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Dispersive estimates and asymptotic expansions for Schrödinger equations in dimension one

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Abstract. We study the time decay of scattering solutions to one-dimensional Schrödinger equations and prove a weighted dispersive estimate with stronger time decay than the case of unweighted estimates for the non-resonant state. Furthermore asymptotic expansions in time of scattering solutions are given. The key of the proof is the study of the Fourier properties of the Jost functions. We improve the Fourier properties of the Jost functions obtained by D'Ancona and Fanelli [**2**].

1. Introduction.

This paper is concerned with dispersive estimates for scattering solutions $e^{-itH}P_{ac}u$ to Schrödinger equations

$$i\partial_t u = Hu,$$

where

$$H = -\frac{d^2}{dx^2} + V(x), \quad x \in \boldsymbol{R}$$

is a one-dimensional Schrödinger operator and P_{ac} is the projection onto the absolutely continuous subspace for H. We assume that $V(x)$ is a real valued potential such that $V \in L^1_1$ at least. Here L^p_γ is the weighted $L^p(\boldsymbol{R})$ space:

$$L^p_\gamma := \left\{ f \mid \langle x\rangle^\gamma f \in L^p(\boldsymbol{R}) \right\}, \ \|f\|_{L^p_\gamma} := \left\| \langle x\rangle^\gamma f \right\|_{L^p},$$

where $1 \le p \le \infty$, $\gamma \in \boldsymbol{R}$ and $\langle x\rangle$ stands for $\sqrt{1+|x|^2}$. Under the above conditions, H is self-adjoint on $L^2(\boldsymbol{R})$ with form domain $H^1(\boldsymbol{R})$ and the absolutely

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continuous spectrum of H is the half line $[0,\infty)$, the singular continuous spectrum of H is absent, and the eigenvalues of H are strictly negative.

In order to state our results, we introduce a few notations. *The Jost functions* $f_{\pm}(\lambda,x)$ are the solutions to the equation

$$-f''(\lambda,x)+V(x)f(\lambda,x)=\lambda^2 f(\lambda,x),\quad \lambda,x\in\boldsymbol{R}$$

satisfying following asymptotic conditions

$$\left|f_{\pm}(\lambda,x)-e^{\pm i\lambda x}\right|\to 0 \text{ as } x\to\pm\infty.$$

It is well known (see [**3**]) that if $V\in L^1_1$, then the Jost functions are uniquely defined for all $\lambda,x\in\boldsymbol{R}$. We denote by $W(\lambda)$ their Wronskian

$$W(\lambda):=f_+(\lambda,x)\cdot\partial_x f_-(\lambda,x)-\partial_x f_+(\lambda,x)\cdot f_-(\lambda,x).$$

$W(\lambda)$ is independent of x and does not vanish for $\lambda\neq 0$.

Definition 1.1. We say that the potential V is of generic type if $W(0)\neq 0$ and is of exceptional type if $W(0)=0$. We also say that zero is a resonance of H if the potential V is of exceptional type.

We note that V is of exceptional type if and only if there exist a non trivial bounded solution to the equation $Hf=0$. Hence the trivial potential $V\equiv 0$ is of exceptional type. Our main result is the following:

Theorem 1.2. *Let m be a positive integer. Suppose that $V\in L^1_{2m}$ and V is of generic type, or $V\in L^1_{2m+2}$ and V is of exceptional type. Let*

$$s=\begin{cases}2m-1 & \text{if } V \text{ is of generic type,}\\ 2m & \text{if } V \text{ is of exceptional type.}\end{cases}$$

Then

$$\left\|\langle x\rangle^{-s}(e^{-itH}P_{ac}-P_{m-1})u\right\|_{L^\infty}\le C|t|^{-1/2-m}\left\|\langle x\rangle^{s}u\right\|_{L^1} \tag{1.1}$$

for all $t\neq 0$, where P_{m-1} is given by

$$P_{m-1}=\sum_{j=0}^{m-1}t^{-1/2-j}C_{j-1}.$$

Furthermore, the coefficients C_{j-1} satisfy the following:

(1) *If V is of generic type, then* $C_{-1} \equiv 0$, $\operatorname{rank} C_{j-1} \le 2j$ *and*

$$\big\| \langle x\rangle^{-2j+1} C_{j-1} u \big\|_{L^\infty} \le C \big\| \langle x\rangle^{2j-1} u \big\|_{L^1}, \quad j = 1, 2, \dots, m-1.$$

In particular, we have

$$\big\| \langle x\rangle^{-1} e^{-itH} P_{ac} u \big\|_{L^\infty} \le C |t|^{-3/2} \big\| \langle x\rangle u \big\|_{L^1}, \quad t \neq 0.$$

(2) *If V is of exceptional type, then* $\operatorname{rank} C_{j-1} \le 2j+1$ *and*

$$\big\| \langle x\rangle^{-2j} C_{j-1} u \big\|_{L^\infty} \le C \big\| \langle x\rangle^{2j} u \big\|_{L^1}, \quad j = 0, 1, \dots, m-1.$$

Remark 1.3. The coefficients C_{j-1} can be computed explicitly. More precisely, the integral kernel of C_{j-1} can be written in the form:

$$\frac{1}{\sqrt{4\pi i}\, j!(4i)^j} \left(\frac{\partial}{\partial \lambda}\right)^{2j} (T(\lambda) f_-(\lambda, x) f_+(\lambda, y)) \bigg|_{\lambda=0},$$

where $f_\pm$ are the Jost functions and $T(\lambda) := -2i\lambda / W(\lambda)$. For example, if V is of exceptional type, then

$$C_{-1} u = \frac{1}{\sqrt{4\pi i}} \langle u, f_0\rangle f_0,$$

where f_0 is a non trivial bounded solution to the equation $Hf = 0$ normalized as

$$\lim_{x \to +\infty} \frac{1}{2} (|f_0(x)|^2 + |f_0(-x)|^2) = 1,$$

(see Section 4).

Theorem 1.2 immediately implies that an asymptotic expansion of $e^{-itH} P_{ac}$ in $\mathscr{B}(L^2_s, L^2_{-s})$. Here $\mathscr{B}(X, Y)$ denotes the Banach space of bounded operators from X to Y.

Corollary 1.4. *Let m be a positive integer. Suppose that $V \in L^1_{2m}$ and V is of generic type, or $V \in L^1_{2m+2}$ and V is of exceptional type. Let*

$$s > \begin{cases} 2m - \dfrac{1}{2} & \textit{if } V \textit{ is of generic type}, \\ 2m + \dfrac{1}{2} & \textit{if } V \textit{ is of exceptional type}. \end{cases}$$

Then $e^{-itH}P_{ac}$ *has the following asymptotic formula in* $\mathscr{B}(L^2_s, L^2_{-s})$:

$$e^{-itH}P_{ac} = \sum_{j=1}^{m-1} t^{-1/2-j}C_{j-1} + O(t^{-1/2-m}), \quad t \to \infty,$$

where C_{j-1} *are given by Theorem* 1.2.

