0) = h_n(x)|u_0|^p \operatorname{sgn} u_0 \left\{ G\left(\frac{u_0}{\rho}\right) + (p+1)^{-1} G'\left(\frac{u_0}{\rho}\right)\frac{u_0}{\rho} \right\}. \tag{4.27}$$

By (4.5), we have

$$-4 \le G(t) + \frac{1}{p+1} t G'(t) \le 1 \quad \text{for } t \in \boldsymbol{R}.$$

By this inequality with Lemma 3.1, (4.27) is estimated as

$$|f_n(x, u_0)| \le 4|h_n||u_0(x)|^p \le 4a(q)^p \|u_0\|_{2,q}^p |h(x)|\rho(x)^p.$$

Using this inequality, we evaluate (4.26) as

$$\|u_0\|_{2,q} \le b(q)\|\Delta u_0\|_q \le 4a(q)^p b(q)\|u_0\|_{2,q}^p \|h\rho^p\|_q.$$

This is reduced to

$$\|u_0\|_{2,q} \le (4a(q)^p b(q)\|h\rho^p\|_q)^{1/(1-p)} = R_0.$$

Then $|u_0(x)| \le R\rho(x)$ in Ω by (3.2). Hence $G(u_0/\rho) = 1$, $G'(u_0/\rho) = 0$, $I_n(u_0, W) = J_n(u_0, W)$ and $f_n(x, u_0) = h_n|u_0|^p \operatorname{sgn} u_0$. Then (4.26) is reduced to

$$-\Delta u_0 = h_n|u_0|^p \operatorname{sgn} u_0.$$

Thus u_0 is a critical point of $I_n(\cdot, W)$. □

We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2. We take a critical point $u_{n,k} \in H$ corresponding to $c_{n,k}$. i.e.,

$$J_n'(u_{n,k}, H) = 0, \quad \underline{c_k} \le J_n(u_{n,k}) = c_{n,k} \le \bar{c}_k < 0. \tag{4.28}$$

By Lemma 4.7, $u_{n,k} \in W$, $\|u_{n,k}\|_{2,q} \le R_0$ and

$$I_n'(u_{n,k}, W) = 0, \quad \underline{c_k} \le I_n(u_{n,k}) = c_{n,k} \le \bar{c}_k < 0. \tag{4.29}$$

Fix $k \in \boldsymbol{N}$. We extract a subsequence (again denoted by $\{u_{n,k}\}$) from $\{u_{n,k}\}$ which weakly converges to a limit u_k in $W^{2,q}(\Omega)$ as $n \to \infty$. Since $W^{2,q}(\Omega)$ is compactly imbedded in $C^1(\overline{\Omega})$, $\{u_{n,k}\}$ strongly converges to u_k in $C^1(\overline{\Omega})$. Let $\phi \in C_0^\infty(\Omega)$ be any test function. We use the same method as in (3.17) to get

$$\int_\Omega h_n |u_{n,k}|^p (\operatorname{sgn} u_{n,k}) \phi dx \longrightarrow \int_\Omega h |u_k|^p (\operatorname{sgn} u_k) \phi dx, \tag{4.30}$$

as $n \to \infty$. Since $I'_n(u_{n,k}, W) = 0$, we have

$$\int_\Omega \big(\nabla u_{n,k} \cdot \nabla \phi - h_n(x) |u_{n,k}|^p (\operatorname{sgn} u_{n,k}) \phi \big) dx = 0.$$

Letting $n \to \infty$, we get

$$\int_\Omega \big(\nabla u_k \cdot \nabla \phi - h(x) |u_k|^p (\operatorname{sgn} u_k) \phi \big) dx = 0. \tag{4.31}$$

Thus u_k is a $W^{2,q}(\Omega)$-solution of (1.1). Letting $n \to \infty$ in (4.29), we have

$$\underline{c}_k \le \int_\Omega \left(\frac{1}{2} |\nabla u_k|^2 - \frac{1}{p+1} h(x) |u_k|^{p+1} \right) dx \le \overline{c}_k < 0. \tag{4.32}$$

