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Width of shape resonances for non globally analytic potentials

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Abstract. We consider the semiclassical Schrödinger operator with a *well in an island* potential, on which we assume smoothness only, except near infinity. We give the asymptotic expansion of the imaginary part of the shape resonance at the bottom of the well. This is a generalization of the result by Helffer and Sjöstrand [**HeSj1**] in the globally analytic case. We use an almost-analytic extension in order to continue the WKB solution coming from the well beyond the caustic set, and, for the justification of the accuracy of this approximation, we develop some refined microlocal arguments in h-dependent small regions.

1. Introduction.

This paper is concerned with the quantum resonances of the semiclassical Schrödinger operator in $\boldsymbol{R}^n$,

$$P = -h^2\Delta + V(x). \tag{1.1}$$

From the physical point of view, such resonances are associated to metastable states, that is, states with a finite life-time, and the life-time is given by the inverse of the absolute value of the imaginary part (width) of the corresponding resonance.

In the literature, many geometrical situations have been studied where the location of the resonances of P has been determined up to errors of order $\mathscr{O}(h^\infty)$: see, e.g., [**BaMn**], [**BCD**], [**GeSj**], [**HiSi**], [**Sj2**]. In particular, when the approximated location is at a distance $\sim h^{N_0}$ of the real line with some fixed $N_0 > 0$,

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then such results automatically produce good estimates on the widths of such resonances, too. However, in the physically most interesting situations where the resonances are exponentially close to the real axis, only very few results give a lower bound on the width. In [**Bu**], a very nice and general estimate is obtained, extending the results by [**Ha**], [**FeLa**], [**Vo**], independently of the geometrical situation presented by the potential V. In particular, because of its wide generality, such a result does not give any precise asymptotic of the width. On the contrary, the result of [**HeSj1**] gives a full asymptotic of the width when the (globally analytic) potential V presents the geometric situation of a *well in an island*, at the origin of the creation of the so-called *shape resonances*.

The purpose of this article is to extend the results of [**HeSj1**] to smooth potentials that are not assumed to be globally analytic, but only near infinity (allowing the definition of resonances by analytic distortion). In that case, we obtain a classical expansion of the first resonance ρ just as in the globally analytic case of [**HeSj1**].

Roughly speaking, a quantum resonance is a complex energy ρ for which the Schrödinger equation

$$Pu = \rho\, u$$

admits a non-trivial solution called *resonant state*, which is *outgoing* for large x (and thus not L^2 in general: see Section 2 for a rigorous definition by analytic distortion). In the case of shape resonances, the resonant state describes a quantum particle, concentrated in the well for a long period, but then escaping to the *sea* (classically allowed region outside the island) by tunneling effect. This effect is reflected by the width of the resonance ρ.

More precisely, let Γ be the set of points on the boundary of the island $\ddot{O}$ where the Agmon distance from the bottom of the well x_0 reaches its minimum S. In the semiclassical regime, it happens that almost all the tunneling occurs along a small neighborhood of the geodesic curves from x_0 to Γ. Then, the width of the resonance ρ is determined by the amplitude of u near Γ, and the result is,

$$\operatorname{Im} \rho(h) \sim -h^{(1-n_\Gamma)/2} \Big(\sum_{j\geq 0} h^j f_j \Big) e^{-2S/h} \mod \mathscr{O}(h^\infty) e^{-2S/h},$$

where n_Γ is some geometrical constant, and $f_0 > 0$ (see Theorem 2.3 below, and [**HeSj1**, Theorem 10.12]).

In order to prove this result for non globally analytic potentials, we mainly follow the strategy of [**HeSj1**], with some additional technical difficulties that we explain now.

Let W be a bounded domain and $(\cdot,\cdot)_W$, $\|\cdot\|_W$ the scalar product and the norm in $L^2(W)$. Then, $\operatorname{Im}\rho$ can be represented in terms of the corresponding resonant state u on W, by applying Green's formula to the identity $((P-\rho)u,u)_W = 0$. This leads to,

$$\operatorname{Im}\rho = -\frac{h^2}{\|u\|_W^2}\operatorname{Im}\int_{\partial W}\frac{\partial u}{\partial n}\bar{u}dS, \tag{1.2}$$

where $\partial/\partial n$ denote the exterior normal derivative on ∂W. The point is that if we take for W a small neighborhood of $\ddot{O}$, then the contribution from the integral of the RHS of (1.2) is concentrated near Γ, and the problem is mainly reduced to the study of the asymptotic behavior of u near Γ.

In the globally analytic case studied by Helffer and Sjöstrand in [**HeSj1**], the first step in order to obtain this asymptotic behavior consisted in extending to the complex domain the WKB solutions coming from the well, in such a way that one could go round the caustic set (see [**HeSj1**, Proposition 10.9]). In our case, such an analytic extension is no longer possible but, by means of almost-analytic extensions, it is still possible to go round the caustic set on the condition of staying close enough to the real domain, namely, at a distance $\mathscr{O}((h\ln h^{-1})^{2/3})$ of the caustic set. In this way, one can recover an (outgoing) WKB expression out of the island, but still at a distance of order $(h\ln h^{-1})^{2/3}$ from the boundary (see Proposition 4.6).

The next step in the proof of [**HeSj1**] was to use an argument of microlocal analytic propagation that, thanks to a suitable a priori estimate, permitted to compare the solution u with the previous WKB constructions near Γ. In our case, the analogous a priori estimate can be obtained only at a distance $\mathscr{O}((h\ln h^{-1})^{2/3})$ of Γ (see Proposition 5.2). For this reason, the usual results of propagation do not apply, and we need to refine them in order to be able to work in h-dependent open sets (see Lemma 6.4). Then, after a suitable change of scale, and still using almost-analytic extensions, the comparison between u and the WKB constructions is obtained up to a distance of order $(h\ln h^{-1})^{2/3}$ of Γ (see Proposition 6.8, and Proposition 6.1).

The final step in [**HeSj1**] consisted in replacing u by its WKB approximation into (1.2) (with W a fixed small enough neighborhood of $\ddot{O}$), and in using a stationary phase argument in order to obtain the asymptotic of $\operatorname{Im}\rho$. In our case, a similar argument can be used with $W=\{d(x,\ddot{O})<|h\ln h|^{2/3}\}$, but this makes appear terms in $h^j(\ln h)^k$ in the integrand of the right-hand side of (1.2). However, by slightly changing the choice of W, and by observing that the left-hand side of (1.2) does not depend on this choice, we can prove that, indeed, the final expansion does not contain terms in $(\ln h)^k$, $k\neq 0$ (see (7.2) and Lemma 7.3).

Summing up, the method consists in the three following parts:

(1) Extension of the WKB solution constructed at the bottom of the well x_0 up to a neighborhood of Γ.
(2) Estimate on the difference between the WKB solution w and the resonant state u.
(3) Computation of the asymptotics of the integral (1.2) using w instead of u.

Point 1 is performed by solving transport equations along the minimal geodesics, and then, by using an Airy integral representation of the solution near Γ. In order to recover a WKB asymptotic expansion outside $\ddot{O}$, almost-analytic extensions (or, more precisely, holomorphic approximations) are used as well as a stationary-phase expansion at a distance of order $|h\ln h|^{2/3}$ from Γ.

Point 2 is obtained by using three propagation arguments. At first, a standard C^∞-propagation, exploiting the fact that u is outgoing, gives a microlocal information on u in the incoming region up to a small distance of Γ. Then, a refined C^∞ propagation result permits to extend this microlocal information up to a distance of order $|h\ln h|^{2/3}$ of Γ. Finally, performing a change of scale both in the variables and in the semiclassical parameter, an analytic-type propagation argument gives the required estimate in a full neighborhood of Γ of size $|h\ln h|^{2/3}$.

For Point 3, we take $W=\{d(x,\ddot{O})<|h\ln h|^{2/3}\}$ and we use Green's formula as we explained before. Then, by a deformation argument, we show that the final expansion does not contain terms in $\ln h$, and actually coincides with the expansion obtained in [**HeSj1**].

The article is organized as follows: In Section 2, we assume conditions on the potential V and state the main results. In Section 3, we prove Theorem 2.2, especially the global a priori estimate (2.7) of u. Section 4 is devoted to the above point 1, and Section 5, Section 6 and Section 7 are devoted to the point 2. In Section 8, we study the point 3, and prove the main Theorem 2.3.

We also refer to [**FLM**] for a shorter version of some aspects of this work.

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2. Assumptions and main results.

Let us state precisely the assumptions on the potential.