Dispersive estimates for Schrödinger equations have been studied by many authors. Journé, Soffer and Sogge [**12**] proved usual dispersive estimates:

$$\left\| e^{-itH}P_{ac}u \right\|_{L^\infty} \le C|t|^{-d/2}\|u\|_{L^1}, \tag{1.2}$$

in dimension $d \ge 3$, under the suitable decay and regularity assumptions for V. Such estimates are very important since (1.2) implies Strichartz estimates which can be applied to prove well-posedness for nonlinear Schrödinger equations. Weder [**22**] proved (1.2) for $d = 1$ under the assumption that $V \in L^1_\gamma$ for some $\gamma > 5/2$, or else that $V \in L^1_\gamma$, $\gamma > 3/2$ and V is of generic type. Later, Goldberg and Schlag [**9**] proved (1.2) under the assumption that $V \in L^1_2$, or else that $V \in L^1_1$ and V is of generic type. In dimensions $d \ge 2$ dispersive estimates (1.2) have recently been proved under various assumptions on the potential V and the assumption that zero is neither an eigenvalue nor a resonance of H. Schlag [**17**] proved dispersive estimates in dimension two. In dimension three, dispersive estimates was proved by Rodnianski and Schlag [**16**], Goldberg and Schlag [**9**], Yajima [**24**], Goldberg [**6**], [**7**] and Vodev [**19**]. In higher dimension Journé, Soffer and Sogge [**12**], Yajima [**23**] and Vodev [**20**] proved dispersive estimates. When zero is either an eigenvalue or a resonance of H, Erdoğan and Schlag [**4**] and Yajima [**24**] proved dispersive estimates in dimension three. Moreover Erdoğan and Schlag [**5**] proved dispersive estimates for matrix Schrödinger operators in dimension three. On the other hand, Yajima [**23**], Weder [**22**], Artbazar and Yajima [**1**] and D'Ancona and Fanelli [**2**] proved the L^p-boundedness of wave operators which implies (1.2). The time decay $t^{-1/2}$ in $d = 1$ is not integrable at infinity and is unsuitable for applying to NLS. We hence are interested in a dispersive estimate whose time decay is integrable at infinity. Schlag [**18**] first proved the estimate (1.1) with $m = 1$ under the assumptions $V \in L^1_4$ and V is of generic type. Goldberg [**8**] also proved (1.1) with $m = 1$ under the assumptions that $V \in L^1_3$ and V is of generic type, or else that

$V \in L^1_4$ and V is of exceptional type. Compared to his results, our assumptions on the potential $V(x)$, which are used in Theorem 1.2, are weaker. The following non self-adjoint matrix Schrödinger operators

$$\mathscr{H} = \mathscr{H}_0 + V := \begin{pmatrix} -\dfrac{d^2}{dx^2} + 1 & 0 \\ 0 & \dfrac{d^2}{dx^2} - 1 \end{pmatrix} + \begin{pmatrix} U & W \\ -W & -U \end{pmatrix}$$

are considered by Krieger and Schlag [**14**]. Here U and W are real-valued functions. Such a system arises in the study of the stability or instability of the standing wave to the NLS

$$i\partial_t u + \partial_x^2 u = -F(|u|^2)u$$

where F is a non negative function. Krieger and Schlag proved some dispersive estimates with time decay $t^{-1/2}$ or $t^{-3/2}$ for the system $\mathscr{H}$ under suitable assumptions for the potentials and the spectrum of $\mathscr{H}$ (*e.g.,* U, W and all derivatives are exponentially decaying and ± 1 are not resonances of $\mathscr{H}$). The proof for the matrix case is similar for the scalar case. We hence expect that the decay and regularity assumptions for the potentials U and W can be relaxed and similar expansions to (1.1) hold for the system $\mathscr{H}$, but it is not clear to the author at the moment.

On the other hand, asymptotic expansions of $e^{-itH}P_{ac}$ as $t \to \infty$ in $\mathscr{B}(L^2_s(\boldsymbol{R}^d), L^2_{-s}(\boldsymbol{R}^d))$ were proved by Jensen and Kato [**11**] ($d = 3$), Jensen [**10**] ($d \geq 5$) and Murata [**15**] ($d \geq 1$). Here $H = -\triangle + V$ in $L^2(\boldsymbol{R}^d)$ with $|V(x)| \leq C\langle x\rangle^{-\sigma}$ for sufficiently large $\sigma > 0$. Compared to the result [**15**], our assumptions on the potential $V(x)$ are weaker and the weight in the generic case is better.

We give here the outline of the proof. We may assume that $t > 0$ without loss of generality. To prove Theorem 1.2, we use the spectral decomposition of $e^{itH}P_{ac}$:

$$\langle e^{-itH}P_{ac}u, v\rangle = \int_0^\infty e^{-it\lambda} d\langle E_{ac}(\lambda)u, v\rangle$$

where $E_{ac}(\lambda)$ is the absolutely continuous part of the spectral measure of H. Since $d\langle E_{ac}(\lambda)u, v\rangle$ satisfies the Stone formula, namely

$$d\langle E_{ac}(\lambda)u, v\rangle = \frac{1}{2\pi i}\langle (R(\lambda + i0) - R(\lambda - i0))u, v\rangle d\lambda,$$

we obtain

$$\left\langle e^{-itH}P_{ac}u,v\right\rangle=\frac{1}{2\pi i}\int_0^\infty e^{-it\lambda}\langle (R(\lambda+i0)-R(\lambda-i0))u,v\rangle d\lambda,$$

where $R(\lambda\pm i0):=\lim_{\varepsilon\to+0}(H-(\lambda\pm i\varepsilon))^{-1}$ are the boundary value of the perturbed resolvent. In order to make the change of variables $\lambda\mapsto\lambda^2$, we define an extended resolvent as follows

$$R_\lambda:=\begin{cases}R(\lambda^2+i0), & \lambda>0,\\ R(\lambda^2-i0), & \lambda<0.\end{cases}\tag{1.3}$$

Using this definition, we have

$$\left\langle e^{-itH}P_{ac}u,v\right\rangle=\frac{1}{\pi i}\int_{\boldsymbol{R}} e^{-it\lambda^2}\lambda\langle R_\lambda u,v\rangle d\lambda.$$

Let $\tilde{K}(\lambda,x,y)$ be the integral kernel of $-2i\lambda R_\lambda$. Applying the stationary phase method, we have

$$\begin{aligned}&\frac{1}{2\pi}\int_{\boldsymbol{R}} e^{-it\lambda^2}\tilde{K}(\lambda,x,y)d\lambda\\ &\quad=\frac{1}{\sqrt{4\pi i}}\sum_{j=0}^{m-1}\frac{t^{-1/2-j}}{j!(4i)^j}\big(\partial_\lambda^{2j}\tilde{K}\big)(0,x,y)+t^{-1/2-m}\tilde{S}_m(t,\tilde{K}).\end{aligned}$$

To estimate the remainder, we split the propagator into high and low energy parts. We prove dispersive estimates for the high energy part in Section 2. In Section 3, we study some properties of the Jost functions. We give the proof of the low energy part in Section 4.

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2. The high energy estimates.

In this section we prove a weighted dispersive estimate for the high energy part under assumptions on the potential V and weights that are employed in Theorem 1.2. The following proposition was essentially proved by Goldberg and Schlag [**9**], but we give full details of the proof.