Putting $\phi = u_k$ in (4.31), we have

$$\int_\Omega h |u_k|^{p+1} dx = \|\nabla u_k\|_2^2.$$

Substituting this relation into (4.32), we obtain

$$\underline{c}_k \le -\frac{1-p}{2(p+1)} \|\nabla u_k\|_2^2 \le \overline{c}_k < 0.$$

Thus $u_k \not\equiv 0$ and $\|\nabla u_k\|_2$ converges to zero as $k \to \infty$. To show that $\|u_k\|_{2,q}$ also converges to zero, we use the Gagliardo-Nirenberg inequality (see [**1**, p. 140, Theorem 5.9]),

$$\|u\|_\infty \le C \|u\|_{1,q}^\theta \|u\|_2^{1-\theta},$$

with $\theta = Nq/(Nq + 2q - 2N)$. Substituting ∇u_k instead of u, we have

$$\|\nabla u_k\|_\infty \le C\|u_k\|_{2,q}^{\theta}\|\nabla u_k\|_2^{1-\theta} \le CR_0^{\theta}\|\nabla u_k\|_2^{1-\theta} \to 0,$$

as $k \to \infty$. Thus the $C^1(\overline{\Omega})$-norm of u_k converges to zero. We use (3.1) to get

$$\|\Delta u_k\|_q \le \big\||h||u_k|^p\big\|_q \le \|h\rho^p\|_q\|\nabla u_k\|_\infty^p \to 0.$$

Consequently, $\|u_k\|_{2,q}$ converges to 0. □

5. Regularity.

In this section, we prove Theorems 2.5 and 2.6. To this end, we need the exact definition of the smoothness of $\partial\Omega$.

Definition 5.1. We say that $\partial\Omega$ belongs to $C^{m,\theta}$ if $\partial\Omega$ is locally represented as a graph of a $C^{m,\theta}$-function. More precisely, for each $x_0 \in \partial\Omega$, we translate and rotate the coordinate system such that $x_0 = 0$ and the inward unit normal vector at $x_0 = 0$ is equal to $(0, \dots, 0, 1)$. Then there exist $r_0 > 0$, an open set V and a function ϕ such that V is an open neighborhood of $x_0 = 0$, $\phi \in C^{m,\theta}(B_{N-1}(0, r_0), \boldsymbol{R})$ and

$$V \cap \partial\Omega = \big\{(x', \phi(x')) : x' \in B_{N-1}(0, r_0)\big\},$$
$$B_{N-1}(0, r_0) = \big\{x' \in \boldsymbol{R}^{N-1} : |x'| < r_0\big\}.$$

Definition 5.1 gives us a $C^{m-1,\theta}$-diffeomorphism from a neighborhood of x_0 to a neighborhood of the origin. To prove it, we prepare cubic domains for $r > 0$,

$$C(r) \equiv \{(x_1, \dots, x_N) : |x_i| < r \ (1 \le i \le N)\},$$
$$C^+(r) \equiv \{x \in C(r) : x_N > 0\},$$
$$C^0(r) \equiv \{x \in C(r) : x_N = 0\}.$$

The next two propositions play the most important roles to get the regularity of solutions.

Proposition 5.2. *Let $\partial\Omega \in C^{m+1,\theta}$. For any $x_0 \in \partial\Omega$, there exist $r > 0$, an open neighborhood U of x_0 and a $C^{m,\theta}$-diffeomorphism Φ from $\overline{U}$ to $\overline{C(r)}$ such that*

$$\Phi(U \cap \Omega) = C^{+}(r), \quad \Phi(U \cap \partial\Omega) = C^{0}(r), \tag{5.1}$$

$$\rho(x) = |y_N| \quad \textit{if } \Phi(x) = y \textit{ and } x \in U. \tag{5.2}$$

In many papers or books (c.f. [**5**, p. 94]), the smoothness of $\partial\Omega$ is defined by the existence of Φ satisfying (5.1) without (5.2). However, we need (5.2) in the proof of Theorem 2.5.

