

Classification of plane curves with infinitely many Galois points

Dedicated to the memory of my mother Yoko Fukasawa

By Satoru FUKASAWA

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Abstract. For a plane curve, a point in the projective plane is said to be Galois when the point projection induces a Galois extension of function fields. We completely classify plane curves with infinitely many outer Galois points.

1. Introduction.

Let $C \subset \mathbf{P}^2$ be an irreducible plane curve of degree $d \geq 3$ over an algebraically closed field K of characteristic $p \geq 0$ and let $K(C)$ be the function field of C . The point projection $\pi_R : C \dashrightarrow \mathbf{P}^1$ from a point $R \in \mathbf{P}^2$ induces a field extension $K(C)/K(\mathbf{P}^1)$ of function fields. When the extension is Galois, we call the point R a *Galois point* for C . This notion was introduced by H. Yoshihara ([13], [22]). A Galois point $R \in \mathbf{P}^2$ is said to be inner (resp. outer) if $R \in C$ (resp. $R \in \mathbf{P}^2 \setminus C$). It is one of important problems to determine the distribution of Galois points for a given curve. In many cases, the distribution has been determined ([2], [7], [12], [13], [22], [23]). In most of such settled cases, the number of Galois points is finite. However, recently T. Hasegawa and the present author [3] found an example of a plane curve having infinitely many inner and outer Galois points. Also, they classified a curve whose almost all inner points are Galois.

The purpose of this paper is to classify plane curves with infinitely many outer Galois points. We will prove the following classification theorem.

THEOREM 1. *Let $C \subset \mathbf{P}^2$ be an irreducible plane curve of degree $d \geq 3$ over an algebraically closed field K of characteristic $p \geq 0$. We denote by Δ' the set of all outer Galois points for C . Then, the following conditions are equivalent.*

- (1) *The set Δ' is infinite.*

- (2) The curve C is a rational strange curve with a center Q and there exists a line $L \subset \mathbf{P}^2$ which contains Q and infinitely many outer Galois points.
- (3) $p > 0$, $d = p^e$ for some $e \geq 1$ and the curve C is projectively equivalent to the image of a morphism $\psi(u : 1) = (\psi_1(u) : \psi_2(u) : 1) : \mathbf{P}^1 \rightarrow \mathbf{P}^2$ such that ψ is birational onto its image and

$$\psi_1 = a_1 u^p + \cdots + a_e u^{p^e}, \quad \psi_2 = u + b_1 u^p + \cdots + b_e u^{p^e},$$

where $a_1, \dots, a_e, b_1, \dots, b_e \in K$.

- (4) $p > 0$, $d = p^e$ for some $e \geq 1$ and the curve C is projectively equivalent to an irreducible plane curve whose equation is of the form

$$\alpha_e x^{p^e} + \alpha_{e-1} x^{p^{e-1}} + \cdots + \alpha_1 x^p + x + \beta_e y^{p^e} + \cdots + \beta_1 y^p = 0,$$

where $\alpha_1, \dots, \alpha_e, \beta_1, \dots, \beta_e \in K$.

Moreover, if one of the conditions holds, then Δ' is a Zariski open set of a line (see Proposition 2) and the Galois group G_R at R is isomorphic to $(\mathbf{Z}/p\mathbf{Z})^{\oplus e}$ for any $R \in \Delta'$, where $p^e = d$.

In Section 2, we introduce some notation and recall some notions. In Section 3, we give several lemmas for plane curves having infinitely many outer Galois points. In Section 4, we prove the rationality when Δ' is infinite. In Section 5, we prove the assertions (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) in our Theorem 1. Finally, in Section 6, we prove the assertion (4) \Rightarrow (1) and discuss the distribution of Galois points.

2. Preliminaries.

Let $(X : Y : Z)$ be a system of homogeneous coordinates of the projective plane \mathbf{P}^2 and let $C \subset \mathbf{P}^2$ be an irreducible plane curve of degree $d \geq 3$. We denote by C_{sm} the smooth locus of C and by $\text{Sing}(C)$ the singular locus of C . If $P \in C_{\text{sm}}$, we denote by $T_P C \subset \mathbf{P}^2$ the (projective) tangent line at P . For a projective line $l \subset \mathbf{P}^2$ and a point $P \in C \cap l$, we denote by $I_P(C, l)$ the intersection multiplicity of C and l at P .

A tangent line at a singular point $P \in \text{Sing}(C)$ is defined as follows. Let $f(x, y)$ be the defining polynomial of C in the affine plane defined by $Z \neq 0$, and let $P = (0 : 0 : 1)$. We can write $f = f_m + f_{m+1} + \cdots + f_d$, where f_i is the i -th homogeneous component. A tangent line at P is the line defined by an irreducible component of f_m . Therefore, a line l passing through P is a tangent line at P if and only if $I_P(C, l) > m$.

Let $r : \hat{C} \rightarrow C$ be the normalization. We denote by \overline{RP} the line passing

through points R and P when $R \neq P$, and by $\pi_R : C \dashrightarrow \mathbf{P}^1; P \mapsto \overline{RP}$ the point projection from a point $R \in \mathbf{P}^2$. We write $\hat{\pi}_R = \pi_R \circ r$. If $P \in C_{\text{sm}}$, we denote by e_P the ramification index of $\hat{\pi}_R$ at (the fiber $r^{-1}(P)$ of) P . It is not difficult to check the following