(A1) $V\in C^\infty$ is real-valued and there exists some compact set $K_0\subset \boldsymbol{R}^n$ such that V is analytic on $K_0^C=\boldsymbol{R}^n\backslash K_0$ and can be extended as a holomorphic function in a sector

$$D_0=\left\{x\in \boldsymbol{C}^n;\, |\operatorname{Im} x|<\sigma_0|\operatorname{Re} x|,\ \operatorname{Re} x\in K_0^C\right\}$$

for some constant $\sigma_0 > 0$. Moreover,

$$V(x) \to 0 \quad \text{as} \quad |\operatorname{Re} x| \to \infty, \quad x \in D.$$

This assumption enables us to define resonances near the real axis as complex eigenvalues of the distorted operator P_θ of P. Let $F(x) \in C^\infty(\boldsymbol{R}^n, \boldsymbol{R}^n)$ such that $F(x) = 0$ on the compact set K_0, $F(x) = x$ for large enough $|x|$, and $|F(x)| \le |x|$ everywhere. For small $\nu > 0$, we define a unitary operator on $L^2(\boldsymbol{R}^n)$ by

$$U_\nu \phi(x) = \det(1 + \nu dF(x))^{1/2} \phi(x + \nu F(x)). \tag{2.1}$$

The conjugate operator $\tilde{P}_\nu = U_\nu P U_{-\nu}$ is analytic with respect to ν and we can define the distorted operator by $P_\theta = \tilde{P}_{i\theta}$ for positive small θ. Then the essential spectrum is given by $\sigma_{ess}(P_\theta) = e^{-2i\theta}\boldsymbol{R}_+$ (Weyl's perturbation theorem) and the spectrum $\sigma(P_\theta)$ in the sector $S_\theta = \{E \in \boldsymbol{C}; -2\theta < \arg E < 0\}$ is discrete (see [**Hu**]). The elements of $\sigma(P_\theta) \cap S_\theta$ are called *resonances*. This definition is independent of θ in the sense that $\sigma(P_{\theta'}) \cap S_\theta = \sigma(P_\theta) \cap S_\theta$ if $\theta' > \theta$, and also of the function $F(x)$. Moreover, if u_θ is an eigenfunction of P_θ, it can be proved (see, e.g., [**HeMa**]) that there exists $u \in C^\infty(\boldsymbol{R}^n)$, holomorphic in D_0, such that $u_\theta = U_{i\theta} u$. Such functions u are called *resonant states* of P.

The next assumption describes the shape of $V(x)$ in the *island*:

(A2) There exist a bounded open domain $\ddot{O}$ with smooth boundary, a point x_0 in $\ddot{O}$ and a positive number E_0 such that

$$V(x_0) = E_0, \quad \frac{\partial V}{\partial x}(x_0) = 0, \quad \frac{\partial^2 V}{\partial x^2}(x_0) > 0,$$

and

$$V(x) > E_0 \quad \text{in} \quad \ddot{O} \backslash \{x_0\}, \qquad V(x) = E_0 \quad \text{on} \quad \partial \ddot{O}.$$

To the *well* $\{x_0\}$ of the potential, we can associate a Dirichlet problem. Let us denote by $d(x,y)$ the Agmon distance associated with the pseudo-metric $ds^2 = \max(V(x),0)dx^2$, $S = d(x_0, \partial\ddot{O})$ the minimal distance from x_0 to the boundary of $\ddot{O}$, and $B_d(x_0,S) := \{x\,;\, d(x,x_0) < S\}$ the open ball centered at x_0 of radius S with respect to the distance d. We consider a Dirichlet realization P_D of the operator P on the domain $\overline{B_d(x_0, S-\eta)}$ for sufficiently small η. The following result is due to Helffer and Sjöstrand [**HeSj2**] (see also Simon [**Si**] for a partial version):

Theorem 2.1 (Helffer-Sjöstrand). *Let* $\lambda_D(h)$ *be the first eigenvalue of* P_D *and* $u_D(x,h)$ *the corresponding normalized eigenfunction. Then* $\lambda_D(h)$ *has a complete classical asymptotic expansion with respect to* h:

$$\lambda_D(h) = E_0 + E_1 h + E_2 h^2 + \cdots ,$$

where $E_1 = \operatorname{tr}\sqrt{(1/2)(\partial^2 V/\partial x^2)(x_0)}$ *is the first eigenvalue of the corresponding harmonic oscillator* $-\Delta + (1/2)\langle(\partial^2 V/\partial x^2)(x_0)x, x\rangle$. *Moreover, in a neighborhood* ω *of* x_0, $u_D(x,h)$ *can be written in the WKB form*:

$$u_D(x,h) = h^{-n/4} a(x,h) e^{-d(x_0,x)/h}, \tag{2.2}$$

where a *is a realization of a classical symbol*:

$$a(x,h) \sim \sum_{j=0}^{\infty} a_j(x) h^j, \quad a_0(0) > 0. \tag{2.3}$$

Above the *sea* $\ddot{O}^C$, on the other hand, we assume that p has no *trapped trajectories* of energy E_0, in the sense that,

(A3) For any $(x,\xi) \in p^{-1}(E_0)$ with $x \in \ddot{O}^C$, the quantity $|\exp tH_p(x,\xi)|$ tends to infinity as $|t|$ tends to ∞.

Here H_p is the Hamilton vector field

$$H_p = \frac{\partial p}{\partial \xi} \cdot \frac{\partial}{\partial x} - \frac{\partial p}{\partial x} \cdot \frac{\partial}{\partial \xi} = 2\xi \cdot \frac{\partial}{\partial x} - \frac{\partial V}{\partial x} \cdot \frac{\partial}{\partial \xi}.$$

If $x \in \partial \ddot{O}$, in particular, the only $\xi \in \boldsymbol{R}^n$ such that $p(x,\xi) = E_0$ is 0, and $H_p = -\nabla V(x) \cdot \partial/\partial \xi$. Hence (A3) also implies,

$$\nabla V(x) \neq 0 \quad \text{on} \quad \partial \ddot{O}. \tag{2.4}$$

Under the conditions (A1), (A2), (A3), we have the following theorem (it is an analog to our situation of a result due to Heffer and Sjöstrand in the globally analytic case):

Theorem 2.2. *Assume* (A1)–(A3). *Then, there exists a unique resonance* $\rho(h)$ *of* P *such that* $h^{-1}|\rho(h) - \lambda_D(h)| \to 0$ *as* $h \to 0_+$, *and it verifies,*

$$|\lambda_D(h) - \rho(h)| = \mathscr{O}\big(e^{-(2S-\epsilon(\eta))/h}\big). \tag{2.5}$$

Moreover, denoting by $u(x,h)$ the corresponding (conveniently normalized) resonant state, one has,

$$|u_D(x,h) - u(x,h)| = \mathscr{O}\big(e^{-(2S-d(x_0,x)-\epsilon(\eta))/h}\big), \tag{2.6}$$

uniformly in $\overline{B_d(x_0, S-\eta)}$, where $\epsilon(\eta) \to 0$ as $\eta \to 0$, and, for any $K \subset \boldsymbol{R}^n$ compact, there exists $N_K \in \boldsymbol{N}$ such that,

$$\big\| e^{s(x)/h} u(x,h) \big\|_{H^1(K)} = \mathscr{O}(h^{-N_K}), \tag{2.7}$$

uniformly as $h \to 0$, where $s(x) = d(x_0, x)$ if $x \in B_d(x_0, S)$ and $s(x) = S$ otherwise.

Finally we assume some conditions on the set $\partial\ddot{O} \cap \overline{B_d(x_0,S)}$ and on the *caustic set* $\mathscr{C} = \overline{\{x \in \ddot{O}; d(x_0,x) = d(x,\partial\ddot{O}) + S\}}$.

The points of the set $\partial\ddot{O} \cap \overline{B_d(x_0,S)}$ are called *points of type* 1 in [**HeSj1**]. Since they mainly concentrate the interactions between the well and the sea, here we prefer to call them *points of interaction*.

Our additional assumption is,

(A4) $\partial\ddot{O} \cap \overline{B_d(x_0,S)}$ is a submanifold Γ of $\partial\ddot{O}$, and $\mathscr{C}$ has a contact of order exactly 2 with $\partial\ddot{O}$ along Γ.

We denote by n_Γ ($\leq n-1$) the dimension of Γ. Then, our main result is,

THEOREM 2.3. *Under Assumptions* (A1)–(A4)*, there exists a classical symbol*

$$f(h) \sim \sum_{j\geq 0} h^j f_j,$$

with $f_0 > 0$, such that

$$\operatorname{Im} \rho(h) = -h^{(1-n_\Gamma)/2} f(h) e^{-2S/h}.$$

REMARK 2.4. In the globally analytic case, it has been shown in [**HeSj1**] that $f(h)$ is an analytic classical symbol. In the general case, however, such a property is probably no longer satisfied.

Remark 2.5. Our arguments can also be applied to resonances of P with a larger real part, and give, as in [**HeSj1**, Theorem 10.14], a lower bound on their width when the corresponding eigenvalue of P_D is asymptotically simple.

3. Proof of Theorem 2.2.

Proceeding as in [**CMR**] and [**Ma2**], we consider the distorted operator P_θ with $\theta = h\ln(1/h)$, constructed with the unitary operator U_ν as in (2.1), and with $F(x) = 0$ on some arbitrary large compact set $K \subset \boldsymbol{R}^n$, that we can assume to contain $\ddot{O}$. Moreover, following the same idea as in [**HeSj1**, Section 9], we also consider the corresponding operator $\tilde{P}_\theta$ where the well has been filled-up, that is,

$$\tilde{P}_\theta = P_\theta + W(x),$$

where $W \in C_0^\infty(\ddot{O})$, $W(x_0) > 0$, $W \geq 0$ everywhere, $\operatorname{Supp} W$ arbitrarily small around x_0 (in particular, the Hamilton flow of $p + W$ has no trapped trajectories with energy E_0).

Then, proceeding as in [**CMR**, Section 7] (or [**Ma2**, Section 4]), one can construct a function $\psi_0 \in C_0^\infty((\boldsymbol{R}^n \setminus \operatorname{Supp} W) \times \boldsymbol{R}^n)$ such that,

$$-\operatorname{Im} \tilde{p}_\theta\big(x - t\partial_x\psi_0 - it\partial_\xi\psi_0, \xi - t\partial_\xi\psi_0 + it\partial_x\psi_0\big) \geq \frac{1}{C_0} h \ln\frac{1}{h}, \tag{3.1}$$

for some constant $C_0 > 0$ large enough, where $t := 2C_0\theta$, $\tilde{p}_\theta$ is an almost-analytic extension of the semiclassical principal symbol of $\tilde{P}_\theta$, and the inequality holds for (x,ξ) such that $|\operatorname{Re}\tilde{p}_\theta(x,\xi) - E_0| \leq \langle\xi\rangle^2/C_0$.

