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Height functions on surfaces with three critical values

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Abstract. For a given closed surface, we study height functions with three critical values associated with immersions of the surface into 3-space, where the critical points may not be non-degenerate. We completely characterize the numbers of critical points corresponding to the three critical values that can be realized by such height functions. We also study the cases where the immersion can be replaced by an embedding or the critical points are all non-degenerate. Similar problems are studied for distance functions as well.

1. Introduction.

In [**1**], Banchoff and Takens studied height functions on closed surfaces with exactly three critical points, and completely characterized those closed surfaces which admit such height functions. Namely, a closed connected surface admits such a height function if and only if it is non-orientable or orientable of genus 0 or 1. In this paper, we extend their result to height functions with three critical values.

More precisely, we consider the sequence (n_1, n_2, n_3) of positive integers, where n_j is the number of critical points corresponding to the j-th critical value, $j = 1, 2, 3$, with respect to the natural order in $\boldsymbol{R}$. Our main result completely characterizes those sequences of three positive integers that can be realized as the sequence of critical point numbers of a height function with three critical values on a given closed surface. As a corollary, we can give a similar characterization of those sequences $(n_1, n_2, \ldots, n_\ell)$ of positive integers that can be realized as the critical point number sequence of a height function with ℓ critical values for an arbitrary $\ell \geq 3$.

The paper is organized as follows. In Section 2, we state our main theorem concerning height functions with three critical values. As a corollary, we prove the characterization of the critical point number sequences for an arbitrary number

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of critical values. In Section 3, we prove the main theorem by using a method similar to that in [**1**]. We will see that for $n_2 \geq 2$ and M orientable, the sequence (n_1, n_2, n_3) can be realized by the height function associated with an embedding of M. In Section 4, we consider the realization by height functions which have only non-degenerate critical points. In Section 5, we study the same problem for the distance function. It is interesting to see that any sequence (n_1, n_2, n_3) can be realized by a distance function on any closed surface, which is different from the height function case.

Throughout the paper, manifolds and maps between them are differentiable of class C^∞ unless otherwise specified. All surfaces are assumed to be connected.

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2. Results.

Let $\iota : M \to \boldsymbol{R}^3$ be an immersion of a closed surface M into the 3-space. Let $Z : \boldsymbol{R}^3 \to \boldsymbol{R}$ be the projection defined by $Z(x, y, z) = z$. Then, the smooth function h on M defined by the composition $h = Z \circ \iota : M \to \boldsymbol{R}$ is called a *height function* associated with the immersion ι.

In this paper, we consider the critical points and critical values of such height functions. Throughout the paper, we will always assume that the number of critical points is finite. Note that the critical points that we consider may not necessarily be non-degenerate.

Recall also that there exist smooth functions on surfaces that can never be realized as a height function (for example, see [**1**], [**3**]).

Let $h : M \to \boldsymbol{R}$ be a height function on a closed surface M as above, and let $v_1 < v_2 < \cdots < v_\ell$ be its critical values. If $\ell = 2$, then it is known that M is diffeomorphic to S^2 and h has exactly two critical points. Therefore, throughout this paper, we assume $\ell \geq 3$. For j with $1 \leq j \leq \ell$, we denote by n_j the number of critical points of h corresponding to the critical value v_j. The sequence of positive integers $(n_1, n_2, \ldots, n_\ell)$ is called the *critical point number sequence* of the height function h associated with the immersion $\iota : M \to \boldsymbol{R}^3$.

Banchoff and Takens [**1**] showed that the sequence $(1, 1, 1)$ can be realized as the critical point number sequence of the height function associated with an immersion of a closed surface M if and only if M is non-orientable or is orientable of genus 0 or 1. Our first result is a generalization of their result.

THEOREM 2.1. *For a closed surface M and a sequence (n_1, n_2, n_3) of positive integers, we have the following.*

(1) *When* $n_2 = 1$*, the sequence* $(n_1, 1, n_3)$ *can be realized as the critical point number sequence of the height function associated with an immersion of* M *into* $\boldsymbol{R}^3$ *if and only if* M *is non-orientable or is orientable of genus* 0 *or* 1*.*

(2) *When* $n_2 \geq 2$*, the sequence* (n_1, n_2, n_3) *can always be realized as the critical point number sequence of the height function associated with an immersion of* M *into* $\boldsymbol{R}^3$*. Furthermore, if* M *is orientable, then it can be realized even by an embedding.*

We will prove Theorem 2.1 in Section 3.

As a corollary, we have the following.