LEMMA 1. *Let $R \in \mathbf{P}^2 \setminus C$ and let $P \in C_{\text{sm}}$. Then for $\hat{\pi}_R$ we have*

$$e_P = I_P(C, \overline{RP}).$$

Let $\check{\mathbf{P}}^2$ be the dual projective plane, which parameterizes lines on \mathbf{P}^2 . Let $\gamma : C_{\text{sm}} \rightarrow \check{\mathbf{P}}^2$ be the dual map of C , which assigns a smooth point $P \in C_{\text{sm}}$ to the tangent line $T_PC \in \check{\mathbf{P}}^2$ at P . We denote by $s(\gamma)$ the separable degree of (the function field extension defined by) the dual map γ of C onto its image, by $q(\gamma)$ the inseparable degree of γ , and by $M(C)$ the generic order of contact (i.e. $I_P(C, T_PC) \geq M(C)$ for any point $P \in C_{\text{sm}}$ and the equality holds for a general point (see [20, p. 5]).), throughout this paper. If the dual map γ is separable onto its image, then $s(\gamma) = 1$ and $M(C) = 2$ (see, for example, [14, Proposition 1.5]). If the dual map γ of C is not separable, then it follows from a theorem of Hefez-Kleiman ([6, (3.5)]) that $M(C) = q(\gamma)$. Using this theorem and Bézout's theorem, we find that $d \geq s(\gamma)q(\gamma)$.

We recall the definition of strangeness. The strangeness can be defined for a curve in projective space, in general. Let $C \subset \mathbf{P}^N$ be an irreducible curve of degree $d \geq 3$ with $N \geq 2$. If there exists a point $Q \in \mathbf{P}^N$ such that almost all tangent lines of C pass through Q , then C is said to be *strange* and Q is called a *strange center* (see [1], [9], [16]). It is easily checked that a strange center is unique for a strange curve. The following fact was proved by E. Lluís [10, p. 51] and reproved by P. Samuel [16, p. 76] (see also [1, (2.2) Remark], [9, p. 167]).

FACT 1. *If a curve $C \subset \mathbf{P}^N$ of degree $d \geq 3$ is strange, then C is singular.*

We consider the case where $N = 2$. Using Lemma 1, we find that the projection $\hat{\pi}_Q$ from a point Q is not separable if and only if C is strange and Q is the strange center. Therefore, if Q is the strange center, then Q is not Galois. If C is strange, then we can identify the dual map γ with the projection π_Q from the strange center Q . Therefore, the dual map γ is not separable. Furthermore, for a strange curve, $d = s(\gamma)q(\gamma)$ if and only if the strange center Q is not contained in C (see also [1, (2.3) Theorem]).

We denote by $\Delta' \subset \mathbf{P}^2$ the set of all outer Galois points for a plane curve $C \subset \mathbf{P}^2$ and by G_R the group of birational maps from C to itself corresponding to the Galois group $\text{Gal}(K(C)/\pi_R^*K(\mathbf{P}^1))$ when R is Galois. We find easily that the group G_R is isomorphic to a subgroup of the automorphism group $\text{Aut}(\hat{C})$ of

\hat{C} . Frequently, we identify G_R with the subgroup. If a Galois covering $\theta : C \rightarrow C'$ between smooth curves is given, then the Galois group G acts on C naturally. We denote by $G(P)$ the stabilizer subgroup of P and by e_P the ramification index at P . The following fact is useful to find Galois points (see [19, III. 7.1, 7.2 and 8.2]).

FACT 2. Let $\theta : C \rightarrow C'$ be a Galois covering of degree d with a Galois group G . Then we have the following.

- (1) For any $\sigma \in G$, we have $\theta(\sigma(P)) = \theta(P)$.
- (2) If $\theta(P) = \theta(Q)$, then there exists an element $\sigma \in G$ such that $\sigma(P) = Q$.
- (3) The order of $G(P)$ is equal to e_P at P for any point $P \in C$.
- (4) If $\theta(P) = \theta(Q)$, then $e_P = e_Q$.
- (5) The index e_P divides the degree d .

Especially, for a strange curve C and an outer Galois point R with respect to C , we can classify ramification points of $\hat{\pi}_R$, as follows.

LEMMA 2. Let C be a strange curve with a center Q . Assume that $R \in \mathbf{P}^2 \setminus C$ is Galois and $P \in \hat{C}$ is a ramification point of $\hat{\pi}_R$. Then, we have the following assertions.

- (1) If $r(P) \in C_{\text{sm}}$, then the line $\overline{r(P)R}$ coincides with the tangent line $T_{r(P)}C$ at $r(P)$. Especially, $r(P)$ should be contained in \overline{RQ} .
- (2) If $r(P) \in \text{Sing}(C)$ and $r(P) \notin \overline{RQ}$, then the line $\overline{r(P)R}$ does not contain a smooth point. Especially, the line $\overline{r(P)R}$ is a tangent line at $r(P)$, or contains two or more singular points.

PROOF. We prove the assertion (1). The first assertion is obvious by Lemma 1. Now, $Q \in T_{r(P)}C$ by the definition of the strangeness. Then, $r(P) \in T_{r(P)}C = \overline{RQ}$.

We prove the assertion (2). If $\overline{r(P)R}$ contains a smooth point P' , then $\overline{r(P)R}$ is tangent to C at P' by our assumption, Fact 2(4) and Lemma 1. Then, $Q \in \overline{r(P)R}$ since Q is the strange center. This is a contradiction. \square

3. Plane curves having infinitely many outer Galois points.

Throughout this section, we assume that the set Δ' of outer Galois points is infinite.

LEMMA 3. If Δ' is infinite, then C is strange.

PROOF. Let $\gamma(C_{\text{sm}})$ be the image of dual map γ on the smooth locus. (We do not take the closure of $\gamma(C_{\text{sm}})$.) Let $\Lambda \subset \gamma(C_{\text{sm}})$ be the set of tangent lines T

at smooth points satisfying one of the following conditions. (i) The tangent line T contains a singular point, (ii) the cardinality of the fiber $\gamma^{-1}(T)$ is strictly less than $s(\gamma)$, or (iii) the tangent line T contains a smooth point P with $I_P(C, T) > M(C)$.

If Λ is one-dimensional, then C is strange by the condition (i) of Λ , since tangent lines which satisfy the condition (ii) or (iii) are finitely many. We may assume that Λ is a finite set.

