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The geometry of symmetric triad and orbit spaces of Hermann actions

By Osamu Ikawa

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Abstract. We introduce the notion of symmetric triad, which is a generalization of the notion of irreducible root system, and study its fundamental properties. Applying these results, we study the orbit spaces of Hermann actions on compact symmetric spaces.

1. Introduction.

Let (G, K_1, K_2) be a compact symmetric triad. The isometric action of K_2 on a compact Riemannian symmetric space $M_1 = G/K_1$ is called a Hermann action (see Section 4.1 for the detail). When $K_1 = K_2$, then the Hermann action is nothing but the isotropy action on M_1. In this paper, we describe the orbit spaces of Hermann actions using the fact that a Hermann action is hyperpolar. Here an isometric action of a compact Lie group on a Riemannian manifold is called hyperpolar if there exists a connected closed flat submanifold, called a section, that meets all orbits orthogonally. A section is automatically totally geodesic. We mainly deal with (G, K_1, K_2) in the case where G is simple, and θ_1 and θ_2 commute each other, where θ_i is an involutive automorphism of G which defines K_i for $i = 1, 2$.

In order to describe the orbit space of such a Hermann action, we introduce the notion of a symmetric triad $(\tilde{\Sigma}, \Sigma, W)$ of a finite dimensional vector space $\mathfrak{a}$ with an inner product, which is a generalization of the notion of irreducible root system (Definition 2.2). For a given symmetric triad, we can define a point in $\mathfrak{a}$ to be a regular point or a totally geodesic point (Definitions 2.5 and 2.8). A connected component of the set of regular points is called a cell. We can define an Affine Weyl group and see that the Affine Weyl group acts transitively on the set of cells (Definition 2.9 and Proposition 2.10). The closure of a cell is a simplex. A totally geodesic point is a vertex of a simplex, but the converse is not true in

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general. We will define a symmetric triad with multiplicities (Definition 2.13). For a given symmetric triad with multiplicities we can define a point in $\mathfrak{a}$ to be an austere point or a minimal point (Definitions 2.15 and 2.16). A totally geodesic point is an austere point for any given multiplicities, and an austere point is a minimal point (Proposition 2.17). We can stratify a cell and see that each strata has a unique minimal point (Theorem 2.24). We classify the set of all symmetric triads (Theorem 2.19) and determine which point is totally geodesic or austere (Corollaries 2.22 and 2.23).

We construct a symmetric triad with multiplicities from a compact symmetric triad mentioned above and can identify the orbit space with the closure of its cell. Under the identification, regular, minimal, austere and totally geodesic point correspond to regular, minimal, austere and totally geodesic orbit, respectively. In particular each strata of the orbit space has a unique minimal orbit. When $K_1 = K_2$, this is a result of D. Hirohashi, H. Tasaki, H. Song and R. Takagi ([**8**]).

The notion of austere submanifold was introduced by Harvey-Lawson [**5**], which is a minimal submanifold whose second fundamental form has a certain symmetry (Definition 4.8). A totally geodesic submanifold is austere. The ambient spaces of known explicit austere submanifolds, except totally geodesic submanifolds and complex submanifolds, are only Euclidean spaces and spheres. In fact Harvey-Lawson constructed some austere submanifolds in spheres. Bryant [**2**] solved the local problem of describing the austere submanifolds of three dimension in Euclidean space in all dimension. Recently we determined austere orbits in the hypersphere of the tangent space among the orbits of s-representations. If an isotropy orbit in a compact Riemannian symmetric space is austere, then it is totally geodesic (Theorem 4.31). On the other hand, many austere orbits which are not totally geodesic can be constructed when we consider Hermann actions which are not isotropy actions.

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2. The geometry of symmetric triad.

We begin with recalling the definition of root system. Let $\mathfrak{a}$ be a finite dimensional vector space over $\boldsymbol{R}$ with an inner product $\langle\ ,\ \rangle$.

Definition 2.1. A finite subset $\Sigma \subset \mathfrak{a} - \{0\}$ is a *root system* of $\mathfrak{a}$, if it satisfies the following three conditions:

(1) $\mathfrak{a} = \mathrm{span}(\Sigma)$.

(2) If $\alpha, \beta \in \Sigma$, then $s_\alpha \beta \in \Sigma$, where we define an orthogonal transformation s_α of $\mathfrak{a}$ by

$$s_\alpha \beta = \beta - 2\frac{\langle \alpha, \beta \rangle}{\|\alpha\|^2}\alpha.$$

(3) If $\alpha, \beta \in \Sigma$, then $2(\langle \alpha, \beta \rangle / \|\alpha\|^2) \in \boldsymbol{Z}$.

A root system Σ of $\mathfrak{a}$ is called *irreducible* if it cannot be decomposed into two disjoint nonempty orthogonal subsets.

We will define a symmetric triad.

DEFINITION 2.2. A triple $(\tilde{\Sigma}, \Sigma, W)$ is a *symmetric triad* of $\mathfrak{a}$, if it satisfies the following six conditions:

(1) $\tilde{\Sigma}$ is an irreducible root system of $\mathfrak{a}$.
(2) Σ is a root system of $\mathfrak{a}$.
(3) W is a nonempty subset of $\mathfrak{a}$, which is invariant under the multiplication by -1, and $\tilde{\Sigma} = \Sigma \cup W$.
(4) If we put $l = \max\{\|\alpha\| \mid \alpha \in \Sigma \cap W\}$, then $\Sigma \cap W = \{\alpha \in \tilde{\Sigma} \mid \|\alpha\| \leq l\}$.
(5) For $\alpha \in W, \lambda \in \Sigma - W$,

$$2\frac{\langle \alpha, \lambda \rangle}{\|\alpha\|^2} \text{ is odd if and only if } s_\alpha \lambda \in W - \Sigma.$$

(6) For $\alpha \in W, \lambda \in W - \Sigma$,

$$2\frac{\langle \alpha, \lambda \rangle}{\|\alpha\|^2} \text{ is odd if and only if } s_\alpha \lambda \in \Sigma - W.$$

When $(\tilde{\Sigma}, \Sigma, W)$ is a symmetric triad of $\mathfrak{a}$, then $\Sigma \cap W$ is a root system of $\mathfrak{a}$ by (4) of Definition 2.2. When $\tilde{\Sigma}$ is an irreducible root system of $\mathfrak{a}$, then the triple $(\tilde{\Sigma}, \tilde{\Sigma}, \tilde{\Sigma})$ is a symmetric triad of $\mathfrak{a}$. Hence the definition of a symmetric triad is a generalization of the notion of an irreducible root system.

LEMMA 2.3. *Let $(\tilde{\Sigma}, \Sigma, W)$ be a symmetric triad of $\mathfrak{a}$. For any $\lambda \in (\Sigma - W) \cup (W - \Sigma)$, there exist $\alpha, \beta \in \Sigma \cap W$ such that $\lambda = \alpha + \beta$.*

PROOF. Any root which is not shortest in an irreducible root system is a sum of two roots which are shortest. Hence the assertion follows from the condition (4) of Definition 2.2. □

For a symmetric triad $(\tilde{\Sigma}, \Sigma, W)$ of $\mathfrak{a}$, we denote by $\tilde{\Pi}$ a fundamental system of $\tilde{\Sigma}$. We denote by $\tilde{\Sigma}^+$ the set of positive roots in $\tilde{\Sigma}$ with respect to $\tilde{\Pi}$. Set $\Sigma^+ = \Sigma \cap \tilde{\Sigma}^+$ and $W^+ = W \cap \tilde{\Sigma}^+$. Denote by Π the set of simple roots of Σ.

Lemma 2.4. *Any element of $\tilde{\Sigma}$ can be expressed as a linear combination of elements in Π, whose coefficients are integers.*

Proof. By the condition (3) of Definition 2.2, $\tilde{\Sigma} = (W - \Sigma) \cup \Sigma$. Hence the assertion follows from Lemma 2.3. $\square$

For a symmetric triad $(\tilde{\Sigma}, \Sigma, W)$ of $\mathfrak{a}$, put

$$\Gamma = \left\{ X \in \mathfrak{a} \mid \langle \lambda, X \rangle \in \frac{\pi}{2}\boldsymbol{Z} \quad (\lambda \in \tilde{\Sigma}) \right\},$$

$$\Gamma_{\Sigma \cap W} = \left\{ X \in \mathfrak{a} \mid \langle \alpha, X \rangle \in \frac{\pi}{2}\boldsymbol{Z} \quad (\alpha \in \Sigma \cap W) \right\}.$$

We have $\Gamma = \Gamma_{\Sigma \cap W}$ by Lemma 2.3.

Definition 2.5. A point in Γ is called a *totally geodesic point*.

Definition 2.6. Let $(\tilde{\Sigma}, \Sigma, W)$ and $(\tilde{\Sigma}', \Sigma', W')$ be symmetric triads of $\mathfrak{a}$ and $\mathfrak{a}'$, respectively. Then $(\tilde{\Sigma}, \Sigma, W)$ and $(\tilde{\Sigma}', \Sigma', W')$ are *equivalent*, if there exist a linear isometric isomorphism $f : \mathfrak{a} \to \mathfrak{a}'$ and $Y \in \Gamma$ such that $f(\tilde{\Sigma}) = \tilde{\Sigma}'$ and

$$\begin{cases} \Sigma' - W' = \{f(\alpha) \mid \alpha \in \Sigma - W, \langle \alpha, 2Y \rangle \in 2\pi\boldsymbol{Z}\} \\ \qquad\qquad \cup \{f(\alpha) \mid \alpha \in W - \Sigma, \langle \alpha, 2Y \rangle \in \pi + 2\pi\boldsymbol{Z}\}, \\ W' - \Sigma' = \{f(\alpha) \mid \alpha \in W - \Sigma, \langle \alpha, 2Y \rangle \in 2\pi\boldsymbol{Z}\} \\ \qquad\qquad \cup \{f(\alpha) \mid \alpha \in \Sigma - W, \langle \alpha, 2Y \rangle \in \pi + 2\pi\boldsymbol{Z}\}. \end{cases} \tag{2.1}$$

We write $(\tilde{\Sigma}, \Sigma, W) \sim (\tilde{\Sigma}', \Sigma', W')$ if $(\tilde{\Sigma}, \Sigma, W)$ and $(\tilde{\Sigma}', \Sigma', W')$ are equivalent. In this case $f(\Sigma \cap W) = \Sigma' \cap W'$ holds. The relation $\sim$ is an equivalent relation.

Proposition 2.7. *W is invariant under the action of the Weyl group $W(\Sigma)$ of Σ.*

Proof. By the condition (4) of Definition 2.2, $W \cap \Sigma$ is invariant under $W(\Sigma)$. Hence $(W - \Sigma) \cup (\Sigma - W)$ is also invariant under $W(\Sigma)$. Since $\Sigma - W = \{\alpha \in \Sigma \mid \|\alpha\| > l\}$, the two subset $\Sigma - W$ and $W - \Sigma$ are invariant under $W(\Sigma)$. The assertion follows from $W = (W \cap \Sigma) \cup (W - \Sigma)$. $\square$

Definition 2.8. We define an open subset $\mathfrak{a}_r$ in $\mathfrak{a}$ by

$$\mathfrak{a}_r = \bigcap_{\lambda\in\Sigma,\alpha\in W} \left\{ H \in \mathfrak{a} \mid \langle\lambda, H\rangle \notin \pi\boldsymbol{Z}, \langle\alpha, H\rangle \notin \frac{\pi}{2} + \pi\boldsymbol{Z} \right\}.$$

A point in $\mathfrak{a}_r$ is called a *regular point*, and a point in $\mathfrak{a} - \mathfrak{a}_r$ *a singular point*. A connected component of $\mathfrak{a}_r$ is called a *cell*.

Definition 2.9. An *Affine Weyl group* $\tilde{W}(\tilde{\Sigma}, \Sigma, W)$ of a symmetric triad $(\tilde{\Sigma}, \Sigma, W)$ is a subgroup of the semidirect product $O(\mathfrak{a}) \ltimes \mathfrak{a}$ whose generator set is given by $\{(s_\lambda, (2n\pi/\|\lambda\|^2)\lambda) \mid \lambda \in \Sigma, n \in \boldsymbol{Z}\} \cup \{(s_\alpha, ((2n+1)\pi/\|\alpha\|^2)\alpha) \mid \alpha \in W, n \in \boldsymbol{Z}\}$.

The action of $(s_\lambda, (2n\pi/\|\lambda\|^2)\lambda)$ to $\mathfrak{a}$ is a reflection with respect to the hyperplane $\langle\lambda, H\rangle = n\pi$, and the action of $(s_\alpha, ((2n+1)\pi/\|\alpha\|^2)\alpha)$ is a reflection with respect to the hyperplane $\langle\alpha, H\rangle = ((2n+1)/2)\pi$.

Proposition 2.10. *The Affine Weyl group $\tilde{W}(\tilde{\Sigma}, \Sigma, W)$ acts transitively on the set of cells.*

Proof. First we will prove that $\tilde{W}(\tilde{\Sigma}, \Sigma, W)$ maps any regular point to a regular point. Let $H \in \mathfrak{a}$ be a regular point. For $\lambda \in \Sigma, n \in \boldsymbol{Z}$, set $H_1 = (s_\lambda, (2n\pi/\|\lambda\|^2)\lambda) \cdot H \in \mathfrak{a}$. We will prove that H_1 is a regular point. For $\mu \in \Sigma$ we have

$$\langle\mu, H_1\rangle = \langle s_\lambda\mu, H\rangle + \frac{2n\pi}{\|\lambda\|^2}\langle\lambda, \mu\rangle, \quad \frac{2n\pi}{\|\lambda\|^2}\langle\lambda, \mu\rangle \in \pi\boldsymbol{Z}.$$

Since H is regular, $\langle s_\lambda\mu, H\rangle \notin \pi\boldsymbol{Z}$. Hence we get $\langle\mu, H_1\rangle \notin \pi\boldsymbol{Z}$. For $\alpha \in W$, we have

$$\langle\alpha, H_1\rangle = \langle s_\lambda\alpha, H\rangle + \frac{2n\pi}{\|\lambda\|^2}\langle\lambda, \alpha\rangle, \quad \frac{2n\pi}{\|\lambda\|^2}\langle\lambda, \alpha\rangle \in \pi\boldsymbol{Z}.$$

Since $s_\lambda\alpha \in W$ by Proposition 2.7 and H is regular, we have $\langle s_\lambda\alpha, H\rangle \notin \pi/2 + \pi\boldsymbol{Z}$. Hence $\langle\alpha, H\rangle \notin \pi/2 + \pi\boldsymbol{Z}$, which implies that H_1 is regular.

For $\alpha \in W, n \in \boldsymbol{Z}$, set $H_2 = (s_\alpha, ((2n+1)\pi/\|\alpha\|^2)\alpha) \cdot H \in \mathfrak{a}$. We will prove that H_2 is regular. Let $\lambda \in \Sigma$, then

$$\langle\lambda, H_2\rangle = \langle s_\alpha\lambda, H\rangle + \frac{2n+1}{2}\frac{2\langle\alpha, \lambda\rangle}{\|\alpha\|^2}\pi, \quad \frac{2n+1}{2}\frac{2\langle\alpha, \lambda\rangle}{\|\alpha\|^2}\pi \in \frac{\pi}{2}\boldsymbol{Z}.$$

When $\lambda \in \Sigma \cap W$, we have $s_\alpha \lambda \in \Sigma \cap W$ by (4) of Definition 2.2. Since H is regular, $\langle s_\alpha \lambda, H\rangle \notin (\pi/2)\boldsymbol{Z}$. Hence $\langle \lambda, H_2\rangle \notin (\pi/2)\boldsymbol{Z}$. Consider the case where $\lambda \in \Sigma - W$. If $2\langle \alpha, \lambda\rangle/\|\alpha\|^2$ is even, then $((2n+1)/2)(2\langle \alpha, \lambda\rangle/\|\alpha\|^2)\pi \in \pi\boldsymbol{Z}$. In this case $s_\alpha\lambda \in \Sigma - W$ by (5) of Definition 2.2. Since H is regular, $\langle s_\alpha\lambda, H\rangle \notin \pi\boldsymbol{Z}$. Hence $\langle \lambda, H_2\rangle \notin \pi\boldsymbol{Z}$. If $2\langle \alpha, \lambda\rangle/\|\alpha\|^2$ is odd, then $((2n+1)/2)(2\langle \alpha, \lambda\rangle/\|\alpha\|^2)\pi \in \pi/2 + \pi\boldsymbol{Z}$. In this case $s_\alpha\lambda \in W - \Sigma$ by (5) of Definition 2.2. Since H is regular, $\langle s_\alpha\lambda, H\rangle \notin \pi/2 + \pi\boldsymbol{Z}$. If $\beta \in W$, then

$$\langle \beta, H_2\rangle = \langle s_\alpha\beta, H\rangle + \frac{2n+1}{2}\frac{2\langle \alpha, \beta\rangle}{\|\alpha\|^2}\pi, \qquad \frac{2n+1}{2}\frac{2\langle \alpha, \beta\rangle}{\|\alpha\|^2}\pi \in \frac{\pi}{2}\boldsymbol{Z}.$$

We may assume that $\beta \in W - \Sigma$. If $2\langle \alpha, \beta\rangle/\|\alpha\|^2$ is even, then $((2n+1)/2)(2\langle \alpha, \beta\rangle/\|\alpha\|^2) \in \pi\boldsymbol{Z}$. In this case $s_\alpha\beta \in W - \Sigma$ by (6) of Definition 2.2. Since H is regular, $\langle s_\alpha\beta, H\rangle \notin \pi/2 + \pi\boldsymbol{Z}$. Hence $\langle \beta, H_2\rangle \notin \pi/2 + \pi\boldsymbol{Z}$. If $2\langle \alpha, \beta\rangle/\|\alpha\|^2$ is odd, then $((2n+1)/2)(2\langle \alpha, \beta\rangle/\|\alpha\|^2) \in \pi/2 + \pi\boldsymbol{Z}$. In this case $s_\alpha\beta \in \Sigma - W$ by (6) of Definition 2.2. Since H is regular, $\langle s_\alpha\beta, H\rangle \notin \pi\boldsymbol{Z}$. Hence $\langle \beta, H_2\rangle \notin \pi/2 + \pi\boldsymbol{Z}$.

From the above argument, we see that the Affine Weyl group maps any regular point to a regular point.

Since the action of $\tilde{W}(\tilde{\Sigma}, \Sigma, W)$ is a homeomorphism, $\tilde{W}(\tilde{\Sigma}, \Sigma, W)$ maps any cell to a cell.

Let P_1, P_2 be two cells and select $H_i \in P_i$ $(i = 1, 2)$. If the segment $\overline{H_1H_2}$ intersects a hyperplane $\langle \lambda, H\rangle = n\pi$ $(\lambda \in \Sigma, n \in \boldsymbol{Z})$, then

$$\|H_2 - H_1\| > \left\| H_2 - \left(s_\lambda, \frac{2n\pi}{\|\lambda\|^2}\lambda \right) \cdot H_1 \right\|.$$

If $\overline{H_1H_2}$ intersects a hyperplane $\langle \alpha, H\rangle = ((2n+1)/2)\pi$ $(\alpha \in W, n \in \boldsymbol{Z})$, then

$$\|H_2 - H_1\| > \left\| H_2 - \left(s_\alpha, \frac{(2n+1)\pi}{\|\alpha\|^2}\alpha \right) \cdot H_1 \right\|.$$

Take $s_0 \in \tilde{W}(\tilde{\Sigma}, \Sigma, W)$ such that

$$\|H_2 - s_0H_1\| = \min\left\{ \|H_2 - sH_1\| \mid s \in \tilde{W}(\tilde{\Sigma}, \Sigma, W) \right\},$$

then the segment from s_0H_1 to H_2 intersects no hyperplane of the form mentioned above. Hence $s_0P_1 = P_2$. $\square$

Corollary 2.11. *$\mathfrak{a} = \bigcup_{s \in \tilde{W}(\tilde{\Sigma}, \Sigma, W)} s\overline{P_0}$ for any fixed cell P_0.*

For a symmetric triad $(\tilde{\Sigma}, \Sigma, W)$ of $\mathfrak{a}$, set

$$P_0 = \left\{ H \in \mathfrak{a} \,\middle|\, \begin{array}{ll} 0 < \langle \lambda, H \rangle & (\lambda \in \Pi), \\ \langle \lambda, H \rangle < \frac{\pi}{2} & (\lambda \in \Sigma^+ \cap W^+), \\ \langle \lambda, H \rangle < \pi & (\lambda \in \Sigma^+ - W^+), \\ -\frac{\pi}{2} < \langle \alpha, H \rangle < \frac{\pi}{2} & (\alpha \in W^+ - \Sigma^+) \end{array} \right\}, \tag{2.2}$$

then P_0 is a cell. Put

$$W_0 = \{\alpha \in W^+ \mid \alpha + \lambda \notin W \quad (\lambda \in \Pi)\}, \tag{2.3}$$

then clearly we have $W_0 \neq \emptyset$.

Lemma 2.12. $P_0 = \{H \in \mathfrak{a} \mid 0 < \langle \lambda, H \rangle \ (\lambda \in \Pi),\ \langle \alpha, H \rangle < \pi/2 \ (\alpha \in W_0)\}$.

Proof. By Lemma 2.3 we have

$$P_0 = \left\{ H \in \mathfrak{a} \,\middle|\, \begin{array}{ll} 0 < \langle \lambda, H \rangle & (\lambda \in \Pi), \\ \langle \lambda, H \rangle < \frac{\pi}{2} & (\lambda \in \Sigma^+ \cap W^+), \\ -\frac{\pi}{2} < \langle \alpha, H \rangle < \frac{\pi}{2} & (\alpha \in W^+ - \Sigma^+) \end{array} \right\}.$$

By the definition of W_0, for any $\alpha \in W^+$ there exist $\lambda_1, \dots, \lambda_k \in \Pi$ such that $\alpha + \lambda_1 + \cdots + \lambda_k \in W_0$ (k may be equal to 0). Denote by P_1 the right-hand side of the equation in Lemma 2.12, then

$$P_0 = \left\{ H \in P_1 \,\middle|\, -\frac{\pi}{2} < \langle \lambda, H \rangle \quad (\lambda \in W^+ - \Sigma^+) \right\} \subset P_1.$$

Take $H \in P_1$ arbitrarily. For $\lambda \in W^+ - \Sigma^+$, there exist $\alpha, \beta \in \Sigma \cap W$ such that $\lambda = \alpha + \beta$ by Lemma 2.3. We may assume that $\beta > 0$ since one of α and β is positive. Then $\langle \lambda, H \rangle > \langle \alpha, H \rangle$. If $\alpha > 0$, then $\langle \lambda, H \rangle > \langle \alpha, H \rangle > 0$. If $\alpha < 0$, then $\langle \lambda, H \rangle > -\langle -\alpha, H \rangle > -\pi/2$. Hence $P_0 = P_1$. $\square$

Definition 2.13. Let $(\tilde{\Sigma}, \Sigma, W)$ be a symmetric triad of $\mathfrak{a}$. Put $\boldsymbol{R}_{\geq 0} = \{x \in \boldsymbol{R} \mid x \geq 0\}$. Consider two mappings $m, n : \tilde{\Sigma} \to \boldsymbol{R}_{\geq 0}$ which satisfy the following four conditions:

(1) $m(\lambda) = m(-\lambda)$, $n(\alpha) = n(-\alpha)$ and

$$m(\lambda) > 0 \Leftrightarrow \lambda \in \Sigma, \quad n(\alpha) > 0 \Leftrightarrow \alpha \in W.$$

(2) When $\lambda \in \Sigma$, $\alpha \in W$, $s \in W(\Sigma)$ then $m(\lambda) = m(s\lambda)$, $n(\alpha) = n(s\alpha)$.
(3) When $\sigma \in W(\tilde{\Sigma})$, the Weyl group of $\tilde{\Sigma}$, and $\lambda \in \tilde{\Sigma}$ then $n(\lambda) + m(\lambda) = n(\sigma\lambda) + m(\sigma\lambda)$.
(4) Let $\lambda \in \Sigma \cap W$ and $\alpha \in W$.
 If $2\langle \alpha, \lambda \rangle / \|\alpha\|^2$ is even then $m(\lambda) = m(s_\alpha \lambda)$.
 If $2\langle \alpha, \lambda \rangle / \|\alpha\|^2$ is odd then $m(\lambda) = n(s_\alpha \lambda)$.