Proof of Proposition 5.2. Let $x_0 \in \partial\Omega$ and T be the tangent space of $\partial\Omega$ at x_0. By translating and rotating the coordinate system, we assume that $x_0 = 0$ and

$$T = \{(x_1, \dots, x_N) : x_N = 0\}.$$

By Definition 5.1, there exist $r_0 > 0$, $\phi \in C^{m+1,\theta}$ and a neighborhood V of the origin such that

$$V \cap \partial\Omega = \big\{(y', \phi(y')) : y' \in \boldsymbol{R}^{N-1},\ |y'| < r_0\big\},$$

with $y' = (y_1, \dots, y_{N-1})$. Then $\phi(0) = 0$ and $\nabla\phi(0) = 0$. Put $\zeta(y') = (y', \phi(y'))$. Then $\zeta(y') \in \partial\Omega$. We denote the inward unit normal vector at $\zeta(y')$ by $n(y')$, which is computed as

$$n(y') = \frac{1}{\sqrt{|\nabla\phi(y')|^2 + 1}}(-\phi_{y_1}, \dots, -\phi_{y_{N-1}}, 1). \tag{5.3}$$

For $t \in \boldsymbol{R}$, we define

$$\psi(y', t) \equiv \zeta(y') + t n(y'). \tag{5.4}$$

Since $\phi \in C^{m+1,\theta}$, ψ lies in $C^{m,\theta}$. We shall show that the Jacobian of ψ is positive at $(0, t)$ for small $|t|$. Let ψ_i denote the i-th element of ψ. Then

$$\begin{aligned} \psi_i(y', t) &= y_i - t\phi_{y_i}(y')(|\nabla\phi|^2 + 1)^{-1/2} \quad \text{for } 1 \le i \le N-1, \\ \psi_N(y', t) &= \phi(y') + t(|\nabla\phi|^2 + 1)^{-1/2}. \end{aligned}$$

Since $\phi(0) = 0$ and $\nabla\phi(0) = 0$, we get

$$\frac{\partial\psi_i}{\partial y_j}(0, t) = \delta_{ij} - t\phi_{y_i y_j}(0),$$

$$\frac{\partial\psi_N}{\partial y_j}(0,t)=0,\qquad \frac{\partial\psi_i}{\partial t}(0,t)=0,\qquad \frac{\partial\psi_N}{\partial t}(0,t)=1.$$

Hence the Jacobian is

$$\frac{\partial\psi}{\partial(y',t)}(0,t)=\det(I-tH). \tag{5.5}$$

Here I stands for the $(N-1)\times(N-1)$ unit matrix and H denotes the Hessian matrix, whose (i,j)-th element is $(\partial^2\phi/\partial y_i\partial y_j)(0)$. We denote the eigenvalues of H by λ_i with $1\le i\le N-1$. Choose an orthogonal matrix S which diagonalizes H into the form, $S^{-1}HS=\mathrm{diag}(\lambda_1,\dots,\lambda_{N-1})$. Then (5.5) is computed as

$$\det(I-tH)=\det(I-tS^{-1}HS)=\prod_{i=1}^{N-1}(1-t\lambda_i). \tag{5.6}$$

Put $\Lambda=\max_i|\lambda_i|$. For $|t|<1/\Lambda$, the Jacobian (5.6) is positive. Since ψ is smooth in a neighborhood of $(0,t)$, the Jacobian is positive near $(0,t)$. Put $x=\psi(y',t)$. By the inverse function theorem, (y',t) is a $C^{m,\theta}$-function of x. We define $(y',t)=\Phi(x)$ and put $y_N=t$. Then $\Phi(x)=y$ and Φ is a $C^{m,\theta}$-diffeomorphism from a neighborhood of $x_0=0$ to a neighborhood of the origin. By (5.4), we see that

$$|t|=|x-\zeta(y')|=\mathrm{dist}(x,\partial\Omega)\quad \text{if } |t| \text{ is small.}$$

Hence $\rho(x)=\mathrm{dist}(x,\partial\Omega)=|y_N|$. Choose an $r>0$ small enough and put $U=\Phi^{-1}(C(r))$. Then all the assertions of Proposition 5.2 hold. □

The proofs of Theorems 2.5 and 2.6 are based on the Schauder estimate with the help of Proposition 5.2 and the proposition below.

Proposition 5.3. *Let $\partial\Omega\in C^{m+2,\theta}$ and σ satisfy (2.2). If $u\in C^{m+1,\theta}(\overline{\Omega})$ and $u=0$ on $\partial\Omega$, then $u(x)/\sigma(x)\in C^{m,\theta}(\overline{\Omega})$.*

To prove the proposition above, we begin with a function of one variable.