We prove that there exists a tangent line $T \in \gamma(C_{\text{sm}}) \setminus \Lambda$ such that $T \cap \Delta' \neq \emptyset$, if C is not strange. If $\Delta' \cap (\bigcup_{T \in \gamma(C_{\text{sm}}) \setminus \Lambda} T) \neq \emptyset$, then we have nothing to prove. Assume that $\Delta' \cap (\bigcup_{T \in \gamma(C_{\text{sm}}) \setminus \Lambda} T) = \emptyset$. Then, Δ' is contained in a finite union of lines. Therefore, there exists a line $L \in \check{\mathbf{P}}^2$ such that $L \cap \Delta'$ is infinite. Considering a morphism $\gamma(C_{\text{sm}}) \setminus (\Lambda \cup \{L\}) \rightarrow L$; $T \mapsto T \cap L$, we find that the set $L \cap \bigcup_{T \in \gamma(C_{\text{sm}}) \setminus (\Lambda \cup \{L\})} T$ is a Zariski open subset of L or consists of a unique point, since $\gamma(C_{\text{sm}}) \setminus (\Lambda \cup \{L\})$ is irreducible. If $L \cap \bigcup_{T \in \gamma(C_{\text{sm}}) \setminus (\Lambda \cup \{L\})} T$ consists of a unique point, then C is strange. If $L \cap \bigcup_{T \in \gamma(C_{\text{sm}}) \setminus (\Lambda \cup \{L\})} T$ is a Zariski open subset of L , then $L \cap \Delta' \cap \bigcup_{T \in \gamma(C_{\text{sm}}) \setminus (\Lambda \cup \{L\})} T \neq \emptyset$, since the set $L \cap \Delta'$ is infinite. Then, there exists a tangent line $T \in \gamma(C_{\text{sm}}) \setminus (\Lambda \cup \{L\})$ such that $T \cap \Delta' \neq \emptyset$.

Let $T \in \gamma(C_{\text{sm}}) \setminus \Lambda$ satisfy that $T \cap \Delta' \neq \emptyset$. Then, all points of $C \cap T$ are smooth and T contains exactly $s(\gamma)$ points of contact $Q_1, \dots, Q_{s(\gamma)}$ such that the intersection multiplicity $I_{Q_i}(C, T) = M(C)$ for $i = 1, \dots, s(\gamma)$. Because T contains an outer Galois point, we have $\{Q_1, \dots, Q_{s(\gamma)}\} = C \cap T$ by Lemma 1 and Fact 2(4). Since $\deg C > 2$, it follows from Bézout's theorem that $s(\gamma) > 1$ or $M(C) > 2$. Therefore, the dual map γ is not separable, and hence $M(C) = q(\gamma)$ by a theorem of Hefez-Kleiman. A plane curve with $d = s(\gamma)q(\gamma)$ is said to be extremal ([1], [5]) and it follows from a theorem of Hefez ([5, (7.16) Corollary]) that such curves are strange (see also [8, Corollary 1]). \square

By Lemma 3, we consider a strange curve. Let Q be the strange center. We define the set $\Sigma \subset \check{\mathbf{P}}^2$ which consists of lines l satisfying one of the following conditions.

- (i) $Q \in C$ and the line l is a tangent line at Q .
- (ii) The line l contains Q and a singular point not equal to Q .
- (iii) The line l is the tangent line $T_P C$ at a smooth point P with $I_P(C, T_P C) > M(C) = q(\gamma)$.
- (iv) The line l is a tangent line at a singular point.
- (v) The line l contains two or more singular points.

Note that the set Σ is a finite set. We consider the case where there exists a point R such that $R \in \Delta'$ but $R \notin \bigcup_{l \in \Sigma} l$.

LEMMA 4. *Let $R \in \Delta' \setminus (\Delta' \cap \bigcup_{l \in \Sigma} l)$. Then, the ramification locus of*

$\hat{\pi}_R : \hat{C} \rightarrow \mathbf{P}^1$ coincides with the set $r^{-1}(C \cap \overline{RQ})$ and each ramification index is equal to $q(\gamma)$. Furthermore, $d > q(\gamma)$ and $C \cap \overline{RQ}$ contains two distinct points.

PROOF. Firstly note that there exists a ramification point for $\hat{\pi}_R$ by Riemann-Hurwitz formula. Let $P \in \hat{C}$ be a ramification point of $\hat{\pi}_R$ with $r(P) \neq Q$. Now we prove that $P \in r^{-1}(C \cap \overline{RQ})$. If $r(P) \in C_{\text{sm}}$, then $r(P) \in \overline{RQ}$, because of Lemma 2(1). If $r(P) \in \text{Sing}(C)$, then it follows from Lemma 2(2) and the conditions (iv) and (v) of Σ that $r(P)$ should be contained in \overline{RQ} . Using Fact 2(4), the ramification locus of $\hat{\pi}_R$ coincides with $r^{-1}(C \cap \overline{RQ})$.

By the conditions (i) and (ii) of Σ , $C_{\text{sm}} \cap (\overline{RQ} \setminus \{Q\}) \neq \emptyset$. Let $P \in C_{\text{sm}} \cap (\overline{RQ} \setminus \{Q\})$. By the condition (iii) of Σ , the intersection multiplicity $I_P(C, \overline{RP})$ at P is equal to $M(C) = q(\gamma)$. By Lemma 1 and Fact 2(4), the ramification index at each point of $r^{-1}(C \cap \overline{RP})$ is equal to $q(\gamma)$.