PROPOSITION 2.1. *Suppose $V \in L^1_N, N \geq 1$ and set $\lambda_0 := \|V\|_{L^1_N}$. Let χ be an even smooth cut-off function such that $\chi(\lambda) = 1$ for $|\lambda| \leq \lambda_0$ and $\chi(\lambda) = 0$ for $|\lambda| \geq 2\lambda_0$. Then for $u \in L^2 \cap L^1_N$,*

$$\big\|\langle x\rangle^{-N} e^{-itH}\big(1-\chi\big(\sqrt{H}\big)\big)u\big\|_{L^\infty} \leq Ct^{-1/2-N}\big\|\langle x\rangle^N u\big\|_{L^1}, \quad t > 0.$$

PROOF. Set $\tilde{\chi}(\lambda) := 1-\chi(\lambda)$. Let η be an even smooth function on $\boldsymbol{R}$ such that $\eta(\lambda) = 1$ if $|\lambda| \leq 1$, $\eta(\lambda) = 0$ if $|\lambda| \geq 2$ and let $\tilde{\chi}_L(\lambda) := \eta(\lambda/L)\tilde{\chi}(\lambda)$ for $L \geq 1$. We want to show that

$$\sup_{L\geq 1}\big|\big\langle e^{-itH}\tilde{\chi}_L\big(\sqrt{H}\big)u, v\big\rangle\big| \leq Ct^{-1/2-N}\big\|\langle x\rangle^N u\big\|_{L^1}\big\|\langle x\rangle^N v\big\|_{L^1}$$

for all $t > 0$ and for any Schwartz functions u and v. We recall the Born series expansion of the resolvent $R(\lambda^2 \pm i0)$:

$$R(\lambda^2 \pm i0) = \sum_{n=0}^{\infty} R_0(\lambda^2 \pm i0)\big(-VR_0(\lambda^2 \pm i0)\big)^n, \quad \pm\lambda > 0, \tag{2.1}$$

where $R_0(\lambda^2 \pm i0) := (-d^2/dx^2 - (\lambda^2 \pm i0))^{-1}$ is the free resolvent which has the distribution kernel

$$R_0(\lambda^2 \pm i0)(x) = -\frac{e^{i\lambda|x|}}{2i\lambda}, \quad \pm\lambda > 0. \tag{2.2}$$

Since $\big\|VR_0(\lambda^2 \pm i0)\big\|_{L^1\to L^1} \leq (2|\lambda|)^{-1}\|V\|_{L^1}$, we have

$$\big|\langle R_0(\lambda^2 \pm i0)(-VR_0(\lambda^2 \pm i0))^n u, v\rangle\big| \leq \frac{\|V\|^n_{L^1}}{(2|\lambda|)^{n+1}}\|u\|_{L^1}\|v\|_{L^1}.$$

Hence the series (2.1) converges in the sense of the operator norm from L^1 to L^∞, provided $|\lambda| \geq \lambda_0$. Then using (1.3), (2.1) and (2.2), we can write the kernel of R_λ explicitly as

$$R_\lambda(x,y) = \sum_{n=0}^{\infty} \frac{1}{(-2i\lambda)^{n+1}} \int_{\boldsymbol{R}^n} e^{i\lambda(|x-x_1|+\sum_{j=2}^n |x_j-x_{j-1}|+|x_n-y|)} \times \prod_{j=1}^n V(x_j)dx_1\dots dx_n$$

which converges, provided $|\lambda| \geq \lambda_0$. Applying the above formula to the kernel of R_λ, we have

$$\langle e^{itH}\tilde{\chi}_L(\sqrt{H})u, v\rangle = \frac{1}{\pi i}\sum_{n=0}^{\infty}\frac{1}{(-2i)^{n+1}}\int_{\boldsymbol{R}^{n+3}} e^{-it\lambda^2+i\lambda\sum_{j=0}^{n}|x_{j+1}-x_j|}\frac{\tilde{\chi}_L(\lambda)}{\lambda^n}$$
$$\times\, u(x_0)\prod_{j=1}^{n}V(x_j)v(x_{n+1})d\lambda dx_0\dots dx_{n+1}.$$

In the previous equality summation and integration can be exchanged because $\tilde{\chi}_L(\lambda)$ are compactly supported. Let us consider the oscillatory integral

$$\Phi(t,a) = \int_{\boldsymbol{R}} e^{-it\lambda^2+ia\lambda}\frac{\tilde{\chi}_L(\lambda)}{\lambda^n}d\lambda, \quad a \in \boldsymbol{R}.$$

Using integration by parts and the Fourier inversion formula, we have

$$\begin{aligned}\Phi(t,a) &= \frac{1}{(-2it)^N}\int_{\boldsymbol{R}} e^{-it\lambda^2}P_\lambda^N\left(e^{ia\lambda}\frac{\tilde{\chi}_L(\lambda)}{\lambda^n}\right)d\lambda\\ &= \frac{1}{\sqrt{4\pi it}}\frac{1}{(-2it)^N}\int_{\boldsymbol{R}} e^{-|\xi|^2/4it}\mathscr{F}\left[P_\lambda^N\left(e^{ia\lambda}\frac{\tilde{\chi}_L(\lambda)}{\lambda^n}\right)\right](\xi)d\xi,\end{aligned}$$

where $P_\lambda = (\partial/\partial_\lambda)(1/\lambda)$ and $\mathscr{F}$ is the Fourier transform with respect to λ. Thus we obtain

$$\begin{aligned}|\Phi(t,a)| &\leq Ct^{-1/2-N}\big\|\mathscr{F}P_\lambda^N(e^{ia\lambda}\tilde{\chi}_L(\lambda)\lambda^{-n})\big\|_{L^1}\\ &\leq Ct^{-1/2-N}\sum_{k=0}^{N}|a|^{N-k}\big\|\mathscr{F}\partial_\lambda^k(\tilde{\chi}_L(\lambda)\lambda^{-n-N})\big\|_{L^1}.\end{aligned}$$

Since $\sum_{j=0}^{n}|x_{j+1}-x_j| \leq \prod_{j=1}^{n}(1+|x_j|)$, if the estimate

$$\sup_{L\geq 1}\sup_{0\leq k\leq N}\big\|\mathscr{F}\partial_\lambda^k(\tilde{\chi}_L(\lambda)\lambda^{-n-N})\big\|_{L^1} \leq C_N n^N\lambda_0^{-n-N}, \quad n \geq 0, \tag{2.3}$$

holds true, then we conclude that

$$\begin{aligned}
&\sup_{L\geq 1}\left|\left\langle e^{itH}\tilde{\chi}_L\big(\sqrt{H}\big)u,v\right\rangle\right| \\
&\quad \leq Ct^{-1/2-N}\sum_{n=0}^{\infty}2^{-n}n^N\lambda_0^{-n-N}\left\|(1+|x|)^N V\right\|_{L^1}^n\left\|(1+|x|)^N u\right\|_{L^1}\left\|(1+|x|)^N v\right\|_{L^1} \\
&\quad \leq Ct^{-1/2-N}\left\|\langle x\rangle^N u\right\|_{L^1}\left\|\langle x\rangle^N v\right\|_{L^1}
\end{aligned}$$

for all $t>0$. We now check (2.3). To prove this it is sufficient to show that

$$\sup_{L\geq 1}\left\|\mathscr{F}(\tilde{\chi}_L(\lambda)\lambda^{-n})\right\|_{L^1}\leq \lambda_0^{-n}\text{ for all } n\geq 0. \tag{2.4}$$

We note that since $\tilde{\chi}(\lambda)=1-\chi(\lambda)$ is not integrable, (2.4) with $n=0$ hence holds in the distribution sense. Let $n=0$, then

$$\begin{aligned}
\left\|\mathscr{F}\tilde{\chi}_L\right\|_{L^1} &\leq \left\|\mathscr{F}(\eta(\cdot/L))\right\|_{L^1}\left(1+\left\|\mathscr{F}\chi\right\|_{L^1}\right) \\
&= \left\|\mathscr{F}\eta\right\|_{L^1}\left(1+\left\|\mathscr{F}\chi\right\|_{L^1}\right)<\infty
\end{aligned} \tag{2.5}$$

uniformly in $L\geq 1$ since $\chi\in C_0^\infty([-2\lambda_0,2\lambda_0])$. Let $n\geq 1$, then we have

$$\left\|\mathscr{F}(\tilde{\chi}_L(\lambda)\lambda^{-n})(\xi)\cdot\xi^2\right\|_{L^\infty}\leq\left\|(\tilde{\chi}_L\lambda^{-n})''\right\|_{L^1}\leq C\lambda_0^{-n}$$

where C is independent of n and L. Moreover, we obtain that

$$\begin{aligned}
&\left\|\mathscr{F}(\tilde{\chi}_L(\lambda)\lambda^{-1})\right\|_{L^\infty}\leq\left\|\mathscr{F}\tilde{\chi}_L\right\|_{L^1}\left\|\lambda^{-1}\right\|_{L^\infty}<\infty, \\
&\left\|\mathscr{F}(\tilde{\chi}_L(\lambda)\lambda^{-n})\right\|_{L^\infty}\leq\left\|\tilde{\chi}_L(\lambda)\lambda^{-n}\right\|_{L^1}\leq C\lambda_0^{-n},\quad n\geq 2.
\end{aligned}$$