Lemma 5.4. *Let $v\in C^{m+1}[0,1]$ with $v(0)=0$. Then $v(t)/t\in C^m[0,1]$ and*

$$\left(\frac{v(t)}{t}\right)^{(k)}=t^{-k-1}\int_0^t v^{(k+1)}(\tau)\tau^k d\tau\quad \textit{for } 0\le k\le m. \tag{5.7}$$

Here $v^{(k)}(t)$ denotes the k-th derivative of $v(t)$ and $v^{(0)}(t)=v(t)$.

Proof. We use induction. Let $m = 0$ and suppose that $v \in C^1[0,1]$ with $v(0) = 0$. Then $v(t)/t \in C[0,1]$ and (5.7) with $k = 0$ holds clearly.

We suppose that the lemma holds for $v \in C^m[0,1]$ with $v(0) = 0$. Let $v \in C^{m+1}[0,1]$ with $v(0) = 0$. By the assumption of induction, we have $v(t)/t \in C^{m-1}[0,1]$ and

$$\left(\frac{v(t)}{t}\right)^{(m-1)} = t^{-m} \int_0^t v^{(m)}(\tau)\tau^{m-1} d\tau. \tag{5.8}$$

Since $v(t)/t$ is of class C^{m+1} except for $t = 0$, we differentiate (5.8) to get

$$\left(\frac{v(t)}{t}\right)^{(m)} = -mt^{-m-1} \int_0^t v^{(m)}(\tau)\tau^{m-1} d\tau + v^{(m)}(t)t^{-1}. \tag{5.9}$$

Integration by parts yields

$$\begin{aligned} m \int_0^t v^{(m)}(\tau)\tau^{m-1} d\tau &= \int_0^t v^{(m)}(\tau) \frac{d}{d\tau}(\tau^m) d\tau \\ &= v^{(m)}(t)t^m - \int_0^t v^{(m+1)}(\tau)\tau^m d\tau. \end{aligned}$$

Substituting this identity into (5.9), we obtain (5.7) with $k = m$. By assumption, $v(t)/t$ belongs to $C^{m+1}(0,1] \cap C^{m-1}[0,1]$. To show $v(t)/t \in C^m[0,1]$, it is enough to prove that $(v(t)/t)^{(m)}$ has a limit as $t \to 0+$. In (5.7) with $k = m$, letting $t \to 0+$ and using L'Hospital's rule, we have

$$\lim_{t \to 0+} \left(\frac{v(t)}{t}\right)^{(m)} = \lim_{t \to 0+} \frac{v^{(m+1)}(t)t^m}{(m+1)t^m} = \frac{1}{m+1} v^{(m+1)}(0).$$

Thus $v(t)/t \in C^m[0,1]$, and the proof is complete. □

We extend the lemma above to a Hölder function of N-variables.

Lemma 5.5. *Suppose that $u \in C^{m+1,\theta}(\overline{C^+(r)})$ and $u(x) = 0$ for $x_N = 0$. Then $(\partial^k/\partial x_N^k)(u(x)/x_N) \in C^\theta(\overline{C^+(r)})$ for $0 \le k \le m$.*

Proof. We use (5.7) with $v(t) = u(x', t)$ to get

$$\frac{\partial^k}{\partial x_N^k}\left(\frac{u(x', x_N)}{x_N}\right) = x_N^{-k-1} \int_0^{x_N} \frac{\partial^{k+1} u}{\partial x_N^{k+1}}(x', t) t^k dt, \tag{5.10}$$

for $x = (x', x_N) \in C^+(r)$. We put

$$V(x', x_N) \equiv \frac{\partial^k}{\partial x_N^k}\left(\frac{u(x', x_N)}{x_N}\right), \quad U(x', x_N) \equiv \frac{\partial^{k+1}}{\partial x_N^{k+1}} u(x', x_N).$$