We have $d > q(\gamma)$, by Lemma 5 below. Then, $C \cap \overline{RQ}$ contains two points. \square

LEMMA 5. *Let C be a strange curve of degree $d = q(\gamma) \geq 3$ with a center Q . Then, $Q \in \mathbf{P}^2 \setminus C$ and $\text{Sing}(C) \neq \emptyset$. If $R \in \Delta'$ and $P \in \text{Sing}(C)$, then $\hat{\pi}_R$ is ramified at a point in the fiber $r^{-1}(P)$ and $\overline{RP} \in \Sigma$.*

PROOF. It follows from a theorem of Hefez-Kleiman that $M(C) = q(\gamma)$. Hence, by Bézout's theorem, $Q \in \mathbf{P}^2 \setminus C$. We consider the projection $\pi_Q : C \rightarrow \mathbf{P}^1$ from Q . Since we can identify π_Q with the dual map γ , π_Q is generically one-to-one, by the assumption $d = q(\gamma)$. Therefore, $\hat{\pi}_Q$ is Frobenius and hence, C is rational and $\text{Sing}(C) \neq \emptyset$.

Let $R \in \Delta'$ and let $P \in \text{Sing}(C)$. Then, $r^{-1}(P)$ consists of a unique point, because $\hat{\pi}_Q$ is one-to-one. It follows from Bézout's theorem that $\hat{\pi}_R$ is ramified at a point in $r^{-1}(P)$. If $P \in \overline{RQ}$, then the line $\overline{RP} = \overline{PQ}$ satisfies the condition (ii) of Σ . If $P \notin \overline{RQ}$, then the line \overline{RP} satisfies the condition (iv) or (v) of Σ , by Lemma 2(2). Therefore, $\overline{RP} \in \Sigma$. \square

Finally in this section, we consider the case where the set Δ' is infinite and $\Delta' \subset \bigcup_{l \in \Sigma} l$. Since Σ is a finite set, there exists a line L containing infinitely many outer Galois points. We define $\Sigma(L) := \Sigma \setminus \{L\}$. Note that $(L \cap \Delta') \setminus (L \cap \bigcup_{l \in \Sigma(L)} l)$ is an infinite set. Then, we have the following

LEMMA 6. *Let $R \in (L \cap \Delta') \setminus (L \cap \bigcup_{l \in \Sigma(L)} l)$.*

- (1) *If $Q \notin L$, then we have the following assertions: (i) the set $C \cap (\overline{RQ} \setminus \{Q\})$ is non-empty and is contained in C_{sm} ; (ii) the ramification index at any point in $r^{-1}(C \cap \overline{RQ})$ is equal to $q(\gamma)$; and (iii) the ramification locus of $\hat{\pi}_R$ is contained in the set $r^{-1}(C \cap \overline{RQ}) \cup r^{-1}(C \cap L)$.*
- (2) *If $Q \in L$, then the ramification locus of $\hat{\pi}_R$ coincides with $r^{-1}(C \cap L)$.*

PROOF. Firstly note that there exist a ramification point for $\hat{\pi}_R$ by Riemann-Hurwitz formula.

We prove the assertion (1). The assertion (i) is derived from the conditions (i) and (ii) of Σ . The assertion (ii) follows from the condition (iii) of Σ and Fact 2(4). We prove the assertion (iii). Let $P \in \hat{C}$ be a ramification point of $\hat{\pi}_R$. If $r(P) \in C_{\text{sm}}$, then $r(P) \in \overline{RQ}$ by Lemma 2(1). If $r(P) \in \text{Sing}(C)$ and $r(P) \notin \overline{RQ}$, then $r(P)$ should be contained in L by Lemma 2(2) and the conditions (iv) and (v) of Σ .

We consider the assertion (2). Let P be a ramification point of $\hat{\pi}_R$. If $r(P) \in C_{\text{sm}}$, then $r(P) \in \overline{RQ} = L$ by Lemma 2(1). If $r(P) \in \text{Sing}(C)$, it follows from Lemma 2(2) and the conditions (iv) and (v) of Σ that $r(P) \in \overline{RQ} = L$. Using Fact 2(4), the ramification locus of $\hat{\pi}_R$ coincides with $r^{-1}(C \cap L)$. \square

4. Rationality.

In this section, we discuss non-existence of curves of genus ≥ 1 having infinitely many outer Galois points. Firstly we note the following

LEMMA 7. *Let R_1, R_2 be two distinct outer Galois points for a plane curve C . Then, $G_{R_1} \cap G_{R_2} = \{1\}$ in the automorphism group $\text{Aut}(\hat{C})$ of \hat{C} .*

PROOF. Let $B \subset C_{\text{sm}}$ be the set of smooth points such that

$$P \in B \Leftrightarrow R_1 \in T_P C \text{ or } R_2 \in T_P C.$$

The set B is finite, since $R_i \in T_P C$ if and only if the point P is a ramification point of $\hat{\pi}_{R_i}$ for $i = 1, 2$ by Lemma 1.

Let $\sigma \in G_{R_1} \cap G_{R_2}$ and let $P \in C_{\text{sm}} \setminus (B \cup \overline{R_1 R_2})$. It follows from Fact 2(1) that $\sigma(P) \in \overline{R_1 P} \cap \overline{R_2 P} = \{P\}$. Since P is not a ramification point of $\hat{\pi}_{R_i}$ for $i = 1, 2$, it follows from Fact 2(3) that $\sigma = 1$. \square

It follows from a generalization of Hurwitz's theorem ([15], [17]) that the automorphism group of a smooth curve of genus > 1 is finite. Using this theorem, we have the following

LEMMA 8. *If $C \subset \mathbf{P}^2$ has infinitely many outer Galois points, then the automorphism group $\text{Aut}(\hat{C})$ is infinite. Furthermore, \hat{C} is rational or elliptic.*

PROOF. Assume that $\text{Aut}(\hat{C})$ is finite. Then, the set $\bigcup_{R \in \Delta'} G_R \subset \text{Aut}(\hat{C})$ is finite. Hence, there exist finitely many Galois points R_1, \dots, R_r such that $\bigcup_i G_{R_i} = \bigcup_{R \in \Delta'} G_R$. Then, there exists $R \in \Delta'$ such that $R \neq R_i$ for any i , because Δ' is infinite. Since $G_R \subset \bigcup_i G_{R_i}$, there exists i such that $G_R \cap G_{R_i} \neq \{1\}$.