We hence have

$$\left|\mathscr{F}(\tilde{\chi}_L(\lambda)\lambda^{-n})(\xi)\right|\leq C\lambda_0^{-n}\frac{1}{1+|\xi|^2},\quad n\geq 1 \tag{2.6}$$

uniformly in $L\geq 1$. (2.4) follows from (2.5) and (2.6). □

3. Jost functions.

In this section, we collect results on the Jost functions $f_\pm(\lambda,x)$ needed later and improve the Fourier properties of $f_\pm$ obtained by Deift and Trubowitz [**3**], D'Ancona and Fanelli [**2**]. Given a potential $V\in L_1^1$, *the Jost functions* $f_\pm(\lambda,x)$ are the unique solutions to the equation

$$-f''(\lambda,x)+V(x)f(\lambda,x)=\lambda^2 f(\lambda,x)$$

satisfying the asymptotic conditions

$$\left|f_\pm(\lambda,x)-e^{\pm i\lambda x}\right|\to 0 \text{ as } x\to\pm\infty. \tag{3.1}$$

We recall a few properties of the Jost functions. $f_\pm(\lambda,x)$ satisfies $\overline{f_\pm(\lambda,x)}=f_\pm(-\lambda,x)$. Using the Jost functions and their Wronskian, the kernel of the resolvent R_λ can be written by

$$R_\lambda(x,y)=\begin{cases}\dfrac{f_+(\lambda,y)f_-(\lambda,x)}{W(\lambda)} & \text{for } x<y,\\[2ex] \dfrac{f_-(\lambda,y)f_+(\lambda,x)}{W(\lambda)} & \text{for } x>y.\end{cases} \tag{3.2}$$

$f_+(\lambda,x)$ and $f_+(-\lambda,x)$ are independent for $\lambda\neq 0$ since their Wronskian

$$\begin{aligned}W[f_+(\lambda,\cdot),f_+(-\lambda,\cdot)] &:= f_+(\lambda,x)\cdot\partial_x f_+(-\lambda,x)-\partial_x f_+(\lambda,x)\cdot f_+(-\lambda,x)\\ &=\lim_{x\to+\infty}\left[e^{i\lambda x}(-i\lambda)e^{-i\lambda x}-i\lambda e^{i\lambda x}e^{-i\lambda x}\right]\\ &=-2i\lambda\neq 0.\end{aligned}$$

Similarly $W[f_-(\lambda,\cdot),f_-(-\lambda,\cdot)]=2i\lambda$. These imply the relations

$$\begin{aligned}T(\lambda)f_-(\lambda,x)&=R_1(\lambda)f_+(\lambda,x)+f_+(-\lambda,x),\\ T(\lambda)f_+(\lambda,x)&=R_2(\lambda)f_-(\lambda,x)+f_-(-\lambda,x),\end{aligned} \tag{3.3}$$

where $T(\lambda)$, $R_1(\lambda)$ and $R_2(\lambda)$ are the *transmission* and *reflection* coefficients, respectively. The modified Jost functions are given by $m_\pm(\lambda,x):=e^{\mp i\lambda x}f_\pm(\lambda,x)$. The $m_\pm(\lambda,x)$ are the unique solutions to the Volterra integral equations

$$m_\pm(\lambda,x)=1\pm\int_x^{\pm\infty}D_\lambda(\pm(y-x))V(y)m_\pm(\lambda,y)dy,$$

where $D_\lambda(x)=\int_0^x e^{2i\lambda z}dz$. It is well known that if $V\in L^1_1$, then $m_\pm(\cdot,x)-1$ belongs to the Hardy space $H^{2\pm}$ and if in addition $V\in L^1_2$, then $m_\pm(\lambda,x)\in C^1(\boldsymbol{R}^2)$ (see [**3**]). Moreover the following two lemmas hold (see [**3**], [**13**] and [**1**]).

Lemma 3.1. *Let $N\in\boldsymbol{N}$, $N\geq 2$ and suppose $V\in L^1_N$. Then $\partial_\lambda^k m_\pm(\lambda,x)$*

exist for $0 \le k \le N-1$ *and are continuous in* $(\lambda, x) \in \boldsymbol{R}^2$. *Moreover,* $m_\pm(\lambda, x)$ *satisfy*

$$\left|\partial_\lambda^k m_\pm(\lambda, x)\right| \le C(1 + \max(\mp x, 0))^{k+1}, \ (\lambda, x) \in \boldsymbol{R}^2, \quad 0 \le k \le N-1.$$

LEMMA 3.2. *Suppose* $V \in L_1^1$, *at least. Then the followings hold*:

(1)

$$\frac{1}{T(\lambda)} = \frac{W(\lambda)}{-2i\lambda} = 1 - \frac{1}{2i\lambda}\int_{\boldsymbol{R}} V(\sigma)m_+(\lambda,\sigma)d\sigma,$$
$$\frac{R_1(\lambda)}{T(\lambda)} = \frac{W[f_-(\lambda,\cdot), f_+(-\lambda,\cdot)]}{-2i\lambda} = \frac{1}{2i\lambda}\int_{\boldsymbol{R}} e^{-2i\lambda\sigma}V(\sigma)m_-(\lambda,\sigma)d\sigma,$$
$$\frac{R_2(\lambda)}{T(\lambda)} = \frac{W[f_-(-\lambda,\cdot), f_+(\lambda,\cdot)]}{-2i\lambda} = \frac{1}{2i\lambda}\int_{\boldsymbol{R}} e^{2i\lambda\sigma}V(\sigma)m_+(\lambda,\sigma)d\sigma.$$

(2)

$$|T(\lambda)|^2 + |R_j(\lambda)|^2 = 1, \quad j = 1, 2,$$
$$R_1(\lambda)\overline{T(\lambda)} + \overline{R_2(\lambda)}T(\lambda) = 0,$$
$$\overline{T(\lambda)} = T(-\lambda), \ \overline{R_j(\lambda)} = R_j(-\lambda), \quad j = 1, 2.$$

(3) (*The generic case*) *Suppose that* V *is of generic type and* $V \in L_N^1$, $N \ge 1$. *Then* T, R_1 *and* $R_2 \in C^{N-1}(\boldsymbol{R})$ *and for* $1 \le k \le N-1$,

$$\left|\partial_\lambda^k T(\lambda)\right| + \left|\partial_\lambda^k R_1(\lambda)\right| + \left|\partial_\lambda^k R_2(\lambda)\right| \le C\langle\lambda\rangle^{-1}, \quad \lambda \in \boldsymbol{R}. \tag{3.4}$$

Furthermore, we have

$$T(\lambda) = \alpha\lambda + o(1), \quad \alpha \ne 0, \quad \lambda \to 0,$$
$$R_1(0) = R_2(0) = -1.$$

(*The exceptional case*) *Suppose that* V *is of exceptional type and* $V \in L_N^1$, $N \ge 2$. *Then* T, R_1 *and* $R_2 \in C^{N-2}(\boldsymbol{R})$ *and* (3.4) *holds for* $1 \le k \le N-2$. *Furthermore, as* $\lambda \to 0$, *we have*

$$T(\lambda) = \frac{2a}{1+a^2} + o(1),$$
$$R_1(\lambda) = \frac{1-a^2}{1+a^2} + o(1), \quad R_2(\lambda) = \frac{a^2-1}{1+a^2} + o(1),$$

with $a := \lim_{x \to -\infty} f_+(0,x) \neq 0$.