COROLLARY 2.2. *Let* M *be an arbitrary closed surface. Then, an arbitrary sequence of positive integers* $(n_1, n_2, \ldots, n_\ell)$ *with* $\ell \geq 4$ *can be realized as the critical point number sequence of the height function associated with an immersion of* M *into* $\boldsymbol{R}^3$*.*

PROOF. Set

$$n = \sum_{j=2}^{\ell-1} n_j.$$

Then, since $n \geq 2$, by Theorem 2.1, the sequence (n_1, n, n_ℓ) can be realized as the critical point number sequence of the height function associated with an immersion of M into $\boldsymbol{R}^3$. By slightly and appropriately changing the heights of the critical points that are not minimal nor maximal, we can realize the sequence $(n_1, n_2, \ldots, n_{\ell-1}, n_\ell)$. This completes the proof. □

In Corollary 2.2, if M is orientable, then the sequence can be realized by an embedding of M into $\boldsymbol{R}^3$. Note that closed connected non-orientable surfaces cannot be embedded in $\boldsymbol{R}^3$.

3. Proof of Theorem 2.1.

In this section, we prove Theorem 2.1.

PROOF OF THEOREM 2.1.

(1) Let M be a closed orientable surface of genus g, and let $\iota : M \to \boldsymbol{R}^3$ be an immersion such that the associated height function $h = Z \circ \iota$ has exactly three critical values $v_1 < v_2 < v_3$ and that there is exactly one critical point, say p, corresponding to the middle critical value v_2. Without loss of generality, we may assume $v_1 = -1$, $v_2 = 0$ and $v_3 = 1$. We denote by n_1 (or n_3) the number of critical points of h corresponding to the critical value -1 (resp. 1).

By the same argument as in [**1**], we may assume that around p, ι is locally equivalent to the graph of a function of the form

$$(x,y) \mapsto \mathrm{Re}\big((x+\sqrt{-1}y)^k\big)$$

for some positive integer k, and $h^{-1}(0)$ is homeomorphic to the bouquet of k copies of S^1.

For $t \in (-1,0)\cup(0,1)$, set $M_t = h^{-1}(t)$ and $j_t = \iota|M_t : M_t \to \boldsymbol{R}^2 \times \{t\}$. Note that M_t is a finite disjoint union of circles and j_t is an immersion, since so is ι. As has been observed in [**1**], for $t<0$ (or $t>0$), j_t is an immersion of n_1 copies (resp. n_3 copies) of S^1 and its restriction to each component has winding number ± 1. Therefore, we have

$$|R_- - R_+| = n_1 + n_3 - 2r \tag{3.1}$$

for some non-negative integer r, where R_- (or R_+) is the sum of the winding numbers of j_t with $t<0$ (resp. $t>0$) restricted to the components.

On the other hand, we can prove

$$|R_- - R_+| = k-1 \tag{3.2}$$

as in [**1**] by examining the behavior of j_t with $|t| \ll 1$ near the critical point p.

Furthermore, since M has the structure of a cell complex consisting of one vertex p, k 1-cells and n_1+n_3 2-cells, we have

$$2-2g = \chi(M) = 1 + n_1 + n_3 - k,$$

where χ denotes the Euler characteristic.

By (3.1) and (3.2), we have $n_1+n_3-2r = k-1$, and therefore we have

$$2-2g = 2r$$

for some non-negative integer r. Thus, we must have $g=0$ or 1.

Let us now show that $(n_1, 1, n_3)$ can be realized as the critical point number sequence of a height function, provided that M is non-orientable or M is orientable of genus 0 or 1. We proceed by induction on n_1+n_3. In the following, $M_{(s)}$ denotes the surface M with s open disks removed.

When $n_1=n_3=1$, this has been proved in [**1**] by explicitly constructing an immersion $\iota_1 : M_{(2)} \to \boldsymbol{R}^2 \times [-\varepsilon, \varepsilon]$ for some sufficiently small $\varepsilon > 0$ such that $Z \circ \iota_1$

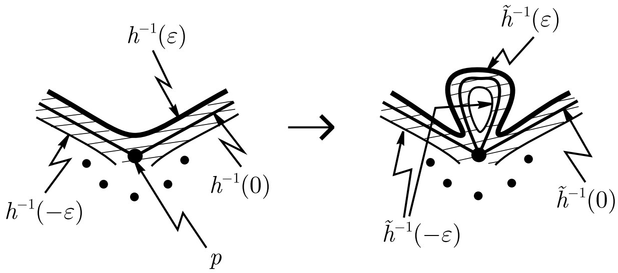

Figure 1. Increasing n_1 by one.