We call $m(\lambda)$ and $n(\alpha)$ the *multiplicities* of λ and α, respectively. If multiplicities are given, we call $(\tilde{\Sigma}, \Sigma, W)$ *the symmetric triad with multiplicities.* For $H \in \mathfrak{a}$, set

$$m_H = -\sum_{\substack{\lambda \in \Sigma^+ \\ \langle \lambda, H \rangle \notin (\pi/2)\boldsymbol{Z}}} m(\lambda) \cot(\langle \lambda, H \rangle)\lambda + \sum_{\substack{\alpha \in W^+ \\ \langle \alpha, H \rangle \notin (\pi/2)\boldsymbol{Z}}} n(\alpha) \tan(\langle \alpha, H \rangle)\alpha.$$

We call m_H the *mean curvature vector* of H. Set

$$F(H) = -\sum_{\substack{\lambda \in \Sigma^+ \\ \langle \lambda, H \rangle \notin (\pi/2)\boldsymbol{Z}}} m(\lambda) \log|\sin(\langle \lambda, H \rangle)| - \sum_{\substack{\alpha \in W^+ \\ \langle \alpha, H \rangle \notin (\pi/2)\boldsymbol{Z}}} n(\alpha) \log|\cos(\langle \alpha, H \rangle)|,$$

and $\mathrm{Vol}(H) = \exp(-F(H))(> 0)$. We call $\mathrm{Vol}(H)$ the *volume* of H.

Remark. Let $(\tilde{\Sigma}, \Sigma, W)$ be a symmetric triad with multiplicities. When $\lambda \in \Sigma \cap W, \alpha \in W$, then we have the following by (3) and (4) of Definition 2.13:
 If $2\langle \alpha, \lambda \rangle / \|\alpha\|^2$ is even, then $n(\lambda) = n(s_\alpha \lambda)$.
 If $2\langle \alpha, \lambda \rangle / \|\alpha\|^2$ is odd, then $n(\lambda) = m(s_\alpha \lambda)$.
When $\sigma \in W(\tilde{\Sigma})$, then we have the following by (3) of Definition 2.13:
 If $\lambda, \sigma\lambda \in \Sigma - W$, then $m(\lambda) = m(\sigma\lambda)$.
 If $\lambda, \sigma\lambda \in W - \Sigma$, then $n(\lambda) = n(\sigma\lambda)$.
 If $\lambda \in \Sigma - W, \sigma\lambda \in W - \Sigma$, then $m(\lambda) = n(\sigma\lambda)$.

Proposition 2.14. *Let $(\tilde{\Sigma}, \Sigma, W)$ be a symmetric triad of $\mathfrak{a}$ with multiplicities. For $H \in \mathfrak{a}$ and $\sigma = (s, X) \in \tilde{W}(\tilde{\Sigma}, \Sigma, W)$ set $H' = \sigma H \in \mathfrak{a}$, then*

$$\mathrm{Vol}(H') = \mathrm{Vol}(H), \quad m_{H'} = s m_H.$$

Proof. For $\mu \in \Sigma, n \in \boldsymbol{Z}, H \in \mathfrak{a}$ set $H_1 = (s_\mu, (2n\pi/\|\mu\|^2)\mu) \cdot H \in \mathfrak{a}$. We will prove that $m_{H_1} = s_\mu m_H$. Since

$$\langle \lambda, H_1 \rangle = \langle s_\mu \lambda, H \rangle + \frac{2\langle \mu, \lambda \rangle}{\|\mu\|^2} n\pi$$

for $\lambda \in \tilde{\Sigma}$, we have

$$|\sin(\langle \lambda, H_1 \rangle)| = |\sin(\langle s_\mu \lambda, H \rangle)|, \quad |\cos(\langle \lambda, H_1 \rangle)| = |\cos(\langle s_\mu \lambda, H \rangle)|.$$

Since $2\langle \mu, \lambda \rangle / \|\mu\|^2 \in \boldsymbol{Z}$, we get

$$\langle \lambda, H_1 \rangle \notin \frac{\pi}{2}\boldsymbol{Z} \Leftrightarrow \langle s_\mu \lambda, H \rangle \notin \frac{\pi}{2}\boldsymbol{Z}.$$

By Proposition 2.7 and Definition 2.13, we have

$$\begin{aligned} 2m_{H_1} &= - \sum_{\substack{\lambda \in \Sigma \\ \langle s_\mu \lambda, H \rangle \notin (\pi/2)\boldsymbol{Z}}} m(\lambda) \cot(\langle s_\mu H \rangle)\lambda + \sum_{\substack{\alpha \in W \\ \langle s_\mu \alpha, H \rangle \notin (\pi/2)\boldsymbol{Z}}} n(\alpha) \tan(\langle s_\mu \alpha, H \rangle)\alpha \\ &= 2s_\mu m_H. \end{aligned}$$

Similarly we have $\mathrm{Vol}(H_1) = \mathrm{Vol}(H)$.

For $\beta \in W, n \in \boldsymbol{Z}, H \in \mathfrak{a}$ set $H_2 = (s_\beta, ((2n+1)\pi/\|\beta\|^2)\beta) \cdot H \in \mathfrak{a}$. We will prove that $m_{H_2} = s_\beta m_H$. Since

$$\langle \lambda, H_2 \rangle = \langle s_\beta \lambda, H \rangle + \frac{(2n+1)\pi}{2} \frac{2\langle \beta, \lambda \rangle}{\|\beta\|^2}, \quad \frac{(2n+1)\pi}{2} \frac{2\langle \beta, \lambda \rangle}{\|\beta\|^2} \in \frac{\pi}{2}\boldsymbol{Z}$$

for $\lambda \in \tilde{\Sigma}$, we have

$$\langle \lambda, H_2 \rangle \notin \frac{\pi}{2}\boldsymbol{Z} \Leftrightarrow \langle s_\beta, H \rangle \notin \frac{\pi}{2}\boldsymbol{Z}.$$

When $2\langle \beta, \lambda \rangle / \|\beta\|^2$ is even, then

$$\begin{gathered} |\sin(\langle \lambda, H_2 \rangle)| = |\sin(\langle s_\beta \lambda, H \rangle)|, \quad |\cos(\langle \lambda, H_2 \rangle)| = |\cos(\langle s_\beta \lambda, H \rangle)|, \\ \tan(\langle \lambda, H_2 \rangle) = \tan(\langle s_\beta \lambda, H \rangle). \end{gathered}$$

When $2\langle \beta, \lambda \rangle / \|\beta\|^2$ is odd, then

$$\begin{gathered} |\sin(\langle \lambda, H_2 \rangle)| = |\cos(\langle s_\beta \lambda, H \rangle)|, \quad |\cos(\langle \lambda, H_2 \rangle)| = |\sin(\langle s_\beta \lambda, H \rangle)|, \\ \tan(\langle \lambda, H_2 \rangle) = -\cot(\langle s_\beta \lambda, H \rangle). \end{gathered}$$

By the definition of mean curvature vector, we have

$$
\begin{aligned}
2m_{H_2} = &- \sum_{\substack{\lambda\in\Sigma-W,\langle\lambda,H_2\rangle\notin(\pi/2)\boldsymbol{Z}\\ (2\langle\beta,\lambda\rangle)/\|\beta\|^2:\mathrm{even}}} m(\lambda)\cot(\langle\lambda,H_2\rangle)\lambda \\
&- \sum_{\substack{\lambda\in\Sigma-W,\langle\lambda,H_2\rangle\notin(\pi/2)\boldsymbol{Z}\\ (2\langle\beta,\lambda\rangle)/\|\beta\|^2:\mathrm{odd}}} m(\lambda)\cot(\langle\lambda,H_2\rangle)\lambda \\
&+ \sum_{\substack{\alpha\in W-\Sigma,\langle\alpha,H_2\rangle\notin(\pi/2)\boldsymbol{Z}\\ (2\langle\beta,\alpha\rangle)/\|\beta\|^2:\mathrm{even}}} n(\alpha)\tan(\langle\alpha,H_2\rangle)\alpha \\
&+ \sum_{\substack{\alpha\in W-\Sigma,\langle\alpha,H_2\rangle\notin(\pi/2)\boldsymbol{Z}\\ (2\langle\beta,\alpha\rangle)/\|\beta\|^2:\mathrm{odd}}} n(\alpha)\tan(\langle\alpha,H_2\rangle)\alpha \\
&+ \sum_{\substack{\lambda\in\Sigma\cap W,\langle\lambda,H_2\rangle\notin(\pi/2)\boldsymbol{Z}\\ (2\langle\beta,\lambda\rangle)/\|\beta\|^2:\mathrm{even}}} \big(n(\lambda)\tan(\langle\lambda,H_2\rangle)\lambda - m(\lambda)\cot(\langle\lambda,H_2\rangle)\lambda\big) \\
&+ \sum_{\substack{\lambda\in\Sigma\cap W,\langle\lambda,H_2\rangle\notin(\pi/2)\boldsymbol{Z}\\ (2\langle\beta,\lambda\rangle)/\|\beta\|^2:\mathrm{odd}}} \big(n(\lambda)\tan(\langle\lambda,H_2\rangle)\lambda - m(\lambda)\cot(\langle\lambda,H_2\rangle)\lambda\big).
\end{aligned}
$$

By the definition of multiplicities, we have

$$
\begin{aligned}
2m_{H_2} = &- \sum_{\substack{\lambda\in\Sigma-W,\langle s_\beta\lambda,H\rangle\notin(\pi/2)\boldsymbol{Z}\\ (2\langle\beta,\lambda\rangle)/\|\beta\|^2:\mathrm{even}}} m(s_\beta\lambda)\cot(\langle s_\beta\lambda,H\rangle)\lambda \\
&+ \sum_{\substack{\lambda\in\Sigma-W,\langle s_\beta\lambda,H\rangle\notin(\pi/2)\boldsymbol{Z}\\ (2\langle\beta,\lambda\rangle)/\|\beta\|^2:\mathrm{odd}}} n(s_\beta\lambda)\tan(\langle s_\beta\lambda,H\rangle)\lambda \\
&+ \sum_{\substack{\alpha\in W-\Sigma,\langle s_\beta\alpha,H\rangle\notin(\pi/2)\boldsymbol{Z}\\ (2\langle\beta,\alpha\rangle)/\|\beta\|^2:\mathrm{even}}} n(s_\beta\alpha)\tan(\langle s_\beta\alpha,H\rangle)\alpha \\
&- \sum_{\substack{\alpha\in W-\Sigma,\langle s_\beta\alpha,H\rangle\notin(\pi/2)\boldsymbol{Z}\\ (2\langle\beta,\alpha\rangle)/\|\beta\|^2:\mathrm{odd}}} m(s_\beta\alpha)\cot(\langle s_\beta\alpha,H\rangle)\alpha \\
&+ \sum_{\substack{\lambda\in\Sigma\cap W,\langle s_\beta\lambda,H\rangle\notin(\pi/2)\boldsymbol{Z}\\ (2\langle\beta,\lambda\rangle)/\|\beta\|^2:\mathrm{even}}} \big(n(s_\beta\lambda)\tan(\langle s_\beta\lambda,H\rangle)\lambda - m(s_\beta\lambda)\cot(\langle s_\beta\lambda,H\rangle)\lambda\big) \\
&+ \sum_{\substack{\lambda\in\Sigma\cap W,\langle s_\beta\lambda,H\rangle\notin(\pi/2)\boldsymbol{Z}\\ (2\langle\beta,\lambda\rangle)/\|\beta\|^2:\mathrm{odd}}} \big(-m(s_\beta\lambda)\cot(\langle s_\beta\lambda,H\rangle)\lambda + n(s_\beta\lambda)\tan(\langle s_\beta\lambda,H\rangle)\lambda\big) \\
= &\, 2s_\beta m_H.
\end{aligned}
$$

Similarly we have $\mathrm{Vol}(H_2) = \mathrm{Vol}(H)$. □

Definition 2.15. Let $(\tilde{\Sigma}, \Sigma, W)$ be a symmetric triad of $\mathfrak{a}$ with multiplicities. Then $H \in \mathfrak{a}$ is a *minimal point* if $m_H = 0$.

Definition 2.16. Let $(\tilde{\Sigma}, \Sigma, W)$ be a symmetric triad of $\mathfrak{a}$ with multiplicities. Then $H \in \mathfrak{a}$ is an *austere point* if the finite subset of $\mathfrak{a}$ with multiplicities defined by

$$\begin{aligned}&\left\{ -\lambda \cot(\langle \lambda, H \rangle) \ (\text{multiplicity} = m(\lambda)) \mid \lambda \in \Sigma^+, \langle \lambda, H \rangle \notin \frac{\pi}{2}\boldsymbol{Z} \right\} \\ &\qquad \cup \left\{ \alpha \tan(\langle \alpha, H \rangle) \ (\text{multiplicity} = n(\alpha)) \mid \alpha \in W^+, \langle \alpha, H \rangle \notin \frac{\pi}{2}\boldsymbol{Z} \right\} \end{aligned} \tag{2.4}$$

is invariant with multiplicities under the multiplication by -1.

The following proposition is clear.

Proposition 2.17.
(1) *Any totally geodesic point is austere for any given multiplicities.*
(2) *Any austere point is minimal.*

Theorem 2.18. *A point $H \in \mathfrak{a}$ is austere if and only if the following three conditions hold*:

(1) $\langle \lambda, H \rangle \in (\pi/2)\boldsymbol{Z}$ *for any* $\lambda \in (\Sigma - W) \cup (W - \Sigma)$.
(2) $2H \in \Gamma_{\Sigma \cap W}$.
(3) $m(\lambda) = n(\lambda)$ *for any* $\lambda \in \Sigma \cap W$ *with* $\langle \lambda, H \rangle \in \pi/4 + (\pi/2)\boldsymbol{Z}$.

Proof. Assume that H satisfies the above conditions (1), (2) and (3). By the conditions (1) and (2), the set defined by (2.4) is given by

$$\begin{aligned}&\left\{ -\lambda \cot(\langle \lambda, H \rangle) \ (\text{multiplicity} = m(\lambda)) \mid \lambda \in \Sigma^+ \cap W^+, \langle \lambda, H \rangle \in \frac{\pi}{4} + \frac{\pi}{2}\boldsymbol{Z} \right\} \\ &\qquad \cup \left\{ \lambda \tan(\langle \lambda, H \rangle) \ (\text{multiplicity} = n(\lambda)) \mid \lambda \in \Sigma^+ \cap W^+, \langle \lambda, H \rangle \in \frac{\pi}{4} + \frac{\pi}{2}\boldsymbol{Z} \right\}.\end{aligned}$$

By (3), the set above is invariant under the multiplication by -1 with multiplicities.

Conversely assume that H is austere. We shall show that the conditions (1), (2) and (3) hold.

First consider in the case where $\tilde{\Sigma} \neq BC$. Since H is austere, $\langle \lambda, H \rangle \in (\pi/2)\boldsymbol{Z}$ for any $\lambda \in (\Sigma - W) \cup (W - \Sigma)$, and $m(\alpha) = n(\alpha)$ and $\cot(\langle \alpha, H \rangle) = \tan(\langle \alpha, H \rangle)$ for any $\alpha \in \Sigma^+ \cap W^+$ with $\langle \alpha, H \rangle \notin (\pi/2)\boldsymbol{Z}$. Hence H satisfies the conditions (1), (2) and (3).

Second consider when $\tilde{\Sigma} = BC$, $\Sigma \cap W = \{e_i\}$. $\langle e_i \pm e_j, H\rangle \in (\pi/2)\boldsymbol{Z}$ since $e_i \pm e_j \in (\Sigma - W) \cup (W - \Sigma)$. Hence $\langle 2e_i, H\rangle \in (\pi/2)\boldsymbol{Z}$. Set $m = m(e_i), n = n(e_i)$. Since H is austere, the set

$$\left\{ - e_i \cot(\langle e_i, H\rangle) \text{ (multiplicity} = m) \mid \langle e_i, H\rangle \in \frac{\pi}{4} + \frac{\pi}{2}\boldsymbol{Z} \right\}$$
$$\cup \left\{ e_i \tan(\langle e_i, H\rangle) \text{ (multiplicity} = n) \mid \langle e_i, H\rangle \in \frac{\pi}{4} + \frac{\pi}{2}\boldsymbol{Z} \right\}$$

is invariant under multiplication by -1 with multiplicities. Hence $m = n$. H satisfies the conditions (1), (2) and (3).

Last consider the case where $\tilde{\Sigma} = BC$, $\Sigma \cap W \supset \{e_i, e_i \pm e_j\}$. $\langle e_i + e_j, H\rangle \in (\pi/4)\boldsymbol{Z}$ since $\langle e_i + e_j, H\rangle \in (\pi/2)\boldsymbol{Z}$ or $\cot(\langle e_i + e_j, H\rangle) = \tan(\langle e_i + e_j, H\rangle)$. Similarly $\langle e_i - e_j, H\rangle \in (\pi/4)\boldsymbol{Z}$. Hence $\langle 2e_i, H\rangle \in (\pi/4)\boldsymbol{Z}$. If there existed i such that $\langle e_i, H\rangle \in \pi/8 + (\pi/4)\boldsymbol{Z}$, by Definition 2.16, the following equation would hold:

$$-e_i \tan(\langle e_i, H\rangle) = 2e_i \tan(2\langle e_i, H\rangle), \quad -e_i \cot(\langle e_i, H\rangle), \text{ or } - 2e_i \cot(\langle 2\langle e_i, H\rangle).$$

Hence one of the following three equations would hold:

$$\tan(\langle e_i, H\rangle) = \begin{cases} -2\tan(2\langle e_i, H\rangle) = \pm 2, \\ 2\cot(2\langle e_i, H\rangle) = \pm 2, \\ \cot(\langle e_i, H\rangle). \end{cases}$$

Since $\langle e_i, H\rangle \in \pi/8 + (\pi/4)\boldsymbol{Z}$, we would have $\tan(\langle e_i, H\rangle) \neq \cot(\langle e_i, H\rangle)$, which would imply $\tan(\langle e_i, H\rangle) = \pm 2$. Since $\tan((2n+1)\pi/8) = \pm(\sqrt{2} \pm 1)$ for $n \in \boldsymbol{Z}$, this would be a contradiction. Hence we can reduce when $\tilde{\Sigma} = B$, the point H satisfies the above conditions (1), (2) and (3). $\square$

In order to state the theorem below, we shall follow the notations of irreducible root systems and the set of positive roots in [**1**]. For instance,

$$\begin{aligned} A_r^+ &= \{e_i - e_j \mid 1 \leq i < j \leq r+1\}, \\ B_r^+ &= \{e_i \mid 1 \leq i \leq r\} \cup \{e_i \pm e_j \mid 1 \leq i < j \leq r\}, \\ C_r^+ &= \{2e_i \mid 1 \leq i \leq r\} \cup \{e_i \pm e_j \mid 1 \leq i < j \leq r\}, \\ BC_r^+ &= \{e_i, 2e_i \mid 1 \leq i \leq r\} \cup \{e_i \pm e_j \mid 1 \leq i < j \leq r\}, \\ D_r^+ &= \{e_i \pm e_j \mid 1 \leq i < j \leq r\}. \end{aligned}$$

For the sets of positive roots above, the sets of simple roots are given as follows:

$$\Pi(A_r^+) = \{\alpha_1 = e_1 - e_2, \dots, \alpha_r = e_r - e_{r+1}\},$$
$$\Pi(B_r^+) = \Pi(BC_r^+) = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{r-1} = e_{r-1} - e_r, \alpha_r = e_r\},$$
$$\Pi(C_r^+) = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{r-1} = e_{r-1} - e_r, \alpha_r = 2e_r\},$$
$$\Pi(D_r^+) = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{r-1} = e_{r-1} - e_r, \alpha_r = e_{r-1} + e_r\}.$$

THEOREM 2.19. *Each symmetric triad* $(\tilde{\Sigma}, \Sigma, W)$ *of* $\mathfrak{a}$ *is one of the forms mentioned below. The set* W_0 *defined by* (2.3) *consists of only one element* $\tilde{\alpha}$. *We list* $(\tilde{\Sigma}, \Sigma, W)$ *and* $\tilde{\alpha}$ *in the table below.*

(I) *In the case where* $\Sigma \supset W$, $\Sigma \neq W$:

type	Σ^+	W^+	$\tilde{\alpha}$
(I-B_r)	B_r^+	$\{e_i \mid 1 \leq i \leq r\}$	e_1
(I-C_r)	C_r^+	D_r^+	$e_1 + e_2$
(I-BC_r-A_1^r)	BC_r^+	$\{e_i \mid 1 \leq i \leq r\}$	e_1
(I-BC_r-B_r)	BC_r^+	B_r^+	$e_1 + e_2$
(I-F_4)	F_4^+	$\{$*short roots in* $F_4^+\} \cong D_4^+$	e_1

(II) *In the case where* $\Sigma \subset W$, $\Sigma \neq W$:

type	Σ^+	W^+	$\tilde{\alpha}$
(II-BC_r) ($r \geq 1$)	B_r^+	BC_r^+	$2e_1$
(I'-C_r)	D_r^+	C_r^+	$2e_1$

(I') *In the case where* $\Sigma \neq W$ *except for* (I) *and* (II):
Type (I'-F_4):

$$\Sigma^+ = \{\textit{short roots of } F_4^+\} \cup \{e_1 \pm e_2, e_3 \pm e_4\} \cong C_4,$$
$$W^+ = \{\textit{short roots of } F_4^+\} \cup \{e_1 \pm e_3, e_1 \pm e_4, e_2 \pm e_3, e_2 \pm e_4\},$$
$$\tilde{\alpha} = e_1 + e_3.$$

Type (I'-B_r) ($r \geq 3$):

$$\Sigma^+ = B_s^+ \cup B_{r-s}^+, \quad W^+ = (B_r^+ - \Sigma) \cup \{e_i\}, \quad \tilde{\alpha} = e_1 + e_{s+1}.$$

Type (I'-BC_r-A_1^r):

$$\Sigma^+ = BC_s^+ \cup BC_{r-s}^+, \quad W^+ = (BC_r^+ - \Sigma) \cup \{e_i\}, \quad \tilde{\alpha} = e_1 + e_{s+1}.$$

(III) *In the case where* $\tilde{\Sigma} = \Sigma = W$, $\tilde{\alpha}$ *is a highest root of the irreducible root system* $\tilde{\Sigma}$.

The following equivalent relation holds:

$$(\text{I-}F_4) \sim (\text{I'-}F_4), \quad (\text{I-}BC_r\text{-}A_1^r) \sim (\text{I'-}BC_r\text{-}A_1^r),$$
$$(\text{I-}C_r) \sim (\text{I'-}C_r), \quad (\text{I-}B_r) \sim (\text{I'-}B_r).$$

Proof. It is clear that the $(\tilde{\Sigma}, \Sigma, W)$'s in the above list are symmetric triads. The assertion is clear when $(\tilde{\Sigma}, \Sigma, W)$ is of type (I), (II) and (III). Hence we assume that $(\tilde{\Sigma}, \Sigma, W)$ is a symmetric triad of type (I'), that is, $\Sigma \neq W$, $\Sigma \not\subset W$ and $W \not\subset \Sigma$.

When $\tilde{\Sigma} = F_4$, then by (4) of Definition 2.2 we have

$$\Sigma \cap W = \{\text{short roots of } F_4\} \cong D_4.$$

Since the only root system Σ between $D_4 (\cong \Sigma \cap W)$ and F_4 is C_4, we have $\Sigma \cong C_4$. Hence $(\tilde{\Sigma}, \Sigma, W)$ is of type (I'-F_4).

Remark that root systems G_2 and C_r have the following property: If α is a long root, then $\{\text{long roots}\} = \{s_\beta \alpha \mid \beta \text{ is a short root}\}$, which implies that $\tilde{\Sigma} \neq G_2, C_r$ by Proposition 2.7. Hence $\tilde{\Sigma} = B_r$ or BC_r.