Since $f_\pm(\lambda,x) = e^{\pm i\lambda x} m_\pm(\lambda,x)$, Lemma 3.1 implies

$$\left|\partial_\lambda^k f_\pm(\lambda,x)\right| \le C\langle x\rangle^k (1+\max(\mp x,0)), \quad (\lambda,x)\in \boldsymbol{R}^2, \quad 0\le k\le N-1. \quad (3.5)$$

Using Lemma 3.2, we can improve the above estimates. More precisely we prove that if $\lambda \neq 0$, then $\partial_\lambda^k f_\pm(\lambda,x)$ are bounded by $\langle x\rangle^k$, at most. Furthermore if V is of exceptional type or k is an odd number, then this holds for any $\lambda \in \boldsymbol{R}$.

Lemma 3.3.

(1) *Suppose that V is of generic type and $V \in L^1_N$, $N \ge 1$. Then*

$$\left|\partial_\lambda^k (T(\lambda) f_\pm(\lambda,x))\right| \le C\langle x\rangle^k, \quad \lambda\neq 0,\ x\in\boldsymbol{R}, \quad 0\le k\le N-1.$$

If in addition $N \ge 2$, then

$$\left|\partial_\lambda^k f_\pm(0,x)\right| \le C\langle x\rangle^k, \quad x\in\boldsymbol{R},$$

for $1 \le k \le N-1$ and k odd.

(2) *Suppose that V is of exceptional type and $V \in L^1_N$, $N \ge 2$, then*

$$\left|\partial_\lambda^k f_\pm(\lambda,x)\right| \le C\langle x\rangle^k, \quad (\lambda,x)\in\boldsymbol{R}^2, \quad 0\le k\le N-2.$$

Proof. We give the proof for f_+ only and the proof for f_- is analogous. Suppose V is of generic type. For $\lambda \neq 0$, the assertion follows from (3.3), (3.5) and Lemma 3.2 (3). By (3.3),

$$-2i\lambda f_-(\lambda,x) = W(\lambda)R_1(\lambda)f_+(\lambda,x) + W(\lambda)f_+(\lambda,x), \quad \lambda\in\boldsymbol{R}.$$

Since $f_+(\cdot,x), W, R_1 \in C^{N-1}(\boldsymbol{R}_\lambda)$ and $W(0)\neq 0$, $R_1(0) = -1$, a direct computation yields

$$\begin{aligned}-2ik\partial_\lambda^{k-1} f_-(0,x) = \sum_{j=0}^{k-1}\binom{k}{j}\big(\partial_\lambda^{k-j}(WR_1)(0) + (-1)^j\partial_\lambda^{k-j}W(0)\big)\partial_\lambda^j f_+(0,x) \\ + \{-1+(-1)^k\}W(0)\partial_\lambda^k f_+(0,x),\end{aligned}$$

for $1 \le k \le N-1$. Hence $\partial_\lambda^k f_+(0,x)$ is a linear combination of $f_+(0,x)$,

$\partial_\lambda f_+(0,x), \dots, \partial_\lambda^{k-1} f_+(0,x)$ and $\partial_\lambda^{k-1} f_-(0,x)$, provided k is an odd number. By induction, we can see that for any $1 \le k \le N-1$ and k odd, $\partial_\lambda^k f_+(0,x)$ is a linear combination of $f_\pm(0,x)$, $\partial_\lambda^2 f_\pm(0,x)$, $\partial_\lambda^4 f_\pm(0,x), \dots$, and $\partial_\lambda^{k-1} f_\pm(0,x)$. This implies

$$\left|\partial_\lambda^k f_\pm(0,x)\right| \le C\langle x\rangle^k, \ x \in \boldsymbol{R}.$$

We next consider the exceptional case. Suppose that V is of exceptional type. Since $f_+(\cdot,x), T, R_1 \in C^{N-2}(\boldsymbol{R}_\lambda)$ and $T(0) \ne 0$, the assertion follows from (3.3), (3.5) and Lemma 3.2, immediately. □

We next study Fourier properties of the Jost functions. Set

$$B_\pm(\xi,x) := \int_{\boldsymbol{R}} e^{2i\lambda\xi}(m_\pm(\lambda,x)-1)d\lambda.$$

Since $m_+(\lambda,x)-1 \in H^{2\pm}$, the support of $B_+(\xi,x)$ with respect to ξ is contained in the half line $[0,\infty)$. The function $B_+(\xi,x)$ satisfies the Marchenko equation:

$$B_+(\xi,x) = \int_{x+\xi}^\infty V(\sigma)d\sigma + \int_0^\xi d\zeta \int_{x+\xi-\zeta}^\infty V(\sigma)B_+(\zeta,\sigma)d\sigma \tag{3.6}$$

and $B_-(\xi,x)$ also satisfies a corresponding equation. It is well known (see [**3**]) that if $V \in L^1_1$, then the function $B_+(\xi,x)$ is well defined for $\xi \ge 0$, $x \in \boldsymbol{R}$ and satisfies the following estimates

$$|B_+(\xi,x)| \le e^{\gamma(x)}\eta(x+\xi), \quad \xi \ge 0, \ x \in \boldsymbol{R}, \tag{3.7}$$

where $\eta(x) = \int_x^\infty |V(\sigma)|d\sigma$, $\gamma(x) = \int_x^\infty (\sigma-x)|V(\sigma)|d\sigma$. $B_-(\xi,x)$ also satisfies a similar inequalities. Moreover, we obtain the following lemma:

Lemma 3.4. *Let $N \in \boldsymbol{N}$, $N \ge 1$ and suppose $V \in L^1_N$. Then $B_\pm(\xi,x)$ satisfy the estimates*

$$\left\|B_\pm(\cdot,x)\right\|_{L^1_{N-1}} \le C(1+\max(\mp x,0))^N, \quad x \in \boldsymbol{R}, \tag{3.8}$$

where C depends on $\|V\|_{L^1_N}$.

We set $n_\pm(\lambda,x) := (m_\pm(\lambda,x) - m_\pm(0,x))/\lambda$ and denote by $C_\pm(\xi,x)$ the Fourier transform with respect to λ of $n_\pm$:

$$C_\pm(\xi,x)=\int_{\boldsymbol{R}} e^{2i\lambda\xi}n_\pm(\lambda,x)d\lambda.$$

Then the following estimate holds for $C_\pm$ as well as $B_\pm$. The proof is obvious by Lemma 3.4 and we omit the details.

Corollary 3.5. *Let $N\in\boldsymbol{N}$, $N\ge 2$ and suppose $V\in L^1_N$. Then $C_\pm(\xi,x)$ satisfy the estimates*

$$\left\|C_\pm(\cdot,x)\right\|_{L^1_{N-2}}\le C(1+\max(\mp x,0))^N,\quad x\in\boldsymbol{R},\tag{3.9}$$

where C depends on $\|V\|_{L^1_N}$.

Remark 3.6. The estimates (3.8) with $N=1,2$ and (3.9) with $N=2$ were proved by D'Ancona and Fanelli [**2**].