In particular, by [**CMR**, Proposition 7.2], we easily obtain,

$$\big\|e^{t\psi_0/h}Tv\big\|_{L^2(\boldsymbol{R}^{2n})} = \mathscr{O}(|h\ln h|^{-1})\big\|e^{t\psi_0/h}T(\tilde{P}_\theta - z)v\big\|_{L^2(\boldsymbol{R}^{2n})}, \tag{3.2}$$

uniformly for $h > 0$ small enough, $v \in L^2(\boldsymbol{R}^n)$, and $|z - E_0| \ll h\ln(1/h)$. Here $T = T_1$ is the so-called FBI-Bargmann transform defined by (8.5).

This means that the norm of $(\tilde{P}_\theta - z)^{-1}$ is uniformly $\mathscr{O}(|h\ln h|^{-1})$ when we consider $\tilde{P}_\theta$ as acting on the space $H_t := L^2(\boldsymbol{R}^n)$ endowed with the norm,

$$\|v\|_t := \big\|e^{t\psi_0/h}Tv\big\|_{L^2(\boldsymbol{R}^{2n})}. \tag{3.3}$$

From this point, one can proceed exactly as in [**HeSj1**, Proof of Proposition 9.6] (that is, by considering a Grushin problem for P_θ, the inverse of which is obtained by using the corresponding Grushin problem for P_D, and by using Agmon-type estimates inside the island $\ddot{O}$), and, working with $u_\theta := U_{i\theta}u$ ($= u$ on K), one concludes (2.5). Indeed, this proof never uses the analyticity of V, but just the fact that one has a good enough control on the resolvent of the "filled-up well"-operator.

Similarly, (2.6) is obtained as in the proof of [**HeSj1**, Theorem 9.9], mainly by considering the spectral projector of P_θ as in [**HeSj1**, Formula (9.37)] (in that case, u must be normalized, e.g., by requiring that $\|u_\theta\|_t = 1$).

Moreover, the same arguments also show that,

$$\left\|e^{t\psi_0/h}Tu_\theta\right\|_{L^2((\boldsymbol{R}^n\setminus B_d(x_0,S-\eta))\times\boldsymbol{R}^n)} = \mathscr{O}\left(e^{-(S-\epsilon(\eta))/h}\right),$$

and thus, also,

$$\|u_\theta\|_{L^2(\boldsymbol{R}^n\setminus B_d(x_0,S-\eta))} = \mathscr{O}\left(e^{-(S-\epsilon'(\eta))/h}\right).$$

As a consequence, up to some constant factor of the type $1+\mathscr{O}(e^{-\delta/h})$ ($\delta>0$), the normalization of u does not depend on the particular choices of K, F, and ψ_0.

Now, we come to the proof of (2.7).

Let $\chi_1\in C_0^\infty(B_d(x_0,S-\eta))$, such that $\chi_1=1$ in $B_d(x_0,S-2\eta)$ ($\eta>0$ fixed arbitrarily small), and let $\chi_2\in C_0^\infty(B_d(x_0,(1/2)(S+\eta)))$, such that $\chi_2=1$ in $B_d(x_0,(1/2)(S-\eta))$. Setting, for $z\in\gamma:=\{z\in\boldsymbol{C};|z-\lambda_D|=h^2\}$,

$$R_\theta(z) := \chi_1(P_D-z)^{-1}\chi_2 + (\tilde{P}_\theta-z)^{-1}(1-\chi_2), \tag{3.4}$$

one has (see [**HeSj1**, Formula (9.39)]),

$$(P_\theta-z)R_\theta(z) = I+K_\theta(z), \tag{3.5}$$

with,

$$\begin{aligned}K_\theta(z) &:= [P_\theta,\chi_1](P_D-z)^{-1}\chi_2 - W(\tilde{P}_\theta-z)^{-1}(1-\chi_2)\\ &= [-h^2\Delta,\chi_1](P_D-z)^{-1}\chi_2 - W(\tilde{P}_\theta-z)^{-1}(1-\chi_2).\end{aligned}$$

Now, if W is taken in such a way that $\operatorname{Supp}W\subset B_d(x_0,\eta)$, then, as in [**HeSj1**] (in particular the proof of Lemma 9.4), Agmon estimates show that,

$$\|K_\theta\| = \mathscr{O}\left(e^{-(S-\varepsilon(\eta))/2h}\right),$$

uniformly with respect to $h > 0$ small enough and $z \in \gamma$, and where $\varepsilon(\eta) \to 0$ as $\eta \to 0_+$ (here, K_θ is considered as an operator acting on H_t). In particular, we deduce from (3.5),

$$(P_\theta - z)^{-1} = R_\theta(z)\big(I + \mathscr{O}(e^{-(S-\varepsilon(\eta))/2h})\big).$$

Moreover, setting

$$k := h \ln \frac{1}{h}, \tag{3.6}$$

by (3.2) and (3.4), for $z \in \gamma$, we have,

$$\|R_\theta(z)\|_{\mathscr{L}(H_t)} = \mathscr{O}(h^{-2} + k^{-1}) = \mathscr{O}(h^{-2}),$$

and thus, we obtain,

$$\|(P_\theta - z)^{-1}\|_{\mathscr{L}(H_t)} = \mathscr{O}(h^{-2}), \tag{3.7}$$

uniformly for $z \in \gamma$ and $h > 0$ small enough.

Now, in order to obtain estimates on u even very close to $\partial\ddot{O}$, we consider the Dirichlet realization P_h of P on the h-dependent domain,

$$M_h := \big\{x \in \ddot{O}; \operatorname{dist}(x, \partial\ddot{O}) \geq k^{2/3}\big\}.$$

We denote by λ_h its first eigenvalue, and by v_h the corresponding normalized eigenstate. We also denote by d_h the Agmon distance on M_h, associated with the pseudo-metric $(V - \lambda_h)_+ dx^2$ (so that, in particular, d_h depends on h, too), and we set,

$$\varphi_h(x) := d_h(x, x_0).$$

At first, with χ_1 as before, we observe that $(P_h - \lambda_D)\chi_1 u_D = (P_D - \lambda_D)\chi_1 u_D = \mathscr{O}(h^\infty)$, and thus, by the Min-max principle (and since $V \geq E_0$ on M_h), we have $E_0 \leq \lambda_h \leq \lambda_D + \mathscr{O}(h^\infty)$ (in particular, $\lambda_h = E_0 + \mathscr{O}(h)$). Moreover, since x_0 is the only point of M_h where V reaches its (non-degenerate) minimum E_0, and since $V - E_0 \geq \delta k^{2/3} \gg h$ near ∂M_h, standard techniques (see, e.g., [**HeSj2**, Section 3], and [**Si**]) show that, actually, $\lambda_h = \lambda_D + \mathscr{O}(h^\infty)$, and the gap between λ_h and the second eigenvalue of P_h behaves like h.

Since $V - \lambda_h \leq V - E_0$, we also have,

$$\varphi_h(x) \le d(x_0, x). \tag{3.8}$$

LEMMA 3.1. *There exists a constant $C_1 \ge 0$, such that, for $x \in M_h$ and $h > 0$ small enough, one has,*

$$\varphi_h(x) \ge s(x) - C_1 k,$$

where, as before, $k = h\ln(1/h)$.

PROOF. We set $U_h^{\pm} := \{x \in M_h\,;\, \pm V(x) \ge \pm\lambda_h\}$. Then, since $V(x) - E_0 \sim k^{2/3} \gg \lambda_h - E_0$ on ∂M_h, by definition, we have,

$$\varphi_h(x) = d_h\big(U_h^-, x\big) = \inf_{\ell \in L_x} \int_\ell \sqrt{V(y) - \lambda_h}|dy|,$$

where L_x stands for the set of C^1 curves $\ell : [0,1] \to U_h^+$ with $\ell(0) \in U_h^-$ and $\ell(1) = x$. Moreover, since $\lambda_h = E_0 + E_1 h + \mathscr{O}(h^2) > E_0$, Assumptions (A2)–(A3) imply that $\nabla V \ne 0$ on $\{V = \lambda_h\}$ for $h > 0$ small enough. Then, if $\varphi_h(x) < S_h := d_h(x_0, \ddot{O})$, standard arguments of Riemannian geometry (exploiting the fact that, in that case, $\dot\ell(t)$ remains colinear to $\nabla\varphi_h(\ell(t))$ for any minimal geodesic ℓ: see, e.g., [**Mi**], [**HeSj2**], and [**Ma3**, Section 3]) show that, for $V(x) > \lambda_h$, $\varphi_h(x)$ is reached at some minimal geodesic ℓ that can be re-parametrized in such a way that the map $t \mapsto (\ell(t), (1/2)\dot\ell(t))$ becomes a null bicharacteristic of $q_h(x,\xi) := \xi^2 - (V(x) - \lambda_h)$. In particular, such an ℓ verifies $|\dot\ell(t)| = 2\sqrt{V(\ell(t)) - \lambda_h}$, and thus, we obtain,

$$\varphi_h(x) = 2\int_0^{T_x} (V(\ell(t)) - \lambda_h)dt,$$

where $T_x = T_x(h) > 0$ represents the time necessary for going from U_h^- to x in the new parametrization. Writing $\lambda_h = E_0 + h\mu_h$, where $\mu_h = E_1 + \mathscr{O}(h)$, this gives,

$$\begin{aligned}
\varphi_h(x) &= 2\int_0^{T_x} (V(\ell(t)) - E_0)dt - 2T_x h\mu_h \\
&= \int_0^{T_x} \sqrt{V(\ell(t)) - E_0}\sqrt{|\dot\ell(t)|^2 + 4h\mu_h}dt - 2T_x h\mu_h \\
&\ge \int_\ell \sqrt{V(y) - E_0}d|y| - 2T_x h\mu_h \\
&\ge d(U_h^-, x) - 2T_x h\mu_h.
\end{aligned} \tag{3.9}$$