This is a contradiction to Lemma 7. □

More strongly, we prove the rationality of \hat{C} .

PROPOSITION 1. *We assume that Δ' is an infinite set. Then, \hat{C} is rational.*

To prove this proposition, we use a linear system. For simplicity, we assume that \hat{C} is rational or elliptic. Let $R \in \mathbf{P}^2 \setminus C$ be a Galois point. We take a general line $l \subset \mathbf{P}^2$ passing through R . Then, $l \cap C$ consists of exactly d smooth points Q_1, \dots, Q_d . We have a divisor $D := r^{-1}(Q_1) + \dots + r^{-1}(Q_d)$ on \hat{C} , which is very ample because $\deg D \geq 3 \geq 2g + 1$, where g is the genus of \hat{C} (under the assumption that \hat{C} is rational or elliptic). Let $\phi : \hat{C} \rightarrow \mathbf{P}^N$ be a morphism induced from a complete linear system $|D|$, which is isomorphic onto its image. It follows from the assertions (1) and (3) in Fact 2 that $\sigma^*(D) = D$ for any $\sigma \in G_R$. Therefore, an automorphism $\sigma \in G_R$ induces a linear transformation $\hat{\sigma}$ on \mathbf{P}^N such that $\hat{\sigma}(\phi(\hat{C})) = \phi(\hat{C})$. The plane curve C is given by a linear projection π_V of $\phi(\hat{C})$ for a linear space $V \subset \mathbf{P}^N$ of dimension $N - 3$. Let W_R be the closure of the set $\pi_V^{-1}(R) \subset \mathbf{P}^N$, which is a linear space of dimension $N - 2$, and let π_{W_R} be a projection $\mathbf{P}^N \dashrightarrow \mathbf{P}^1$ with a center W_R . Then, we can identify $\pi_R \circ r$ with $\pi_{W_R} \circ \phi$. (See also [23, Section 2], [24, Section 3].)

PROOF OF PROPOSITION 1. It follows from Lemma 3 that C is a strange curve. We can also assume that \hat{C} is rational or elliptic, by Lemma 8. Let $Q \in \mathbf{P}^2$ be the strange center and let W_Q be the closure of the set $\pi_V^{-1}(Q) \subset \mathbf{P}^N$.

Now we consider $\phi(\hat{C}) \subset \mathbf{P}^N$. By abuse of terminology, we denote $\phi(\hat{C})$ also by \hat{C} here. We prove that there exists a point $P \in \hat{C}$ such that $T_P \hat{C} \cap V \neq \emptyset$. If there exists a point P such that $T_P \hat{C} \subset W_Q$, then $T_P \hat{C} \cap V \neq \emptyset$ by counting dimensions. If $T_P \hat{C} \not\subset W_Q$ for any point $P \in \hat{C}$, then the set $\bigcup_{P \in \hat{C}} (T_P \hat{C} \cap W_Q) \subset W_Q$ is closed and irreducible. If $\bigcup_{P \in \hat{C}} (T_P \hat{C} \cap W_Q)$ is zero-dimensional, then \hat{C} is strange in \mathbf{P}^N . Since \hat{C} is smooth in \mathbf{P}^N , this is a contradiction to Fact 1 due to Lluís and Samuel. Therefore, $\bigcup_{P \in \hat{C}} (T_P \hat{C} \cap W_Q)$ is one-dimensional. Then, there exists a point $P \in \hat{C}$ such that $T_P \hat{C} \cap V \neq \emptyset$ by counting dimensions.

For each point $R \in \Delta'$, $\pi_{W_R}|_{\hat{C}}$ is ramified at P . It follows from Fact 2(3) and Lemma 7 that there are infinitely many automorphisms σ on \hat{C} such that $\sigma(P) = P$. Therefore, \hat{C} should be rational, because of the following basic property of an elliptic curve: the cardinality of the automorphism group fixing a given point is finite (see, for example, [18, Theorem 10.1]). □

5. Case of rational curves.

We start with the following

LEMMA 9. *Let $\theta : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ be a Galois covering of degree d with a Galois group G . Then we have the following.*

- (1) *If the index e_P at a point P is a power of p at least 2, then the stabilizer subgroup $G(P)$ of P is a Sylow p -group. Furthermore, if $e_P = d$, then P is the unique ramification point.*
- (2) *If $\theta(P)$ is the unique branch point, then θ is ramified only at P and $e_P = d$. Furthermore, d is a power of p .*

PROOF. We may assume that $P = (1 : 0)$. First we consider the assertion (1). Let $e_P = q$, where q is a power of p . It follows from Fact 2(3) that the index of $G(P)$ is equal to q . Let $\sigma \in G(P)$. Then, by direct computations, σ is represented by a matrix

$$A_\sigma = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \tag{A}$$

as an automorphism of \mathbf{P}^1 , where $\alpha \in K$, because $\sigma(P) = P$ and $\sigma^q = 1$. Then, note that

$$G(P) \cap G(R) = \{1\} \tag{B}$$

for any $R \neq P$ because a fixed point by σ of the form (A) is uniquely determined if σ is not identity.

Let S be a Sylow p -group containing $G(P)$. Now we prove a claim that the normalizer $N_S(G(P)) := \{\tau \in S \mid \tau^{-1}(G(P))\tau = G(P)\}$ is equal to $G(P)$, because this claim implies $S = G(P)$ by using a lemma of Matsuyama (see [11, Lemma 2], [21, p. 88, Theorem 1.6]).

We assume that there exists $\tau \in N_S(G(P)) \setminus G(P)$. We take $\sigma \in G(P) \setminus \{1\}$. Then, there exists $\eta \in G(P)$ such that $\tau^{-1}\sigma\tau = \eta$. We have $\sigma(\tau(P)) = \tau(\eta(P)) = \tau(P)$, hence $\sigma = 1$ because $P \neq \tau(P)$ and the above equation (B) holds. This is a contradiction.