Proof of Lemma 3.4. We prove the estimates for B_+ only and the proof for B_- is similar. We first prove the case $x\ge 0$. Since $\operatorname{supp}B_+(\cdot,x)\subset[0,\infty)$, a direct computation yields

$$\begin{aligned}\left\|\xi^{N-1}B_+(\xi,x)\right\|_{L^1(\boldsymbol{R}_\xi)}&\le e^{\gamma(x)}\int_0^\infty\xi^{N-1}\int_{x+\xi}^\infty|V(\sigma)|d\sigma d\xi\\&=\frac{1}{N}e^{\gamma(x)}\int_x^\infty(\sigma-x)^N|V(\sigma)|d\sigma\\&\le Ce^{\|V\|_{L^1_1}}\|V\|_{L^1_N}\end{aligned}$$

for all $N\ge 1$, provided $x\ge 0$. This implies (3.8) with $x\ge 0$. The proof for $x<0$ is by induction on N. Multiplying the Marchenko equation (3.6) by ξ^{N-1} and integrating in ξ from 0 to ∞, we have

$$\begin{aligned}&\left\|\xi^{N-1}B_+(\xi,x)\right\|_{L^1(\boldsymbol{R}_\xi)}\\&\quad\le\int_0^\infty\xi^{N-1}d\xi\int_{x+\xi}^\infty|V(\sigma)|d\sigma+\int_0^\infty\xi^{N-1}d\xi\int_0^\xi d\zeta\int_{x+\xi-\zeta}^\infty|V(\sigma)||B_+(\zeta,\sigma)|d\sigma\\&=:B_1+B_2.\end{aligned}\tag{3.10}$$

It is clear that B_1 is dominated by $C\|V\|_{L^1_N}\langle x\rangle^N$. Changing the order of integration of B_2, we obtain

$$
\begin{aligned}
B_2 &= \int_0^\infty d\zeta \int_x^\infty d\sigma \int_\zeta^{\zeta+\sigma-x} \xi^{N-1}|V(\sigma)||B_+(\zeta,\sigma)|d\xi \\
&= \frac{1}{N}\sum_{k=1}^{N}\binom{N}{k}\int_0^\infty d\zeta \int_x^\infty \zeta^{N-k}(\sigma-x)^k|V(\sigma)||B_+(\zeta,\sigma)|d\sigma.
\end{aligned}
$$

If $N=1$, then

$$
B_2 \le \int_x^\infty (\sigma-x)|V(\sigma)|\big\|B_+(\cdot,\sigma)\big\|_{L^1}d\sigma.
$$

Since

$$
\int_x^0 \sigma|V(\sigma)|\big\|B_+(\cdot,\sigma)\big\|_{L^1}d\sigma \le 0,
$$

and $\big\|B_+(\cdot,\sigma)\big\|_{L^1}$ is bounded uniformly in $\sigma \ge 0$, we obtain

$$
\begin{aligned}
\big\|B_+(\cdot,x)\big\|_{L^1} &\le C\|V\|_{L^1_1}\langle x\rangle + C\int_0^\infty \sigma|V(\sigma)|d\sigma + \langle x\rangle \int_x^\infty |V(\sigma)|\big\|B_+(\cdot,\sigma)\big\|_{L^1}d\sigma \\
&\le C\langle x\rangle + \langle x\rangle \int_x^\infty |V(\sigma)|\big\|B_+(\cdot,\sigma)\big\|_{L^1}d\sigma.
\end{aligned}
$$

We now can apply Gronwall's lemma for $x<0$, and have the bound (3.8) for $N=1$. For $N\ge 2$, we see that

$$
\begin{aligned}
B_2 &\le C\sum_{k=2}^{N}\int_x^\infty (\sigma-x)^k|V(\sigma)|\big\|\zeta^{N-k}B_+(\zeta,\sigma)\big\|_{L^1(\boldsymbol{R}_\zeta)}d\sigma \\
&\quad + C\int_x^\infty (\sigma-x)|V(\sigma)|\big\|\zeta^{N-1}B_+(\zeta,\sigma)\big\|_{L^1(\boldsymbol{R}_\zeta)}d\sigma \\
&=: B_{21}+B_{22}.
\end{aligned}
$$

By hypothesis for the induction and the trivial inequality

$$
(\sigma-x)^k \le |x|^k, \quad x\le\sigma\le 0,
$$

we have

$$\int_x^0 (\sigma - x)^k |V(\sigma)| \big\| \zeta^{N-k} B_+(\zeta,\sigma) \big\|_{L^1(\boldsymbol{R}_\zeta)} d\sigma \le C \int_x^0 |x|^k |V(\sigma)| \langle \sigma \rangle^{N-k+1} d\sigma$$
$$\le C \|V\|_{L^1_{N-k+1}} \langle x \rangle^k$$

for all $2 \le k \le N$. By the inequality $(\sigma - x)^k \le C(|x|^k + |\sigma|^k)$ for $0 \le \sigma$, we also obtain

$$\int_0^\infty (\sigma - x)^k |V(\sigma)| \big\| \zeta^{N-k} B_+(\zeta,\sigma) \big\|_{L^1(\boldsymbol{R}_\zeta)} d\sigma \le C \|V\|_{L^1_k} \langle x \rangle^k, \quad 2 \le k \le N,$$

since $\big\| \zeta^{N-k} B_+(\zeta,\sigma) \big\|_{L^1(\boldsymbol{R}_\zeta)}$ is bounded uniformly in $\sigma \ge 0$. We thus have

$$B_{21} \le C \sum_{k=2}^N \big(\|V\|_{L^1_{N-k+1}} \langle x \rangle^k + \|V\|_{L^1_k} \langle x \rangle^k \big)$$
$$\le C\big(\|V\|_{L^1_N}\big) \langle x \rangle^N. \tag{3.11}$$

On the other hand, since

$$\int_x^0 \sigma |V(\sigma)| \big\| \zeta^{N-1} B_+(\zeta,\sigma) \big\|_{L^1(\boldsymbol{R}_\zeta)} d\sigma \le 0$$

and $\big\| \zeta^{N-1} B_+(\zeta,\sigma) \big\|_{L^1(\boldsymbol{R}_\xi)}$ is bounded uniformly in $\sigma \ge 0$, B_{22} satisfies

$$B_{22} \le C \int_0^\infty \sigma |V(\sigma)| \big\| \zeta^{N-1} B_+(\zeta,\sigma) \big\|_{L^1(\boldsymbol{R}_\zeta)} d\sigma$$
$$- C \int_x^\infty x |V(\sigma)| \big\| \zeta^{N-1} B_+(\zeta,\sigma) \big\|_{L^1(\boldsymbol{R}_\zeta)} d\sigma$$
$$\le C\big(\|V\|_{L^1_N}\big) + C\langle x \rangle \int_x^\infty |V(\sigma)| \big\| \zeta^{N-1} B_+(\zeta,\sigma) \big\|_{L^1(\boldsymbol{R}_\zeta)} d\sigma. \tag{3.12}$$

By (3.10), (3.11) and (3.12), we obtain

$$\langle x \rangle^{-N} \big\| \xi^{N-1} B_\pm(\xi,x) \big\|_{L^1(\boldsymbol{R}_\xi)}$$
$$\le C\big(\|V\|_{L^1_N}\big) + C \int_x^\infty |V(\sigma)| \langle \sigma \rangle^N \langle \sigma \rangle^{-N} \big\| \zeta^{N-1} B_+(\zeta,\sigma) \big\|_{L^1(\boldsymbol{R}_\zeta)} d\sigma.$$

Applying the Gronwall's lemma for $x \le 0$, we have

$$\langle x\rangle^{-N} \big\| \xi^{N-1} B_{\pm}(\xi, x) \big\|_{L^1(\boldsymbol{R}_\xi)} \le C\big(\|V\|_{L^1_N}\big) \tag{3.13}$$

for $x \le 0$, $N \ge 2$ and we conclude the proof. □

Using Lemma 3.4 and (3.4), we obtain the following Lemma which is a natural generalization of Lemma 5 in [**9**].