Moreover, since, for x close to x_0, we have $d(x_0,x) = \mathscr{O}(|x-x_0|^2)$, we see that $d(U_h^-, x_0) = \mathscr{O}(h)$, and thus, by the triangular inequality, $d(x_0,x) \leq d(U_h^-,x)+Ch$ for some constant $C>0$. Combining with (3.9), we obtain,

$$\varphi_h(x) \geq d(x_0,x) - Ch - 2T_x h\mu_h \geq s(x) - Ch - 2T_x h\mu_h, \tag{3.10}$$

still for x verifying $\varphi_h(x) < S_h$. Thus, in that case, it remains only to prove that $T_x = \mathscr{O}(\ln(1/h))$. To do so, we set $(x(t),\xi(t)) := (\ell(t),(1/2)\dot{\ell}(t)) = \exp tH_q(x_h,0)$, where $H_q(x,\xi) := (2\xi, \nabla V(x))$ is the Hamilton field of q_h, and $x_h := \ell(0) \in U_h^-$. If $\varepsilon > 0$ is any arbitrarily small fixed number, we see that $H_q(x,\xi)$ remains outside some fixed neighborhood of 0 on $\{(x,\xi); |x-x_0| \geq \varepsilon,\, q_h(x,\xi)=0\}$. As a consequence, if $|x-x_0| \geq \varepsilon$, the time needed by ℓ to go from x to the set $\{y; |y-x_0| = \varepsilon\}$ is bounded, uniformly with respect to h. Therefore, it remains to estimate the time $\tilde{T}$ employed by ℓ to go from $\ell(0) \in U_h^-$ to $\{y; |y-x_0|=\varepsilon\}$. Since we stay in an arbitrarily small neighborhood of x_0, we can assume $x(t)\cdot\nabla V(x(t)) \geq 4\delta^2|x(t)|^2$ for $t\in[0,\tilde{T}]$ and with some $\delta>0$ constant. Then, setting $f(t) := x(t)\cdot\xi(t)/|x(t)|^2$, we compute,

$$\dot{f}(t) + 4(f(t))^2 = \frac{2|\xi(t)|^2 + x(t)\cdot\nabla V(x(t))}{|x(t)|^2} \geq 4\delta^2.$$

Therefore, on its domain of definition, the function $g(t) := (\delta - f(t))^{-1}$ verifies,

$$\dot{g} \geq 4\frac{\delta^2 - f^2}{(\delta-f)^2} = 4\frac{\delta+f}{\delta-f} = 8\delta g - 4,$$

and thus, since $g(0) = \delta^{-1}$, we easily deduce,

$$g(t) \geq \frac{1}{2\delta}(1+e^{8\delta t}),$$

as long as $f(t) < \delta$, and thus,

$$f(t) \geq \delta - \frac{2\delta}{1+e^{8\delta t}},$$

on the same interval. Now, if $f(t_1) = \delta$ for some $t_1 \in [0,\tilde{T})$, we fix $\delta_1 < \delta$ arbitrary, and, for t close to t_1, we set $g_1(t) := (\delta_1 - f)^{-1}$. Using that $\dot{f}(t) + 4(f(t))^2 \geq 4\delta_1^2$, the same procedure gives,

$$g_1(t) \geq \frac{1}{2\delta_1} - \left(\frac{1}{2\delta_1} + \frac{1}{\delta - \delta_1}\right)e^{8\delta_1(t-t_1)},$$

as long as $f(t) > \delta_1$, that is,

$$\frac{1}{f(t) - \delta_1} \leq \left(\frac{1}{2\delta_1} + \frac{1}{\delta - \delta_1}\right)e^{8\delta_1(t-t_1)} - \frac{1}{2\delta_1}.$$

In particular, $(f(t) - \delta_1)^{-1}$ remains bounded on any finite interval $[t_1, T_1]$ where $f(t) > \delta_1$, and this means that $f(t)$ cannot take the value δ_1 on $[t_1, \tilde{T}]$. Thus, in this case, $f(t)$ necessarily remains $\geq \delta$ on $[t_1, \tilde{T}]$. Summing up, we have proved,

$$f(t) \geq \delta - \frac{2\delta}{1 + e^{8\delta t}}, \tag{3.11}$$

on the whole interval $[0, \tilde{T}]$. Since,

$$\frac{d}{dt} \ln |x(t)| = \frac{x(t) \cdot \dot{x}(t)}{|x(t)|^2} = 2f(t),$$

and $|x(0)| \geq \delta' \sqrt{h}$ for some $\delta' > 0$ constant, we deduce from (3.11),

$$\ln |x(t)| \geq \ln\left(\delta'\sqrt{h}\right) + 2\delta t - \int_0^t \frac{2\delta ds}{1 + e^{8\delta s}} \geq \ln(\delta'\sqrt{h}) + 2\delta t - \int_0^{+\infty} \frac{2\delta ds}{1 + e^{8\delta s}},$$

and thus, on $[0, \tilde{T}]$,

$$|x(t)| \geq \delta'' \sqrt{h} e^{2\delta t}, \tag{3.12}$$

with $\delta'' = \delta' e^{-C_2}$, $C_2 := \int_0^{+\infty}(2\delta ds/(1 + e^{8\delta s}))$. Since $\delta''\sqrt{h}e^{2\delta t} = \varepsilon$ when $t = (2\delta)^{-1} \ln(\varepsilon/\delta''\sqrt{h})$, we deduce from (3.12) that, necessarily, one has $\tilde{T} \leq (2\delta)^{-1} \ln(\varepsilon/\delta''\sqrt{h})$, and, by (3.10), it follows that,

$$\varphi_h(x) \geq s(x) - C_1' k, \tag{3.13}$$

for x verifying $\varphi_h(x) < S_h$, and some constant $C_1' > 0$, independent of x. On the other hand, $\varphi_h(x)$ reaches S_h at some point x_h verifying $V(x_h) = \lambda_0 = E_0 + \mathscr{O}(h)$ (and x_h away from some fix neighborhood of x_0), and thus $\text{dist}(x_h, \ddot{O}) = \mathscr{O}(h)$ (where dist stands for the Euclidian distance). As a consequence, the Agmon distance $d(x_h,\ddot{O})=\mathscr{O}(h)$, too, and thus, $d(x_0,x_h)\geq S-Ch$ for some constant $C>0$. Therefore, by continuity, (3.13) also proves that $S_h\geq S-C_1k$ with $C_1=C_1'+C$, and it follows that (3.13) is still valid if $\varphi_h(x)\geq S_h$ (since, by definition, $s(x)\leq S$ everywhere). □

Lemma 3.2. *There exists a constant $N_0\geq 0$, such that,*

$$\left\| e^{\varphi_h/h}v_h\right\|_{H^1(M_h)}=\mathscr{O}(h^{-N_0}),$$

uniformly for $h>0$ small enough.

Proof. Following [**HeSj2**], we set,

$$\phi(x):=\begin{cases}\varphi_h(x)-Ch\ln\left[\dfrac{\varphi_h(x)}{h}\right] & \text{if } \varphi_h(x)\geq Ch;\\ \varphi_h(x)-Ch\ln C & \text{if } \varphi_h(x)\leq Ch,\end{cases}$$

where $C\geq 1$ is some constant that will be fixed large enough later on. Then, we use the following identity (that is at the origin of Agmon estimates: see, e.g., Lemma 8.2 below, and [**HeSj2**, Theorem 1.1]),

$$\begin{aligned}&\operatorname{Re}\left\langle e^{\phi/h}(P-\lambda_h)v_h,e^{\phi/h}v_h\right\rangle\\&\quad=h^2\left\|\nabla(e^{\phi/h}v_h)\right\|^2+\left\langle(V-\lambda_h-(\nabla\phi)^2)e^{\phi/h}v_h,e^{\phi/h}v_h\right\rangle.\end{aligned}\tag{3.14}$$

Now, if C is taken large enough, by (3.8) we see that $M_h\cap\{\varphi_h(x)\geq Ch\}\subset\{V\geq\lambda_h\}$. Moreover, on this set we have,

$$\nabla\phi=\left(1-\frac{Ch}{\varphi_h}\right)\nabla\varphi_h,$$

and thus, using that $(\nabla\varphi_h)^2\leq(V-\lambda_h)_+$,

$$V-\lambda_h-(\nabla\phi)^2\geq V-\lambda_h-\left(1-\frac{Ch}{\varphi_h}\right)^2(V-\lambda_h)\geq Ch\frac{V-\lambda_h}{\varphi_h}.$$

Writing again $\lambda_h=E_0+\mu_h h$ with $\mu_h=E_1+\mathscr{O}(h)$, and using again (3.8) and the fact that $d(x_0,x)\leq C_0(V(x)-E_0)$ near x_0 (for some $C_0>0$ constant), we deduce,

$$V(x)-\lambda_h-(\nabla\phi(x))^2\geq Ch\frac{V(x)-E_0}{d(x_0,x)}-Ch\frac{\mu_h h}{\varphi_h}\geq\frac{Ch}{C_0}-\mu_h h,$$

on $M_h\cap\{\varphi_h(x)\geq Ch\}\cap\{|x-x_0|\ll 1\}$, and thus, taking C large enough,

$$V(x)-\lambda_h-(\nabla\phi(x))^2\geq\frac{Ch}{2C_0},$$

on the same set. On the other hand, away from some neighborhood of x_0, in M_h we have (possibly by increasing C again),

$$V(x)-\lambda_h-(\nabla\phi(x))^2\geq\frac{hk^{2/3}}{C}.$$

Inserting these estimates into (3.14), and using that the left-hand side is 0, we obtain,

$$h^2\left\|\nabla(e^{\phi/h}v_h)\right\|^2+hk^{2/3}\left\|e^{\phi/h}v_h\right\|^2_{\{\varphi_h\geq Ch\}}=\mathscr{O}\big(\|e^{\phi/h}v_h\|^2_{\{\varphi_h\leq Ch\}}\big),$$

and thus, since $\phi\leq Ch(1-\ln C)\leq 0$ on $\{\varphi_h\leq Ch\}$, we conclude,

$$h^2\left\|\nabla(e^{\phi/h}v_h)\right\|^2+hk^{2/3}\left\|e^{\phi/h}v_h\right\|^2=\mathscr{O}(1).$$