When $\tilde{\Sigma} = B_r$, then $\Sigma^+ \cap W^+ = \{e_i\}$ by (4) of Definition 2.2. Since Σ is a root system and $s_{e_j}(e_i + e_j) = e_i - e_j$, we have

$$e_i - e_j \in \Sigma \Leftrightarrow e_i + e_j \in \Sigma.$$

Hence

$$e_i - e_j \in W \Leftrightarrow e_i + e_j \in W.$$

If i, j and k are mutually distinct, then

$$\frac{2\langle e_i - e_j, e_j - e_k \rangle}{\|e_i - e_j\|^2} = 1, \quad s_{e_i - e_j}(e_j - e_k) = e_i - e_k.$$

Hence by (5) of Definition 2.2, we get

$$e_i - e_j \in W, \quad e_j - e_k \in \Sigma \Rightarrow e_i - e_k \in W.$$

By (6) of Definition 2.2, we get

$$e_i - e_j, \quad e_j - e_k \in W \Rightarrow e_i - e_k \in \Sigma.$$

If $e_i - e_j, e_j - e_k$ were in Σ and $e_k - e_i$ were in W, then $e_j - e_i = (e_j - e_k) + (e_k - e_i)$ would be in W, which would be a contradiction. Thus, $e_i - e_k \in \Sigma$ when $e_i - e_j, e_j - e_k \in \Sigma$. Hence $\Sigma = B_{r_1} \cup \cdots \cup B_{r_k}$. If k were greater than or equal to 3, then we could take $e_a \in B_{r_1}$, $e_b \in B_{r_2}$ and $e_c \in B_{r_3}$. Then we would have $e_a - e_b, e_b - e_c, e_a - e_c \in W - \Sigma$. On the other hand, we would have $e_a - e_c = (e_a - e_b) + (e_b - e_c) \in \Sigma - W$ by the above argument, which would be a contradiction. Hence $k = 2$ and

$$\Sigma^+ = B_s^+ \cup B_{r-s}^+, \quad W^+ = \left(B_r^+ - \Sigma\right) \cup \{e_i\}.$$

If r were equal to 2, then we would have $\Sigma \subset W$. Hence $r \geq 3$.

When $\tilde{\Sigma} = BC_r$, then $\Sigma^+ \cap W^+ = \{e_i\}$ since $s_{e_i \pm e_j}(2e_i) = \pm 2e_j$. When $i \neq j$, then $s_{e_i}(e_i + e_j) = -e_i + e_j$. Thus by Proposition 2.7

$$e_i - e_j \in \Sigma \Leftrightarrow e_i + e_j \in \Sigma, \quad e_i - e_j \in W \Leftrightarrow e_i + e_j \in W.$$

Assume that i, j and k are mutually distinct. If $e_i - e_j, e_j - e_k \in \Sigma$ then $e_i - e_k \in \Sigma$ by a similar argument to the above. Thus

$$\Sigma^+ = B_{r_1}^+ \cup \cdots \cup B_{r_k}^+ \cup \{2e_i \mid 2e_i \in \Sigma\}.$$

We get $k \leq 2$ by a similar argument to the above. If k were equal to 1, we would have $\Sigma = B_r, W^+ = \{e_i, 2e_i\}$ since $s_{e_i + e_j}(2e_i) = -2e_j$. Moreover if r were greater than or equal to 2, then for $\alpha = 2e_1 \in W - \Sigma$ and $\lambda = e_1 + e_2 \in \Sigma - W$ we would have $2\langle \alpha, \lambda \rangle / \|\alpha\|^2 = 1$. Since $s_\alpha \lambda = -e_1 + e_2 \in \Sigma - W$, this would contradict to (5) of Definition 2.2. Thus we would have $r = 1$. Hence $(\tilde{\Sigma}, \Sigma, W)$ would be of type (II-BC_1). Thus $k = 2$ and

$$\Sigma^+ = \{e_i\} \cup D_s^+ \cup D_{r-s}^+ \cup \{2e_i \mid 2e_i \in \Sigma\}.$$

Since $s_{e_i + e_j}(2e_i) = -2e_j$,

$$\Sigma = BC_s \cup BC_{r-s}, \quad B_s \cup BC_{r-s}, \quad \text{or} \quad B_s \cup B_{r-s}.$$

If Σ were equal to $B_s \cup BC_{r-s}$ or $B_s \cup B_{r-s}$, we would have $2\langle \alpha, \beta \rangle / \|\alpha\|^2 = 1$ for $\alpha = 2e_1 \in W$ and $\beta = e_1 + e_{s+1} \in W - \Sigma$. This would contradict to (6) of Definition 2.2 since $s_\alpha \beta = -e_1 + e_{s+1} \in W - \Sigma$. Hence $(\tilde{\Sigma}, \Sigma, W)$ is of type (I'-BC_r-A_1^r).

It is clear that W_0 consists of only one element $\tilde{\alpha}$.

To show (I-F_4) $\sim$ (I'-F_4), let $(\tilde{\Sigma}, \Sigma, W)$ and $(\tilde{\Sigma}', \Sigma', W')$ be of types (I-F_4) and (I'-F_4), respectively. Set $Y = (\pi/2\|e_1\|^2)(e_1 + e_2)$, then $Y \in \Gamma$. By the identity mapping of $\mathfrak{a}$ and Y, $(\tilde{\Sigma}, \Sigma, W)$ maps to $(\tilde{\Sigma}', \Sigma', W')$. Hence (I-$F_4$)$\sim$ (I'-F_4).

To show (I-BC_r-A_1^r) $\sim$ (I'-BC_r-A_1^r), let $(\tilde{\Sigma}, \Sigma, W)$ and $(\tilde{\Sigma}', \Sigma', W')$ be of type (I-BC_r-A_1^r) and type (I'-BC_r-A_1^r), respectively. Set $Y = (\pi/2\|e_1\|^2) \sum_{i=1}^{s} e_i$ for $1 \leq s \leq r$, then $Y \in \Gamma$. By the identity mapping of $\mathfrak{a}$ and Y, $(\tilde{\Sigma}, \Sigma, W)$ maps to $(\tilde{\Sigma}', \Sigma', W')$.

To show (I-C_r) $\sim$ (I'-C_r), let $(\tilde{\Sigma}, \Sigma, W)$ and $(\tilde{\Sigma}', \Sigma', W')$ be of type (I'-C_r) and type (I'-C_r), respectively. Set $Y = (\pi/4\|e_1\|^2) \sum_{i=1}^{r} e_i \in \Gamma$, then $(\tilde{\Sigma}, \Sigma, W)$ maps to $(\tilde{\Sigma}', \Sigma', W')$ by the identity mapping of $\mathfrak{a}$ and Y.

To show (I-B_r) $\sim$ (I'-B_r), let $(\tilde{\Sigma}, \Sigma, W)$ and $(\tilde{\Sigma}', \Sigma', W')$ be of types (I'-B_r) and (I-B_r), respectively. Set $Y = (\pi/2\|e_1\|^2) \sum_{i=1}^{s} e_i \in \Gamma$ for $1 \leq s \leq r$, then $(\tilde{\Sigma}, \Sigma, W)$ maps to $(\tilde{\Sigma}', \Sigma', W')$ by the identity mapping of $\mathfrak{a}$ and Y.

Hence the assertion is proved. □

For $\alpha = \sum_{\lambda \in \Pi} n_\lambda \lambda \in W^+$, set $h(\alpha) = \sum_{\lambda \in \Pi} n_\lambda$ then $h(\alpha) \in \boldsymbol{Z}$ by Lemma 2.4. Put $h = \max\{h(\alpha) \mid \alpha \in W^+\}$.

Lemma 2.20. *Let $(\tilde{\Sigma}, \Sigma, W)$ be a symmetric triad of $\mathfrak{a}$, then*

$$\{\tilde{\alpha}\} = \{\alpha \in W^+ \mid h(\alpha) = h\}.$$

Proof. By the definition of W_0 and Theorem 2.19, we have

$$\{\tilde{\alpha}\} = W_0 \supset \{\alpha \in W^+ \mid h(\alpha) = h\} \neq \emptyset,$$

which implies the assertion. □

The following corollary immediately follows from Lemma 2.12 and Theorem 2.19.

Corollary 2.21. *Let $(\tilde{\Sigma}, \Sigma, W)$ be a symmetric triad of $\mathfrak{a}$, then*

$$P_0 = \left\{ H \in \mathfrak{a} \mid \langle \tilde{\alpha}, H \rangle < \frac{\pi}{2},\ 0 < \langle \lambda, H \rangle \quad (\lambda \in \Pi) \right\}.$$

Set $\Pi = \{\alpha_1, \dots, \alpha_r\}$, then there exist integers $m_i \in \boldsymbol{Z}$ such that $\tilde{\alpha} = \sum m_i \alpha_i$ by Lemma 2.4. We have $m_i \geq 1$ by Theorem 2.19. Hence for any i with $1 \leq i \leq r$, the set $\{\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_r, \tilde{\alpha}\}$ is linearly independent. We can define $H_i \in \mathfrak{a}$ by the following equations:

$$\langle H_i, \tilde{\alpha} \rangle = \frac{\pi}{2}, \quad \langle H_i, \alpha_j \rangle = 0 \quad (j \neq i).$$

Then

$$P_0 = \left\{ \sum_{i=1}^{r} t_i H_i \middle| 0 < t_i,\ \sum_{i=1}^{r} t_i < 1 \right\}.$$

The following corollary immediately follows from Corollary 2.21.

Corollary 2.22. *A point $H \in \overline{P_0}$ is totally geodesic if and only if $H = 0$ or $H = H_i$ with $m_i = 1$.*

Proof. Since $\pi/2 = \langle H_i, \tilde{\alpha} \rangle = m_i \langle H_i, \alpha_i \rangle$, we have $\langle H_i, \alpha_i \rangle = \pi/2m_i$. Express H as $H = \sum_{i=1}^{r} t_i H_i$, then $0 \leq t_i \leq 1$, $\sum_{i=1}^{r} t_i \leq 1$. Hence

$$H \in \Gamma \Leftrightarrow \langle H, \alpha_i \rangle \in \frac{\pi}{2}\boldsymbol{Z} \Leftrightarrow \frac{t_i}{m_i} \in \boldsymbol{Z} \Leftrightarrow H = 0 \text{ or } H = H_i \text{ with } m_i = 1. \qquad \square$$

Corollary 2.23. *Let $(\tilde{\Sigma}, \Sigma, W)$ be a symmetric triad of $\mathfrak{a}$ with multiplicities. Then $H \in \overline{P_0}$ is austere if and only if the following four conditions hold*:

(1) $\langle \lambda, H \rangle = 0,\ \pi/2,\ \pi$ *for* $\lambda \in \Sigma^+ - W^+$.
(2) $\langle \alpha, H \rangle = 0,\ \pm\pi/2$ *for* $\alpha \in W^+ - \Sigma^+$.
(3) $\langle \alpha, H \rangle = 0,\ \pi/4,\ \pi/2$ *for* $\alpha \in \Sigma^+ \cap W^+$.
(4) $m(\alpha) = n(\alpha)$ *for* $\alpha \in \Sigma^+ \cap W^+$ *with* $\langle \alpha, H \rangle = \pi/4$.

Let $H \in \overline{P_0}$ be an austere point which is not totally geodesic, then H can be expressed as one of the following forms.

$$H = \begin{cases} H_i & (m_i = 2), \\ \frac{1}{2} H_i & (m_i = 1), \\ \frac{1}{2}(H_i + H_j) & (m_i = m_j = 1) \end{cases}$$

Proof. Let H be in $\overline{P_0}$ then $0 \leq \langle \lambda, H \rangle \leq \pi$ for any $\lambda \in \Sigma^+$ and $-\pi/2 \leq \langle \alpha, H \rangle \leq \pi/2$ for any $\alpha \in W^+$. Hence the first part of the assertion follows from

Theorem 2.18.

We shall prove the second part. Let $H \in \overline{P_0}$ be an austere point which is not totally geodesic. Express H as $H = \sum t_i H_i$, then for any $\alpha_i \in \Pi$, we have

$$\langle \alpha_i, H \rangle = t_i \langle \alpha_i, H_i \rangle = t_i \frac{\pi}{2m_i} \leq \frac{\pi}{2m_i} \leq \frac{\pi}{2}.$$

By the first part, $\langle \alpha_i, H \rangle = 0, \pi/4, \pi/2$ for each $\alpha_i \in \Pi$. If there were to exist i such that $\langle \alpha_i, H \rangle = \pi/2$, then we would have $m_i = t_i = 1$ and $H = H_i$. Hence H would be totally geodesic by Corollary 2.22, which would be a contradiction. Thus $\langle \alpha_i, H \rangle = 0, \pi/4$ for each $\alpha_i \in \Pi$. For i with $\langle \alpha_i, H \rangle = \pi/4$, we have

$$t_i = \frac{m_i}{2} = \begin{cases} \frac{1}{2} & (m_i = 1), \\ 1 & (m_i = 2) \end{cases}$$

Hence we get the second part. □

For a subset $\Delta \subset \Pi \cup \{\tilde{\alpha}\}$, set

$$P_0^{\Delta} = \left\{ H \in \overline{P_0} \;\middle|\; \begin{array}{l} \langle \lambda, H \rangle > 0 \quad (\lambda \in \Delta \cap \Pi), \\ \langle \lambda, H \rangle = 0 \quad (\lambda \in \Pi - \Delta), \\ \langle \tilde{\alpha}, H \rangle \begin{cases} < \frac{\pi}{2} & (\text{if } \tilde{\alpha} \in \Delta), \\ = \frac{\pi}{2} & (\text{if } \tilde{\alpha} \notin \Delta) \end{cases} \end{array} \right\},$$

then

$$\overline{P_0} = \bigcup_{\Delta \subset \Pi \cup \{\tilde{\alpha}\}} P_0^{\Delta} \quad (\text{disjoint union}).$$

$\Delta_1 \subset \Delta_2$ if and only if $P_0^{\Delta_1} \subset \overline{P_0^{\Delta_2}}$ for $\Delta_1, \Delta_2 \subset \Pi \cup \{\tilde{\alpha}\}$.

Theorem 2.24. *For any subset $\Delta \subset \Pi \cup \{\tilde{\alpha}\}$, there exists a unique minimal point $H \in P_0^{\Delta}$.*

The proof of Theorem 2.24 is divided into some steps.

For a subset $\Delta \subset \Pi \cup \{\tilde{\alpha}\}$, we define subsets $\Sigma_{\Delta}^{+} \subset \Sigma^{+}$ and $W_{\Delta}^{+} \subset W^{+}$ as follows: When $\tilde{\alpha} \in \Delta$, then set

$$\Sigma_{\Delta}^{+} = \Sigma^{+} \cap (\Pi - \Delta)_{\boldsymbol{Z}}, \quad W_{\Delta}^{+} = \emptyset.$$

When $\tilde{\alpha} \notin \Delta$, then set

$$\Sigma_\Delta^+ = \Sigma^+ \cap (\Pi - \Delta)_{\boldsymbol{Z}} \cup \Big\{ 2\tilde{\alpha} - \sum_{\lambda \in \Pi - \Delta} n_\lambda \lambda \in \Sigma^+ - W^+ \mid n_\lambda \geq 0 \Big\},$$

$$W_\Delta^+ = \Big\{ \tilde{\alpha} - \sum_{\lambda \in \Pi - \Delta} m_\lambda \lambda \in W^+ \mid m_\lambda \geq 0 \Big\} \cup \Big\{ -\tilde{\alpha} + \sum_{\lambda \in \Pi - \Delta} m_\lambda \lambda \in W^+ - \Sigma^+ \mid m_\lambda \geq 0 \Big\}.$$

LEMMA 2.25. *Let H be in P_0^Δ.*

(1) $\Sigma_\Delta^+ = \{\lambda \in \Sigma^+ \mid \langle \lambda, H \rangle \in \pi \boldsymbol{Z}\}$. *In particular, the right hand side of the above equation does not depend on $H \in P_0^\Delta$, but only on Δ. The value $\langle \lambda, H \rangle$ does not depend on $H \in P_0^\Delta$, but only on $\lambda \in \Sigma_\Delta^+$.*
(2) $W_\Delta^+ = \{\alpha \in W^+ \mid \langle \alpha, H \rangle \in \pi/2 + \pi \boldsymbol{Z}\}$. *In particular, the right hand side of the above equation does not depend on $H \in P_0^\Delta$, but only on Δ. The value $\langle \alpha, H \rangle$ does not depend on $H \in P_0^\Delta$ but only on $\alpha \in W_\Delta^+$.*

PROOF.
(1) Since $H \in P_0^\Delta \subset \overline{P_0}$, we have

$$\begin{aligned}\{\lambda \in \Sigma^+ \mid \langle \lambda, H \rangle \in \pi \boldsymbol{Z}\} &= \{\lambda \in \Sigma^+ \mid \langle \lambda, H \rangle = 0\} \cup \{\lambda \in \Sigma^+ \mid \langle \lambda, H \rangle = \pi\} \\ &= (\Sigma^+ \cap (\Pi - \Delta)_{\boldsymbol{Z}}) \cup \{\lambda \in \Sigma^+ - W^+ \mid \langle \lambda, H \rangle = \pi\}.\end{aligned}$$

When $\tilde{\alpha} \in \Delta$, then $\langle \tilde{\alpha}, H \rangle < \pi/2$ for $H \in P_0^\Delta$. For $\lambda \in \Sigma^+ - W^+$, there exist $\alpha, \beta \in W^+ \cap \Sigma^+$ such that $\lambda = \pm\alpha \pm \beta$ by Lemma 2.3. Then $\langle \lambda, H \rangle = \pm\langle \alpha, H \rangle \pm \langle \beta, H \rangle$, which implies that

$$\langle \lambda, H \rangle \leq |\langle \lambda, H \rangle| \leq |\langle \alpha, H \rangle| + |\langle \beta, H \rangle| \leq 2\langle \tilde{\alpha}, H \rangle < \pi.$$

Hence $\{\lambda \in \Sigma^+ - W^+ \mid \langle \lambda, H \rangle = \pi\} = \emptyset$.

When $\tilde{\alpha} \notin \Delta$ then, for any $\lambda \in \Sigma^+ - W^+$ with $\langle \lambda, H \rangle = \pi$, there exist $\alpha, \beta \in W^+ \cap \Sigma^+$ such that

$$\lambda = \alpha + \beta, \quad \langle \alpha, H \rangle = \langle \beta, H \rangle = \frac{\pi}{2}.$$

By Lemma 2.20, λ is expressed as

$$\lambda = \alpha + \beta = 2\tilde{\alpha} - \sum_{\mu \in \Pi - \Delta} n_\mu \mu \quad (0 \leq n_\mu \in \boldsymbol{Z}).$$

Hence

$$\{\lambda \in \Sigma^+ - W^+ \mid \langle \lambda, H \rangle = \pi\} = \left\{ 2\tilde{\alpha} - \sum_{\mu \in \Pi - \Delta} n_\mu \mu \in \Sigma^+ - W^+ \mid n_\mu \geq 0 \right\},$$

which implies the assertion.

(2) Since $H \in P_0^\Delta \subset \overline{P_0}$, we have

$$\begin{aligned} &\left\{ \alpha \in W^+ \mid \langle \alpha, H \rangle \in \frac{\pi}{2} + \pi \boldsymbol{Z} \right\} \\ &\quad = \left\{ \alpha \in W^+ - \Sigma^+ \mid \langle \alpha, H \rangle = -\frac{\pi}{2} \right\} \cup \left\{ \alpha \in W^+ \mid \langle \alpha, H \rangle = \frac{\pi}{2} \right\}. \end{aligned}$$

When $\tilde{\alpha} \in \Delta$, then $\langle \tilde{\alpha}, H \rangle < \pi/2$ for $H \in P_0^\Delta$. By Lemma 2.20, we have

$$\left\{ \alpha \in W^+ \mid \langle \alpha, H \rangle = \frac{\pi}{2} \right\} = \emptyset.$$

For $\alpha \in W^+ - \Sigma^+$, there exist $\lambda, \mu \in \Sigma \cap W$ such that $\alpha = \lambda + \mu$ by Lemma 2.3. We may assume that $\mu > 0$ since one of λ and μ is positive. Then $\langle \alpha, H \rangle \geq \langle \lambda, H \rangle$. If $\lambda > 0$, then $\langle \alpha, H \rangle \geq \langle \lambda, H \rangle \geq 0$. If $-\lambda > 0$, then

$$\langle \alpha, H \rangle \geq -\langle -\lambda, H \rangle \geq -\langle \tilde{\alpha}, H \rangle > \frac{\pi}{2}.$$

Hence

$$\left\{ \alpha \in W^+ - \Sigma^+ \mid \langle \alpha, H \rangle = -\frac{\pi}{2} \right\} = \emptyset.$$

When $\tilde{\alpha} \notin \Delta$, then by Lemma 2.20 we have

$$\left\{ \alpha \in W^+ \mid \langle \alpha, H \rangle = \frac{\pi}{2} \right\} = \left\{ \tilde{\alpha} - \sum_{\lambda \in \Pi - \Delta} m_\lambda \lambda \in W^+ \mid m_\lambda \geq 0 \right\}.$$

For $\alpha \in W^+ - \Sigma^+$ with $\langle \alpha, H \rangle = -\pi/2$, there exist $\lambda, \mu \in \Sigma \cap W$ such that $\alpha = \lambda + \mu$ by Lemma 2.3. We may assume that $\mu > 0$. Then $-\pi/2 = \langle \alpha, H \rangle \geq \langle \lambda, H \rangle$. Since

$-\lambda > 0$, we have

$$-\frac{\pi}{2} = \langle \alpha, H \rangle \geq -\langle -\lambda, H \rangle \geq -\langle \tilde{\alpha}, H \rangle = -\frac{\pi}{2}.$$

Hence $\langle \alpha, H \rangle = -\langle \tilde{\alpha}, H \rangle = -\pi/2$. By Lemma 2.20, we have

$$\left\{ \alpha \in W^+ - \Sigma^+ \mid \langle \alpha, H \rangle = -\frac{\pi}{2} \right\} = \left\{ -\tilde{\alpha} + \sum_{\lambda \in \Pi - \Delta} m_\lambda \lambda \in W^+ - \Sigma^+ \mid m_\lambda \geq 0 \right\},$$

which implies the assertion. □

Corollary 2.26. *Let $H, H' \in P_0^\Delta$. If $\langle \lambda, H \rangle = \langle \lambda, H' \rangle$ for any $\lambda \in (\Sigma^+ - \Sigma_\Delta^+) \cup (W^+ - W_\Delta^+)$, then $H = H'$.*

Proof. By Lemma 2.25, $\langle \alpha, H \rangle = \langle \alpha, H' \rangle$ for any $\alpha \in \Sigma_\Delta^+ \cup W_\Delta^+$. Hence $\langle \lambda, H \rangle = \langle \lambda, H' \rangle$ for any $\lambda \in \tilde{\Sigma}$ by the assumption. Hence $H = H'$ since $\mathrm{span}(\tilde{\Sigma}) = \mathfrak{a}$. □

For $\Delta \subset \Pi \cup \{\tilde{\alpha}\}$, define an affine subspace $\mathfrak{a}^\Delta$ of $\mathfrak{a}$ by

$$\mathfrak{a}^\Delta = \left\{ H \in \mathfrak{a} \,\middle|\, \begin{array}{l} \langle \lambda, H \rangle = 0 \ \ (\lambda \in \Pi - \Delta), \\ \langle \tilde{\alpha}, H \rangle = \frac{\pi}{2} \ (\text{if } \tilde{\alpha} \notin \Delta) \end{array} \right\},$$

then

$$P_0^\Delta = \left\{ H \in \mathfrak{a}^\Delta \,\middle|\, \begin{array}{l} \langle \lambda, H \rangle > 0 \ \ (\lambda \in \Delta \cap \Pi), \\ \langle \tilde{\alpha}, H \rangle < \frac{\pi}{2} \ (\text{if } \tilde{\alpha} \in \Delta) \end{array} \right\}.$$

Hence P_0^Δ is an open subset of $\mathfrak{a}^\Delta$. Since, for $H \in P_0^\Delta$,

$$F(H) = -\sum_{\lambda \in \Sigma^+ - \Sigma_\Delta^+} m(\lambda) \log|\sin\langle \lambda, H \rangle)| - \sum_{\alpha \in W^+ - W_\Delta^+} n(\alpha) \log|\cos(\langle \alpha, H \rangle)|,$$

the function F is differentiable on P_0^Δ. Since m_H is tangent to P_0^Δ for $H \in P_0^\Delta$ by Proposition 2.14, we can regard m_H as a differentiable vector field on P_0^Δ. The mean curvature vector m_H can be expressed as

$$m_H = -\sum_{\lambda \in \Sigma^+ - \Sigma_\Delta^+} m(\lambda) \cot(\langle \lambda, H \rangle)\lambda + \sum_{\alpha \in W^+ - W_\Delta^+} n(\alpha) \tan(\langle \alpha, H \rangle)\alpha.$$

Lemma 2.27.