Then (5.10) is rewritten as

$$V(x', x_N) = x_N^{-k-1} \int_0^{x_N} U(x', t) t^k \, dt. \tag{5.11}$$

Fix $x = (x', x_N)$, $y = (y', y_N) \in C^+(r)$ arbitrarily. We suppose that $0 < x_N \leq y_N$. If $x_N > y_N$, we exchange x with y. Making the change of variable $t = (x_N/y_N)s$, we rewrite (5.11) as

$$V(x', x_N) = y_N^{-k-1} \int_0^{y_N} U\left(x', \frac{x_N}{y_N} s\right) s^k \, ds. \tag{5.12}$$

On the other hand, substituting (y', y_N) into (5.11), we get

$$V(y', y_N) = y_N^{-k-1} \int_0^{y_N} U(y', t) t^k \, dt. \tag{5.13}$$

Subtracting (5.13) from (5.12) and using the Hölder continuity of U, we obtain

$$\begin{aligned}
&\big|V(x', x_N) - V(y', y_N)\big| \\
&\quad \leq y_N^{-k-1} \int_0^{y_N} \left| U\left(x', \frac{x_N}{y_N} t\right) - U(y', t) \right| t^k \, dt \\
&\quad \leq y_N^{-k-1} \int_0^{y_N} C\left(|x' - y'|^\theta + \left| \frac{x_N}{y_N} t - t \right|^\theta \right) t^k \, dt \\
&\quad \leq C|x' - y'|^\theta + C|x_N - y_N|^\theta.
\end{aligned}$$

Therefore V is Hölder continuous in $C^+(r)$ with exponent θ. Hence it is uniquely extended on $\overline{C^+(r)}$ as a Hölder continuous function. □

Although Lemma 5.5 deals with the partial derivative with respect to x_N only, we extend it to all derivatives up to order m.

Lemma 5.6. *Let u satisfy the assumption of Lemma* 5.5. *Then $u(x)/x_N \in C^{m,\theta}(\overline{C^+(r)})$.*

PROOF. Let $\gamma = (\gamma_1, \dots, \gamma_N)$ denote the multi-index whose elements are non-negative integers. We use a notation,

$$D^\gamma = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \cdots \partial x_N^{\gamma_N}} \quad \text{with } |\gamma| = \gamma_1 + \cdots + \gamma_N.$$

Let $|\gamma| \leq m$. We put $\gamma' = (\gamma_1, \dots, \gamma_{N-1}, 0)$ and define $v = D^{\gamma'} u$. Then

$$v \in C^{m+1-|\gamma'|,\theta}(\overline{C^+(r)}),$$

and $v(x) = 0$ for $x_N = 0$ because $D^{\gamma'}$ is a partial derivative with respect to the variables $x_1, \dots, x_{N-1}$. Therefore Lemma 5.5 asserts that

$$\frac{\partial^{\gamma_N}}{\partial x_N^{\gamma_N}} \left(\frac{v(x)}{x_N} \right) \in C^\theta(\overline{C^+(r)}).$$

Observe that $D^{\gamma'}(u(x)/x_N) = v(x)/x_N$. Then we obtain

$$D^\gamma \left(\frac{u(x)}{x_N} \right) = \frac{\partial^{\gamma_N}}{\partial x_N^{\gamma_N}} \left(\frac{v(x)}{x_N} \right) \in C^\theta(\overline{C^+(r)}).$$

Therefore $u(x)/x_N$ lies in $C^{m,\theta}(\overline{C^+(r)})$. □

Combining Lemma 5.6 with Proposition 5.2, we obtain the next lemma.

LEMMA 5.7. *Let $\partial\Omega \in C^{m+2,\theta}$ in the sense of Definition* 5.1 *and let u be in $C^{m+1,\theta}(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$. Then $u/\rho \in C^{m,\theta}(\overline{\Omega}_d)$ with a small $d > 0$. Here Ω_d is defined by* (3.4).