Then, observing that $e^{\phi/h}\geq(h/M)^Ce^{\varphi_h/h}$ with $M:=\sup\varphi_h=\mathscr{O}(1)$, the result follows. $\square$

Now, let $\chi_h\in C_0^\infty(M_h)$, such that $\chi_h=1$ on $\{x\in\ddot{O};\operatorname{dist}(x,\partial\ddot{O})\geq 2k^{2/3}\}$ and, for all α, $\partial^\alpha\chi_h=\mathscr{O}(k^{-2|\alpha|/3})$ (such a χ_h exists because $\partial\ddot{O}$ is a hypersurface of $\boldsymbol{R}^n$). In particular, $\chi_h v_h$ is in the domain of P_θ, and one has,

Lemma 3.3. *There exists a constant $N_1\geq 0$, such that, for the positively oriented contour $\gamma=\{z\in\boldsymbol{C};|z-\lambda_D|=h^2\}$,*

$$\left\|\frac{1}{2i\pi}\int_\gamma(z-P_\theta)^{-1}\chi_h v_h dz-\chi_h v_h\right\|_t=\mathscr{O}\big(h^{-N_1}e^{-S/h}\big),$$

uniformly for $h>0$ small enough.

Proof. Setting $w_h:=\chi_h v_h$, we have,

$$P_\theta w_h=P_h w_h=\lambda_h w_h+[P,\chi_h]v_h=\lambda_h w_h-2h^2(\nabla\chi_h)(\nabla v_h)-h^2(\Delta\chi_h)v_h. \quad (3.15)$$

Moreover, on the support of $\nabla\chi_h$, we have $\text{dist}(x, \partial\ddot{O}) \leq 2k^{2/3}$, and thus, by standard properties of the Agmon distance near $\partial\ddot{O}$ (see, e.g., below Lemma 4.1, and [**HeSj1**, Section 10]), we see that $s(x) \geq S - Ck$ for some constant $C > 0$. Therefore, by Lemma 3.1, we also have $\varphi_h(x) \geq S - C'k$ ($C' = C + C_0$), and by Lemma 3.2, we deduce from (3.15),

$$\|(P_\theta - \lambda_h)w_h\|_{L^2} = \mathscr{O}\big(h^{-N}e^{-S/h}\big),$$

with some constant $N \geq 0$. As a consequence, since ψ_0 is bounded and T is an isometry, $r_h := (P_\theta - \lambda_h)w_h$ verifies,

$$\|r_h\|_t = \big\|h^{-2C_0\psi_0}Tr_h\big\| = \mathscr{O}(h^{-M})\|r_h\|_{L^2} = \mathscr{O}\big(h^{-M'}e^{-S/h}\big), \tag{3.16}$$

with $M, M' > 0$ constant. Then, writing,

$$\begin{aligned}\frac{1}{2i\pi}\int_\gamma (z - P_\theta)^{-1}w_h dz - w_h &= \frac{1}{2i\pi}\int_\gamma \big[(z - P_\theta)^{-1} - (z - \lambda_h)^{-1}\big]w_h dz \\ &= \frac{1}{2i\pi}\int_\gamma \frac{1}{z - \lambda_h}(z - P_\theta)^{-1}r_h dz, \end{aligned}\tag{3.17}$$

and using (3.7) and (3.16), the result immediately follows. □

Using the equation $(P_h - \lambda_h)v_h = 0$ and the ellipticity (in the standard sense) of P_h, it is not difficult to deduce from (3.16) and Lemma 3.2, that, for all $\ell \geq 0$, there exists $M_\ell \geq 0$ such that,

$$\big\|P_\theta^\ell r_h\big\|_t = \mathscr{O}\big(h^{-M_\ell}e^{-S/h}\big). \tag{3.18}$$

As a consequence, applying P_θ^ℓ to (3.17), we deduce from (3.18),

$$\left\|P_\theta^\ell\left(\frac{1}{2i\pi}\int_\gamma (z - P_\theta)^{-1}\chi_h v_h dz - \chi_h v_h\right)\right\|_t = \mathscr{O}\big(h^{-N_\ell}e^{-S/h}\big), \tag{3.19}$$

for all $\ell \geq 0$ and some $N_\ell \geq 0$ constant. Then, using the classical ellipticity of P_θ and the fact that, for all u, $\|u\|_{L^2} = \|Tu\|_{L^2} = \mathscr{O}(h^{-M}\|u\|_t)$, we deduce from (3.19),

$$\left\|\frac{1}{2i\pi}\int_\gamma (z - P_\theta)^{-1}\chi_h v_h dz - \chi_h v_h\right\|_{H^s(\boldsymbol{R}^n)} = \mathscr{O}\big(h^{-N_s}e^{-S/h}\big), \tag{3.20}$$

for all $s \geq 0$ and some constant $N_s \geq 0$.

It also follows from Lemmas 3.2 and 3.3, that,

$$\left\| \frac{1}{2i\pi} \int_\gamma (z - P_\theta)^{-1} \chi_h v_h dz \right\|_t = 1 + \mathscr{O}(e^{-\delta/h}),$$

for some $\delta > 0$ constant, and therefore, we necessarily have,

$$u_\theta = \frac{\alpha'}{2i\pi} \int_\gamma (z - P_\theta)^{-1} \chi_h v_h dz,$$

with $|\alpha'| = 1 + \mathscr{O}(e^{-\delta/h})$. In particular, by (3.20),

$$\left\| u_\theta - \alpha' \chi_h v_h \right\|_{H^s} = \mathscr{O}\big(h^{-N_s} e^{-S/h}\big). \tag{3.21}$$

Then, since $u_\theta = u$ on K, (2.7) easily follows from Lemma 3.1, Lemma 3.2, and (3.21).

4. Extension of the WKB solution.

Let $x^1 \in \Gamma$. In this section, we will extend to a neighborhood of x^1 the WKB solution,

$$w(x,h) \approx h^{-n/4} a(x,h) e^{-d(x_0,x)/h}, \tag{4.1}$$

that approximates both the eigenfunction $u_D(x,h)$ of P_D near x_0 (see (2.2)) and the resonant state $u(x,h)$ (Theorem 2.2).

4.1. Extension up to the caustic set.

The extension in the island will be done along the geodesic with respect to the Agmon distance from x_0 to x^1.

Let $q(x,\xi) = \xi^2 - V(x)$ and $H_q = 2\xi \cdot \partial/\partial x + \partial_x V(x) \cdot \partial/\partial \xi$ its Hamilton vector field. Then, one can see as in [**HeSj1**] that the integral curve $\tilde{\gamma}(t) = (\tilde{x}(t), \tilde{\xi}(t))$ of H_q starting at $(\tilde{x}(0), \tilde{\xi}(0)) = (x^1, 0)$ verifies,

$$(x(-\infty), \xi(-\infty)) = (x_0, 0).$$

Moreover, its projection on the x-space is the unique minimal geodesic between x_0 and x^1 staying in $\ddot{O} \cup \{x^1\}$ (see [**HeSj1**, Section 10] and [**HeSj2**]).

If $\phi(x) := d(x_0, x) - S$, we learn from [**HeSj2**] that ϕ is C^∞ in a neighborhood

Ω of $\tilde{x}([-\infty,0))$, and the Lagrangian manifold,

$$\Lambda = \{(x, \nabla\phi(x)); x \in \Omega\},$$

is the outgoing stable manifold of dimension n associated to the fixed point $(x_0, 0)$ of H_q.

In particular, Λ is H_q-invariant and contains $\tilde{\gamma}([-\infty,0))$. Moreover, since H_q does not vanish at $\tilde{\gamma}(0) = (x^1, 0)$ by (A3), Λ can be extended by the flow of H_q to a larger Lagrangian manifold (that we still denote by Λ) that contains $\tilde{\gamma}([-\infty,0])$. The natural projection $\Pi : \Lambda \to \boldsymbol{R}^n_x$ is singular at $\tilde{\gamma}(0)$, and, as shown in [**HeSj1**, Lemma 10.1], the kernel of $d\Pi(x^1, 0)$ is a one-dimensional vector space generated by $H_q(x^1, 0)$.