has exactly one critical point, which has the value zero, and that on $\boldsymbol{R}^2 \times \{\pm\varepsilon\}$, ι_1 corresponds to an immersion of the circle of winding number ± 1. Since the latter immersion is regularly homotopic to an embedding, we can extend ι_1 to obtain an immersion $\iota_2 : M_{(2)} \to \boldsymbol{R}^2 \times [-2\varepsilon, 2\varepsilon]$ such that on $\boldsymbol{R}^2 \times \{\pm 2\varepsilon\}$, ι_2 corresponds to an embedding of the circle. Then, the required immersion $\iota : M \to \boldsymbol{R}^3$ can be constructed by gluing an embedding of the 2-disk whose height function has exactly one critical point at the center, to ι_2 along each of the two boundary components of $M_{(2)}$.

If $n_1 > 1$, then by our induction hypothesis, $(n_1 - 1, 1, n_3)$ can be realized by a height function, say $h : M \to \boldsymbol{R}$, where we may assume that the middle critical value is equal to zero. Then, consider the operation as depicted in Figure 1, where p is the unique critical point of h corresponding to the middle critical value and $\varepsilon > 0$ is sufficiently small. Note that around p, the immersion can be identified with the graph of the height function. In Figure 1, the left-hand side figure explains the immersion restricted to $h^{-1}([-\varepsilon, \varepsilon])$ in a neighborhood of p. The right-hand side figure explains the new height function $\tilde{h}$, whose source is the surface $M_{(n_1+n_3)}$. Then, we see that the sequence $(n_1, 1, n_3)$ is realized by the new height function $\tilde{h} : M \to \boldsymbol{R}$.

When $n_3 > 1$, a similar argument can be applied. This completes the proof of Theorem 2.1 (1).

(2) As has been seen above, when M is non-orientable or orientable of genus 0 or 1, the sequence $(n_1, 1, n_3)$ can be realized. In order to increase the number of critical points corresponding to the middle critical value, we consider the operation as in Figure 2.

At the new critical point, the immersion is locally the graph of the function $(x, y) \mapsto y^3 - x^2$. Repeating this procedure, we see that the sequence (n_1, n_2, n_3) can be realized.

Let us now assume that M is orientable of genus $g \geq 1$. When $(n_1, n_2, n_3) = (1, 2, 1)$, we consider the construction as in Figure 3, where the number of bands

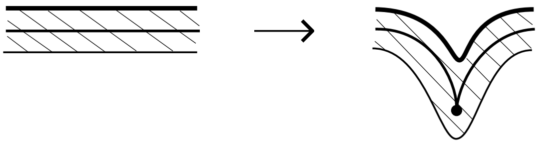

Figure 2. Increasing n_2 by one.

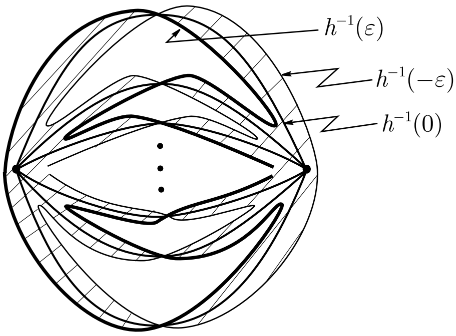

Figure 3. Realizing $(1, 2, 1)$.

is equal to $2g + 2$. This explains the construction of an *embedding* $\iota_1 : M_{(2)} \to \boldsymbol{R}^2 \times [-\varepsilon, \varepsilon]$ for $\varepsilon > 0$ sufficiently small, where $h = Z \circ \iota_1$ is the height function associated with the embedding ι_1.

Note that each band is twisted so that the surface is orientable. By an Euler characteristic argument, we see that the source surface is diffeomorphic to $M_{(2)}$. Around each of the two critical points of the associated height function, ι_1 is identified with the graph of the function $(x, y) \mapsto \mathrm{Re}((x + \sqrt{-1}y)^{g+1})$. Furthermore, on $\boldsymbol{R}^2 \times \{\pm\varepsilon\}$, ι_1 corresponds to an embedding of the circle. Therefore, we get an *embedding* $\iota : M \to \boldsymbol{R}^3$ whose associated height function realizes $(1, 2, 1)$.

Then, we can increase the number of critical points corresponding to the top or the bottom critical value by using the same argument as in the proof of (1). We can also increase the number of critical points corresponding to the middle critical value as in Figure 2.