LEMMA 3.7. *Let* $\chi \in C_0^\infty(\boldsymbol{R})$.

(1) *Suppose that* $V \in L^1_N$, $N \ge 1$ *and* V *is of generic type, then*

$$\mathscr{F}\left(\frac{\chi}{W}\right) \in L^1_{N-1}.$$

(2) *Suppose that* $V \in L^1_N$, $N \ge 2$ *and* V *is of exceptional type, then*

$$\mathscr{F}\left(\frac{\lambda\chi}{W}\right) \in L^1_{N-2}.$$

Here $W(\lambda)$ *is the Wronskian of the Jost functions.*

PROOF. We first prove the generic case. By Lemma 3.2 (1),

$$\chi(\lambda)W(\lambda) = -2i\lambda\chi(\lambda) + \int_{\boldsymbol{R}} V(\sigma)\chi(\lambda)m_+(\lambda,\sigma)d\sigma. \tag{3.14}$$

Taking the Fourier transform with respect to λ of (3.14) and using the Minkowski inequality, we have

$$\begin{aligned}\big\|\mathscr{F}(\chi W)\big\|_{L^1_{N-1}} &\le 2\big\|\mathscr{F}(\lambda\chi)\big\|_{L^1_{N-1}} + \int_{\boldsymbol{R}} |V(\sigma)| \big\|\mathscr{F}(\chi m_+(\lambda,\sigma))\big\|_{L^1_{N-1}} d\sigma \\ &\le C\|V\|_{L^1_N}.\end{aligned}$$

For the last inequality, we used Lemma 3.4 and support property of χ. Now choosing a smooth cut-off $\tilde\chi$ such that $\chi\tilde\chi \equiv \chi$, we can realize that $\chi/W \equiv \chi/\tilde\chi W$. Since $\mathscr{F}\chi$ and $\mathscr{F}(\tilde\chi W)$ are in L^1_{N-1} and $\tilde\chi W$ does not vanish in $\operatorname{supp}\chi$, we can apply Wiener's lemma for $\partial_\lambda^{N-1}(\chi/\tilde\chi W)$ and we conclude that $\mathscr{F}(\chi/\tilde\chi W) \in L^1_{N-1}$.

For the exceptional case, we note that V is of exceptional type *i.e.* $W(0) = 0$ if and only if

$$\int_{\boldsymbol{R}} V(\sigma)m_+(0,\sigma)d\sigma = 0.$$

We set

$$\begin{aligned} p(\lambda) := \frac{1}{T(\lambda)} = \frac{W(\lambda)}{-2i\lambda} &= 1 - \frac{1}{2i\lambda}\int_{\boldsymbol{R}} V(\sigma)m_+(\lambda,\sigma)d\sigma \\ &= 1 - \frac{1}{2i}\int_{\boldsymbol{R}} V(\sigma)n_+(\lambda,\sigma)d\sigma \end{aligned} \tag{3.15}$$

where $n_+(\lambda,x) = (m_+(\lambda,x) - m_+(0,x))/\lambda$. Since $T(\lambda)$ is continuous on $\boldsymbol{R}$, $p(\lambda) \neq 0$ for all $\lambda \in \boldsymbol{R}$. Taking the Fourier transform with respect to λ of (3.15) and using the Minkowski inequality, we obtain

$$\begin{aligned} \left\|\mathscr{F}(\chi p)\right\|_{L^1_{N-2}} &\le \left\|\mathscr{F}\chi\right\|_{L^1_{N-2}} + \frac{1}{2}\int_{\boldsymbol{R}} |V(\sigma)| \left\|\mathscr{F}(\chi n_+(\lambda,\sigma))\right\|_{L^1_{N-2}} d\sigma \\ &\le C\|V\|_{L^1_N}. \end{aligned}$$

For the last inequality, we use Corollary 3.5 and support property of χ. By a similar argument as in the generic case, we can see that $2i\lambda\chi/W \equiv \chi/\tilde{\chi}p$. Since $\mathscr{F}\chi$ and $\mathscr{F}(\tilde{\chi}p)$ are in L^1_{N-2} and $\tilde{\chi}p$ does not vanish in $\operatorname{supp}\chi$, we can apply Wiener's lemma and we conclude that $\mathscr{F}(\chi/\tilde{\chi}p) \in L^1_{N-2}$. $\square$

Recall that $f_\pm(\lambda,x) = e^{\pm i\lambda x}m_\pm(\lambda,x)$, $T(\lambda) = -W(\lambda)/2i\lambda$. By the same argument as in the proof of Lemma 3.3, we obtain the following.

Corollary 3.8. *Let $\chi \in C_0^\infty(\boldsymbol{R})$. Suppose that $V \in L^1_N$, $N \ge 1$. Then*

$$\left\|\mathscr{F}(\chi(\cdot)f_\pm(\cdot,x))\right\|_{L^1_{N-1}} \le C\langle x\rangle^{N-1}(1+\max(\mp x,0)), \quad x \in \boldsymbol{R}, \tag{3.16}$$

Furthermore,

(1) *If V is of generic type, then*

$$\left\|\mathscr{F}(\chi(\cdot)T(\cdot)f_\pm(\cdot,x))\right\|_{L^1_{N-1}} \le C\langle x\rangle^{N-1}, \quad x \in \boldsymbol{R}.$$

(2) *If V is of exceptional type and $V \in L^1_N$, $N \ge 2$, then*

$$\left\|\mathscr{F}(\chi(\cdot)f_\pm(\cdot,x))\right\|_{L^1_{N-2}} \le C\langle x\rangle^{N-2}, \quad x \in \boldsymbol{R}.$$

4. The low energy estimates.

To complete the proof of Theorem 1.2, we prove the following.

PROPOSITION 4.1. *Let m be a positive integer and let χ be an even smooth cut-off function such that $\chi(\lambda) = 1$ close to zero. Suppose that $V \in L^1_{2m}$ and V is of generic type, or else that $V \in L^1_{2m+2}$ and V is of exceptional type. Let*

$$s = \begin{cases} 2m-1 & \text{if } V \text{ is of generic type,} \\ 2m & \text{if } V \text{ is of exceptional type.} \end{cases}$$

Then there exists an operator P_{m-1} such that

$$\big\|\langle x\rangle^{-s}\big(e^{-itH}\chi\big(\sqrt{H}\big)P_{ac} - P_{m-1}\big)u\big\|_{L^\infty} \le Ct^{-1/2-m}\big\|\langle x\rangle^s u\big\|_{L^1}, \quad t > 0.$$

Moreover P_{m-1} has the following expansion:

$$P_{m-1} = \sum_{j=0}^{m-1} t^{-1/2-j} C_{j-1}, \quad t > 0,$$

where the coefficients C_{j-1} are given by

$$C_{j-1}u(x) = \frac{1}{\sqrt{4\pi i}j!(4i)^j}\int_{\boldsymbol{R}} \big(\partial_\lambda^{2j}K\big)(0,x,y)u(y)dy,$$
$$K(\lambda,x,y) := T(\lambda)f_+(\lambda,y)f_-(\lambda,x),$$

and satisfy the conditions as in Theorem 1.2 *and Remark* 1.3.