Let us choose Euclidian coordinates x centered at x^1 such that $T_{x^1}(\partial\ddot{O})$ is given by $x_n = 0$, and $\partial/\partial x_n$ is the exterior normal of $\ddot{O}$ at this point. Then by Assumption (A3),

$$V(x) - E_0 = -C_0 x_n + W(x), \tag{4.2}$$

where $C_0 > 0$ is a constant and $W(x) = \mathscr{O}(|x|^2)$. In a neighborhood of the point $\tilde{\gamma}(0)$, the Lagrangian manifold Λ is defined by a real-valued C^∞ function $g(x', \xi_n)$ with $g(0) = 0$, $dg(0) = 0$, that is,

$$\Lambda = \left\{(x,\xi); \xi' = \frac{\partial g}{\partial x'}(x', \xi_n),\ x_n = -\frac{\partial g}{\partial \xi_n}(x', \xi_n)\right\}. \tag{4.3}$$

Moreover, there exist real-valued smooth functions $\xi_n^c(x')$, $a(x')$, $b(x')$ and $\nu_0(x', \xi_n)$, $\nu_1(x', \xi_n)$, such that,

$$\left|\xi_n^c(x')\right| + |a(x')| + |b(x')| = \mathscr{O}(|x'|^2) \text{ as } |x'| \to 0; \tag{4.4}$$

$$\nu_0 = \nu_1 + \mathscr{O}\big(|\xi_n - \xi_n^c(x')|\big) = \frac{1}{C_0} + \mathscr{O}(|x'| + |\xi_n|); \tag{4.5}$$

$$g(x', \xi_n) = a(x') + b(x')\big(\xi_n - \xi_n^c(x')\big) + \frac{1}{3}\nu_0(x', \xi_n)\big(\xi_n - \xi_n^c(x')\big)^3; \tag{4.6}$$

$$\frac{\partial g}{\partial \xi_n}(x', \xi_n) = b(x') + \nu_1(x', \xi_n)\big(\xi_n - \xi_n^c(x')\big)^2. \tag{4.7}$$

All these properties are proved in [**HeSj1**, pages 136–148] and do not require the analyticity of the potential.

Then, near $(x^1, 0)$, the caustic set $\mathscr{C}$ (that, by definition, is the set where ϕ

fails to be smooth, and thus, the set of x for which the roots $\xi_n(x)$ of the equation $x_n = -\partial_{\xi_n} g(x', \xi_n)$ are not smooth) is given by,

$$\mathscr{C} = \{x; x_n + b(x') = 0\}. \tag{4.8}$$

It is shown in [**HeSj1**, Lemma 10.2], that there exists a positive constant C such that

$$\phi(x)|_{\mathscr{C}} \geq C(V(x) - E_0). \tag{4.9}$$

(The proof of (4.9) does not use the analyticity, but rather the fact that, since $V|_{\mathscr{C}} \geq 0$, one has $|\nabla(V\ |_{\mathscr{C}})| = \mathscr{O}(\sqrt{V\ |_{\mathscr{C}}})$.)

This estimate together with Assumptions (A3) and (A4) mean that $\phi(x)|_{\mathscr{C}}$ is non-negative and quadratic along Γ, with $\phi\ |_{\Gamma} = 0$.

Let $\tilde{\Omega}$ be a small neighborhood of $\gamma([-\infty, 0])$ and let

$$\tilde{\Omega}_+ = \big\{x \in \tilde{\Omega}; x_n + b(x') > 0\big\}, \quad \tilde{\Omega}_- = \tilde{\Omega}\backslash\big(\tilde{\Omega}_+ \cup \mathscr{C}\big). \tag{4.10}$$

Then the phase function $-d(x_0, x) = -\phi(x) - S$ of (4.1) is a C^∞ function defined in $\tilde{\Omega}_-$. The symbol $a(x, h)$ can also be extended to $\tilde{\Omega}_-$ by solving successively for $a_j(x)$ in (2.3) the transport equations (that are first-order ordinary differential equations along the integral curves of H_q), and by re-summing the series $\sum_{j\leq 0} h^j a_j$.

More generally, the previous arguments also permit us to extend w in the open set defined as the union of all smooth minimal geodesics included in $\ddot{O}$ and starting from x_0. We denote this set by Ω.

4.2. Extension beyond the caustic set.

In order to extend the WKB solution w beyond the caustic set, we follow the idea of [**HeSj1**] and represent $h^{n/4}e^{S/h}w$ in the integral form,

$$I[c](x, h) = h^{-1/2} \int_{\gamma(x)} e^{-(x_n\xi_n + g(x', \xi_n))/h} c\big(x', \xi_n, h\big) d\xi_n. \tag{4.11}$$

For x in $\tilde{\Omega}_-$ close to x^1, the phase function $x_n\xi_n + g(x', \xi_n)$ has two real critical points (see (4.7)). The steepest descent method at one of these points gives us the asymptotic expansion of $I[c]$. Comparing this with the symbol a, we can determine $c(x', \xi_n, h)$ so that the asymptotic expansion of $e^{S/h}w$ coincides with that of $I[c]$ in $\tilde{\Omega}_-$.

Hence, in order to determine a possible asymptotic expansion of $e^{S/h}w$ in $\tilde{\Omega}_+$, it is enough to compute it for $I[c]$.

If g was analytic with respect to ξ_n, we would find two complex critical points for $x \in \tilde{\Omega}_+$, one of them corresponding to an outgoing solution (i.e. resonant state). In our C^∞ case, however, g is only defined for real ξ_n. So we will extend $g(x', \cdot)$ *almost-analytically* (see Appendix for the definition) to a (h-dependent) small complex neighborhood of $\xi_n = 0$. Then, we will apply the steepest descent method for $x \in \tilde{\Omega}_+$ sufficiently close to x^1 so that the imaginary part of the critical point ξ_n remains sufficiently small.

In the following, we carry out the above procedure in several steps.

4.2.1. Integral representation in $\tilde{\Omega}_-$.

We first determine the C^∞ classical symbol $c(x', \xi_n, h) \sim \sum_{j=0}^{\infty} c_j(x', \xi_n)h^j$ and the integration contour $\gamma(x)$ for $x \in \tilde{\Omega}_-$.

Let x^3 be in $\tilde{\Omega}_-$ close to x^1 and $\tilde{U}$ a small neighborhood of x^3. The critical points of the phase function $x_n\xi_n + g(x', \xi_n)$ are the zeros of the function $x_n + (\partial g/\partial \xi_n)(x', \xi_n)$. From (4.7), we see that for $x \in \tilde{U}$, there are two real critical points $\xi_n^+(x)$, $\xi_n^-(x)$, and they verify,

$$\xi_n^\pm(x) \sim \xi_n^c(x') \pm \sqrt{\frac{-(x_n + b(x'))}{\nu_1(x', \xi_n^c(x'))}},$$

as $|x_n + b(x')| \to 0$. We define a sufficiently small real open interval $\gamma(x)$ so that it contains $\xi_n^+(x)$ inside as the only non-degenerate minimal point of $x_n\xi_n + g(x', \xi_n)$. The minimal value is

$$\phi(x) = x_n\xi_n^+(x) + g\big(x', \xi_n^+(x)\big). \tag{4.12}$$

Then by the steepest descent method, we obtain the asymptotic expansion as $h \to 0$ of $I[c]$,

$$I[c](x, h) \sim e^{-\phi(x)/h} \sum_{j=0}^{\infty} b_j(x)h^j, \tag{4.13}$$

for some C^∞ functions $b_j(x)$ defined on $\tilde{U}$. In particular,

$$b_0(x) = \sqrt{\frac{\pi}{r(x)}}c_0\big(x', \xi_n^+(x)\big),$$

where

$$r(x) = \frac{1}{2}\frac{\partial^2 g}{\partial \xi_n^2}\big(x', \xi_n^+(x)\big).$$

Moreover, the map (defined by (4.13)) that associates a sequence of functions $\{b_j(x)\}_{j=0}^{\infty}$ on $\tilde{U}$ to a sequence of functions $\{c_j(x', \xi_n)\}_{j=0}^{\infty}$ on $U = \{(x', \xi_n^+(x)); x \in \tilde{U}\}$ is bijective, and we define the function $c(x', \xi_n, h)$ as a realization of the inverse image of $\{a_j(x)\}_{j=0}^{\infty}$ by this map. In particular,

$$c_0\big(x', \xi_n^+(x)\big) = \sqrt{\frac{r(x)}{\pi}}\, a_0(x). \tag{4.14}$$

4.2.2. Extension of $c(x', \xi_n, h)$ to a neighborhood of $(x', \xi_n) = (0, 0)$.

The symbol $c(x', \xi_n, h)$, previously defined in U, formally verifies,

$$e^{g/h}(\hat{P} - \rho(h))\big(e^{-g/h}c\big) \sim 0. \tag{4.15}$$

Here $\hat{P} = -h^2\Delta_{x'} - \xi_n^2 + V(x', h(\partial/\partial\xi_n))$, where $V(x', h(\partial/\partial\xi_n))$ is considered as a pseudodifferential operator whose action on $e^{-g/h}c$ is defined by the standard asymptotic expansion,

$$\begin{aligned} &V\left(x', h\frac{\partial}{\partial \xi_n}\right)\big(e^{-g/h}c\big) \\ &\quad := e^{-g/h}\sum_{\ell \geq 0}\frac{h^\ell}{\ell!}\partial_{x_n}^\ell V\big(-\partial_{\xi_n} g\big)\partial_\eta^\ell\big(c(x', \eta)e^{-\kappa(x', \xi_n, \eta)/h}\big)\Big|_{\eta = \xi_n}, \end{aligned}$$

where $\kappa(x', \xi_n, \eta) := g(x', \eta) - g(x', \xi_n) - (\eta - \xi_n)\partial_{\xi_n} g(x', \xi_n)$.

(4.15) leads us to transport equations, which are also differential equations along the integral curves of H_q on Λ. The flows emanating from U covers a full neighborhood of $(x', \xi_n) = (0, 0)$, and thus we have extended c there.