Finally, let us observe that when $n_2 \geq 2$ and M is orientable, the immersion of M into $\boldsymbol{R}^3$ realizing (n_1, n_2, n_3) can be chosen to be an embedding. In fact, when increasing n_1 (or n_3), we perform the operation as in Figure 1 along the curve $h^{-1}(\varepsilon)$ (resp. $h^{-1}(-\varepsilon)$) in Figure 3 near the left-hand side vertex (resp. the right-hand side vertex) so that on each level $\boldsymbol{R}^2 \times \{\pm\varepsilon\}$, the components bound disjointly embedded disks.

We can similarly construct a desired embedding for the 2-sphere case as well. We just start with an embedding of the annulus as in Figure 3 with two twisted bands, whose associated height function has no critical points.

This completes the proof. □

REMARK 3.1. It would also be an interesting problem to consider similar problems in higher dimensions. Note that in [**5**], Morse functions with three critical values are thoroughly studied.

4. Morse function case.

In this section, we determine those sequences (n_1, n_2, n_3) of three positive integers which can be realized as the critical point number sequence of a height function on a given closed connected surface M with only *non-degenerate* critical points. A smooth function with only non-degenerate critical points is called a *Morse function*.

THEOREM 4.1. *Let M be a closed connected surface and (n_1, n_2, n_3) a sequence of three positive integers. Then, the sequence can be realized as the critical point number sequence of a height function associated with an immersion $\iota : M \to \boldsymbol{R}^3$ with only non-degenerate critical points if and only if $n_1 - n_2 + n_3 = \chi(M)$. Furthermore, when M is orientable, the immersion can be chosen to be an embedding.*

PROOF. If (n_1, n_2, n_3) is realized, then the indices of the n_2 critical points corresponding to the middle critical value must be all equal to 1. Therefore, we have $n_1 - n_2 + n_3 = \chi(M)$ by the Morse equality (for example, see [**4**]).

Conversely, suppose that $n_1 - n_2 + n_3 = \chi(M)$ holds. When M is orientable, we can construct the desired embedding starting from the embedding $M_{(n_1+n_3)} \to \boldsymbol{R}^2 \times [-\varepsilon, \varepsilon]$ explained in Figure 4, where there are $2g + 2$ twisted bands in the middle and g is the genus of M. Since each vertex of $h^{-1}(0)$ has degree four, we can arrange the embedding so that the associated height function is a Morse function. Since the components of $h^{-1}(\pm\varepsilon)$ bound disjointly embedded disks in $\boldsymbol{R}^2 \times \{\pm\varepsilon\}$, we can get a desired embedding of the whole M.

When M is non-orientable, we use the immersion $M_{(2)} \to \boldsymbol{R}^2 \times [-\varepsilon, \varepsilon]$ as explained in Figure 5 when $n_1 = n_3 = 1$, where g is the genus of M and the non-degenerate critical points are indicated by small black dots. Note that the winding number of each of the boundary curves is equal to ± 1. We can easily increase n_1 and n_3 as in the orientable case. This completes the proof. □

Compare the above theorem with the results obtained in [**2**], [**3**].

REMARK 4.2. In the case of a sequence consisting of four or more positive integers, the authors do not know a complete characterization.

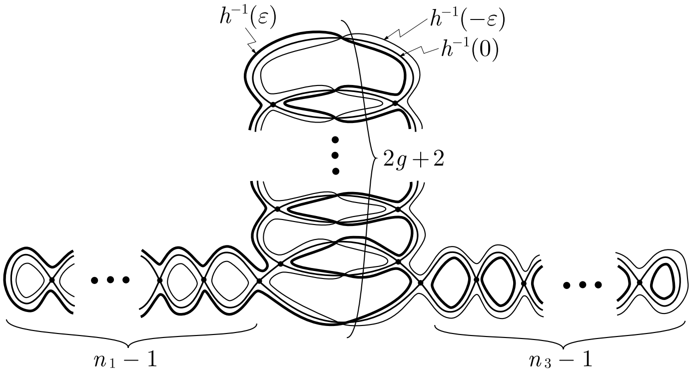

Figure 4. Orientable case.

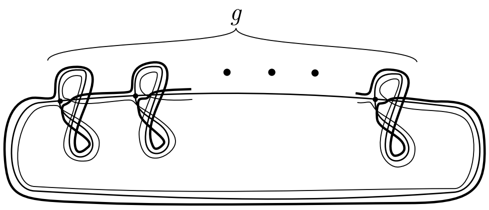

Figure 5. Non-orientable case.