PROOF. Let $x_0 \in \partial\Omega$. Define U, Φ and $r > 0$ by Proposition 5.2. Then Φ is a $C^{m+1,\theta}$-diffeomorphism. Let u satisfy the assumption of the lemma. We define $v(y) = u(\Phi^{-1}(y))$ for $y \in C^+(r)$. Put $w(x) = u(x)/\rho(x)$ and $W(y) = v(y)/y_N$. Then $w(x) = W(y)$ if $\Phi(x) = y$ and $x \in U \cap \Omega$. Since $v \in C^{m+1,\theta}(\overline{C^+(r)})$ and $v(y) = 0$ for $y_N = 0$, $W(y)$ belongs to $C^{m,\theta}(\overline{C^+(r)})$ by Lemma 5.6. Hence $w(x) = W(\Phi(x))$ also lies in $C^{m,\theta}(\overline{U \cap \Omega})$. Since x_0 is arbitrary, we have an open neighborhood $U(x_0)$ of $x_0 \in \partial\Omega$ such that

$$\partial\Omega \subset \bigcup_{x_0 \in \partial\Omega} U(x_0), \quad w(x) \in C^{m,\theta}(\overline{U(x_0) \cap \Omega}).$$

By the compactness of $\partial\Omega$, we have a finite covering $U(x_i)$ with $1 \le i \le m$. Hence there is a $d > 0$ such that $\overline{\Omega}_d$ is covered by the union of $U(x_i)$ with $1 \le i \le m$. Then $w(x) \in C^{m,\theta}(\overline{\Omega}_d)$. □

Since $\rho \notin C^1(\Omega)$, Lemma 5.7 can not be altered so as to assert that $u/\rho \in C^{m,\theta}(\overline{\Omega})$ with $m \ge 1$. However, the C^θ-regularity of u/ρ is assured, as is shown in the next lemma.

Lemma 5.8. *Let $\partial\Omega \in C^3$ and let $u = 0$ on $\partial\Omega$. If $u \in C^{1,\theta}(\overline{\Omega})$, then $u/\rho \in C^\theta(\overline{\Omega})$. If $u \in C^2(\overline{\Omega})$, then $u(x)/\rho(x)$ is Lipschitz continuous on $\overline{\Omega}$.*

Proof. By Lemma 5.7, u/ρ lies in $C^\theta(\overline{\Omega}_d)$. Since ρ is Lipschitz continuous on $\overline{\Omega}$ and $\rho(x) \ge d$ in $\Omega \setminus \Omega_d$, we have $u/\rho \in C^\theta(\overline{\Omega})$. For $\theta = 1$, we define $C^{m,1}(\overline{\Omega})$ by the set of u whose m-th derivatives are Lipschitz continuous on $\overline{\Omega}$. Observing the proofs of Lemmas 5.5–5.7, we can verify that these lemmas are valid for $\theta = 1$ also. Therefore $u(x)/\rho(x)$ is Lipschitz continuous on $\overline{\Omega}$ if $u \in C^2(\overline{\Omega})$. □

Using Lemma 5.7, we prove Proposition 5.3.

Proof of Proposition 5.3. Let $\partial\Omega$, σ and u satisfy the assumptions of the proposition. By Lemma 5.7, u/ρ, $\sigma/\rho \in C^{m,\theta}(\overline{\Omega}_d)$. Since $\partial\sigma/\partial\nu < 0$ on $\partial\Omega$, we have $\sigma(x)/\rho(x) \ge c_0$ on $\overline{\Omega}$ with some $c_0 > 0$. Then $\rho/\sigma \in C^{m,\theta}(\overline{\Omega}_d)$. Thus we have

$$\frac{u}{\sigma(x)} = \frac{u}{\rho(x)} \frac{\rho(x)}{\sigma(x)} \in C^{m,\theta}(\overline{\Omega}_d).$$

Since $u, \sigma > 0$ in $\Omega \setminus \Omega_d$, we have $u/\sigma \in C^{m,\theta}(\overline{\Omega})$. □

Proof of Theorem 2.5. Suppose that $\partial\Omega \in C^4$. Let u be a $W^{2,q}(\Omega)$-solution. By the Sobolev imbedding, u lies in $C^{1,\beta}(\overline{\Omega})$ with $\beta = 1 - N/q$. Put $v(x) = u(x)/\rho(x)$, which belongs to $C^\beta(\overline{\Omega})$ by Lemma 5.8. Then (1.1) is rewritten as

$$-\Delta u = h|u|^p \operatorname{sgn} u = h\rho^p|v|^p \operatorname{sgn} v \in C^\gamma(\overline{\Omega}), \tag{5.14}$$

where we have put $\gamma = \min(\theta, \beta p)$. Since $\partial\Omega \in C^4$, Φ is a C^3-diffeomorphism. Then the Schauder estimate (see [**5**, Theorem 6.8]) means that $u \in C^{2,\gamma}(\overline{\Omega})$. By Lemma 5.8, $v = u/\rho$ is Lipschitz continuous on $\overline{\Omega}$. Then $h\rho^p|v|^p \operatorname{sgn} v \in C^\alpha(\overline{\Omega})$ with $\alpha = \min(\theta, p)$. The Schauder estimate again gives that $u \in C^{2,\alpha}(\overline{\Omega})$. □