4.2.3. Critical points and the extension of ϕ.

Let $N \geq 1$, $k = h\ln(1/h)$, and let $\tilde{\nu}_0$ be a holomorphic $(Nk)^{1/3}$-approximation of ν_0 with respect to ξ_n (in the sense of Lemma 8.1), where ν_0 is the function appearing in (4.6). Then, setting,

$$\tilde{g}(x', \xi_n) := a(x') + b(x')\big(\xi_n - \xi_n^c(x')\big) + \frac{1}{3}\tilde{\nu}_0(x', \xi_n)\big(\xi_n - \xi_n^c(x')\big)^3 \tag{4.16}$$

we see that $\tilde{g}$ is a holomorphic $(Nk)^{1/3}$-approximation of g with respect to ξ_n, and we look for the critical points of $\xi_n \mapsto x_n\xi_n + \tilde{g}(x', \xi_n)$, that is, the roots of the equation with respect to ξ_n,

$$x_n + \frac{\partial \tilde{g}}{\partial \xi_n}(x', \xi_n) = 0. \tag{4.17}$$

Recalling the definition of $\nu_1(x', \xi_n)$ in (4.7), we fix a small enough neighborhood Ω' of $(x', \xi_n) = (0, 0)$, such that $\inf_{\Omega'} \nu_1 \geq c_1$ for some constant $c_1 > 0$. Then, possibly by shrinking a little bit $\tilde{\Omega}_+$, we have,

Lemma 4.1. *Let $x \in \tilde{\Omega}_+ \cap \{x_n + b(x') \leq c_1(Nk)^{2/3}\}$. Then, the equation (4.17) has two complex roots $\xi_n^{-i}(x)$, $\xi_n^{+i}(x)$ satisfying*

$$\xi_n^{\pm i}(x) \sim \xi_n^c(x') \pm i\sqrt{\frac{x_n + b(x')}{\tilde{\nu}_1(x', \xi_n^c(x'))}}$$

as $x_n + b(x')$ tends to 0, where $\tilde{\nu}_1$ is a holomorphic $(Nk)^{1/3}$-approximation in the ξ_n-variable of $\nu_1(x', \xi_n)$. Moreover, setting,

$$\tilde{\phi}(x) = x_n\xi_n^{-i}(x) + \tilde{g}\big(x', \xi_n^{-i}(x)\big), \tag{4.18}$$

one has,

$$\operatorname{Im} \nabla_x \tilde{\phi}(x) = -\frac{1}{\sqrt{\tilde{\nu}_1(x', \xi_n^c(x'))}}\big(x_n + b(x')\big)^{1/2}\nabla\big(x_n + b(x')\big) + \mathscr{O}\big(x_n + b(x')\big), \tag{4.19}$$

and there exists $\varepsilon(h) = \mathscr{O}(h^\infty)$ real, such that,

$$\operatorname{Re} \tilde{\phi}(x) \geq \varepsilon(h), \tag{4.20}$$

for all $x \in \tilde{\Omega}_+ \cap \{x_n + b(x') \leq c_1(Nk)^{2/3}\}$ and $\alpha \in \boldsymbol{N}^n$.

Proof. From (4.16), we have,

$$\begin{aligned} x_n\xi_n + \tilde{g}(x', \xi_n) = {} & a(x') + x_n\xi_n^c(x') + \big(x_n + b(x')\big)\big(\xi_n - \xi_n^c(x')\big) \\ & + \frac{1}{3}\tilde{\nu}_0(x', \xi_n)\big(\xi_n - \xi_n^c(x')\big)^3, \end{aligned} \tag{4.21}$$

$$x_n + \frac{\partial \tilde{g}}{\partial \xi_n}(x', \xi_n) = x_n + b(x') + \tilde{\nu}_1(x', \xi_n)\big(\xi_n - \xi_n^c(x')\big)^2, \tag{4.22}$$

where, actually, $\tilde{\nu}_1(x', \xi_n)$ is a holomorphic $(Nk)^{1/3}$-approximations in the ξ_n-variable of $\nu_1(x', \xi_n)$.

We set,

$$x_n + b(x') = -z^2.$$

If ξ_n is a critical point of the phase, the left-hand side of (4.22) vanishes, and one has,

$$z = \sqrt{\tilde{\nu}_1(x', \xi_n)}\big(\xi_n - \xi_n^c(x')\big).$$

Since $\tilde{\nu}_1(x', \xi_n^c(x')) = 1/C_0 + \mathscr{O}(|x'|)$, and $\tilde{\nu}_1(x', \xi_n)$ is holomorphic with respect to ξ_n in $\{|\operatorname{Im}\xi_n| \leq (Nk)^{1/3}\}$, for z and x' small enough this equation is solvable with respect to ξ_n, and the solution is given by the Lagrange inversion formula,

$$\xi_n = \xi_n^c(x') + Y(x', z),$$

with,

$$Y(x', z) := \sum_{k=1}^{\infty} \frac{d^{k-1}}{d\xi_n^{k-1}} \big(\tilde{\nu}_1(x', \xi_n)\big)^{-k/2} \,|_{\xi_n = \xi_n^c(x')}\, \frac{z^k}{k!}, \tag{4.23}$$

that is holomorphic with respect to z in $\{|\operatorname{Im} z| \leq \sqrt{c_1}(Nk)^{1/3}\}$. Then, taking the sign into account, we have,

$$\xi_n^{\pm}(x) = \xi_n^c(x') + Y\big(x', \pm\sqrt{-x_n - b(x')}\big),$$

for $x \in \tilde{\Omega}_-$, and,

$$\xi_n^{\pm i}(x) = \xi_n^c(x') + Y\big(x', \pm i\sqrt{x_n + b(x')}\big),$$

for $x \in \tilde{\Omega}_+ \cap \{x_n + b(x') \leq c_1(Nk)^{2/3}\}$.

We again suppose ξ_n is a critical point. Then $x_n\xi_n + \tilde{g}$ can be represented in terms of x' and z, as,

$$x_n\xi_n + \tilde{g}(x', \xi_n) = a(x') - b(x')\xi_n^c(x') - \xi_n^c(x')z^2 - \tilde{\nu}(x', z)z^3 \tag{4.24}$$

where $\tilde\nu(x',z)$ is smooth in x', holomorphic in z for $\{|\operatorname{Im} z|\leq \sqrt{c_1}(Nk)^{1/3}\}$, and,

$$\tilde\nu(x',z)=\frac{2}{3\sqrt{\tilde\nu_1(x',\xi_n^c(x'))}}+\mathscr{O}(z) \tag{4.25}$$

as $z\to 0$. Let $\Phi(x',z)$ be the right-hand side of (4.24). Then, for $x\in\tilde\Omega_-$, the critical value is,

$$\tilde\phi(x)=\Phi\big(x',\sqrt{-x_n-b(x')}\big)=\phi(x)+\mathscr{O}(h^\infty), \tag{4.26}$$

and, for $x\in\tilde\Omega_+\cap\{x_n+b(x')\leq c_1(Nk)^{2/3}\}$,

$$\tilde\phi(x)=\Phi\big(x',-i\sqrt{x_n+b(x')}\big). \tag{4.27}$$

In particular, since the functions a, b and ξ_n^c are all real valued, we have,

$$\operatorname{Im}\tilde\phi(x)=-\big(x_n+b(x')\big)^{3/2}\operatorname{Re}\tilde\nu\big(x',-i\sqrt{x_n+b(x')}\big),$$

for $x\in\tilde\Omega_+\cap\{x_n+b(x')\leq c_1(Nk)^{2/3}\}$, and thus, in view of (4.4) and (4.25), the estimate (4.19) easily follows.

In order to prove (4.20), recall that ϕ is solution of the eikonal equation,

$$q\Big(x,\frac{\partial\phi}{\partial x}\Big)+E_0=\Big(\frac{\partial\phi}{\partial x}\Big)^2-V(x)+E_0=0.$$

By (4.26), this implies that Φ verifies,

$$\Big(\frac{\partial\Phi}{\partial x'}-\frac{\partial b}{\partial x'}\frac{1}{2z}\frac{\partial\Phi}{\partial z}\Big)^2+\Big(\frac{1}{2z}\frac{\partial\Phi}{\partial z}\Big)^2-V\big(x',-z^2-b(x')\big)+E_0=\mathscr{O}(h^\infty),$$

for z real close enough to 0. Since in addition $\partial\Phi/\partial z=\mathscr{O}(|z|)$ by (4.24), we see that the left-hand side is holomorphic in z for $\{|\operatorname{Im} z|\leq\sqrt{c_1}(Nk)^{1/3}\}$, and thus, returning to the x variable, we easily deduce that $\tilde\phi$ verifies,

$$\Big(\frac{\partial\tilde\phi}{\partial x}\Big)^2-V(x)+E_0=\mathscr{O}(h^\infty),$$

uniformly for $x\in\tilde\Omega_+\cap\{x_n+b(x')\leq c_1(Nk)^{2/3}\}$ and $h>0$ small enough. In particular, taking the imaginary part, we obtain,

$$\nabla \operatorname{Im} \tilde{\phi}(x) \cdot \nabla \operatorname{Re} \tilde{\phi}(x) = \mathscr{O}(h^\infty). \tag{4.28}$$

Then, following [**HeSj1**], we take local coordinates (y', y_n) such that $\{x_n + b = 0\} = \{y_n = 0\}$ and $\nabla(x_n + b) \cdot \nabla = \partial/\partial y_n$. In view of (4.19), we obtain,

$$\nabla \operatorname{Im} \tilde{\phi}(x) \cdot \nabla = -\frac{1}{\sqrt{\nu_1(x', \xi_n^c(x'))}} y_n^{1/2} \frac{\partial}{\partial y_n} + \sum_{j=1}^{n} \mathscr{O}(y_n) \frac{\partial}{\partial y_j},$$

and the vector field $\nabla \operatorname{Im} \tilde{\phi}(x) \cdot \nabla$ can be desingularized at $y_n = 0$ by setting $(z', z_n) := (y', y_n^{1/2})$, leading to,

$$\nabla \operatorname{Im} \tilde{\phi}(x) \cdot \nabla = \Big(-\frac{1}{2\sqrt{\nu_1(x', \xi_n^c(x'))}} + \mathscr{O}(z_n) \Big) \frac{\partial}{\partial z_n} + \sum_{j=1}^{n-1} \mathscr{O}(z_n^2) \frac{\partial}{\partial z_j}.$$

Therefore, using (4.9) and (4.28), we immediately deduce (4.20). □

4.2.4. Modification of $I[c]$.

Let us introduce the notation:

$$\Omega(\varepsilon_1, \varepsilon_2) = \big\{ x \in \tilde{\Omega}; \varepsilon_1 < x_n + b(x') < \varepsilon_2 \big\} \tag{4.29}$$

for two real small numbers $\varepsilon_1 < \varepsilon_2$. We want to extend $I[c]$, which is so far defined in $\Omega(-\delta, 0)$ for some $\delta > 0$, to $\Omega(-\delta, c_1(Nk)^{2/3})$.