(1) $(\mathrm{grad}\, F)(H) = m_H$ *for any* $H \in P_0^\Delta$.
(2) *For* $H, H_1 \in P_0^\Delta$ *with* $H \neq H_1$, *we have*

$$\frac{d^2}{dt^2} F\big(H + t\overrightarrow{HH_1}\big)_{|t=0} > 0.$$

Proof.

(1) Let $H, H_1 \in P_0^\Delta$. A simple calculation implies that

$$\frac{d}{dt} F\big(H + t\overrightarrow{HH_1}\big)_{|t=0} = \langle m_H, \overrightarrow{HH_1}\rangle.$$

Hence $(\mathrm{grad}\, F)(H) = m_H$.
(2) A simple calculation implies that

$$\begin{aligned}&\frac{d^2}{dt^2} F\big(H + t\overrightarrow{HH_1}\big)_{|t=0}\\ &\quad = \sum_{\lambda \in \Sigma^+ - \Sigma^+_\Delta} \frac{m(\lambda)\langle \lambda, \overrightarrow{HH_1}\rangle^2}{\sin^2(\langle \lambda, H\rangle)} + \sum_{\alpha \in W^+ - W^+_\Delta} \frac{n(\alpha)\langle \alpha, \overrightarrow{HH_1}\rangle^2}{\cos^2(\langle \alpha, H\rangle)}.\end{aligned}$$

Since $H \neq H'$ there exists $\lambda \in (\Sigma^+ - \Sigma^+_\Delta) \cup (W^+ - W^+_\Delta)$ such that $\langle \lambda, \overrightarrow{HH_1}\rangle \neq 0$ by Corollary 2.26. Hence $(d^2/dt^2)F(H + t\overrightarrow{HH_1})_{|t=0} > 0$. □

Lemma 2.28. *The boundary* ∂P_0^Δ *of* P_0^Δ *is given as follows*: *When* $\tilde{\alpha} \in \Delta$, *then*

$$\partial P_0^\Delta = \bigcup_{\mu \in \Pi \cap \Delta} \left\{ H \in \mathfrak{a} \;\middle|\; \begin{array}{l} \langle \lambda, H\rangle = 0 \quad (\lambda \in \Pi - \Delta), \\ \langle \lambda, H\rangle \geq 0 \quad (\lambda \in \Pi \cap \Delta), \\ \langle \mu, H\rangle = 0,\ \langle \tilde{\alpha}, H\rangle \leq \frac{\pi}{2} \end{array} \right\}$$
$$\cup \left\{ H \in \mathfrak{a} \;\middle|\; \begin{array}{l} \langle \lambda, H\rangle = 0 \quad (\lambda \in \Pi - \Delta), \\ \langle \lambda, H\rangle \geq 0 \quad (\lambda \in \Pi \cap \Delta), \\ \langle \tilde{\alpha}, H\rangle = \frac{\pi}{2} \end{array} \right\}.$$

When $\tilde{\alpha} \notin \Delta$, *then*

$$\partial P_0^\Delta = \bigcup_{\mu \in \Pi \cap \Delta} \left\{ H \in \mathfrak{a} \;\middle|\; \begin{array}{ll} \langle \lambda, H \rangle \geq 0 & (\lambda \in \Pi \cap \Delta), \\ \langle \lambda, H \rangle = 0 & (\lambda \in \Pi - \Delta), \\ \langle \mu, H \rangle = 0, \ \langle \tilde{\alpha}, H \rangle = \frac{\pi}{2} & \end{array} \right\}.$$

Proof of Theorem 2.24. Take a sequence $\{H_n\} \subset P_0^\Delta$ with $\lim_{n\to\infty} H_n = H_\infty \in \partial P_0^\Delta$. It is sufficient to prove that $\lim_{n\to\infty} F(H_n) = \infty$ by Lemma 2.27.

When $\tilde{\alpha} \in \Delta$, there exists $\lambda \in \Pi \cap \Delta$ such that $\langle \lambda, H_n \rangle \to +0$ or $\langle \tilde{\alpha}, H_n \rangle \to \pi/2$ by Lemma 2.28. Since $\Sigma_\Delta^+ = \Sigma^+ \cap (\Pi - \Delta)_{\boldsymbol{Z}}$, we have

$$\Sigma_\Delta^+ \cap (\Pi \cap \Delta) = \Pi \cap \Delta \cap (\Pi - \Delta)_{\boldsymbol{Z}} = \emptyset,$$

which implies that $\Pi \cap \Delta \subset \Sigma^+ - \Sigma_\Delta^+$. Hence when $\langle \lambda, H_n \rangle \to +0$, then $F(H_n) \to \infty$. Since $W_\Delta^+ = \emptyset$, $\tilde{\alpha}$ is in $W^+ - W_\Delta^+$. Hence when $\langle \tilde{\alpha}, H_n \rangle \to \pi/2$, then $F(H_n) \to \infty$.

When $\tilde{\alpha} \notin \Delta$, there exists $\lambda \in \Pi \cap \Delta$ such that $\langle \lambda, H_n \rangle \to +0$ by Lemma 2.28. In this case we have $\lambda \notin \{2\tilde{\alpha} - \sum_{\mu \in \Pi - \Delta} n_\mu \mu \in \Sigma^+ - W^+ \mid n_\mu \geq 0\}$. In fact if it were not, then $\langle \lambda, H_n \rangle = \pi$, which would be a contradiction. Hence $\lambda \notin \Sigma_\Delta^+$ and $\lambda \in \Sigma^+ - \Sigma_\Delta^+$. We get $F(H_n) \to \infty$.

Hence the assertion is proved. □

3. Totally geodesic points and austere points.

In this section we shall classify the totally geodesic points and the austere points for each (representative of) symmetric triad with multiplicities using Corollaries 2.22 and 2.23.

3.1. Type (I-B_r).

Since $\tilde{\alpha} = e_1 = \sum_{i=1}^r \alpha_i$, a point $H \in \overline{P_0}$ is totally geodesic if and only if H is a vertex of $\overline{P_0}$. By Definition 2.13, we have

$$0 < m(\pm e_i) = \text{const}, \quad 0 < m(\pm e_i \pm e_j) = \text{const} \ (i \neq j), \quad 0 < n(\pm e_i) = \text{const}.$$

When $m(\pm e_i) = n(\pm e_i)$, a point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = (1/2)H_r$. When $m(\pm e_i) \neq n(\pm e_i)$, if $H \in \overline{P_0}$ is austere then it is totally geodesic.

3.2. Type (I-C_r).

Since $\tilde{\alpha} = e_1 + e_2 = \alpha_1 + 2\sum_{i=2}^{r-1} \alpha_i + \alpha_r$, a point $H \in \overline{P_0}$ is totally geodesic if and only if $H = 0, H_1, H_r$. When $r \geq 3$, then

$$0 < m(\pm e_i \pm e_j) = n(\pm e_i \pm e_j) = \text{const } (i \neq j), \quad 0 < m(2e_i) = \text{const}.$$

When $r = 2$, then

$$0 < m(\pm e_1 \pm e_2) = \text{const},\ 0 < m(\pm 2e_i) = \text{const},\ 0 < n(\pm e_1 \pm e_2) = \text{const}.$$

A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if

$$H = H_i \ (2 \leq i \leq r-1), \quad \frac{1}{2}H_1$$

3.3. Type (I-$\boldsymbol{BC_r}$-$\boldsymbol{A_1^r}$).

Since $\tilde{\alpha} = \sum_{i=1}^{r} \alpha_i$, a point $H \in \overline{P_0}$ is totally geodesic if and only if H is a vertex of $\overline{P_0}$. By Definition 2.13, we have

$$0 < m(\pm e_i) = \text{const}, \quad 0 < m(\pm e_i \pm e_j) = \text{const } (i \neq j),$$
$$0 < m(\pm 2e_i) = \text{const}, \quad 0 < n(\pm e_i) = \text{const}.$$

When $m(\pm e_i) = n(\pm e_i)$, a point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = (1/2)H_r$. When $m(\pm e_i) \neq n(\pm e_i)$, any austere point is totally geodesic.

3.4. Type (I-$\boldsymbol{BC_r}$-$\boldsymbol{B_r}$).

Since $\tilde{\alpha} = e_1 + e_2 = \alpha_1 + 2\sum_{i=2}^{r} \alpha_i$, a point $H \in \overline{P_0}$ is totally geodesic if and only if $H = 0, H_1$. When $r \geq 3$,

$$0 < m(\pm e_i) = n(\pm e_i) = \text{const}, \quad 0 < m(\pm 2e_i) = \text{const},$$
$$0 < m(\pm e_i \pm e_j) = n(\pm e_i \pm e_j) = \text{const } (i \neq j).$$

When $r = 2$, then $0 < m(e_i) = n(e_i) = \text{const}$, $0 < m(2e_i) = \text{const}$. When $r \geq 3$, a point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = (1/2)H_1$, H_i $(2 \leq i \leq r)$. When $r = 2$, the condition $H \in \overline{P_0}$ to be austere which is not totally geodesic is given as follows:

(1) If $m(\pm e_1 \pm e_2) = n(\pm e_1 \pm e_2)$, then $H = (1/2)H_1,\ H_2$.

(2) If $m(\pm e_1 \pm e_2) \neq n(\pm e_1 \pm e_2)$, then $H = H_2$.

3.5. Type (I-$\boldsymbol{F_4}$).

Since

$$\Pi = \left\{ \alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4) \right\},$$
$$\tilde{\alpha} = e_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4,$$

a point $H \in \overline{P_0}$ is totally geodesic if and only if $H = 0, H_1$. By Definition 2.13, $0 < m(\alpha) = n(\alpha)$ =constant for any $\alpha \in W$ and $0 < m(\lambda)$ =constant for any $\lambda \in \Sigma - W$. A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_4$. The vertexes H_2 and H_3 are minimal which are not austere.

3.6. Type (II-BC_r) ($r \geq 1$).

Since $\tilde{\alpha} = 2e_1 = 2\sum_{i=1}^{r} \alpha_i$, a point $H \in \overline{P_0}$ is totally geodesic if and only if $H = 0$. By Definition 2.13, we have

$$0 < n(\pm e_i) = m(\pm e_i) = \text{const}, \quad 0 < n(\pm 2e_i) = \text{const},$$
$$0 < n(\pm e_i \pm e_j) = m(\pm e_i \pm e_j) = \text{const } (i \neq j).$$

Hence $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_i$ $(1 \leq i \leq r)$.

3.7. Type (III-A_r).

Since $\tilde{\alpha} = e_1 - e_{r+1} = \sum_{i=1}^{r} \alpha_i$, a point $H \in \overline{P_0}$ is totally geodesic if and only if H is a vertex of $\overline{P_0}$. By Definition 2.13, $0 < m(\lambda) = n(\lambda)$ =constant for any $\lambda \in \tilde{\Sigma}$. A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = (1/2)H_i, (1/2)(H_i + H_j)$ $(i < j)$.

3.8. Type (III-B_r).

Since $\tilde{\alpha} = e_1 + e_2 = \alpha_1 + 2\sum_{i=2}^{r} \alpha_i$, a point $H \in \overline{P_0}$ is totally geodesic if and only if $H = 0, H_1$. When $r \geq 3$, then

$$0 < m(\pm e_i) = n(\pm e_i) = \text{const},$$
$$0 < m(\pm e_i \pm e_j) = n(\pm e_i \pm e_j) = \text{const } (i \neq j).$$

When $r = 2$, then

$$0 < m(e_i) = n(e_i) = \text{const}, \quad 0 < m(\pm e_1 \pm e_2) = \text{const},$$
$$0 < n(\pm e_1 \pm e_2) = \text{const}.$$

A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = (1/2)H_1$, H_i $(2 \leq i \leq r)$.

3.9. Type (III-C_r).

Since $\tilde{\alpha} = 2e_1 = 2\sum_{i=1}^{r-1} \alpha_i + \alpha_r$, a point $H \in \overline{P_0}$ is totally geodesic if and only if $H = 0, H_r$. By Definition 2.13,

$$0 < m(\pm e_i \pm e_j) = n(\pm e_i \pm e_j) = \text{const } (i \neq j),$$

$$0 < m(\pm 2e_i) = \text{const}, \quad 0 < n(\pm 2e_i) = \text{const}.$$

The condition $H \in \overline{P_0}$ to be austere which is not totally geodesic is given as follows:

(1) If $m(\pm 2e_i) \neq n(\pm 2e_i)$, then $H = H_i$ $(1 \leq i \leq r-1)$,
(2) If $m(\pm 2e_i) = n(\pm 2e_i)$, then $H = H_i$ $(1 \leq i \leq r-1), (1/2)H_r$.

3.10. Type (III-BC_r).

Since $\tilde{\alpha} = 2e_1 = 2\sum_{i=1}^{r} \alpha_i$, a point $H \in \overline{P_0}$ is totally geodesic if and only if $H = 0$. When $r = 2$, then

$$0 < m(\pm e_i) = n(\pm e_i) = \text{const}, \quad 0 < m(\pm e_1 \pm e_2) = \text{const},$$

$$0 < n(\pm e_1 \pm e_2) = \text{const}, \quad 0 < m(\pm 2e_i) = \text{const}, \quad 0 < n(\pm 2e_i) = \text{const}.$$

When $r \geq 3$, then

$$0 < m(\pm e_i) = n(\pm e_i) = \text{const}, \quad 0 < m(\pm e_i \pm e_j) = n(\pm e_i \pm e_j) = \text{const},$$

$$0 < m(\pm 2e_i) = \text{const}, \quad 0 < n(\pm 2e_i) = \text{const}.$$

Hence $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_i$ $(1 \leq i \leq r)$.

3.11. Type (III-D_r).

Since $\tilde{\alpha} = e_1 + e_2 = \alpha_1 + 2\sum_{i=2}^{r-2} \alpha_i + \alpha_{r-1} + \alpha_r$, a point $H \in \overline{P_0}$ is totally geodesic if and only if $H = 0, H_1, H_{r-1}, H_r$. By Definition 2.13, $0 < m(\lambda) = n(\lambda)$ =constant for any $\lambda \in \tilde{\Sigma}$. A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if

$$H = H_i \ (2 \leq i \leq r-2), \ \frac{1}{2}H_1, \ \frac{1}{2}H_{r-1}, \ \frac{1}{2}H_r,$$

$$\frac{1}{2}(H_1 + H_{r-1}), \ \frac{1}{2}(H_1 + H_r), \ \frac{1}{2}(H_{r-1} + H_r).$$

3.12. Type (III-E_6).

Since $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, a point $H \in \overline{P_0}$ is totally geodesic if and only if $H = 0, H_1, H_6$. By Definition 2.13, $0 < m(\lambda) = n(\lambda) =$constant for any $\lambda \in \tilde{\Sigma}$. A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if

$$H = H_2,\ H_3,\ H_5,\ \frac{1}{2}H_1,\ \frac{1}{2}H_6,\ \frac{1}{2}(H_1 + H_6).$$

The vertex H_4 is minimal which is not austere.

3.13. Type (III-E_7).

Since $\tilde{\alpha} = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$, a point $H \in \overline{P_0}$ is totally geodesic if and only if $H = 0, H_7$. By Definition 2.13, $0 < m(\lambda) = n(\lambda) =$constant for any $\lambda \in \tilde{\Sigma}$. A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_1, H_2, H_6, (1/2)H_7$. The vertexes H_3, H_4 and H_5 are minimal which are not austere.

3.14. Type (III-E_8).

Since $\tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$, a point $H \in \overline{P_0}$ is totally geodesic if and only if $H = 0$. By Definition 2.13, $0 < m(\lambda) = n(\lambda) =$constant for any $\lambda \in \tilde{\Sigma}$. A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_1, H_8$. The vertexes H_2, H_3, H_4, H_5, H_6 and H_7 are minimal which are not austere.

3.15. Type (III-F_4).

Since

$$\Pi = \left\{\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\right\},$$
$$\tilde{\alpha} = e_1 + e_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4,$$

a point $H \in \overline{P_0}$ is totally geodesic if and only if $H = 0$. By Definition 2.13,

$$0 < m(\text{short}) = n(\text{short}) = \text{const}, \quad 0 < m(\text{long}) = n(\text{long}) = \text{const}.$$

A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_1, H_4$. The vertexes H_2 and H_3 are minimal which are not austere.

3.16. Type (III-G_2).

Since

$$\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = -2e_1 + e_2 + e_3\}, \quad \tilde{\alpha} = -e_1 - e_2 + 2e_3 = 3\alpha_1 + 2\alpha_2,$$

a point $H \in \overline{P_0}$ is totally geodesic if and only if $H = 0$. By Definition 2.13,

$$0 < m(\text{short}) = n(\text{short}) = \text{const}, \quad 0 < m(\text{long}) = n(\text{long}) = \text{const}.$$

A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_2$. The vertex H_1 is minimal which is not austere.

4. The orbit spaces of Hermann actions.

4.1. General case.

Let (G, K_1) and (G, K_2) be two compact symmetric pairs: There exist two involutive automorphisms θ_1 and θ_2 on the compact connected Lie group G such that the closed subgroup K_i of G lie between G_{θ_i} and the identity component $(G_{\theta_i})_0$ of G_i. Here we denote by G_{θ_i} $(i = 1, 2)$ the closed subgroup of G consisting of all fixed points of θ_i in G. In this case the triple (G, K_1, K_2) is called a compact symmetric triad. Take an $\mathrm{Aut}(G)$-invariant Riemannian metric $\langle\ ,\ \rangle$ on G. Then the coset manifold $M_i = G/K_i$ $(i = 1, 2)$ is a compact Riemannian symmetric space with respect to the induced G-invariant Riemannian metric, also denoted by $\langle\ ,\ \rangle$, from $\langle\ ,\ \rangle$. The isometric action of K_2 on M_1 is called a Hermann action. We denote by $\mathfrak{g}$, $\mathfrak{k}_1$ and $\mathfrak{k}_2$ the Lie algebras of G, K_1 and K_2, respectively. The involutive automorphisms θ_1 and θ_2 of G induce involutive automorphisms of $\mathfrak{g}$, also denoted by θ_1 and θ_2, respectively. We have two canonical decompositions of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{m}_1 = \mathfrak{k}_2 \oplus \mathfrak{m}_2,$$

where we define a subspace $\mathfrak{m}_i$ of $\mathfrak{g}$ by

$$\mathfrak{m}_i = \{X \in \mathfrak{g} \mid \theta_i(X) = -X\} \ (i = 1, 2).$$

We denote by π_i the natural projection from G onto M_i. In order to consider K_2-orbit space $\{K_2\pi_1(g) \subset M_1 \mid g \in G\}$, we define a equivalent relation $\sim$ on G as follows:

$$g_1 \sim g_2 \Leftrightarrow \text{there exist } k_1 \in K_1, k_2 \in K_2 \text{ such that } g_2 = k_2 g_1 k_1^{-1}.$$

Since

$$g_1 \sim g_2 \Leftrightarrow K_2\pi_1(g_2) = K_2\pi_1(g_1),$$

we can regard $K_2\backslash G/K_1$ as K_2-orbit space. The following mapping is a bijection:

$$K_2\backslash G/K_1 \cong K_1\backslash G/K_2; [g] \leftrightarrow [g^{-1}].$$

Define a closed subgroup G_{12} in G by

$$G_{12} = \{g \in G \mid \theta_1(g) = \theta_2(g)\}.$$

Consider an involutive automorphism $\theta = \theta_1 = \theta_2$ on G_{12}. Define a closed subgroup K_{12} of the identity component $(G_{12})_0$ of G_{12} by

$$K_{12} = \{g \in (G_{12})_0 \mid \theta(g) = g\}.$$

Then $((G_{12})_0, K_{12})$ is a compact symmetric pair. The canonical decomposition of the Lie algebra $\mathfrak{g}_{12}$ of G_{12} is given by

$$\mathfrak{g}_{12} = (\mathfrak{k}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{m}_2).$$

Take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}_1 \cap \mathfrak{m}_2$. Then $\exp \mathfrak{a}$ is closed in $(G_{12})_0$, hence a toral subgroup. The isometric action of K_2 on M_1 is hyperpolar, whose section is $\pi_1(\exp \mathfrak{a})$. The cohomogeneity is equal to $\dim \mathfrak{a}$ ([**4**]). In oder to study K_2-orbit space, we define a group $\tilde{J}$ by

$$\tilde{J} = \left\{([s], Y) \in N_{K_2}(\mathfrak{a})/Z_{K_1\cap K_2}(\mathfrak{a}) \ltimes \mathfrak{a} \mid \exp(-Y)s \in K_1\right\}.$$

The centralizer $Z_{K_2}(\mathfrak{a})$ is a normal subgroup of the normalizer $N_{K_2}(\mathfrak{a})$. We denote by $W_i(\mathfrak{a})$ the quotient group $N_{K_i}(\mathfrak{a})/Z_{K_i}(\mathfrak{a})$ for $i = 1, 2$. Denote by φ_2 the natural homomorphism from $N_{K_2}(\mathfrak{a})/Z_{K_1\cap K_2}(\mathfrak{a})$ onto $W_2(\mathfrak{a})$. The group $\tilde{J}$ naturally acts on $\mathfrak{a}$ by the following:

$$([s], Y)Z = \mathrm{Ad}(\varphi_2(s))Z + Y.$$

Based on the above, we set $[s] = \mathrm{Ad}(\varphi_2(s))$.

PROPOSITION 4.1 ([**16**]). $K_2\backslash G/K_1 \cong \mathfrak{a}/\tilde{J}$.

LEMMA 4.2. *The Lie algebras of $Z_{K_i}(\mathfrak{a})$ and $N_{K_i}(\mathfrak{a})$ are given as follows*:

$$\mathrm{Lie}(Z_{K_i}(\mathfrak{a})) = \mathrm{Lie}(N_{K_i}(\mathfrak{a})) = \{X \in \mathfrak{k}_i \mid [X, \mathfrak{a}] = \{0\}\}.$$

In particular, the group $W_i(\mathfrak{a})$ is finite.

Proof. $\mathrm{Lie}(N_{K_i}(A))$ is given by

$$\begin{aligned}\mathrm{Lie}(N_{K_i}(A)) &= \{X \in \mathfrak{k}_i \mid \mathrm{Ad}(\exp tX)Y \in \mathfrak{a}\ (t \in \boldsymbol{R}, Y \in \mathfrak{a})\}\\ &= \{X \in \mathfrak{k}_i \mid [X, \mathfrak{a}] \subset \mathfrak{a}\}.\end{aligned}$$

Let $X \in \mathrm{Lie}(N_{K_i}(A))$, then $[H, X] \in \mathfrak{a}$ for each $H \in \mathfrak{a}$, so $[H, [H, X]] = 0$. Hence

$$\|[H, X]\|^2 = -\langle H, [H, [H, X]]\rangle = 0,$$

which implies that $[H, X] = 0$. □

We denote by Σ the restricted root system of $(\mathfrak{g}_{12}, \mathfrak{k}_1 \cap \mathfrak{k}_2)$ with respect to $\mathfrak{a}$. Since the Weyl group $N_{K_1\cap K_2}(\mathfrak{a})/Z_{K_1\cap K_2}(\mathfrak{a})$ of $(G_{12}, K_1 \cap K_2)$ is generated by $\{s_\lambda \mid \lambda \in \Sigma\}$, we denote it by $W(\Sigma)$: $W(\Sigma) = N_{K_1\cap K_2}(\mathfrak{a})/Z_{K_1\cap K_2}(\mathfrak{a})$. We can regard $W(\Sigma)$ as a subgroup of $W_1(\mathfrak{a}) \cap W_2(\mathfrak{a})$. For $\lambda \in \Sigma$, we define subspaces $\mathfrak{m}_\lambda$ in $\mathfrak{m}_1 \cap \mathfrak{m}_2$ and $\mathfrak{k}_\lambda$ in $\mathfrak{k}_1 \cap \mathfrak{k}_2$ as follows:

$$\begin{aligned}\mathfrak{m}_\lambda &= \big\{X \in \mathfrak{m}_1 \cap \mathfrak{m}_2 \mid [H, [H, X]] = -\langle \lambda, H\rangle^2 X\ (H \in \mathfrak{a})\big\},\\ \mathfrak{k}_\lambda &= \big\{X \in \mathfrak{k}_1 \cap \mathfrak{k}_2 \mid [H, [H, X]] = -\langle \lambda, H\rangle^2 X\ (H \in \mathfrak{a})\big\}.\end{aligned}$$

Denote by Π a fundamental system of Σ, and by Σ^+ the set of positive roots with respect to Π. Take a maximal abelian subalgebra $\mathfrak{t}$ in $\mathfrak{g}_{12}$ containing $\mathfrak{a}$. Denote by $\tilde{R}$ the root system of $\mathfrak{g}_{12}$ with respect to $\mathfrak{t}$. Let $\mathfrak{t} \to \mathfrak{a}; H \mapsto \bar{H}$ be the orthogonal projection and set $\tilde{R}_0 = \{\alpha \in \tilde{R} \mid \bar{\alpha} = 0\}$. Define a subalgebra $\mathfrak{k}_0$ in $\mathfrak{k}_1 \cap \mathfrak{k}_2$ by

$$\mathfrak{k}_0 = \{X \in \mathfrak{k}_1 \cap \mathfrak{k}_2 \mid [\mathfrak{a}, X] = \{0\}\}.$$

Take a compatible ordering of $\mathfrak{t}$, then we have the following.