5. Number of critical points of distance function.

Let $D : \boldsymbol{R}^3 \setminus \{0\} \to (0, \infty)$ be the function defined by

$$D(x, y, z) = \sqrt{x^2 + y^2 + z^2},$$

which measures the distance from the origin. For an immersion $\iota : M \to \boldsymbol{R}^3 \setminus \{0\}$ of a closed surface M, the composition $d = D \circ \iota$ is called the *distance function* associated with the immersion ι.

Let $d : M \to \boldsymbol{R}$ be a distance function on a closed surface M with finitely many critical points, and let $v_1 < v_2 < \cdots < v_\ell$ be its critical values. For j with $1 \leq j \leq \ell$, we denote by n_j the number of critical points of d corresponding to the critical value v_j. The sequence of positive integers $(n_1, n_2, \ldots, n_\ell)$ is called the *critical point number sequence* of the distance function d associated with the immersion $\iota : M \to \boldsymbol{R}^3$.

Consider the commutative diagram

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[d, "h"'] & \boldsymbol{R}^2 \times \boldsymbol{R} \arrow[r, "\eta\times\exp"] \arrow[d, "Z"'] & S^2 \times (0,\infty) \arrow[r, "\varphi"] \arrow[d, "\pi_2"'] & \boldsymbol{R}^3 \setminus \{0\} \arrow[d, "D"'] \\ \boldsymbol{R} \arrow[r, "="] & \boldsymbol{R} \arrow[r, "\exp"] & (0,\infty) \arrow[r, "="] & (0,\infty), \end{tikzcd}$$

where h is the height function associated with an immersion $\iota : M \to \boldsymbol{R}^3$, $\eta : \boldsymbol{R}^2 \to S^2$ is an embedding, exp is the exponential function, π_2 is the projection to the second factor, and φ is an appropriate diffeomorphism. Therefore, we see that if a sequence $(n_1, n_2, \dots, n_\ell)$ is realized as the critical point number sequence of a height function, then it is also realized as that of a distance function.

In fact, we can prove the following, which should be compared with Corollary 2.2.

THEOREM 5.1. *Let M be an arbitrary closed surface. Then, an arbitrary sequence of positive integers $(n_1, n_2, \dots, n_\ell)$ with $\ell \geq 3$ can be realized as the critical point number sequence of the distance function associated with an immersion of M into $\boldsymbol{R}^3 \setminus \{0\}$.*

PROOF. We have only to consider the case where $\ell = 3$ and M is orientable. We proceed by induction on the genus g of M.

When $g = 0$ or 1, the result follows from Theorem 2.1. In order to increase the genus, let us consider the operation as depicted in Figure 6. The left-hand side figure explains an immersion

$$\iota_1 : M_{(n_1+n_3)} \to S^2 \times [\exp(-\varepsilon), \exp\varepsilon]$$

such that $d = \pi'_2 \circ \iota_1$ has exactly n_2 critical points with value zero, where $\pi'_2 : S^2 \times [\exp(-\varepsilon), \exp\varepsilon] \to [\exp(-\varepsilon), \exp\varepsilon]$ is the projection to the second factor. Furthermore, $d^{-1}(\exp(-\varepsilon))$ (or $d^{-1}(\exp\varepsilon)$) consists of n_1 circles (resp. n_3 circles) of "odd winding numbers".

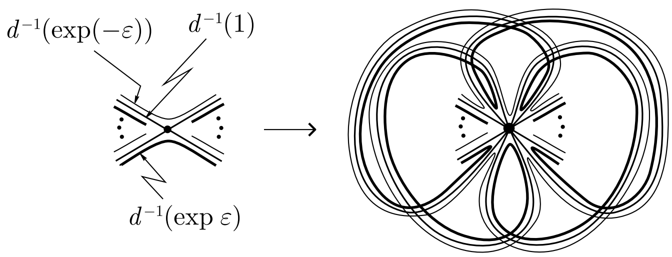

Figure 6. Increasing the genus by two.

Then the source surface of the new immersion ι_2 is orientable and has genus greater than that of $M_{(n_1+n_3)}$ by two. On the level of $S^2\times\{-\varepsilon\}$, it corresponds to n_1 immersed circles, and their winding numbers are all congruent modulo two to those of ι_1. Therefore, as immersed circles in S^2, they are regularly homotopic. For the level of $S^2\times\{\varepsilon\}$, we have the same situation. Therefore, we can construct an immersion $\tilde{M}\to S^2\times(0,\infty)\cong \boldsymbol{R}^3\setminus\{0\}$ whose associated distance function realizes (n_1,n_2,n_3), where $\tilde{M}$ is a closed orientable surface of genus greater than that of M by two. Thus, by induction, we get the desired conclusion. This completes the proof. $\square$

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