PROOF. We first consider the generic case. Set

$$G(\lambda,x,y) := \frac{K(\lambda,x,y)}{\lambda}.$$

We start from the representation

$$\begin{aligned}\big\langle e^{-itH}\chi\big(\sqrt{H}\big)P_{ac}u, v\big\rangle &= \frac{1}{\pi i}\int_{\boldsymbol{R}} e^{-it\lambda^2}\lambda\chi(\lambda)\langle R_\lambda u, v\rangle d\lambda \\ &= \frac{1}{2\pi}\int_{\boldsymbol{R}^2}\bigg(\int_{\boldsymbol{R}} e^{-it\lambda^2}\lambda\chi(\lambda)\tilde{G}(\lambda,x,y)d\lambda\bigg)u(y)\overline{v(x)}dydx,\end{aligned}$$

where $\tilde{G}(\lambda,x,y)$ denotes the kernel of $-2iR_\lambda$ and is given by

$$\tilde{G}(\lambda,x,y)=\begin{cases}G(\lambda,x,y) & \text{for } x<y,\\ G(\lambda,y,x) & \text{for } x>y.\end{cases} \tag{4.1}$$

Consider the integral

$$\begin{aligned} I(t,G) &:= \frac{1}{2\pi}\int_{\boldsymbol{R}} e^{-it\lambda^2}\lambda\chi(\lambda)G(\lambda,x,y)d\lambda \\ &= \frac{1}{4\pi it}\int_{\boldsymbol{R}} e^{-it\lambda^2}\partial_\lambda(\chi(\lambda)G(\lambda,x,y))d\lambda. \end{aligned} \tag{4.2}$$

for $x<y$. The proof for the case $x>y$ is analogous.

The case $m=1$: It suffice to show that

$$|I(t,G)|\le Ct^{-3/2}\langle x\rangle\langle y\rangle,\quad x<y. \tag{4.3}$$

Using the Fourier inversion formula, we obtain

$$|I(t,G)|\le Ct^{-3/2}\big\|(\mathscr{F}\partial_\lambda\chi(\cdot)G(\cdot,x,y)\big\|_{L^1}$$

for all $t>0$ and $x<y$, where $\mathscr{F}$ is the Fourier transform with respect to λ. By Young's inequality, Corollary 3.8 and Lemma 3.7 (1), we have

$$\big\|(\mathscr{F}\partial_\lambda\chi G)(\cdot,x,y)\big\|_{L^1}\le C\langle x\rangle\langle y\rangle,\quad x<y.$$

The case $m\ge 2$: Applying the stationary phase method to (4.2), we have

$$I(t,G)=\frac{1}{\sqrt{\pi i}}\sum_{j=1}^{m-1}\frac{t^{-1/2-j}}{(j-1)!(4i)^j}\big(\partial_\lambda^{2j-1}G\big)(0,x,y)+t^{-1/2-m}S_{m-1}(t,G),$$

where the remainder satisfies

$$|S_{m-1}(t,G)|\le C\big\|\big(\mathscr{F}\partial_\lambda^{2m-1}\chi G\big)(\cdot,x,y)\big\|_{L^1}\le C\langle x\rangle^{2m-1}\langle y\rangle^{2m-1},$$

by Lemma 3.7 (1) and Corollary 3.8. Considering the fact that

$$\big(\partial_\lambda^{2j-1}G\big)(0,x,y)=\frac{1}{2j}\big(\partial_\lambda^{2j}K\big)(0,x,y),$$

$$\big(\partial_\lambda^{2j}K\big)(0,x,y) = \big(\partial_\lambda^{2j}K\big)(0,y,x), \quad x,y \in \boldsymbol{R},\ j = 1,2,\dots,m,$$

we now define C_{j-1} and P_{m-1} by

$$\begin{aligned} C_{j-1}u(x) &:= \frac{1}{\sqrt{4\pi i}j!(4i)^j}\int_{\boldsymbol{R}}\big(\partial_\lambda^{2j}K\big)(0,x,y)u(y)dy,\ x \in \boldsymbol{R}, \\ P_{m-1} &:= \sum_{j=0}^{m-1} t^{-1/2-j}C_{j-1}. \end{aligned} \tag{4.4}$$

We then have

$$\big|\big\langle\big(e^{-itH}\chi(\sqrt{H})P_{ac} - P_{m-1}\big)u,v\big\rangle\big| \le Ct^{-1/2-m}\big\|\langle x\rangle^{2m-1}u\big\|_{L^1}\big\|\langle x\rangle^{2m-1}v\big\|_{L^1}.$$

By the definition and Corollary 3.3 (1), we can see that $\operatorname{rank} C_{j-1} \le 2j$, and

$$\big\|\langle x\rangle^{-2j+1}C_{j-1}u\big\|_{L^\infty} \le C\big\|\langle x\rangle^{2j-1}u\big\|_{L^1}.$$

In particular, $C_{-1} \equiv 0$. These complete the proof of the generic case.

We next consider the exceptional case. Suppose that $V \in L^1_{2m+2}$ and V is of exceptional type. By the stationary phase method, Lemma 3.7 (2) and Corollary 3.8, we have

$$\begin{aligned} &\frac{1}{2\pi}\int_{\boldsymbol{R}} e^{-it\lambda^2}\chi(\lambda)K(\lambda,x,y)d\lambda \\ &\quad = \frac{1}{\sqrt{4\pi i}}\sum_{j=0}^{m-1}\frac{t^{-1/2-j}}{j!(4i)^j}\partial_\lambda^{2j}K(0,x,y) + t^{-1/2-m}S_m(t,K) \end{aligned}$$

and

$$|S_m(t,K)| \le C\big\|(\mathscr{F}\partial_\lambda^{2m}K)(\cdot,x,y)\big\|_{L^1} \le C\langle x\rangle^{2m}\langle y\rangle^{2m},$$

for $x < y$. The same argument as in proof of the generic case implies

$$\big\|\langle x\rangle^{-2m}\big(e^{-itH}\chi(\sqrt{H})P_{ac} - P_{m-1}\big)u\big\|_{L^\infty} \le Ct^{-1/2-m}\big\|\langle x\rangle^{2m}u\big\|_{L^1},$$

where P_{m-1} is given by (4.4). Since $T(0) \ne 0$, $\operatorname{rank} C_{j-1} \le 2j+1$ and

$$\big\|\langle x\rangle^{-2j}C_{j-1}u\big\|_{L^\infty} \le C\big\|\langle x\rangle^{2j}u\big\|_{L^1}.$$

Finally, we define a function $f_0(x)$ by

$$f_+(0,x) = \sqrt{1 + \left(\frac{R_2(0)}{T(0)} + \frac{1}{T(0)}\right)^2} f_0(x),$$

where the coefficient follows from the asymptotic behavior of $f_+(0,x)$:

$$f_+(0,x) \to \begin{cases} 1 & \text{as } x \to +\infty, \\ \dfrac{R_2(0)}{T(0)} + \dfrac{1}{T(0)} & \text{as } x \to -\infty, \end{cases}$$

(see (3.1), (3.3) and (3.4)). $f_0(x)$ is a bounded solution to the equation $Hf = 0$ and satisfies the normalized condition

$$\lim_{x \to +\infty} (|f_0(x)|^2 + |f_0(-x)|^2) = 2,$$

and C_{-1} can be written by

$$\begin{aligned} C_{-1}u(x) &= \frac{1}{\sqrt{4\pi i}} \int T(0) f_-(0,x) f_+(0,y) u(y) dy \\ &= \frac{1}{\sqrt{4\pi i}} \int f_0(y) u(y) dy f_0(x). \end{aligned}$$

We complete the proof. □

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