Consider the latter assertion of (1). If $e_P = d$, then $G(P) = G$ by Fact 2(3). For each $\sigma \in G(P) \setminus \{1\}$, a fixed point of σ is uniquely determined, as is mentioned above. Therefore, we have the conclusion.

We consider the assertion (2). We prove that $\theta^{-1}(\theta(P)) = \{P\}$. We assume that $\theta^{-1}(\theta(P)) = \{P_1, P_2, \dots, P_s\}$ with $s \geq 2$. Then, it follows from Fact 2(3) and (4) that the order of $G(P_i)$ is equal to that of $G(P_j)$ for any i, j . Since $G(P_i) \cap G(P_j)$ is not empty as a set, $\bigcup_i G(P_i) \neq G$ as a set by considering the order. Let $\tau \in G \setminus (\bigcup_i G(P_i))$. Considering a matrix representing τ as an automorphism of \mathbf{P}^1 , there is a fixed point R by τ . It follows from Fact 2(3) that R is a ramifica-

tion point of θ . Now $\theta(R) \neq \theta(P)$ because $\tau \notin \bigcup_i G(P_i)$. This is a contradiction to the uniqueness of the branch point. Therefore, P is the unique ramification point. We also have $e_P = d$ by Fact 2(3).

Now we prove that the index e_P is a power of p . Let $e_P = ql$, where q is a power of p and l is not divisible by p . Let $\sigma \in G(P) = G$ be any element. Then, σ is represented by a matrix

$$A_\sigma = \begin{pmatrix} \zeta & \alpha \\ 0 & 1 \end{pmatrix}$$

as an automorphism of \mathbf{P}^1 , where ζ is an l -th root of unity and $\alpha \in K$, because $\sigma(P) = P$ and $\sigma^{ql} = 1$. If $\zeta \neq 1$, then we find that σ has two fixed points, by direct computations. Therefore, $\zeta = 1$ by our assumption. Then, any element of $G(P) \setminus \{1\}$ is of order p , by direct computations. If $l > 1$ then there exists an element whose order is not divisible by p , by Sylow's theorem. This is a contradiction. Therefore, $l = 1$. \square

PROOF OF (1) \Rightarrow (2) IN THEOREM 1. From Lemma 3, C is a strange curve. Let Q be the strange center. By Proposition 1, we can assume that C is rational. From Lemmas 4 and 9(2), $\Delta' \subset \bigcup_{l \in \Sigma} l$. Since $\bigcup_{l \in \Sigma} l$ is a finite union of lines, there exists a line L containing infinitely many outer Galois points. We define $\Sigma(L)$ as before Lemma 6. We prove that $Q \in L$.

Assume that $Q \notin L$. Now $(L \cap \Delta') \setminus (L \cap \bigcup_{l \in \Sigma(L)} l)$ is an infinite set. It follows from Lemma 6(1)(ii) and Fact 2(5) that d is divisible by $q(\gamma)$. Let $n := d/q(\gamma)$. Then, it follows from Lemma 9(1) that n is not divisible by p . Let $R \in (L \cap \Delta') \setminus (L \cap \bigcup_{l \in \Sigma(L)} l)$. If $n > 1$, then it follows from Lemmas 6(1)(iii) and 9(2) that the ramification locus of $\hat{\pi}_R$ coincides with the set $r^{-1}(C \cap \overline{RQ}) \cup r^{-1}(C \cap L)$. By Sylow's theorem, there exists $\sigma_R \in G_R \setminus \{1\}$ whose order divides n and is not divisible by p . Then, by Fact 2(3) and Lemma 6(1)(ii), σ_R does not fix any point in $r^{-1}(C \cap \overline{RQ})$. Since σ_R is a non-trivial automorphism of \mathbf{P}^1 whose order is not divisible by p , there exist exactly two points $P_R, P'_R \in \hat{C}$ such that $\sigma_R(P_R) = P_R$ and $\sigma_R(P'_R) = P'_R$. Since P_R and P'_R are ramification points not contained in $r^{-1}(C \cap \overline{RQ})$, they should be contained in $r^{-1}(C \cap L)$. Because $r^{-1}(C \cap L)$ is a finite set, there exist points P_1 and P_2 such that $\sigma_R(P_1) = P_1$ and $\sigma_R(P_2) = P_2$ for infinitely many $R \in L \cap \Delta'$. By Lemma 7, the set $\{\sigma_R | R \in \Delta', \sigma_R(P_1) = P_1, \sigma_R(P_2) = P_2\}$ is infinite. However, this is a contradiction to the following property on automorphisms of \mathbf{P}^1 : for an integer n and two points P_1 and P_2 , the number of automorphisms σ , such that the order of σ divides n and σ fixes P_1 and P_2 , is finite.

Let $n = 1$, i.e. $d = q(\gamma)$. It follows from Lemma 5 that $\text{Sing}(C) \neq \emptyset$. Let $P \in \text{Sing}(C)$. Then, by Lemma 6(1)(i), $P \notin \overline{RQ}$. By Lemma 5, $\hat{\pi}_R$ is ramified at

a point in the fiber $\pi^{-1}(P)$. Therefore, $\hat{\pi}_R$ has two ramification points. This is a contradiction to the latter assertion of Lemma 9(1). \square