In the next lemma, we show that the assumption $h\sigma^p \in C^{m,\theta}(\overline{\Omega})$ in Theorem 2.6 does not depend on the choice of $\sigma(x)$.

Lemma 5.9. *Let σ_1, σ_2 satisfy* (2.2). *Then $h\sigma_1^p \in C^{m,\theta}(\overline{\Omega})$ is equivalent to $h\sigma_2^p \in C^{m,\theta}(\overline{\Omega})$.*

Proof. By Proposition 5.3, σ_2/σ_1 lies in $C^{m,\theta}(\overline{\Omega})$. Since $\sigma_2/\sigma_1 \geq c_0 > 0$ on $\overline{\Omega}$ with a certain $c_0 > 0$, $(\sigma_2/\sigma_1)^p$ also belongs to $C^{m,\theta}(\overline{\Omega})$. If $h\sigma_1^p \in C^{m,\theta}(\overline{\Omega})$, then we have

$$h\sigma_2^p = h\sigma_1^p \left(\frac{\sigma_2}{\sigma_1}\right)^p \in C^{m,\theta}(\overline{\Omega}).$$

Replacing σ_1 with σ_2, we obtain the converse assertion. □

We conclude this paper by proving Theorem 2.6.

Proof of Theorem 2.6. The uniqueness of positive solutions has already been proved in Theorem 2.1(ii). Let $h\sigma^p \in C^{m,\theta}(\overline{\Omega})$. By Theorems 2.1 and 2.5, (1.1) has a unique positive solution u in $C^{2,\alpha}(\overline{\Omega})$ with $\alpha = \min(\theta, p)$. We shall show that u has a regularity of class $C^{m+2,\theta}(\overline{\Omega})$. Since $u \in C^{2,\alpha}(\overline{\Omega})$, u/σ belongs to $C^{1,\alpha}(\overline{\Omega})$ by Proposition 5.3. Putting $v(x) = u(x)/\sigma(x)$, we have

$$-\Delta u = hu^p = h\sigma^p v^p. \tag{5.15}$$

Since $v(x) \geq c_0 > 0$ on $\overline{\Omega}$ with some c_0, $v(x)^p$ also lies in $C^{1,\alpha}(\overline{\Omega})$. Then the right-hand side of (5.15) belongs to $C^{1,\alpha}(\overline{\Omega})$. By the Schauder estimate, u has a regularity of class $C^{3,\alpha}(\overline{\Omega})$. Then $v = u/\sigma(x) \in C^{2,\alpha}(\overline{\Omega})$. Hence the right-hand side of (5.15) belongs to $C^{2,\alpha}(\overline{\Omega})$ if $m \geq 2$. By the Schauder estimate again, u is in $C^{4,\alpha}(\overline{\Omega})$. Repeating this argument, we obtain $u \in C^{m+2,\alpha}(\overline{\Omega})$. Then the right-hand side of (5.15) lies in $C^{m,\theta}(\overline{\Omega})$ and therefore $u \in C^{m+2,\theta}(\overline{\Omega})$.

Conversely, let u be a positive solution in $C^{m+2,\theta}(\overline{\Omega})$. Since $\sigma \in C^{m+1,\theta}(\overline{\Omega})$ because of (2.2), σ/u is in $C^{m,\theta}(\overline{\Omega})$. Moreover, since σ/u has a positive lower bound on $\overline{\Omega}$, $(\sigma/u)^p$ also belongs to $C^{m,\theta}(\overline{\Omega})$. Therefore

$$h\sigma^p = hu^p \left(\frac{\sigma}{u}\right)^p = -\Delta u \left(\frac{\sigma}{u}\right)^p \in C^{m,\theta}(\overline{\Omega}).$$

This completes the proof. □

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