Lemma 4.3 ([**18**, p. 89, Lemma 1]).

(1) *We have orthogonal direct sum decompositions*:

$$\mathfrak{k}_1 \cap \mathfrak{k}_2 = \mathfrak{k}_0 \oplus \sum_{\lambda\in\Sigma^+} \mathfrak{k}_\lambda, \quad \mathfrak{m}_1 \cap \mathfrak{m}_2 = \mathfrak{a} \oplus \sum_{\lambda\in\Sigma^+} \mathfrak{m}_\lambda.$$

(2) *For each $\alpha \in \tilde{R}^+ - \tilde{R}_0$ there exist $S_\alpha \in \mathfrak{k}_1 \cap \mathfrak{k}_2$ and $T_\alpha \in \mathfrak{m}_1 \cap \mathfrak{m}_2$ such that*

$$\big\{S_\alpha \mid \alpha \in \tilde{R}^+, \bar{\alpha} = \lambda\big\}, \quad \big\{T_\alpha \mid \alpha \in \tilde{R}^+, \bar{\alpha} = \lambda\big\}$$

are respectively orthonormal bases of $\mathfrak{k}_\lambda$ *and* $\mathfrak{m}_\lambda$ *and for* $H \in \mathfrak{a}$

$$[H, S_\alpha] = \langle \alpha, H\rangle T_\alpha, \quad [H, T_\alpha] = -\langle \alpha, H\rangle S_\alpha, \quad [S_\alpha, T_\alpha] = \bar{\alpha},$$
$$\mathrm{Ad}(\exp H) S_\alpha = \cos\langle \alpha, H\rangle S_\alpha + \sin\langle \alpha, H\rangle T_\alpha,$$
$$\mathrm{Ad}(\exp H) T_\alpha = -\sin\langle \alpha, H\rangle S_\alpha + \cos\langle \alpha, H\rangle T_\alpha.$$

LEMMA 4.4. $(s_\lambda, (2n\pi/\|\lambda\|^2)\lambda) \in \tilde{J}$ *for* $\lambda \in \Sigma$ *and* $n \in \boldsymbol{Z}$.

PROOF. By (2) of Lemma 4.3, $s_\lambda = \mathrm{Ad}(\exp(\pi/\|\lambda\|) S_\alpha)$ where $\exp(\pi/\|\lambda\|) S_\alpha \in K_1 \cap K_2$. Hence it is sufficient to prove that $\exp(2\pi/\|\lambda\|^2)\lambda \in K_1$. Select $Y \in \mathfrak{a}$ such that $\langle Y, \lambda\rangle = \pi$ and set $a = \exp Y \in A$. Since

$$\begin{aligned} a^{-1} \exp\left(-\frac{\pi}{\|\lambda\|} S_\alpha\right) a &= \exp\left(-\frac{\pi}{\|\lambda\|} \mathrm{Ad}(\exp(-Y)) S_\alpha\right) \\ &= \exp\left(-\frac{\pi}{\|\lambda\|}(\cos(\langle \alpha, Y\rangle) S_\alpha - \sin(\langle \alpha, Y\rangle) T_\alpha)\right) \\ &= \exp\left(\frac{\pi}{\|\lambda\|} S_\alpha\right) \in K_1, \end{aligned}$$

we get

$$\begin{aligned} K_1 \ni \left(\exp \frac{\pi}{\|\lambda\|} S_\alpha\right) a^{-1} \exp\left(-\frac{\pi}{\|\lambda\|} S_\alpha\right) a &= \exp(-s_\lambda Y) \exp Y \\ &= \exp(Y - s_\lambda Y) \\ &= \exp\left(\frac{2\pi}{\|\lambda\|^2}\lambda\right). \end{aligned}$$ □

LEMMA 4.5. *For* $\lambda \in \Sigma$, *set* $m(\lambda) = \dim \mathfrak{m}_\lambda$, *then we have*:

(1) $m(\lambda) = m(-\lambda)$.
(2) $m(s_\mu \lambda) = m(\lambda)$ *for* $\mu \in \Sigma$.

PROOF.
(1) Since $\mathfrak{m}_\lambda = \mathfrak{m}_{-\lambda}$, we have $m(\lambda) = m(-\lambda)$.
(2) Take $\alpha \in \tilde{R}$ such that $\bar{\alpha} = \mu$, then $s_\mu = \mathrm{Ad}(\exp(\pi/\|\mu\|) S_\alpha)$. Hence we can regard s_μ as an inner automorphism of $\mathfrak{g}$. Since $s_\mu \mathfrak{m}_\lambda = \mathfrak{m}_{s_\mu \lambda}$, we have $m(s_\mu \lambda) = m(\lambda)$. □

Let g be in G. Since

$$
\begin{aligned}
g_*^{-1}T_{\pi_1(g)}(K_2\pi_1(g)) &= \left\{\frac{d}{dt}\pi_1(\exp t\mathrm{Ad}(g^{-1})X)_{|t=0} \mid X \in \mathfrak{k}_2\right\} \\
&= (\mathrm{Ad}(g^{-1})\mathfrak{k}_2)_{\mathfrak{m}_1},
\end{aligned}
$$

we have

$$
g_*^{-1}T^{\perp}_{\pi_1(g)}(K_2\pi_1(g)) = \{X \in \mathfrak{m}_1 \mid \mathrm{Ad}(g)X \in \mathfrak{m}_2\},
$$

which is a Lie triple system in $\mathfrak{m}_1$. The isotropy subgroup at $\pi_1(g)$ of K_2-action is given by

$$
(K_2)_{\pi_1(g)} = \{k \in K_2 \mid g^{-1}kg \in K_1\}.
$$

LEMMA 4.6. *The slice representation of $K_2\pi_1(g)$ at $\pi_1(g)$ is equivalent to the adjoint representation of $(K_1)_{\pi_2(g^{-1})}$ to the Lie triple system $\{X \in \mathfrak{m}_1 \mid \mathrm{Ad}(g)X \in \mathfrak{m}_2\}$.*

PROOF. We can identify the normal space $T^{\perp}_{\pi_1(g)}(K_2\pi_1(g))$ with $\{X \in \mathfrak{m}_1 \mid \mathrm{Ad}(g)X \in \mathfrak{m}_2\}$ through g_*. Then, for $k \in (K_1)_{\pi_2(g^{-1})}$ and $X \in \mathfrak{m}_1$ with $\mathrm{Ad}(g)X \in \mathfrak{m}_2$,

$$
\begin{aligned}
k \cdot X &= g_*^{-1}k_*g_*X \\
&= \frac{d}{dt}g^{-1}kg\pi_1(\exp tX)_{|t=0} \\
&= \frac{d}{dt}\pi_1(\exp t\mathrm{Ad}(g^{-1}kg)X)_{|t=0} \\
&= \mathrm{Ad}(g^{-1}kg)X,
\end{aligned}
$$

where

$$
g^{-1}(K_2)_{\pi_1(g)}g = \{k \in K_1 \mid gkg^{-1} \in K_2\} = (K_1)_{\pi_2(g^{-1})}.
$$

Hence we get the assertion. □

Denote by τ_x the inner automorphism of G defined by $x \in G$. Let ρ be in $\mathrm{Aut}(G)$. The isometry on G defined by $G \to G; g \mapsto \rho(g)x^{-1}$ induces an isometry between two compact symmetric spaces G/K_1 and $G/\tau_x\rho(K_1)$:

$$
G/K_1 \to G/\tau_x\rho(K_1); gK_1 \mapsto \rho(g)x^{-1}\tau_x\rho(K_1).
$$

Denote by $\pi_x : G \to G/\tau_x\rho(K_1)$ the natural projection, then K_2-orbit $K_2\pi_1(g)$ maps to $\rho(K_2)$-orbit $\rho(K_2)\pi_x(\rho(g)x^{-1})$ by the isometry defined above. Hence we can identify $K_2\backslash G/K_1$ with $\rho(K_2)\backslash G/\tau_x\rho(K_1)$. If $K_i = G_{\theta_i}$, then

$$\rho(K_2) = G_{\rho\theta_2\rho^{-1}}, \quad \tau_x\rho(K_1) = G_{\tau_x\rho\theta_1\rho^{-1}\tau_x^{-1}}.$$

Based on the above, Matsuki introduced the following equivalent relation.

Definition 4.7 ([**17**]). Let (θ_1, θ_2) and (θ_1', θ_2') be two pairs of two involutive automorphisms of G. Then $(\theta_1, \theta_2) \sim (\theta_1', \theta_2')$ means that there exist $\rho \in \mathrm{Aut}(G)$ and $x \in G$ such that $\theta_1' = \tau_x\rho\theta_1\rho^{-1}\tau_x^{-1}$, $\theta_2' = \rho\theta_2\rho^{-1}$.

Since $\pi_1(\exp\mathfrak{a})$ is a section of K_2-action on M_1, in order to consider the orbit $K_2\pi_1(g)$, we may assume $g = \exp H$ for some $H \in \mathfrak{a}$. Then, since

$$g_*^{-1}T^\perp_{\pi_1(g)}(K_2\pi_1(g)) = \{X \in \mathfrak{m}_1 \mid \mathrm{Ad}(\exp H)X \in \mathfrak{m}_2\},$$

$\mathfrak{a}$ is a maximal abelian subspace of $g_*^{-1}T^\perp_{\pi_1(g)}(K_2\pi_1(g))$. Moreover

$$\mathrm{Ad}((K_1)_{\pi_2(g^{-1})})\mathfrak{a} = \{X \in \mathfrak{m}_1 \mid \mathrm{Ad}(g)X \in \mathfrak{m}_2\}. \tag{4.5}$$

Definition 4.8. Let M be a submanifold of a Riemannian manifold $\tilde{M}$. We denote the shape operator of M by A. Then M is called an *austere submanifold* if for each normal vector $\xi \in T_x^\perp M$, the set of eigenvalues with their multiplicities of A_ξ is invariant under the multiplication by -1. It is obvious that an austere submanifold is a minimal submanifold.

The notion of austere submanifold was first given by Harvey-Lawson [**5**]. By (4.5), we have the following.

Lemma 4.9. *The orbit $K_2\pi_1(g) \subset M_1$ is austere if and only if the set of eigenvalues of the shape operator $A_{g_*\xi}$ with multiplicities is invariant under the multiplication by -1 for each $\xi \in \mathfrak{a}$.*

Since $(\mathfrak{k}_1 + \mathfrak{k}_2)^\perp = \mathfrak{m}_1 \cap \mathfrak{m}_2$, we have

$$\mathfrak{g} = (\mathfrak{k}_1 + \mathfrak{k}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{m}_2). \tag{4.6}$$

For $\alpha \in \mathfrak{a}$, we define a subspace $\mathfrak{g}(\mathfrak{a}, \alpha)$ in the complexification $\mathfrak{g}^{\boldsymbol{C}}$ of $\mathfrak{g}$ by

$$\mathfrak{g}(\mathfrak{a}, \alpha) = \left\{X \in \mathfrak{g}^{\boldsymbol{C}} \mid [H, X] = \sqrt{-1}\langle\alpha, H\rangle X \ (H \in \mathfrak{a})\right\}$$

and set

$$\tilde{\Sigma} = \{\alpha \in \mathfrak{a} - \{0\} \mid \mathfrak{g}(\mathfrak{a}, \alpha) \neq \{0\}\},$$

then

$$\mathfrak{g}^{\boldsymbol{C}} = \mathfrak{g}(\mathfrak{a}, 0) \oplus \sum_{\alpha \in \tilde{\Sigma}} \mathfrak{g}(\mathfrak{a}, \alpha). \tag{4.7}$$

Denote by $^{-}$ the complex conjugation of $\mathfrak{g}^{\boldsymbol{C}}$ with respect to $\mathfrak{g}$. Since $\overline{\mathfrak{g}(\mathfrak{a}, \alpha)} = \mathfrak{g}(\mathfrak{a}, -\alpha)$, if $\alpha \in \tilde{\Sigma}$ then $-\alpha \in \tilde{\Sigma}$.

LEMMA 4.10 ([**16**]).

(1) $[\mathfrak{g}(\mathfrak{a}, \alpha), \mathfrak{g}(\mathfrak{a}, \beta)] \subset \mathfrak{g}(\mathfrak{a}, \alpha + \beta)$.
(2) $\theta_i \mathfrak{g}(\mathfrak{a}, \alpha) = \mathfrak{g}(\mathfrak{a}, -\alpha)$ *for* $i = 1, 2$.
(3) $\mathfrak{g}(\mathfrak{a}, \alpha)$ *is invariant under* $\theta_1\theta_2$ *and* $\theta_2\theta_1$.

The absolute value of the eigenvalues of $\theta_1\theta_2$ on $\mathfrak{g}^{\boldsymbol{C}}$ is equal to 1. For $\epsilon \in U(1)$, define a subspace $\mathfrak{g}(\mathfrak{a}, \alpha, \epsilon)$ of $\mathfrak{g}(\mathfrak{a}, \alpha)$ by

$$\mathfrak{g}(\mathfrak{a}, \alpha, \epsilon) = \{X \in \mathfrak{g}(\mathfrak{a}, \alpha) \mid \theta_1\theta_2 X = \epsilon X\},$$

then, by (3) of Lemma 4.10, we have

$$\mathfrak{g}(\mathfrak{a}, \alpha) = \sum_{\epsilon \in U(1)} \mathfrak{g}(\mathfrak{a}, \alpha, \epsilon).$$

LEMMA 4.11 ([**16**]).

(1) $\theta_1 \mathfrak{g}(\mathfrak{a}, \alpha, \lambda) = \mathfrak{g}(\mathfrak{a}, -\alpha, \lambda^{-1})$.
(2) $\overline{\mathfrak{g}(\mathfrak{a}, \alpha, \lambda)} = \mathfrak{g}(\mathfrak{a}, -\alpha, \lambda^{-1})$.
(3) $[\mathfrak{g}(\mathfrak{a}, \alpha, \lambda), \mathfrak{g}(\mathfrak{a}, \beta, \mu)] \subset \mathfrak{g}(\mathfrak{a}, \alpha + \beta, \lambda\mu)$.

LEMMA 4.12. $\tilde{\Sigma}$ *is a root system of* $\mathfrak{a} \cap \mathfrak{z}^{\perp}$, *where* $\mathfrak{z}$ *is the center of* $\mathfrak{g}$.

PROOF. It is known that $\tilde{\Sigma}$ satisfies (2) and (3) of Definition 2.1 by [**16**, p. 60, Proposition 1]. We shall prove $\tilde{\Sigma}$ satisfies (1) of Definition 2.1. If $H \in \mathfrak{a}$ satisfies the condition that $\langle \alpha, H \rangle = 0$ for any $\alpha \in \tilde{\Sigma}$. Then H is in $\mathfrak{z}$ by (4.7). Since the converse is true, for $H \in \mathfrak{a}$, $\langle \alpha, H \rangle = 0$ for any $\alpha \in \tilde{\Sigma}$ if and only if $H \in \mathfrak{z}$. Hence $\mathrm{span}(\tilde{\Sigma}) = \mathfrak{a} \cap \mathfrak{z}^{\perp}$. $\square$

We will close this subsection by explaining the covariant derivatives on M_1. For $X \in \mathfrak{g}$ we define a Killing vector field X^+ on M_1 by

$$(X^+)_q = \frac{d}{dt} \exp tX \cdot q_{|t=0} \in T_q(M_1).$$

By a formula of Koszul ([**6**, p. 48, (2)]), we have the following.

Lemma 4.13. *Denote by $\tilde{\nabla}$ the Levi-Civita connection on M_1.*

(1) *For $g \in G$ and $X, Y \in \mathfrak{g}$,*

$$g_* \tilde{\nabla}_{X^+} Y^+ = \tilde{\nabla}_{(\mathrm{Ad}(g)X)^+} (\mathrm{Ad}(g)Y)^+.$$

(2) *For $X, Y \in \mathfrak{g}$,*

$$(\tilde{\nabla}_{X^+} Y^+)_o = \begin{cases} -[X, Y]_{\mathfrak{m}_1} & (X \in \mathfrak{m}_1), \\ 0 & (X \in \mathfrak{k}_1). \end{cases}$$

(3) *For $p = \pi_1(g) \in M_1$,*

$$(\tilde{\nabla}_{X^+} Y^+)_p = -g_* \big[(\mathrm{Ad}(g^{-1})X)_{\mathfrak{m}_1}, \mathrm{Ad}(g^{-1})Y \big]_{\mathfrak{m}_1}.$$

4.2. When $\boldsymbol{G}$ is semisimple and $\boldsymbol{\theta_1 \theta_2 = \theta_2 \theta_1}$:
Since $\mathfrak{z} = \{0\}$, $\tilde{\Sigma}$ is a root system of $\mathfrak{a}$ by Lemma 4.12. Since

$$\mathfrak{k}_1 + \mathfrak{k}_2 = (\mathfrak{k}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{k}_2),$$

we have, by (4.6),

$$\mathfrak{g} = (\mathfrak{k}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{m}_2).$$

Since $\mathfrak{a} \subset \mathfrak{m}_1 \cap \mathfrak{m}_2$, we have

$$[\mathfrak{a}, \mathfrak{k}_1 \cap \mathfrak{m}_2] \subset \mathfrak{m}_1 \cap \mathfrak{k}_2, \quad [\mathfrak{a}, \mathfrak{m}_1 \cap \mathfrak{k}_2] \subset \mathfrak{k}_1 \cap \mathfrak{m}_2.$$

Since

$$\sum_{\alpha \in \tilde{\Sigma}} \mathfrak{g}(\mathfrak{a}, \alpha, 1) = \Big(\sum_{\lambda \in \Sigma} \mathfrak{k}_\lambda \oplus \sum_{\lambda \in \Sigma} \mathfrak{m}_\lambda \Big)^{\boldsymbol{C}},$$

we get

$$\Sigma = \{\alpha \in \tilde{\Sigma} \mid \mathfrak{g}(\mathfrak{a}, \alpha, 1) \neq \{0\}\}.$$

For $\lambda \in \Sigma$,

$$\mathfrak{g}(\mathfrak{a}, \lambda, 1) \oplus \mathfrak{g}(\mathfrak{a}, -\lambda, 1) = (\mathfrak{k}_\lambda \oplus \mathfrak{m}_\lambda)^{\boldsymbol{C}}.$$

Define subspaces of $\mathfrak{k}_1 \cap \mathfrak{m}_2$ and $\mathfrak{m}_1 \cap \mathfrak{k}_2$ respectively by

$$\begin{aligned} V(\mathfrak{k}_1 \cap \mathfrak{m}_2) &= \{X \in \mathfrak{k}_1 \cap \mathfrak{m}_2 \mid [\mathfrak{a}, X] = 0\}, \\ V(\mathfrak{m}_1 \cap \mathfrak{k}_2) &= \{X \in \mathfrak{m}_1 \cap \mathfrak{k}_2 \mid [\mathfrak{a}, X] = 0\}. \end{aligned}$$

Since $\mathfrak{g}(\mathfrak{a}, 0)$ is θ_i-invariant,

$$\mathfrak{g}(\mathfrak{a}, 0) = (\mathfrak{a} \oplus \mathfrak{k}_0 \oplus V(\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus V(\mathfrak{m}_1 \cap \mathfrak{k}_2))^{\boldsymbol{C}}.$$

Define subspaces of $\mathfrak{k}_1 \cap \mathfrak{m}_2$ and $\mathfrak{m}_1 \cap \mathfrak{k}_2$ respectively by

$$\begin{aligned} V^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) &= \{X \in \mathfrak{k}_1 \cap \mathfrak{m}_2 \mid X \perp V(\mathfrak{k}_1 \cap \mathfrak{m}_2)\}, \\ V^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2) &= \{X \in \mathfrak{m}_1 \cap \mathfrak{k}_2 \mid X \perp V(\mathfrak{m}_1 \cap \mathfrak{k}_2)\}, \end{aligned}$$

then we have the following orthogonal decompositions:

$$\mathfrak{k}_1 \cap \mathfrak{m}_2 = V(\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus V^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2), \quad \mathfrak{m}_1 \cap \mathfrak{k}_2 = V(\mathfrak{m}_1 \cap \mathfrak{k}_2) \oplus V^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2).$$

LEMMA 4.14.

$$\big[\mathfrak{a}, V^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)\big] \subset V^\perp(\mathfrak{k}_2 \cap \mathfrak{m}_1), \quad \big[\mathfrak{a}, V^\perp(\mathfrak{k}_2 \cap \mathfrak{m}_1)\big] \subset V^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2).$$

PROOF. Since

$$\big[\mathfrak{a}, V^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)\big] \subset [\mathfrak{a}, \mathfrak{k}_1 \cap \mathfrak{m}_2] \subset \mathfrak{k}_2 \cap \mathfrak{m}_1$$

and

$$\big\langle V(\mathfrak{k}_2 \cap \mathfrak{m}_1), [\mathfrak{a}, V^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)]\big\rangle = \big\langle [V(\mathfrak{k}_2 \cap \mathfrak{m}_1), \mathfrak{a}], V^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)\big\rangle = \{0\},$$

we have $[\mathfrak{a}, V^{\perp}(\mathfrak{k}_1 \cap \mathfrak{m}_2)] \subset V^{\perp}(\mathfrak{k}_2 \cap \mathfrak{m}_1)$. Similarly we get $[\mathfrak{a}, V^{\perp}(\mathfrak{k}_2 \cap \mathfrak{m}_1)] \subset V^{\perp}(\mathfrak{k}_1 \cap \mathfrak{m}_2)$. $\square$

Define a subspace V of $\mathfrak{g}$ by

$$V = \mathfrak{g} \cap \sum_{\alpha \in \tilde{\Sigma}} \mathfrak{g}(\mathfrak{a}, \alpha, -1) = V^{\perp}(\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus V^{\perp}(\mathfrak{k}_2 \cap \mathfrak{m}_1),$$

then V is a representation space of the torus $\exp \mathfrak{a}$ by Lemma 4.14. For $\alpha \in \mathfrak{a}$ define a subspace $V(\alpha)$ of $V^{\boldsymbol{C}}$ by

$$V(\alpha) = \left\{ X \in V^{\boldsymbol{C}} \mid (\mathrm{ad} H) X = \sqrt{-1} \langle \alpha, H \rangle X \quad (H \in \mathfrak{a}) \right\},$$

then there exists a finite subset $W \subset \mathfrak{a} - \{0\}$ such that

$$V^{\boldsymbol{C}} = \sum_{\alpha \in \tilde{\Sigma}} \mathfrak{g}(\mathfrak{a}, \alpha, -1) = \sum_{\alpha \in W} V(\alpha),$$

since any complex irreducible representation of a torus is one dimensional. Denote by $\overline{X}$ the complex conjugation of $X \in V^{\boldsymbol{C}}$ with respect to V, then $\overline{V(\alpha)} = V(-\alpha)$. Hence, if $\alpha \in W$ then $-\alpha \in W$. Since

$$V(\alpha) \oplus V(-\alpha) = \left\{ X \in V^{\boldsymbol{C}} \mid (\mathrm{ad} H)^2 X = -\langle \alpha, H \rangle^2 X \ (H \in \mathfrak{a}) \right\},$$

we have

$$(V(\alpha) \oplus V(-\alpha)) \cap V = \{ X \in V \mid (\mathrm{ad} H)^2 X = -\langle \alpha, H \rangle^2 X \quad (H \in \mathfrak{a}) \}.$$

Define subspaces $V^{\perp}_{\alpha}(\mathfrak{k}_1 \cap \mathfrak{m}_2)$ and $V^{\perp}_{\alpha}(\mathfrak{k}_2 \cap \mathfrak{m}_1)$ by

$$V^{\perp}_{\alpha}(\mathfrak{k}_1 \cap \mathfrak{m}_2) = (V(\alpha) \oplus V(-\alpha)) \cap V^{\perp}(\mathfrak{k}_1 \cap \mathfrak{m}_2),$$
$$V^{\perp}_{\alpha}(\mathfrak{k}_2 \cap \mathfrak{m}_1) = (V(\alpha) \oplus V(-\alpha)) \cap V^{\perp}(\mathfrak{k}_2 \cap \mathfrak{m}_1),$$

then

$$V^{\perp}_{\alpha}(\mathfrak{k}_1 \cap \mathfrak{m}_2) = V^{\perp}_{-\alpha}(\mathfrak{k}_1 \cap \mathfrak{m}_2), \quad V^{\perp}_{\alpha}(\mathfrak{k}_2 \cap \mathfrak{m}_1) = V^{\perp}_{-\alpha}(\mathfrak{k}_2 \cap \mathfrak{m}_1).$$

Moreover

$$W = \{\alpha \in \tilde{\Sigma} \mid \mathfrak{g}(\mathfrak{a}, \alpha, -1) \neq \{0\}\}, \quad \tilde{\Sigma} = \Sigma \cup W. \tag{4.8}$$

For $\alpha \in W$, we have

$$\mathfrak{g}(\mathfrak{a}, \alpha, -1) \oplus \mathfrak{g}(\mathfrak{a}, -\alpha, -1) = \left(V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus V_\alpha^\perp(\mathfrak{k}_2 \cap \mathfrak{m}_1)\right)^{\boldsymbol{C}}.$$

Set $n(\alpha) = \dim_{\boldsymbol{C}} \mathfrak{g}(\mathfrak{a}, \alpha, -1)$ for $\alpha \in W$.