PROOF OF (2) \Rightarrow (3) IN THEOREM 1. Assume the condition (2). Let Q be the strange center and let L be a line such that $Q \in L$ and $L \cap \Delta'$ is infinite. Let $R \in (L \cap \Delta') \setminus (L \cap \bigcup_{l \in \Sigma(L)} l)$. It follows from Lemmas 6(2) and 9(2) that d is a power of p and $r^{-1}(C \cap L)$ consists of a unique point. Let $d = p^e$ and let P be the point contained in $r^{-1}(C \cap L)$. We use the same notation as in Section 4. Let $H \subset \mathbf{P}^N$ be the closure of $\pi_V^{-1}(L)$, which is a hyperplane. Since $r = \pi_V \circ \phi$ on \hat{C} , $H \cap \phi(\hat{C}) = \{\phi(P)\}$. We may assume that $\phi : \hat{C} = \mathbf{P}^1 \rightarrow \mathbf{P}^d$ is given by $\phi(u : v) = (u^d : u^{d-1}v : \cdots : v^d)$ and $P = (1 : 0)$. Then, H is defined by $X_d = 0$ because $H \cap \phi(\hat{C}) = \{(1 : 0 : \cdots : 0)\}$. Let three linear forms H_1, H_2, X_d define V . Then, $\pi_V = (H_1 : H_2 : X_d)$. We can take a system of coordinates on \mathbf{P}^N such that the form H_2 does not have a term X_0 nor X_d , and H_1 has the term X_0 and does not have X_d . Then, $\pi_V(\phi(P)) = (1 : 0 : 0)$. Now we can write $\pi_V \circ \phi(u : 1) = (\psi_1(u) : \psi_2(u) : 1)$, where $\psi_1(u)$ is a polynomial of degree d in u and $\psi_2(u)$ is a polynomial of degree $< d$ in u with $\psi_1(0) = \psi_2(0) = 0$. We consider a line $l \subset \mathbf{P}^2$ passing through $(1 : 0 : 0)$. Then, by direct computations, $r^{-1}(C \cap l) = \{P\}$ if and only if l is defined by $Z = 0$. Since $r^{-1}(C \cap L) = \{P\}$, the line L is defined by $Z = 0$. Let $R = (1 : t : 0) \in L$. Then, $\pi_R = (Y - tX : Z)$. Hence, $\pi_R \circ \pi_V \circ \phi = (\psi_2 - t\psi_1 : 1)$.

We prove that ψ_1 and ψ_2 have only terms of degree equal to some power of p . If ψ_1 or ψ_2 has a term whose degree is not a power of p , then $\psi_2 - t\psi_1$ has such a term for a general t . Therefore, we have only to prove that, for a Galois covering $\mathbf{P}^1 \rightarrow \mathbf{P}^1; (u : 1) \mapsto (\Psi(u) : 1)$ of degree p^e such that $\Psi(0) = 0$ and $P = (1 : 0)$ is the unique ramification point with $e_P = p^e$, the polynomial Ψ has only terms of degree equal to some power of p . Let G be the Galois group. Note that $G = G(P)$, by Fact 2(3). From the form of A_σ described as the equation (A) in the proof of Lemma 9, for any $\sigma \in G$, there exists $\alpha(\sigma) \in K$ such that $\sigma^*(u) = u + \alpha(\sigma)$. Then, $\Psi(u + \alpha(\sigma)) = \Psi(u)$, since $\Psi(u) = \sigma^*\Psi(u) = \Psi(\sigma^*u)$. Especially $\Psi(\alpha(\sigma)) = \Psi(0) = 0$. Note that the set $\{\alpha(\sigma) | \sigma \in G\}$ forms an additive subgroup of the base field K . It follows from [4, Proposition 1.1.5 and Theorem 1.2.1] that Ψ has only terms of degree equal to some power of p .

Now we have

$$\psi_1(u) = a_0u + a_1u^p + \cdots + a_eu^{p^e}, \quad \psi_2(u) = b_0u + b_1u^p + \cdots + b_eu^{p^e},$$

where $a_i, b_i \in K$ for $i = 0, \dots, e$. Since the morphism $\psi := \pi_V \circ \phi = (\psi_1 : \psi_2 : 1) : \mathbf{P}^1 \rightarrow \mathbf{P}^2$ is birational onto its image, we have $a_0b_0 \neq 0$. We can take $b_0 = 1$ and $a_0 = 0$ for a suitable system of coordinates. We have the conclusion (3). \square

PROOF OF (3) \Rightarrow (4) IN THEOREM 1. Assume the condition (3). Let

$$x := \psi_1(u) = a_1u^p + \cdots + a_eu^{p^e}, \quad y := \psi_2(u) = u + b_1u^p + \cdots + b_eu^{p^e}.$$

Then, we consider $(2e + 1)$ elements $x, x^p, x^{p^2}, \dots, x^{p^e}, y^p, y^{p^2}, \dots, y^{p^e}$, which is contained in $K[u]$. Then, these elements are contained in a vector space V (over K) which is spanned by $2e$ elements $u^p, u^{p^2}, \dots, u^{p^{2e}}$. Since V is of dimension $2e$, there exist $\alpha_0, \dots, \alpha_e, \beta_1, \dots, \beta_e \in K$ such that

$$f(x, y) := \alpha_e x^{p^e} + \cdots + \alpha_1 x^p + \alpha_0 x + \beta_e y^{p^e} + \cdots + \beta_1 y^p = 0$$

in $K[u]$. Since $d = p^e$, the curve C is defined by $f(x, y) = 0$. Then, $\alpha_0 \neq 0$, because $f(x, y)$ is irreducible. We can take $\alpha_0 = 1$ for a suitable system of coordinates. \square

6. Proof of (4) \Rightarrow (1) and the distribution of Galois points.

Now we consider the curve in (4) in our Theorem 1. Let L be the line defined by $Z = 0$.

PROPOSITION 2. *Let $C \subset \mathbf{P}^2$ be an irreducible plane curve of degree p^e defined by $\alpha_e x^{p^e} + \alpha_{e-1} x^{p^{e-1}} + \cdots + \alpha_1 x^p + x + \beta_e y^{p^e} + \cdots + \beta_1 y^p = 0$. Then we have the following.*

- (i) *The curve C is strange with a center $Q = (0 : 1 : 0)$, and there exists a unique singular point P on L . ($P = Q$ is possible.)*
- (ii) $\Delta' = L \setminus \{P, Q\}$.
- (iii) *For each point $R \in \Delta'$, the Galois group G_R is isomorphic to $(\mathbf{Z}/p\mathbf{Z})^{\oplus e}$.*
- (iv) *For each point $R \in \Delta'$ and any automorphism $\sigma \in G_R$, σ can be extended to a linear transformation on \mathbf{P}^2 .*

PROOF. The tangent line at a point $(x_0 : y_0 : 1)$ is defined by $X - x_0 Z = 0$, by direct computations. Therefore, C is strange with a center $Q = (0 : 1 : 0)$. The singular locus is given by $\alpha_e X^{p^e} + \beta_e Y^{p^e} = Z = 0$. Therefore, a singular point is unique and is contained in L . We have the assertion (i). Let P be the singular point.

We prove that $L \setminus \{P, Q\} \subset \Delta'$, and the assertions (iii) and (iv) hold for any point $R \in L \setminus \{P, Q\}$. Let $R = (1 : b : 0) \in L \setminus \{P, Q\}$. Then, $\alpha_e + b^{p^e} \beta_e \neq 0$ and the projection $\pi_R = (y - bx : 1)$. Let $\hat{y} = y - bx$. Then, we have a field extension $K(x, \hat{y})/K(\hat{y})$ with $h(x) := (\alpha_e + b^{p^e} \beta_e)x^{p^e} + \cdots + (\alpha_1 + b^p \beta_1)x^p + x + \beta_e \hat{y}^{p^e} + \cdots + \beta_1 \hat{y}^p = 0$. This gives a Galois extension and the Galois group is isomorphic to $(\mathbf{Z}/p\mathbf{Z})^{\oplus e}$ if $\alpha_e + b^{p^e} \beta_e \neq 0$ (see [19, pp.117–118]). Therefore,

$R \in \Delta'$. For any $\sigma \in G_R$, there exists $c \in K$ such that $\sigma^*(x) = x + c$ and $\sigma^*(y) = y + bc$, since $\sigma^*(\hat{y}) = \hat{y}$ and $h(x) = h(x + c)$ for any $c \in K$ such that $((\alpha_e + b^{p^e}\beta_e)x^{p^e} + \cdots + (\alpha_1 + b^p\beta_1)x^p + x)(c) = 0$. Hence, σ can be extended to a linear transformation on \mathbf{P}^2 .

Finally, we prove that $\Delta' \subset L \setminus \{P, Q\}$. Let $R \in \Delta'$. Then, $R \neq P, Q$. Now, the assertion (1) in Theorem 1 holds. Therefore, by the result (3), normalization r is given by the morphism ψ . We find that $r^{-1}(P)$ consists of one point. Therefore, by Bézout's theorem, $\hat{\pi}_R$ is ramified at $r^{-1}(P)$. Note that a tangent line at P is uniquely determined, which is L .

Assume that $P \neq Q$. If $R \notin L$, then $P \notin \overline{RQ}$. This is a contradiction to Lemma 2(2). Therefore, $R \in L$.

Assume that $P = Q$. Then, it follows from Lemma 2 and Fact 2(4) that the ramification locus coincides with $r^{-1}(C \cap \overline{RQ})$. If $\overline{RQ} \neq L$, then $C \cap \overline{RQ}$ contains two points, since the line $\overline{RQ} = \overline{RP}$ is not a tangent line at P in this case. This is a contradiction to Lemma 9(2). We have $\overline{RQ} = L$. Therefore, $R \in L$. \square

Finally, we mention the distribution of inner Galois points for the curve.

PROPOSITION 3. *Let $C \subset \mathbf{P}^2$ be an irreducible plane curve of degree p^e defined by $\alpha_e x^{p^e} + \alpha_{e-1} x^{p^{e-1}} + \cdots + \alpha_1 x^p + x + \beta_e y^{p^e} + \cdots + \beta_1 y^p = 0$, and let Δ be the set of all inner Galois points in C_{sm} . Then, we find that $\Delta = C_{\text{sm}}$ if C is projectively equivalent to the curve defined by $x - y^{p^e} = 0$, and $\Delta = \emptyset$ otherwise.*

PROOF. As in Proposition 2(i), C is strange with a center Q and $\text{Sing}(C) = \{P\}$ for some point P . By Proposition 2(ii), $\Delta' = L \setminus \{P, Q\}$. Let $R \in \Delta$ and let $R_1 \in C \setminus (C \cap (\overline{RP} \cup \overline{RQ}))$. Then, $R_1 \in C_{\text{sm}}$ and $\overline{RR_1} \cap \Delta' \neq \emptyset$.

Let $R' \in \overline{RR_1} \cap \Delta'$. Consider the projection $\hat{\pi}_{R'}$ from R' . It follows from Fact 2(2) that there exists an automorphism $\sigma \in G_{R'}$ such that $\sigma(R) = R_1$. Since the automorphism $\sigma \in G_{R'}$ can be extended to \mathbf{P}^2 by Proposition 2(iv), the point R_1 is also Galois for C .

By the above discussion, if there exists one inner smooth Galois point, then almost all inner points are Galois. Such a curve is projectively equivalent to the curve given by $x - y^{p^e} = 0$, from a result of the previous paper [3]. For this curve, it is also proved that $\Delta = C_{\text{sm}}$ in [3]. \square

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Satoru FUKASAWA

Department of Mathematics
School of Science and Engineering
Waseda University
Ohkubo 3-4-1, Shinjuku
Tokyo 169-8555, Japan
E-mail: s.fukasawa@kurenai.waseda.jp

Current Address:

Department of Mathematical Sciences
Faculty of Science
Yamagata University
Kojirakawa-machi 1-4-12
Yamagata 990-8560, Japan
E-mail: s.fukasawa@sci.kj.yamagata-u.ac.jp