Lemma 4.15. *$n(\alpha) = n(-\alpha)$ and $n(s\alpha) = n(\alpha)$ for $\alpha \in W, s \in W(\Sigma)$.*

Proof. By Lemma 4.11, $n(\alpha) = n(-\alpha)$. Since there exists $k \in K_1 \cap K_2$ such that $s = \mathrm{Ad}(k)$ on $\mathfrak{a}$, we can regard s as an inner automorphism of $\mathfrak{g}^{\boldsymbol{C}}$. Since s maps $\mathfrak{g}(\mathfrak{a}, \alpha, -1)$ onto $\mathfrak{g}(\mathfrak{a}, s\alpha, -1)$, we have $n(s\alpha) = n(\alpha)$. □

Since $\tilde{\Sigma}$ is a root system of $\mathfrak{a}$ by Lemma 4.12, we can take a fundamental system $\tilde{\Pi}$ of $\tilde{\Sigma}$. We denote by $\tilde{\Sigma}^+$ the set of positive roots in $\tilde{\Sigma}$ with respect to $\tilde{\Pi}$. Set $\Sigma^+ = \Sigma \cap \tilde{\Sigma}^+$ and $W^+ = W \cap \tilde{\Sigma}^+$. Denote by Π the set of simple roots of Σ. Then we have

$$V^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) = \sum_{\alpha \in W^+} V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2), \quad V^\perp(\mathfrak{k}_2 \cap \mathfrak{m}_1) = \sum_{\alpha \in W^+} V_\alpha^\perp(\mathfrak{k}_2 \cap \mathfrak{m}_1).$$

Lemma 4.16.

(1) *For any $\alpha \in W^+$,*

$$\left[\mathfrak{a}, V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)\right] = V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2), \quad \left[\mathfrak{a}, V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2)\right] = V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2).$$

(2) *There exist orthonormal bases $\{X_{\alpha,i}\}_{1 \leq i \leq n(\alpha)}$ and $\{Y_{\alpha,i}\}_{1 \leq i \leq n(\alpha)}$ of $V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)$ and $V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2)$ respectively such that, for any $H \in \mathfrak{a}$,*

$$[H, X_{\alpha,i}] = \langle \alpha, H \rangle Y_{\alpha,i}, \quad [H, Y_{\alpha,i}] = -\langle \alpha, H \rangle X_{\alpha,i}, \quad [X_{\alpha,i}, Y_{\alpha,i}] = \alpha,$$
$$\mathrm{Ad}(\exp H) X_{\alpha,i} = \cos(\langle \alpha, H \rangle) X_{\alpha,i} + \sin(\langle \alpha, H \rangle) Y_{\alpha,i},$$
$$\mathrm{Ad}(\exp H) Y_{\alpha,i} = -\sin(\langle \alpha, H \rangle) X_{\alpha,i} + \cos(\langle \alpha, H \rangle) Y_{\alpha,i}.$$

(3) *For $H \in \mathfrak{a}$ set $g = \exp H$, then*

$$(\mathrm{Ad}(g^{-1})\mathfrak{k}_2)_{\mathfrak{m}_1} = \sum_{\substack{\lambda \in \Sigma^+ \\ \langle \lambda, H \rangle \notin \pi \boldsymbol{Z}}} \mathfrak{m}_\lambda \oplus V(\mathfrak{k}_2 \cap \mathfrak{m}_1) \oplus \sum_{\substack{\beta \in W^+ \\ \langle \beta, H \rangle \notin \pi/2 + \pi \boldsymbol{Z}}} V_\alpha^\perp(\mathfrak{k}_2 \cap \mathfrak{m}_1).$$

(4) *Put* $g = \exp H$ *for* $H \in \mathfrak{a}$. *For* $\beta \in \Sigma$ *with* $\langle \beta, H \rangle \notin \pi \boldsymbol{Z}$,

$$g_* T_\beta = -\frac{1}{\sin(\langle \beta, H \rangle)} \big(S_\beta^+\big)_{\pi_1(g)}.$$

For $\alpha \in W$ *with* $\langle \alpha, H \rangle \notin \pi/2 + \pi \boldsymbol{Z}$,

$$g_* Y_{\alpha,i} = \frac{1}{\cos(\langle \alpha, H \rangle)} (Y_{\alpha,i})^+_{\pi_1(g)}.$$

$Y^+_{\pi_1(g)} = g_* Y$ *for* $Y \in V(\mathfrak{k}_2 \cap \mathfrak{m}_1)$.

Proof.
(1) By Lemma 4.14,

$$\big[\mathfrak{a}, V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)\big] \subset [\mathfrak{a}, V^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)] \subset V^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2).$$

For any $H, H' \in \mathfrak{a}$ and $X \in V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)$,

$$(\mathrm{ad} H')^2 [H, X] = [H, (\mathrm{ad} H')^2 X] = -\langle \alpha, H' \rangle^2 [H, X].$$

Hence $[\mathfrak{a}, V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)] \subset V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2)$. Conversely, for any $X \in V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2)$,

$$\left[\alpha, \left[-\frac{\alpha}{\|\alpha\|^4}, X\right]\right] = X.$$

Since $[-\alpha/\|\alpha\|^4, X] \in V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)$, we have

$$\big[\mathfrak{a}, V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)\big] = V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2).$$

Similarly we have $[\mathfrak{a}, V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2)] = V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)$.
(2) By polarization,

$$V_\alpha^\perp(*) = \big\{X \in V^\perp(*) \mid [H_1, [H_2, X]] = -\langle \alpha, H_1 \rangle \langle \alpha, H_2 \rangle X \ (H_1, H_2 \in \mathfrak{a})\big\},$$

where $* = \mathfrak{m}_1 \cap \mathfrak{k}_2$ or $* = \mathfrak{k}_1 \cap \mathfrak{m}_2$. Denote by $\{X_{\alpha,i}\}_{1 \le i \le n(\alpha)}$ an orthonormal basis of $V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)$ and set

$$Y_{\alpha,i} = \frac{1}{\|\alpha\|^2} [\alpha, X_{\alpha,i}] \in V_\alpha^\perp(\mathfrak{k}_2 \cap \mathfrak{m}_1).$$

Then

$$\langle Y_{\alpha,i}, Y_{\alpha,j}\rangle = -\frac{1}{\|\alpha\|^4}\langle X_{\alpha,i}, [\alpha, [\alpha, X_{\alpha,i}]]\rangle = \langle X_{\alpha,i}, X_{\alpha,j}\rangle = \delta_{ij}.$$

Hence $\{Y_{\alpha,i}\}_{1\leq i\leq n(\alpha)}$ is an orthonormal basis of $V_\alpha^\perp(\mathfrak{k}_2\cap\mathfrak{m}_1)$ by (1). Since

$$[H, Y_{\alpha,i}] = \frac{1}{\|\alpha\|^2}[H, [\alpha, X_{\alpha,i}]] = -\langle\alpha, H\rangle X_{\alpha,i}.$$

for any $H\in\mathfrak{a}$, we have $X_{\alpha,i} = -(1/\|\alpha\|^2)[\alpha, Y_{\alpha,i}]$. Hence $[H, X_{\alpha,i}] = \langle\alpha, H\rangle Y_{\alpha,i}$. Remark that

$$[X_{\alpha,i}, Y_{\alpha,i}] \in [\mathfrak{k}_1\cap\mathfrak{m}_2, \mathfrak{m}_1\cap\mathfrak{k}_2] \subset \mathfrak{m}_1\cap\mathfrak{m}_2.$$

By the Jacobi identity, for any $H\in\mathfrak{a}$, we have $[H, [X_{\alpha,i}, Y_{\alpha,i}]] = 0$. Hence $[X_{\alpha,i}, Y_{\alpha,i}]\in\mathfrak{a}$ by the maximality of $\mathfrak{a}$. Since $\langle H, [X_{\alpha,i}, Y_{\alpha,i}]\rangle = \langle\alpha, H\rangle$, we have $[X_{\alpha,i}, Y_{\alpha,i}] = \alpha$.

(3) We decompose $\mathfrak{k}_2$ as follows:

$$\begin{aligned}\mathfrak{k}_2 &= (\mathfrak{k}_2\cap\mathfrak{k}_1)\oplus(\mathfrak{k}_2\cap\mathfrak{m}_1)\\ &= \mathfrak{k}_0\oplus\sum_{\lambda\in\Sigma^+}\mathfrak{k}_\lambda\oplus V(\mathfrak{k}_2\cap\mathfrak{m}_1)\oplus\sum_{\beta\in W^+}V_\alpha^\perp(\mathfrak{k}_2\cap\mathfrak{m}_1).\end{aligned}$$

According to the above decomposition, we decompose $X\in\mathfrak{k}_2$ as follows:

$$X = X_0 + \sum_{\lambda\in\Sigma^+}\sum_{\bar{\alpha}=\lambda} x_\alpha S_\alpha + X_1 + \sum_{\beta\in W^+}\sum_{i=1}^{n(\beta)} y_{\beta,i}Y_{\beta,i}.$$

Then we have

$$\begin{aligned}\mathrm{Ad}(\exp(-H))X = X_0 &+ \sum_{\lambda\in\Sigma^+}\sum_{\bar{\alpha}=\lambda} x_\alpha(\cos(\langle\alpha, H\rangle)S_\alpha - \sin(\langle\alpha, H\rangle)T_\alpha)\\ &+ X_1 + \sum_{\beta\in W^+}\sum_{i=1}^{n(\beta)} y_{\beta,i}(\sin(\langle\beta, H\rangle)X_{\beta,i} + \cos(\langle\beta, H\rangle)Y_{\beta,i}),\end{aligned}$$

which implies that

$$\begin{aligned}&(\mathrm{Ad}(\exp(-H))X)_{\mathfrak{m}_1}\\&\quad = -\sum_{\lambda\in\Sigma^+}\sin(\langle\lambda,H\rangle)\sum_{\bar{\alpha}=\lambda}x_\alpha T_\alpha + X_1 + \sum_{\beta\in W^+}\sum_{i=1}^{n(\beta)} y_{\beta,i}\cos(\langle\beta,H\rangle)Y_{\beta,i}.\end{aligned}$$

Hence we get the assertion.

(4) By the definition of S_β^+, we have

$$\begin{aligned}\left(S_\beta^+\right)_{\pi_1(g)} &= \frac{d}{dt}\exp tS_\beta\pi_1(g)_{|t=0}\\&= g_*\frac{d}{dt}\pi_1(\exp\mathrm{Ad}(\exp(-H))S_\beta)_{|t=0}\\&= g_*\frac{d}{dt}(\mathrm{Ad}(\exp(-H))S_\beta)_{\mathfrak{m}_1}\\&= g_*(\cos(\langle\beta,H\rangle)S_\beta - \sin(\langle\beta,H\rangle)T_\beta)_{\mathfrak{m}_1}\\&= -\sin(\langle\beta,H\rangle)g_*T_\beta.\end{aligned}$$

Similarly we have

$$(Y_{\alpha,i})^+_{\pi_1(g)} = g_*(\mathrm{Ad}(\exp(-H))Y_{\alpha,i})_{\mathfrak{m}_1} = \cos(\langle\beta,H\rangle)g_*Y_{\alpha,i}.$$

Since $[\mathfrak{a}, V(\mathfrak{k}_2\cap\mathfrak{m}_1)] = 0$, we have $Y^+_{\pi_1(g)} = g_*Y$. □

COROLLARY 4.17. *$s_\alpha \in W_2(\mathfrak{a})\cap W_1(\mathfrak{a})$ for each $\alpha\in\tilde{\Sigma}$.*

PROOF. We already proved that $s_\alpha\in W_2(\mathfrak{a})\cap W_1(\mathfrak{a})$ for $\alpha\in\Sigma$. Assume that $\alpha\in W$. By (2) of Lemma 4.16, for each $H\in\mathfrak{a}$,

$$\mathrm{Ad}(\exp tY_{\alpha,i})H = H + \frac{\langle\alpha,H\rangle}{\|\alpha\|^2}(\cos(t\|\alpha\|)-1)\alpha + \frac{\langle\alpha,H\rangle}{\|\alpha\|}\sin(t\|\alpha\|)X_{\alpha,i}.$$

Select $t\in\boldsymbol{R}$ such that $\cos(t\|\alpha\|) = -1$, then $\mathrm{Ad}(\exp tY_{\alpha,i})H = s_\alpha(H)$. Hence $s_\alpha\in W_2(\mathfrak{a})$. Similarly there exists $t\in\boldsymbol{R}$ such that $\mathrm{Ad}(\exp tX_{\alpha,i})H = s_\alpha(H)$. Hence $s_\alpha\in W_1(\mathfrak{a})$. □

Corollary 4.18 follows from (3) of Lemma 4.16.

COROLLARY 4.18 ([**3**, Corollary 5.2]). *Set $g=\exp H$ for $H\in\mathfrak{a}$, then $K_2\pi_1(g)$ is a regular orbit if and only if $\langle\lambda,H\rangle\notin\pi\boldsymbol{Z}$ for any $\lambda\in\Sigma$ and $\langle\beta,H\rangle\notin\pi/2+\pi\boldsymbol{Z}$ for any $\beta\in W$.*

Corollary 4.19 follows from Corollary 4.18.

COROLLARY 4.19. *Let $g \in G$, $K_2\pi_1(g)$ is regular if and only if $K_1\pi_2(g^{-1})$ is regular.*

LEMMA 4.20. *$n(\lambda) + m(\lambda) = n(\sigma\lambda) + m(\sigma\lambda)$ for $\lambda \in \tilde{\Sigma}$ and $\sigma \in W(\tilde{\Sigma})$.*

PROOF. Since $W(\tilde{\Sigma})$ is generated by s_μ ($\mu \in \Sigma$) and s_α ($\alpha \in W$), we can regard σ as an inner automorphism of $\mathfrak{g}^{\boldsymbol{C}}$ by Corollary 4.17. Then σ maps $\mathfrak{g}(\mathfrak{a}, \lambda)$ onto $\mathfrak{g}(\mathfrak{a}, \sigma\lambda)$. Hence

$$m(\lambda) + n(\lambda) = \dim \mathfrak{g}(\mathfrak{a}, \lambda) = \dim \mathfrak{g}(\mathfrak{a}, \sigma\lambda) = n(\sigma\lambda) + m(\sigma\lambda). \qquad \square$$

Define an open subset $\mathfrak{a}_r$ in $\mathfrak{a}$ by

$$\mathfrak{a}_r = \{H \in \mathfrak{a} \mid K_2\pi_1(\exp H) \text{ is a regular orbit}\}.$$

Each connected component of $\mathfrak{a}_r$ is called a cell. Each cell is a bounded convex open subset of $\mathfrak{a}$. The action of $\tilde{J}$ induces a transformation on the set of cells.

LEMMA 4.21. *$(s_\alpha, ((2n+1)\pi/\|\alpha\|^2)\alpha) \in \tilde{J}$ for $\alpha \in W$ and $n \in \boldsymbol{Z}$.*

PROOF. Since $s_\alpha = \mathrm{Ad}(\exp(\pi/\|\alpha\|)Y_{\alpha,i})$ where $\exp(\pi/\|\alpha\|)Y_{\alpha,i} \in N_{K_2}(\mathfrak{a})$, it is sufficient to show that $\exp(-((2n+1)\pi/\|\alpha\|^2)\alpha)\exp((\pi/\|\alpha\|)Y_{\alpha,i}) \in K_1$. Hence it is sufficient to prove that

$$\exp\left(-\frac{(2n+1)\pi}{\|\alpha\|^2}\alpha\right)\exp\left(\frac{\pi}{\|\alpha\|}Y_{\alpha,i}\right) = \exp\left((-1)^n\frac{\pi}{\|\alpha\|}X_{\alpha,i}\right). \tag{4.9}$$

Take $H \in \mathfrak{a}$ such that $2\langle\alpha, H\rangle = (2n+1)\pi$ and set $a = \exp H$. We calculate $\exp((\pi/\|\alpha\|)Y_{\alpha,i})a^{-1}\exp(-(\pi/\|\alpha\|)Y_{\alpha,i})a$ in the following two ways. The first way is

$$\begin{aligned}
&\exp\left(\frac{\pi}{\|\alpha\|}Y_{\alpha,i}\right)a^{-1}\exp\left(-\frac{\pi}{\|\alpha\|}Y_{\alpha,i}\right)a \\
&\quad = \exp\left(\frac{\pi}{\|\alpha\|}Y_{\alpha,i}\right)\exp\left(-\frac{\pi}{\|\alpha\|}\mathrm{Ad}(\exp(-H))Y_{\alpha,i}\right) \\
&\quad = \exp\left(\frac{\pi}{\|\alpha\|}Y_{\alpha,i}\right)\exp\left((-1)^{n+1}\frac{\pi}{\|\alpha\|}X_{\alpha,i}\right).
\end{aligned}$$

The second is

$$\begin{aligned}\exp\left(\frac{\pi}{\|\alpha\|}Y_{\alpha,i}\right)a^{-1}\exp\left(-\frac{\pi}{\|\alpha\|}Y_{\alpha,i}\right)a &= \exp(-s_\alpha H)\exp H\\ &= \exp(H - s_\alpha H)\\ &= \exp\left(\frac{(2n+1)\pi\alpha}{\|\alpha\|^2}\right).\end{aligned}$$

Hence we get (4.9). □

Denote by $\tilde{W}(\tilde{\Sigma},\Sigma,W)$ the subgroup of $\tilde{J}$ generated by

$$\left\{\left(s_\lambda, \frac{2n\pi}{\|\lambda\|^2}\lambda\right) \mid \lambda\in\Sigma, n\in\boldsymbol{Z}\right\}\cup\left\{\left(s_\alpha, \frac{(2n+1)\pi}{\|\alpha\|^2}\alpha\right) \mid \alpha\in W, n\in\boldsymbol{Z}\right\},$$

then $\tilde{W}(\tilde{\Sigma},\Sigma,W)$ acts transitively on the set of cells. This is obtained in the same way of the proof of Proposition 2.10. In the same way of the proof of Corollary 2.11, we get the following: Select and fix any cell P_0, then $\mathfrak{a} = \bigcup_{s\in\tilde{W}(\tilde{\Sigma},\Sigma,W)} s\overline{P_0}$. Hence any K_2-orbit in M_1 can be expressed as $K_2\pi_1(\exp H)$ for $H\in\overline{P_0}$. By Proposition 4.1, we can identify the orbit space $K_2\backslash G/K_1$ with $\overline{P_0}/\{\sigma\in\tilde{J} \mid \sigma\overline{P_0} = \overline{P_0}\}$. From now on we use $\overline{P_0}$ as a substitute for the orbit space. It is known that $\tilde{J} = \tilde{W}(\tilde{\Sigma},\Sigma,W)$ when G is simply connected ([**17**, Proposition 3.1]). Set P_0 as in (2.2), then P_0 is a cell ([**13**]). This is obtained from Corollary 4.18.

Lemma 4.22. *Set $g=\exp H$ for $H\in\mathfrak{a}$. Denote by h the second fundamental form of $K_2\pi_1(g)$ in M_1, then we have*:

(1) $g_*^{-1}h(g_*T_\alpha, g_*T_\beta) = \cot(\langle\beta,H\rangle)[T_\alpha,S_\beta]^\perp$ *for* $\alpha,\beta\in\Sigma$ *with* $\langle\alpha,H\rangle,\langle\beta,H\rangle\notin\pi\boldsymbol{Z}$.
(2) $g_*^{-1}h(g_*Y_{\alpha,i}, g_*Y_{\beta,j}) = -\tan(\langle\beta,H\rangle)[Y_{\alpha,i},X_{\beta,j}]^\perp$ *for* α *and* β *in* W *with* $\langle\alpha,H\rangle,\langle\beta,H\rangle\notin\pi/2+\pi\boldsymbol{Z}$.
(3) $h(g_*Y_0, g_*Y_1) = 0$ *for* $Y_0,Y_1\in V(\mathfrak{k}_2\cap\mathfrak{m}_1)$.
(4) $h(g_*T_\alpha, g_*Y) = 0$ *for* $\alpha\in\Sigma$ *with* $\langle\alpha,H\rangle\notin\pi\boldsymbol{Z}$ *and* $Y\in V(\mathfrak{k}_2\cap\mathfrak{m}_1)$.
(5) $h(g_*Y_{\alpha,i}, g_*Y) = 0$ *for* $\alpha\in W$ *with* $\langle\alpha,H\rangle\notin\pi/2+\pi\boldsymbol{Z}$ *and* $Y\in V^\perp(\mathfrak{k}_2\cap\mathfrak{m}_1)$.
(6) *For* $\alpha\in\Sigma$, $\beta\in W$ *with* $\langle\alpha,H\rangle\notin\pi\boldsymbol{Z}$, $\langle\beta,H\rangle\notin\pi/2+\pi\boldsymbol{Z}$,

$$g_*^{-1}h(g_*T_\alpha, g_*Y_{\beta,i}) = \tan(\langle\beta,H\rangle)[T_\alpha,X_{\beta,i}]^\perp.$$

Proof. (1) By (4) of Lemma 4.16, (3) of Lemma 4.13 and (2) of Lemma 4.3,

$$\begin{aligned} h(g_*T_\alpha, g_*T_\beta) &= \frac{1}{\sin(\langle \alpha, H\rangle)\sin(\langle \beta, H\rangle)} h\big((S_\alpha^+)_{\pi_1(g)}, (S_\beta^+)_{\pi_1(g)}\big) \\ &= \frac{-g_*}{\sin(\langle \alpha, H\rangle)\sin(\langle \beta, H\rangle)} \big[(\mathrm{Ad}(g^{-1})S_\alpha)_{\mathfrak{m}_1}, \mathrm{Ad}(g^{-1})S_\beta\big]_{\mathfrak{m}_1}^\perp \\ &= \cot(\langle \beta, H\rangle) g_*[T_\alpha, S_\beta]^\perp. \end{aligned}$$

(2) By (4) of Lemma 4.16 and (3) of Lemma 4.13, we have

$$\begin{aligned} h(g_*Y_{\alpha,i}, g_*Y_{\beta,j}) &= \frac{1}{\cos(\langle \alpha, H\rangle)\cos(\langle \beta, H\rangle)} h\big((Y_{\alpha,i})^+_{\pi_1(g)}, (Y_{\beta,j})^+_{\pi_1(g)}\big) \\ &= \frac{-g_*[(\mathrm{Ad}(g^{-1})Y_{\alpha,i})_{\mathfrak{m}_1}, \mathrm{Ad}(g^{-1})Y_{\beta,j}]^\perp_{\mathfrak{m}_1}}{\cos(\langle \alpha, H\rangle)\cos(\langle \beta, H\rangle)} \\ &= \frac{-g_*[Y_{\alpha,i}, \sin(\langle \beta, H\rangle)X_{\beta,j} + \cos(\langle \beta, H\rangle)Y_{\beta,j}]^\perp_{\mathfrak{m}_1}}{\cos(\langle \beta, H\rangle)} \\ &= -\tan(\langle \beta, H\rangle) g_*[Y_{\alpha,i}, X_{\beta,j}]^\perp. \end{aligned}$$

(3) By (4) of Lemma 4.16 and (3) of Lemma 4.13,

$$h(g_*Y_0, g_*Y_1) = h\big(Y_0^+, Y_1^+\big)_{\pi_1(g)} = -g_*[Y_0, Y_1]^\perp_{\mathfrak{m}_1},$$

where $[Y_0, Y_1] \in \mathfrak{k}_2 \cap \mathfrak{k}_1$ since $Y_0, Y_1 \in V(\mathfrak{k}_2 \cap \mathfrak{m}_1) \subset \mathfrak{k}_2 \cap \mathfrak{m}_1$. Hence $[Y_0, Y_1]_{\mathfrak{m}_1} = 0$.

(4) By (4) of Lemma 4.16 and (3) of Lemma 4.13, we have

$$\begin{aligned} h(g_*T_\alpha, g_*Y) &= -\frac{1}{\sin(\langle \alpha, H\rangle)} h\big(S_\alpha^+, Y^+\big)_{\pi_1(g)} \\ &= -\frac{1}{\sin(\langle \alpha, H\rangle)} g_*\big[(\mathrm{Ad}(g^{-1})S_\alpha)_{\mathfrak{m}_1}, Y\big]^\perp_{\mathfrak{m}_1} \\ &= -g_*[T_\alpha, Y]^\perp_{\mathfrak{m}_1} = 0. \end{aligned}$$

(5) By (4) of Lemma 4.16 and (3) of Lemma 4.13,

$$\begin{aligned} h(g_*Y_{\alpha,i}, g_*Y) &= \frac{1}{\cos(\langle \alpha, H\rangle)} h\big(Y_{\alpha,i}^+, Y^+\big)_{\pi_1(g)} \\ &= -\frac{1}{\cos(\langle \alpha, H\rangle)} g_*\big[(\mathrm{Ad}(g^{-1})Y_{\alpha,i})_{\mathfrak{m}_1}, Y\big]^\perp_{\mathfrak{m}_1} \\ &= -g_*[Y_{\alpha,i}, Y]^\perp_{\mathfrak{m}_1} = 0 \end{aligned}$$

(6) By (4) of Lemma 4.16 and (3) of Lemma 4.13, we have

$$\begin{aligned} h(g_* T_\alpha, g_* Y_{\beta,i}) &= -\frac{1}{\sin(\langle\alpha, H\rangle)\cos(\langle\beta, H\rangle)} h\big(S_\alpha^+, Y_{\beta,i}^+\big)_{\pi_1(g)} \\ &= \frac{g_*[(\mathrm{Ad}(g^{-1})S_\alpha)_{\mathfrak{m}_1}, \mathrm{Ad}(g^{-1})Y_{\beta,i}]_{\mathfrak{m}_1}^\perp}{\sin(\langle\alpha, H\rangle)\cos(\langle\beta, H\rangle)} \\ &= -\frac{g_*[T_\alpha, -\sin(\langle\beta, H\rangle)X_{\beta,i} + \cos(\langle\beta, H\rangle)Y_{\beta,i}]_{\mathfrak{m}_1}^\perp}{\cos(\langle\beta, H\rangle)} \\ &= \tan(\langle\beta, H\rangle) g_*[T_\alpha, X_{\beta,i}]^\perp. \end{aligned}$$

□

COROLLARY 4.23. *Set $g = \exp H$ for H in $\mathfrak{a}$. The mean curvature vector of $K_2\pi_1(g)$ in M_1 is given by*

$$g_*^{-1} m_{\pi_1(g)} = -\sum_{\substack{\lambda\in\Sigma^+ \\ \langle\lambda, H\rangle\notin\pi\boldsymbol{Z}}} m(\lambda)\cot(\langle\lambda, H\rangle)\lambda + \sum_{\substack{\alpha\in W^+ \\ \langle\alpha, H\rangle\notin(\pi/2)+\pi\boldsymbol{Z}}} n(\alpha)\tan(\langle\alpha, H\rangle)\alpha.$$

PROOF. By (1), (2) and (3) of Lemma 4.22, we have

$$\begin{aligned} g_*^{-1} m_{\pi_1(g)} &= -\sum \cot(\langle\alpha, H\rangle)\bar{\alpha} - \sum_{\substack{\alpha\in W^+ \\ \langle\alpha, H\rangle\notin(\pi/2)+\pi\boldsymbol{Z}}} \tan(\langle\alpha, H\rangle)[Y_{\alpha,i}, X_{\alpha,i}]^\perp \\ &= -\sum_{\substack{\lambda\in\Sigma^+ \\ \langle\lambda, H\rangle\notin\pi\boldsymbol{Z}}} m(\lambda)\cot(\langle\lambda, H\rangle)\lambda - \sum \tan(\langle\alpha, H\rangle)[Y_{\alpha,i}, X_{\alpha,i}]^\perp \\ &= -\sum_{\substack{\lambda\in\Sigma^+ \\ \langle\lambda, H\rangle\notin\pi\boldsymbol{Z}}} m(\lambda)\cot(\langle\lambda, H\rangle)\lambda + \sum_{\substack{\alpha\in W^+ \\ \langle\alpha, H\rangle\notin(\pi/2)+\pi\boldsymbol{Z}}} n(\alpha)\tan(\langle\alpha, H\rangle)\alpha. \end{aligned}$$

□

In [**10**, Corollary 2.8], we showed that the mean curvature vector of $K_2\pi_1(g)$ in M_1 is parallel with respect to the normal connection. The following corollary is a generalization of a result in [**7**].

COROLLARY 4.24. *Set $g = \exp H$ for H in $\mathfrak{a}$. The orbit $K_2\pi_1(g)$ in M_1 is totally geodesic if and only if $\langle\lambda, H\rangle \in (\pi/2)\boldsymbol{Z}$ for any $\lambda \in \tilde{\Sigma}^+$.*

PROOF. It is sufficient to prove that $K_2\pi_1(g)$ is totally geodesic if and only if the following conditions (1) and (2) hold:

(1) For $\lambda \in \Sigma^+$ with $\langle\lambda, H\rangle \notin \pi\boldsymbol{Z}$,

$$\cot(\langle\lambda, H\rangle) = 0, \text{ that is, } \langle\lambda, H\rangle \in \frac{\pi}{2} + \pi\boldsymbol{Z}.$$

In other words $\langle\lambda, H\rangle \in (\pi/2)\boldsymbol{Z}$ for any $\lambda \in \Sigma^+$.

(2) For $\alpha \in W^+$ with $\langle\alpha, H\rangle \notin (\pi/2) + \pi\boldsymbol{Z}$,

$$\tan(\langle\alpha, H\rangle) = 0, \text{ that is, } \langle\alpha, H\rangle \in \pi\boldsymbol{Z}.$$

In other words $\langle\alpha, H\rangle \in (\pi/2)\boldsymbol{Z}$ for any $\alpha \in W^+$.

By Lemma 4.22, $K_2\pi_1(g)$ is totally geodesic if and only if the following three conditions (A), (B) and (C) hold:

(A) $\cot(\langle\beta, H\rangle)[T_\alpha, S_\beta]^\perp = 0$ for $\alpha, \beta \in \Sigma$ with $\langle\alpha, H\rangle, \langle\beta, H\rangle \notin \pi\boldsymbol{Z}$.
(B) $\tan(\langle\beta, H\rangle)[Y_{\alpha,i}, X_{\beta,j}]^\perp = 0$ for $\alpha, \beta \in W$ with $\langle\alpha, H\rangle, \langle\beta, H\rangle \notin \pi/2 + \pi\boldsymbol{Z}$.
(C) $\tan(\langle\beta, H\rangle)[T_\alpha, X_{\beta,i}]^\perp = 0$ for $\alpha \in \Sigma, \beta \in W$ with $\langle\alpha, H\rangle \notin \pi\boldsymbol{Z}, \langle\beta, H\rangle \notin \pi/2 + \pi\boldsymbol{Z}$.

In this case, set $\beta = \alpha$ in (A), then

$$0 = \cot(\langle\alpha, H\rangle)[T_\alpha, S_\alpha]^\perp = -\cot(\langle\alpha, H\rangle)\bar{\alpha},$$

which implies (1). Conversely (1) implies (A). Set $\beta = \alpha, j = i$ in (B), then, by (2) of Lemma 4.16,

$$0 = \tan(\langle\alpha, H\rangle)[Y_{\alpha,i}, X_{\alpha,i}]^\perp = -\tan(\langle\alpha, H\rangle)\alpha,$$

which implies (2). Conversely (2) implies (B) and (C). □

Corollary 4.25. *$K_2\pi_1(e)$ is a totally geodesic submanifiold in M_1. There exists a totally geodesic submanifold through $\pi_1(e)$ whose tangent space is $T^\perp_{\pi_1(e)}(K_2\pi_1(e))$.*

Proof. By Corollary 4.24, $K_2\pi_1(e)$ is a totally geodesic submanifold in M_1. Since $T^\perp_{\pi_1(e)}(K_2\pi_1(e)) = \mathfrak{m}_2 \cap \mathfrak{m}_1$, this is a Lie triple system in $\mathfrak{m}_1$. Hence the last assertion follows. □

We recall the definition of reflective submanifold given by Leung [**15**]. Let $\tilde{M}$ be a complete Riemannian manifold. A connected component of the fixed point set of an involutive isometry of $\tilde{M}$ is called a *reflective submanifold*.

Remark 4.26. $K_2\pi_1(e)$ is a reflective submanifold of M_1.

Proof. Since $\theta_1\theta_2 = \theta_2\theta_1$, K_1 is θ_2-invariant. Hence θ_2 induces an involutive isometry $\tilde{\theta_2}$ on M_1:

$$\tilde{\theta_2} : M_1 \to M_1; gK_1 \mapsto \theta_2(g)K_1.$$

Then $\tilde{\theta_2}$ is identity on $K_2\pi_1(e)$ and -1 on $T^\perp(\pi_1(K_2)) = \mathfrak{m}_1 \cap \mathfrak{m}_2$. □

A reflective submanifold is totally geodesic ([**12**]). In the case where the orbit of a Hermann action, the converse is true in the following sense.

Proposition 4.27. *If $K_2\pi_1(g) \subset M_1$ is totally geodesic then it is a reflective submanifold.*

Proof. We may assume that $g = \exp H$ for some $H \in \mathfrak{a}$. Define an involutive automorphism θ_2' on G by

$$\theta_2' : G \to G; x \mapsto \exp(-2H)\theta_2(x)\exp(2H),$$

then the fixed point set K_2' of θ_2' is given by

$$K_2' = \exp(-H)K_2 \exp H.$$

Hence $g^{-1}K_2\pi_1(g) = K_2'\pi_1(e)$. Since $K_2\pi(g)$ is totally geodesic, $\langle\lambda, 4H\rangle \in 2\pi\boldsymbol{Z}$ for any $\lambda \in \tilde{\Sigma}$ by Corollary 4.24. Hence $\mathrm{Ad}(\exp 4H) = 1$ by Lemma 4.16. Thus $\exp 4H$ is in the center of G. Hence $\theta_1\theta_2' = \theta_2'\theta_1$, since $\theta_1\theta_2 = \theta_2\theta_1$. By Remark 4.26, $K_2\pi_1(g)$ is a reflective submanifold. □

Corollary 4.28 ([**3**, Theorem 5.3]). *Set $g = \exp H$ for H in $\mathfrak{a}$. For $\xi \in \mathfrak{a}$, the set of eigenvalues of the shape operator $A^{g_*\xi}$ of $K_2\pi_1(g) \subset M_1$ is given by*

$$\begin{aligned}
&\{-\langle\xi,\lambda\rangle\cot(\langle\lambda,H\rangle)\ (\textit{multiplicity} = m(\lambda)) \mid \lambda \in \Sigma^+, \langle\lambda,H\rangle \notin \pi\boldsymbol{Z}\}\\
&\quad\cup\left\{\langle\alpha,\xi\rangle\tan(\langle\alpha,H\rangle)\ (\textit{multiplicity} = n(\alpha)) \mid \alpha \in W^+, \langle\alpha,H\rangle \notin \frac{\pi}{2} + \pi\boldsymbol{Z}\right\}\\
&\quad\cup\{0\ (\textit{multiplicity} = \dim(V(\mathfrak{k}_2 \cap \mathfrak{m}_1)))\}.
\end{aligned}$$

Proof. By Lemma 4.22, we have

$$\begin{aligned}
A^{g_*\xi}g_*T_\alpha &= -\langle\alpha,\xi\rangle\cot(\langle\alpha,H\rangle)g_*T_\alpha,\\
A^{g_*\xi}g_*Y_{\alpha,i} &= \langle\alpha,\xi\rangle\tan(\langle\alpha,H\rangle)g_*Y_{\alpha,i},
\end{aligned}$$

$$A^{g_*\xi} g_* Y = 0 \quad \text{for } Y \in V(\mathfrak{k}_2 \cap \mathfrak{m}_1),$$

which implies the assertion. □

We showed the following in [**11**, p. 459]. Let A be a finite subset of a finite dimensional vector space $\mathfrak{a}$ with an inner product $\langle\ ,\ \rangle$. We consider a condition that, for any $\xi \in \mathfrak{a}$, the set $\{\langle a, \xi\rangle \mid a \in A\}$ with multiplicity is invariant under the multiplication by -1. This condition is equivalent to a condition that A is invariant under the multiplication by -1. By Lemma 4.9, Corollary 4.28 and the mentioned above, we have the following.

Corollary 4.29. *Set $g = \exp H$ for $H \in \mathfrak{a}$, then $K_2\pi_1(g) \subset M_1$ is austere if and only if the finite subset of $\mathfrak{a}$ defined by*

$$\{-\lambda \cot(\langle\lambda, H\rangle)\ (\textit{multiplicity} = m(\lambda)) \mid \lambda \in \Sigma^+, \langle\lambda, H\rangle \notin \pi\boldsymbol{Z}\}$$
$$\cup \left\{\alpha \tan(\langle\alpha, H\rangle)\ (\textit{multiplicity} = n(\alpha)) \mid \alpha \in W^+, \langle\alpha, H\rangle \notin \frac{\pi}{2} + \pi\boldsymbol{Z}\right\}$$

is invariant under the multiplication by -1 with multiplicities.

By Corollaries 4.23, 4.24 and 4.29, we have the following.

Corollary 4.30. *Let g be in G. Then $K_2\pi_1(g) \subset M_1$ is minimal, austere, and totally geodesic if and only if $K_1\pi_2(g^{-1}) \subset M_2$ is minimal, austere, and totally geodesic, respectively.*

Theorem 4.31. *Let (G, K) be a compact symmetric pair. If the orbit Kp in the compact Riemannian symmetric space $M = G/K$ is austere, then it is totally geodesic.*

Proof. Put $\theta_1 = \theta_2$, then we can apply a setup prepared until now. We may assume that $p = \pi(\exp H)$ for some $H \in \mathfrak{a}$. Since Kp in M is austere, the finite subset $\{-\lambda \cot(\langle\lambda, H\rangle) \mid \lambda \in \Sigma^+, \langle\lambda, H\rangle \notin \pi\boldsymbol{Z}\}$ of $\mathfrak{a}$ with multiplicities is invariant under the multiplication by -1 by Corollary 4.29. Hence for any $\lambda \in \Sigma^+$ with $\langle\lambda, H\rangle \notin \pi\boldsymbol{Z}$, there exists $\mu \in \Sigma^+$ with $\langle\mu, H\rangle \notin \pi\boldsymbol{Z}$ such that

$$-\lambda \cot(\langle\lambda, H\rangle) = \mu \cot(\langle\mu, H\rangle).$$

If $\cot(\langle\lambda, H\rangle) \neq 0$ or $\cot(\langle\mu, H\rangle) \neq 0$, then Σ would be of type BC. Moreover $\mu = 2\lambda, m(\lambda) = m(2\lambda)$ or $\lambda = 2\mu, m(\mu) = m(2\mu)$, which would be a contradiction, since $m(2\lambda) < m(\lambda)$ by Lemma 4.32. Hence $\cot(\langle\lambda, H\rangle) = 0$ for any $\lambda \in \Sigma^+$ with

$\langle \lambda, H\rangle \notin \pi \boldsymbol{Z}$. By Corollary 4.24, Kp is totally geodesic. $\square$

LEMMA 4.32. *Let Σ be the restricted root system of a compact symmetric pair (G,K). Assume that there exists $\lambda \in \Sigma$ such that $2\lambda \in \Sigma$, then $m(\lambda) > m(2\lambda)$.*

PROOF. We extend the invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ to a complex symmetric bilinear form on $\mathfrak{g}^{\boldsymbol{C}}$, which is also denoted by $\langle\ ,\ \rangle$. Define a subspace $\mathfrak{g}(\mathfrak{a},\lambda)$ of $\mathfrak{g}^{\boldsymbol{C}}$ by

$$\mathfrak{g}(\mathfrak{a},\lambda) = \left\{X \in \mathfrak{g}^{\boldsymbol{C}} \mid [H,X] = \sqrt{-1}\langle \alpha, H\rangle X \quad (H \in \mathfrak{a})\right\},$$

then $\dim_{\boldsymbol{C}} \mathfrak{g}(\mathfrak{a},\lambda) = m(\lambda)$ since $\mathfrak{g}(\mathfrak{a},\lambda) \oplus \mathfrak{g}(\mathfrak{a},-\lambda) = (\mathfrak{k}_\lambda \oplus \mathfrak{m}_\lambda)^{\boldsymbol{C}}$. Take and fix $X \in \mathfrak{g}(\mathfrak{a},\lambda) - \{0\}$ with $\theta X = \bar{X}$, and define a subspace $\mathfrak{g}(\mathfrak{a},\lambda)'$ of $\mathfrak{g}(\mathfrak{a},\lambda)$ by

$$\mathfrak{g}(\mathfrak{a},\lambda)' = \{Y \in \mathfrak{g}(\mathfrak{a},\lambda) \mid \langle Y, \bar{X}\rangle = 0\}.$$

Then $\dim_{\boldsymbol{C}} \mathfrak{g}(\mathfrak{a},\lambda)' < m(\lambda)$ since $X \notin \mathfrak{g}(\mathfrak{a},\lambda)'$. It is sufficient to show that the linear mapping $\mathrm{ad}X : \mathfrak{g}(\mathfrak{a},\lambda)' \to \mathfrak{g}(\mathfrak{a},2\lambda)$ is surjective. For $Z \in \mathfrak{g}(\mathfrak{a},2\lambda)$, set

$$Y = \frac{-1}{2\|\lambda\|^2\|X\|^2}[\theta X, Z] \in \mathfrak{g}(\mathfrak{a},\lambda) \quad \text{where} \quad \|X\|^2 = \langle X, \bar{X}\rangle.$$

Then $Y \in \mathfrak{g}(\mathfrak{a},\lambda)'$ since $\langle Y, \bar{X}\rangle = \langle Y, \theta X\rangle = 0$. By the Jacobi identity and $3\lambda \notin \Sigma$, we have

$$[X,Y] = \frac{-1}{2\|\lambda\|^2\|X\|^2}[[X,\theta X],Z] = -\frac{-1}{2\|\lambda\|^2}[\lambda, Z] = Z.$$

Hence we get the assertion. $\square$

The totally geodesic submanifolds mentioned in Theorem 4.31 were classified in [**7**].

4.3. When G is simple and $\theta_1\theta_2 = \theta_2\theta_1$:

A main purpose of this subsection is to give a proof of the following theorem.

THEOREM 4.33.

(1) *Let (G,K_1,K_2) be a compact symmetric triad. Assume that G is simple, $\theta_1\theta_2 = \theta_2\theta_1$ and $\theta_1 \not\sim \theta_2$. Denote by $(\tilde{\Sigma},\Sigma,W)$ the triple constructed from (G,K_1,K_2) in the previous subsection, then $(\tilde{\Sigma},\Sigma,W)$ is a symmetric triad of $\mathfrak{a}$. For $\lambda \in \Sigma$ and $\alpha \in W$, set*

$$m(\lambda) = \dim \mathfrak{g}(\mathfrak{a}, \lambda, 1), \quad n(\alpha) = \dim \mathfrak{g}(\mathfrak{a}, \lambda, -1),$$

then $m(\lambda)$ *and* $n(\alpha)$ *satisfy the conditions* (1), (2), (3) *and* (4) *of Definition* 2.13.

(2) *Let* (G, K_1, K_2) *and* (G, K_1', K_2') *be two compact symmetric triads. Assume that* G *is simple,* $\theta_1\theta_2 = \theta_2\theta_1, \theta_1 \not\sim \theta_2, \theta_1'\theta_2' = \theta_2'\theta_1'$ *and* $\theta_1' \not\sim \theta_2'$*. Denote by* $(\tilde{\Sigma}, \Sigma, W)$ *and* $(\tilde{\Sigma}', \Sigma', W')$ *the corresponding symmetric triads. If* $(\theta_1, \theta_2) \sim (\theta_1', \theta_2')$*, then* $(\tilde{\Sigma}, \Sigma, W) \sim (\tilde{\Sigma}', \Sigma', W')$*.*

Lemma 4.34. $\tilde{\Sigma}$ *is an irreducible root system of* $\mathfrak{a}$*.*

Proof. It is already proved that $\tilde{\Sigma}$ is a root system of $\mathfrak{a}$. If $\tilde{\Sigma}$ were to be reducible, there would exist non empty subsets $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ such that

$$\tilde{\Sigma} = \tilde{\Sigma}_1 \cap \tilde{\Sigma}_2 \quad \text{(disjoint union)}, \quad \tilde{\Sigma}_1 \perp \tilde{\Sigma}_2.$$

Denote by $\mathfrak{g}_1^{\boldsymbol{C}}$ the subalgebra of $\mathfrak{g}^{\boldsymbol{C}}$ generated by $\sum_{\alpha \in \tilde{\Sigma}_1} \mathfrak{g}(\mathfrak{a}, \alpha)$ $(\neq \{0\})$. Since

$$\mathfrak{g}_1^{\boldsymbol{C}} \subset \mathfrak{g}(\mathfrak{a}, 0) \oplus \sum_{\alpha \in \tilde{\Sigma}_1} \mathfrak{g}(\mathfrak{a}, \alpha),$$

we would have $\mathfrak{g}_1^{\boldsymbol{C}} \neq \mathfrak{g}^{\boldsymbol{C}}$. Since

$$\begin{aligned}\left[\mathfrak{g}^{\boldsymbol{C}}, \sum_{\alpha \in \tilde{\Sigma}_1} \mathfrak{g}(\mathfrak{a}, \alpha)\right] &= \left[\mathfrak{g}(\mathfrak{a}, 0) \oplus \sum_{\alpha \in \tilde{\Sigma}_1} \mathfrak{g}(\mathfrak{a}, \alpha) \oplus \sum_{\alpha \in \tilde{\Sigma}_2} \mathfrak{g}(\mathfrak{a}, \alpha), \sum_{\alpha \in \tilde{\Sigma}_1} \mathfrak{g}(\mathfrak{a}, \alpha)\right] \\ &\subset \sum_{\alpha \in \tilde{\Sigma}_1} (\mathfrak{g}(\mathfrak{a}, \alpha) + [\mathfrak{g}(\mathfrak{a}, \alpha), \mathfrak{g}(\mathfrak{a}, -\alpha)]) \subset \mathfrak{g}_1^{\boldsymbol{C}},\end{aligned}$$

$\mathfrak{g}_1^{\boldsymbol{C}}$ would be a non trivial ideal of $\mathfrak{g}^{\boldsymbol{C}}$. Since $\mathfrak{g}^{\boldsymbol{C}}$ is simple, this would be a contradiction. Hence $\tilde{\Sigma}$ is an irreducible root system of $\mathfrak{a}$. □

Lemma 4.35. *Let* $\tilde{\Sigma}$ *be an irreducible root system of* $\mathfrak{a}$*. Set* $l = \max\{\|\alpha\| \mid \alpha \in \tilde{\Sigma}\}$*. For any* $\beta \in \tilde{\Sigma}$ *with* $\|\beta\| < l$*, there exists* $\gamma \in \tilde{\Sigma}$ *with* $\|\gamma\| = l$ *such that*

$$-2\frac{\langle \beta, \gamma \rangle}{\|\gamma\|^2} = 1.$$

Proof. First we assume that $\|\beta\| < l$ and $\|\gamma\| = l$. Since the Weyl group maps γ to the highest root, we have, by [**19**],

$$-2\frac{\langle \beta, \gamma \rangle}{\|\gamma\|^2} = \begin{cases} 0 & (\beta \perp \gamma) \\ \pm 2 & (\beta = \mp\gamma) \\ \pm 1 & (\text{otherwise}) \end{cases} \tag{4.10}$$

Since $\operatorname{span}\{\gamma \in \tilde{\Sigma} \mid \|\gamma\| = l\} = \mathfrak{a}$, if it were not to exist such γ satisfying the above condition, then $\beta = 0$, which would be a contradiction. □

It is necessary to recall the finite dimensional complex irreducible representations of $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \boldsymbol{C}) = \mathfrak{su}(2)^{\boldsymbol{C}}$. We choose a basis of $\mathfrak{su}(2)$ as follows:

$$e_1 = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad e_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2}\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

Lemma 4.36 ([**9**, Lemma 5.1]). *Let (ρ, V) be a finite dimensional complex irreducible representation of $\mathfrak{su}(2)$, then there exists a basis $\{f_k\}_{0 \le k \le n}$ of V such that*

$$\begin{aligned}
\rho(e_3) f_k &= \frac{\sqrt{-1}}{2}(-n + 2k) f_k, \\
\rho(e_1) f_k &= \frac{\sqrt{-1}}{2}\big\{\sqrt{(n-k)(k+1)}\, f_{k+1} + \sqrt{k(n-k+1)}\, f_{k-1}\big\}, \\
\rho(e_2) f_k &= \frac{1}{2}\big\{-\sqrt{(n-k)(k+1)}\, f_{k+1} + \sqrt{k(n-k+1)}\, f_{k-1}\big\}.
\end{aligned}$$

In order to state the finite dimensional complex irreducible representations of $\mathfrak{sl}(2, \boldsymbol{C})$, set

$$X = e_1 - \sqrt{-1} e_2 = \sqrt{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \boldsymbol{C})$$

and denote by $^-$ the complex conjugation of $\mathfrak{sl}(2, \boldsymbol{C})$ with respect to the compact real form $\mathfrak{su}(2)$, then

$$\bar{X} = \sqrt{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = e_1 + \sqrt{-1} e_2.$$

Put

$$H = [X, \bar{X}] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = 2\sqrt{-1} e_3,$$

then $\{X, \bar{X}, H\}$ is a basis of $\mathfrak{sl}(2, \boldsymbol{C})$. Lemma 4.37 immediately follows from the lemma above.

Lemma 4.37. *Let (ρ, V) be a finite dimensional complex irreducible representation of $\mathfrak{sl}(2, \boldsymbol{C})$, then there exists a basis $\{f_k\}_{0 \le k \le n}$ of V such that*

$$\begin{aligned} \rho(X) f_k &= \sqrt{-1}\sqrt{(n-k)(k+1)} f_{k+1}, \\ \rho(\bar{X}) f_k &= \sqrt{-1}\sqrt{k(n-k+1)} f_{k-1}, \\ \rho(H) f_k &= (n-2k) f_k \end{aligned}$$

Lemma 4.38 (Matsuki). *Take $X \in \mathfrak{g}(\mathfrak{a}, \alpha, \epsilon) - \{0\}$ such that $\theta_1 X = \bar{X}$, $\epsilon = \pm 1$.*

(1) $\mathfrak{l} = \boldsymbol{C}X \oplus \boldsymbol{C}\bar{X} \oplus \boldsymbol{C}[X, \bar{X}] \cong \mathfrak{sl}(2, \boldsymbol{C})$ *as Lie algebras.*
(2) *When $\langle \alpha, \beta \rangle < 0$, then $\|\beta\| \ge \|\alpha + \beta\|$ and the mapping*

$$(\mathrm{ad}X)^m : \mathfrak{g}(\mathfrak{a}, \beta) \to \mathfrak{g}(\mathfrak{a}, s_\alpha \beta)$$

is a linear isomorphism, where we set $m = -2\langle \alpha, \beta \rangle / \|\alpha\|^2 \in \boldsymbol{N}$. In particular the linear mapping $\mathrm{ad}X : \mathfrak{g}(\mathfrak{a}, \beta) \to \mathfrak{g}(\mathfrak{a}, \beta + \alpha)$ is injective.
(3) *When $\Sigma \cap W = \emptyset, \alpha, \beta, \alpha + \beta \in \tilde{\Sigma}$, then*

$$\mathfrak{g}(\mathfrak{a}, \alpha + \beta) = \mathfrak{g}(\mathfrak{a}, \alpha + \beta, \epsilon_1 \epsilon_2),$$

where we set $\mathfrak{g}(\mathfrak{a}, \alpha) = \mathfrak{g}(\mathfrak{a}, \alpha, \epsilon_1)$, $\mathfrak{g}(\mathfrak{a}, \beta) = \mathfrak{g}(\mathfrak{a}, \beta, \epsilon_2)$.

Proof. For (1), we refer to [**16**, p. 61].

(2) Since $\|\beta\|^2 - \|\alpha + \beta\|^2 = \|\alpha\|^2(m-1) \ge 0$, we have $\|\beta\| \ge \|\alpha + \beta\|$. Denote by $\beta + n\alpha$ $(p \le n \le q)$ the α-series containing β, then $p + q = m$ and

$$\langle \alpha, \beta + n\alpha \rangle = -\frac{1}{2}\|\alpha\|^2(p + q - 2n) \quad (p \le n \le q).$$

Taking this into account, we decompose

$$\oplus_{n \in \boldsymbol{Z}} \mathfrak{g}(\mathfrak{a}, \beta + n\alpha) = \oplus_{n=p}^{q} \mathfrak{g}(\mathfrak{a}, \beta + n\alpha)$$

into $\mathfrak{l}$-irreducible representations. Then Lemma 4.37 implies the assertion.

(3) When $\langle \alpha, \beta \rangle < 0$, then we have, by (2),

$$0 \neq [X, \mathfrak{g}(\mathfrak{a}, \beta, \epsilon_1)] \subset \mathfrak{g}(\mathfrak{a}, \alpha + \beta, \epsilon_1\epsilon_2).$$

Since $\Sigma \cap W = \emptyset$, we have

$$\mathfrak{g}(\mathfrak{a}, \alpha + \beta) = \mathfrak{g}(\mathfrak{a}, \alpha + \beta, \epsilon_1\epsilon_2).$$

When $\langle \alpha, \beta \rangle \geq 0$, then

$$\langle \alpha, -(\alpha + \beta) \rangle = -\|\alpha\|^2 - \langle \alpha, \beta \rangle \leq -\|\alpha\|^2 < 0.$$

Since

$$\mathfrak{g}(\mathfrak{a}, -(\alpha + \beta)) = \mathfrak{g}(\mathfrak{a}, -(\alpha + \beta), \mu) \ (\mu = \pm 1),$$

we have, by (2),

$$0 \neq [X, \mathfrak{g}(\mathfrak{a}, -(\alpha + \beta), \mu)] \subset \mathfrak{g}(\mathfrak{a}, -\beta, \epsilon_1\mu).$$

Hence

$$\mathfrak{g}(\mathfrak{a}, -\beta) = \mathfrak{g}(\mathfrak{a}, -\beta, \epsilon_2) = \mathfrak{g}(\mathfrak{a}, -\beta, \epsilon_1\mu).$$

Thus $\mu = \epsilon_1\epsilon_2$, which implies that

$$\mathfrak{g}(\mathfrak{a}, \alpha + \beta) = \mathfrak{g}(\mathfrak{a}, \alpha + \beta, \epsilon_1\epsilon_2). \qquad \square$$

PROPOSITION 4.39 (Matsuki). *The following four conditions are equivalent.*

(1) *The involutive automorphisms θ_1 and θ_2 of $\mathfrak{g}$ cannot transform each other by any inner automorphism of $\mathfrak{g}$.*
(2) $\Sigma \cap W \neq \emptyset$.
(3) $\Sigma \cap W \cap \tilde{\Pi} \neq \emptyset$.
(4) *Set $l = \max\{\|\alpha\| \mid \alpha \in \Sigma \cap W\}$, then $\Sigma \cap W = \{\beta \in \tilde{\Sigma} \mid \|\beta\| \leq l\}$.*

PROOF. First we shall prove that the negative of (1) implies the negative of (2). Assume that θ_1 and θ_2 transforms each other by an inner automorphism of $\mathfrak{g}$. Since $G = K_1AK_2$, there exists $a \in A$ such that $\theta_1 = \mathrm{Ad}(a)\theta_2\mathrm{Ad}(a^{-1})$. Then, for any $x \in G$, we have $\theta_2\theta_1(x) = a^{-2}xa^2$ and $\theta_1\theta_2(x) = a^2xa^{-2}$. Since $\theta_1\theta_2 = \theta_2\theta_1$, a^4 is in $Z(G)$, the center of G. Select $H \in \mathfrak{a}$ such that $a = \exp H$, then $4\langle \alpha, H \rangle \in 2\pi\boldsymbol{Z}$ for any $\alpha \in \tilde{\Sigma}$. Hence, for $\alpha \in \tilde{\Sigma}$ and $X \in \mathfrak{g}(\mathfrak{a}, \alpha)$, we have

$$\theta_1\theta_2(X) = \mathrm{Ad}(a^2)X = e^{2\mathrm{ad}H}X = e^{2\sqrt{-1}\langle\alpha,H\rangle}X = \pm X,$$

which implies that $\Sigma \cap W = \emptyset$.

Second we shall prove that the negative of (2) implies the negative of (1). Assume that $\Sigma \cap W = \emptyset$, then, for $\alpha \in \tilde{\Sigma}$, there exists $\epsilon_\alpha = \pm 1$ such that $\mathfrak{g}(\mathfrak{a}, \alpha) = \mathfrak{g}(\mathfrak{a}, \alpha, \epsilon_\alpha)$. Hence $\theta_1\theta_2 = \epsilon_\alpha$ on $\mathfrak{g}(\mathfrak{a}, \alpha)$. Select $H \in \mathfrak{a}$ as follows: For $\alpha \in \tilde{\Pi}$,

$$2\langle\alpha, H\rangle = \begin{cases} 0 & (\epsilon_\alpha = 1), \\ \pi & (\epsilon_\alpha = -1). \end{cases}$$

Set $a = \exp H$, then for any $X \in \mathfrak{g}(\mathfrak{a}, \alpha)$ with $\alpha \in \tilde{\Pi}$, we have $\mathrm{Ad}(a^2)X = \epsilon_\alpha X$. Hence, by Lemma 4.38, $\theta_1\theta_2 = \mathrm{Ad}(a^2)$ on $\sum_{\alpha\in\tilde{\Sigma}} \mathfrak{g}(\mathfrak{a}, \alpha)$. The subalgebra generated by $\sum_{\alpha\in\tilde{\Sigma}} \mathfrak{g}(\alpha) (\neq \{0\})$ is an ideal of $\mathfrak{g}^{\boldsymbol{C}}$. Hence it coincides with $\mathfrak{g}^{\boldsymbol{C}}$ since $\mathfrak{g}^{\boldsymbol{C}}$ is simple. Hence $\theta_1\theta_2 = \mathrm{Ad}(a^2)$, which implies that $\theta_1 = \mathrm{Ad}(a)\theta_2\mathrm{Ad}(a^{-1})$.

Hence the conditions (1) and (2) are equivalent. It is sufficient to prove that (2) $\Rightarrow$ (3) $\Rightarrow$ (4) since (4) $\Rightarrow$ (2) is trivial. We shall prove that (2) implies (3). Let α be in $\Sigma^+ \cap W^+$. We will show that there exists $\beta \in \tilde{\Pi}$ such that $\langle\alpha, \beta\rangle > 0$ when $\alpha \notin \tilde{\Pi}$. We were to assume that $\langle\alpha, \beta\rangle \leq 0$ for any $\beta \in \tilde{\Pi}$. Express α as $\alpha = \sum_{\beta\in\tilde{\Pi}} m_\beta \beta$ $(m_\beta \geq 0)$, then

$$\|\alpha\|^2 = \sum_{\beta\in\tilde{\Pi}} m_\beta \langle\alpha, \beta\rangle \leq 0.$$

Hence we would have $\alpha = 0$, which would be a contradiction. Thus when $\alpha \notin \tilde{\Pi}$, there exists $\beta \in \tilde{\Pi}$ such that $\langle\alpha, \beta\rangle > 0$. Then $\alpha - \beta \in \tilde{\Sigma}^+$. We will show that $\alpha - \beta \in \Sigma^+ \cap W^+$. Since $\alpha \in \Sigma^+ \cap W^+$, we have $\mathfrak{g}(\mathfrak{a}, \alpha, \pm 1) \neq \{0\}$. We can take $X \in \mathfrak{g}(\mathfrak{a}, -\beta, 1) - \{0\}$ such that $\theta_1 X = \bar{X}$ by Lemma 4.11. Since $\mathrm{ad}X : \mathfrak{g}(\mathfrak{a}, \alpha, \pm 1) \to \mathfrak{g}(\mathfrak{a}, \alpha - \beta, \pm 1)$ is injective by (2) of Lemma 4.38, $\mathfrak{g}(\mathfrak{a}, \alpha - \beta, \pm 1) \neq \{0\}$. Hence $\alpha - \beta \in \Sigma^+ \cap W^+$. By iteration, we have $\Sigma^+ \cap W^+ \cap \tilde{\Pi} \neq \emptyset$.

Last we will show (3) $\Rightarrow$ (4). When the lengths of elements of $\tilde{\Sigma}$ are a constant, then $\tilde{\Sigma} = \Sigma$ and $W(\Sigma)$ acts transitively on W. Hence $\tilde{\Sigma} = \Sigma = W$. Thus $\Sigma \cap W = \tilde{\Sigma} = \{\beta \in \tilde{\Sigma} \mid \|\beta\| \leq l\}$. Hence we assume that the lengths of elements of Σ are not a constant. Take $\beta \in \tilde{\Sigma}$ such that $\|\beta\| = l$. We will show that $\beta \in \Sigma \cap W$. By the definition of l there exists $\alpha \in \Sigma \cap W$ such that $\|\beta\| = \|\alpha\|$. Since $\tilde{\Sigma}$ is an irreducible root system of $\mathfrak{a}$ by Lemma 4.34, there exists $s \in W(\tilde{\Sigma})$ such that $\beta = s\alpha$. Since $\{s_\gamma \mid \gamma \in \tilde{\Sigma}\}$ generates $W(\tilde{\Sigma})$, it is sufficient to prove $\beta \in \Sigma \cap W$ when $\beta = s_\gamma\alpha$. We may assume that $\langle\alpha, \gamma\rangle \neq 0$, and γ is not proportional to α. We may assume that $\langle\alpha, \gamma\rangle < 0$ since $s_{-\gamma} = s_\gamma$. Take $X \in \mathfrak{g}(\mathfrak{a}, \gamma, \epsilon) - \{0\}$ with $\theta_1 X = \bar{X}, \epsilon = \pm 1$ then

$$(\mathrm{ad}X)^m : \mathfrak{g}(\mathfrak{a}, \alpha) \to \mathfrak{g}(\mathfrak{a}, \beta)$$

is a linear isomorphism by (2) of Lemma 4.38. Since $\alpha \in W \cap \Sigma$ we have $\beta \in \Sigma \cap W$. Thus $\{\beta \in \tilde{\Sigma} \mid \|\beta\| = l\} \subset \Sigma \cap W$. When $l = \min\{\|\alpha\| \mid \alpha \in \tilde{\Sigma}\}$, then $\Sigma \cap W = \{\beta \in \tilde{\Sigma} \mid \|\beta\| \leq l\}$. Hence we may assume that $l > \min\{\|\alpha\| \mid \alpha \in \tilde{\Sigma}\}$. We consider the case where $\tilde{\Sigma}$ is not of type BC_r. Then $l = \max\{\|\alpha\| \mid \alpha \in \tilde{\Sigma}\}$. Take γ in $\tilde{\Sigma}$ such that $\|\gamma\| < l$. We will show that γ is in $\Sigma \cap W$. By Lemma 4.35 there exists β in $\tilde{\Sigma}$ with $\|\beta\| = l$ such that $-2\langle\beta, \gamma\rangle/\|\beta\|^2 = 1$. Then $\beta + \gamma$ is in $\tilde{\Sigma}$. Since $-2(\langle-\beta, \beta + \gamma\rangle/\| - \beta\|^2) = 1$ we have

$$\begin{aligned}
\dim \mathfrak{g}(\mathfrak{a}, \gamma, 1) &= \dim \mathfrak{g}(\mathfrak{a}, \beta + \gamma, 1) \quad (0 \neq X \in \mathfrak{g}(\mathfrak{a}, \beta, 1)) \\
&= \dim \mathfrak{g}(\mathfrak{a}, \gamma, -1) \qquad (0 \neq X' \in \mathfrak{g}(\mathfrak{a}, -\beta, -1)),
\end{aligned}$$

which implies that $\gamma \in \Sigma \cap W$. When $\tilde{\Sigma}$ is of type BC_r, then the assertion reduces to the case when Σ is of type B_r. □

Proof of (1) of Theorem 4.33. We have already proved the condition (1) of Definition 2.2 by Lemma 4.34. We shall prove (2) of Definition 2.2. By (4) of Proposition 4.39, we have $\mathrm{span}(\Sigma \cap W) = \mathfrak{a}$. Hence $\mathrm{span}(\Sigma) = \mathfrak{a}$ and Σ is a root system of $\mathfrak{a}$. We have already proved (3) of Definition 2.2. The condition (4) of Definition 2.2 follows from (4) of Proposition 4.39. The conditions (1) and (2) of Definition 2.13 was proved in Lemmas 4.5 and 4.15. The condition (3) of Definition 2.13 was proved in Lemma 4.20. We shall prove (5) and (6) of Definition 2.2 and (4) of Definition 2.13. Let $\alpha \in W$ and $\lambda \in \tilde{\Sigma}$, then set $m = -2\langle\alpha, \lambda\rangle/\|\alpha\|^2$. We can take $X \in \mathfrak{g}(\mathfrak{a}, \alpha, -1) - \{0\}$ such that $\theta_1 X = \bar{X}$. In order to prove the assertion, we may assume that $\langle\alpha, \lambda\rangle < 0$. By Lemma 4.38,

$$(\mathrm{ad}X)^m : \mathfrak{g}(\mathfrak{a}, \lambda) \to \mathfrak{g}(\mathfrak{a}, s_\alpha\lambda)$$

is a linear isomorphism. Let λ be in $(\Sigma - W) \cup (W - \Sigma)$. When m is even, then

$$\begin{aligned}
(\mathrm{ad}X)^m &: \mathfrak{g}(\mathfrak{a}, \lambda, 1) \to \mathfrak{g}(\mathfrak{a}, s_\alpha\lambda, 1) \qquad (\text{if } \lambda \in \Sigma - W), \\
(\mathrm{ad}X)^m &: \mathfrak{g}(\mathfrak{a}, \lambda, -1) \to \mathfrak{g}(\mathfrak{a}, s_\alpha\lambda, -1) \quad (\text{if } \lambda \in W - \Sigma)
\end{aligned}$$

is a linear isomorphism. When m is odd, then

$$\begin{aligned}
(\mathrm{ad}X)^m &: \mathfrak{g}(\mathfrak{a}, \lambda, 1) \to \mathfrak{g}(\mathfrak{a}, s_\alpha\lambda, -1) \quad (\text{if } \lambda \in \Sigma - W), \\
(\mathrm{ad}X)^m &: \mathfrak{g}(\mathfrak{a}, \lambda, -1) \to \mathfrak{g}(\mathfrak{a}, s_\alpha\lambda, 1) \qquad (\text{if } \lambda \in W - \Sigma)
\end{aligned}$$

is a linear isomorphism. Hence (5) and (6) of Definition 2.2 hold. Thus $(\tilde{\Sigma}, \Sigma, W)$ is a symmetric triad of $\mathfrak{a}$. We shall show (4) of Definition 2.13. When m is even, then

$$(\mathrm{ad} X)^m : \mathfrak{g}(\mathfrak{a}, \lambda, 1) \to \mathfrak{g}(\mathfrak{a}, s_\alpha \lambda, 1)$$

is a linear isomorphism. When m is odd, then

$$(\mathrm{ad} X)^m : \mathfrak{g}(\mathfrak{a}, \lambda, 1) \to \mathfrak{g}(\mathfrak{a}, s_\alpha \lambda, -1)$$

is a linear isomorphism. Hence we get the assertion. □

Proof of (2) of Theorem 4.33. Since $(\theta_1, \theta_2) \sim (\theta_1', \theta_2')$, there exist $x \in G$ and $\rho \in \mathrm{Aut}(G)$ such that

$$\theta_1' = \tau_x \rho \theta_1 \rho^{-1} \tau_x^{-1}, \quad \theta_2' = \rho \theta_2 \rho^{-1}.$$

Then

$$\theta_1' = \rho(\rho^{-1} \tau_x \rho) \theta_1 (\rho^{-1} \tau_x^{-1} \rho) \rho^{-1}.$$

Since $\rho^{-1}\tau_x\rho$ is an inner automorphism of G, there exists $y \in G$ such that $\rho^{-1}\tau_x\rho = \tau_y$. Then $\theta_1' = \rho\tau_y\theta_1\tau_y^{-1}\rho^{-1}$. Since θ_1' and θ_2' commute each other, $\tau_y\theta_1\tau_y^{-1}$ and θ_2 commute each other. In order to give the proof, we may assume that ρ is the identity transformation. Since $G = K_2(\exp \mathfrak{a})K_1$, there exist $k_i \in K_i$ and $Y \in \mathfrak{a}$ such that $y = k_2 \exp H k_1$. Since $\tau_{k_i}\theta_i\tau_{k_i}^{-1} = \theta_i$, we have

$$\theta_1' = \tau_{k_2}\tau_{\exp Y}\theta_1\tau_{\exp Y}^{-1}\tau_{k_2}^{-1}, \quad \theta_2' = \tau_{k_2}\theta_2\tau_{k_2}^{-1}.$$

Since $\theta_1\theta_2 = \theta_2\theta_1$ and $\theta_1'\theta_2' = \theta_2'\theta_1'$, $\exp 4Y$ is in the center of G. Thus $\mathrm{Ad}(\exp 4Y) = 1$, which implies that $Y \in \Gamma$. To complete the proof, we may assume that k_2 is the identity element. Define a subspace $\mathfrak{m}_1'$ of $\mathfrak{g}$ by

$$\mathfrak{m}_1' = \{X \in \mathfrak{g} \mid \theta_1'(X) = -X\},$$

then $\mathfrak{m}_1' = \mathrm{Ad}(\exp Y)\mathfrak{m}_1$. Hence $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{m}_1' \cap \mathfrak{m}_2$. Hence $\tilde{\Sigma}' = \tilde{\Sigma}$. For any $\alpha \in \tilde{\Sigma}$ and $X \in \mathfrak{g}(\mathfrak{a}, \alpha, \epsilon)$,

$$\theta_1'\theta_2'X = \mathrm{Ad}(\exp 2Y)\theta_1\theta_2 X = \epsilon e^{2\sqrt{-1}\langle \alpha, Y\rangle} X (= \pm\epsilon X).$$

Hence (2.1) holds when we put $f = 1$. We have $(\tilde{\Sigma}, \Sigma, W) \sim (\tilde{\Sigma}', \Sigma', W')$ and the proof is completed. $\square$

Let $H \in \mathfrak{a}$. Then $K_2\pi_1(\exp H)$ is a regular orbit, totally geodesic orbit, austere orbit, and minimal orbit if and only if H is a regular point, totally geodesic point, austere point and minimal point, respectively. By Theorem 2.24, for any $\Delta \subset \Pi \cup \{\tilde{\alpha}\}$, there exists a unique element $H \in P_0^{\Delta}$ such that $K_2\pi_1(\exp H)$ is a minimal orbit in M_1. Using the results in Section 3, we can classify the totally geodesic K_2-orbits and the austere K_2-orbits in M_1.

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Osamu IKAWA
Department of General Education
Fukushima National College of Technology
Iwaki
Fukushima 970-8034, Japan
E-mail: ikawa@fukushima-nct.ac.jp