If $x \in \Omega(0, c_1(Nk)^{2/3})$, then by Lemma 4.1, $|\operatorname{Im} \xi_n^{-i}(x)| \leq (Nk)^{1/3}$. We modify the integration contour $\gamma(x)$ in (4.11) within this complex strip so that it remains to be a steepest descent curve passing by $\xi_n^{-i}(x)$ for $x \in \Omega(0, c_1(Nk)^{2/3})$. A careful observation of the real part of the phase as in [**HeSj1**, Remark 10.4], gives the following lemma:

LEMMA 4.2. *Let $\delta > 0$ be small enough. Then, for $x \in \Omega(-\delta, c_1(Nk)^{2/3})$, there exists a piecewise smooth curve $\gamma_N(x, h)$ in a small complex neighborhood of $\xi_n = \xi_n^c(x')$ satisfying the following properties:*

- (i) *$\gamma_N(x, h)$ is included in a band $\{\xi_n \in \boldsymbol{C}; |\operatorname{Im} \xi_n| \leq (Nk)^{1/3}\}$ and the extremities are independent of x (i.e. fixed when h is fixed).*
- (ii) *If $x \in \Omega(-\delta, 0)$, $\gamma_N(x, h)$ contains $\xi_n^+(x)$, and along $\gamma_N(x, h)$, one has,*

$$\begin{aligned} &\operatorname{Re} \big(x_n \xi_n + \tilde{g}(x', \xi_n) \big) - \phi(x) \\ &\quad \geq \delta_1 \big(|x_n + b(x')|^{1/2} + |\xi_n - \xi_n^+(x)| \big) \big| \xi_n - \xi_n^+(x) \big|^2 \end{aligned}$$

for some constant $\delta_1 > 0$. *Moreover,* $|\xi_n - \xi_n^+(x)| \geq \delta_1 (Nk)^{1/3}$ *at the extremities of* $\gamma_N(x,h)$.

(iii) *If* $x \in \Omega(0, c_1(Nk)^{2/3})$, $\gamma_N(x,h)$ *contains* $\xi_n^{-i}(x)$, *and along the contour* $\gamma_N(x,h)$, *one has,*

$$\begin{aligned}&\operatorname{Re}\big(x_n\xi_n + \tilde{g}(x',\xi_n) - \tilde{\phi}(x)\big)\\&\qquad \geq \delta_1\big(|x_n + b(x')|^{1/2} + \big|\xi_n - \xi_n^{-i}(x)\big|\big)\big|\xi_n - \xi_n^{-i}(x)\big|^2\end{aligned}$$

for some constant $\delta_1 > 0$. *Moreover,* $|\xi_n - \xi_n^{-i}(x)| \geq \delta_1 (Nk)^{1/3}$ *at the extremities of* $\gamma_N(x,h)$.

Now, we also define a holomorphic $(Nk)^{1/3}$-approximation $\tilde{c}(x',\cdot,h)$ of the symbol $c(x',\cdot,h)$ by writing $c \sim \sum_{j\geq 0} h^j c_j$, and by taking a holomorphic $(Nk)^{1/3}$-approximation $\tilde{c}_j$ of c_j, and by re-summing the formal symbol $\sum_{j\geq 0} h^j \tilde{c}_j$ (note that here, each $\tilde{c}_j$ depends on h, but in a very well-controlled way).

With these $\tilde{c}(x',\cdot,h)$ and $\gamma_N(x,h)$, we define the modified integral representation of $e^{S/h}w$ by the formula,

$$\tilde{I}_N[\tilde{c}](x,h) = h^{-1/2}\int_{\gamma_N(x,h)} e^{-(x_n\xi_n + \tilde{g}(x',\xi_n))/h}\tilde{c}(x',\xi_n,h)d\xi_n. \tag{4.30}$$

Lemma 4.3. *There exists a constant* $\delta_2 > 0$ *such that, for* $x \in \Omega(-\delta, 0)$, *and for all* $N \geq 1$, *one has,*

$$\partial^\alpha\big(I[c](x,h) - \tilde{I}_N[\tilde{c}](x,h)\big) = \mathscr{O}\big(h^{\delta_2 N - 1/2 - |\alpha|}e^{-\phi(x)/h}\big).$$

Proof. By definition (c,g) and $(\tilde{c},\tilde{g})$ coincide on the real, up to $\mathscr{O}(h^\infty)$. Therefore, substituting the real contour $\gamma(x)$ to $\gamma_N(x,h)$ in the expression of $\tilde{I}_N[\tilde{c}](x,h)$, we obtain an integral $J_N(x)$ that coincides with $I[c](x,h)$ up to $\mathscr{O}(h^\infty e^{-\phi(x)/h})$. Then, modifying continuously $\gamma(x)$ into $\gamma_N(x,h)$ in $J_N(x)$, we recover $\tilde{I}_N[\tilde{c}](x,h)$ up to error terms coming from the fact that $\gamma(x)$ and $\gamma_N(x,h)$ do not have the same extremities. However, in view of Lemma 4.2 (ii) and the fact that, along $\gamma(x)$, the minimum of $\operatorname{Re}(x_n\xi_n + \tilde{g}(x',\xi_n)) - \phi(x)$ is non-degenerate, we see that the deformation can be done in such a way that these error terms are $\mathscr{O}(e^{-(\phi(x)+\delta_2 Nk)/h}) = \mathscr{O}(h^{\delta_2 N}e^{-\phi(x)/h})$, with $\delta_2 = \delta_1^4$. □

Lemma 4.4. *As* $h \to 0$, *one has*

$$(P - \rho(h))\tilde{I}_N[\tilde{c}] = \mathscr{O}\big(h^{\delta_2 N}e^{-\operatorname{Re}\tilde{\phi}(x)/h}\big),$$

uniformly in $\Omega(-\delta, c_1(Nk)^{2/3})$.

Proof. In view of (4.15), it is enough to check,

$$P\big(\tilde{I}_N[\tilde{c}]\big) = h^{-1/2}\int_{\gamma_N(x,h)} e^{-x_n\xi_n/h}\hat{P}\big(e^{-\tilde{g}/h}\tilde{c}\big)d\xi_n + \mathscr{O}\big(h^{\delta_2 N}e^{-\operatorname{Re}\tilde{\phi}(x)/h}\big).$$

This can be done exactly in the same way as Proposition 10.5 in [**HeSj1**], with the only difference that, in our case, the values of $\operatorname{Re}(x_n\xi_n + g)$ at the extremities of $\gamma_N(x,h)$ are greater than $\operatorname{Re}\tilde{\phi}(x) + \delta_2 Nk$. □

4.2.5. Asymptotic expansion of $\tilde{I}_N[\tilde{c}]$.
Here, we fix $c_2 \in (0, c_1)$, and we study the asymptotic behavior of $\tilde{I}_N[\tilde{c}](x,h)$ as h tends to 0, for x in $\Omega(c_2(Nk)^{2/3}, c_1(Nk)^{2/3})$. Setting,

$$\begin{aligned}\tilde{r}(x) &:= \frac{1}{2}\frac{\partial^2\tilde{g}}{\partial\xi_n^2}(x', \xi_n^{-i}(x))\\ &\qquad \big(= -i\sqrt{\tilde{\nu}_1(x', \xi_n^c(x'))(x_n + b(x'))} + \mathscr{O}(|x_n + b(x')|)\big),\end{aligned}$$

we have,

Proposition 4.5. *For all integers L, M and N large enough, and for x in $\Omega(c_2(Nk)^{2/3}, c_1(Nk)^{2/3})$, one has,*

$$\tilde{I}_N[\tilde{c}](x,h) = \frac{e^{-\tilde{\phi}(x)/h}}{\sqrt{\tilde{r}(x)}}\left\{\sum_{m=0}^{L+[M/2]}\beta_m(x)\left\{\frac{h}{\tilde{r}(x)^3}\right\}^m + R_{L,M,N}(x,h)\right\}, \tag{4.31}$$

with, for any $\alpha \in \boldsymbol{N}^n$,

$$\begin{aligned}\big|\partial_x^\alpha R_{L,M,N}(x,h)\big| \le C_{N,\alpha}h^{\delta_\alpha N - 1/2}\sum_{m=0}^{M} C_{L,\alpha}^{m+1}\left(\frac{hm}{Nk}\right)^{m/2}\\ + C_{L,\alpha}h^{L+1/2} + h^{-1/2}C_{L,\alpha}^{M+1}\left(\frac{hM}{Nk}\right)^{M/2},\end{aligned} \tag{4.32}$$

where the positive constant δ_α does not depend on (L,M,N), while $C_{N,\alpha}$ does not depend on (L,M), and $C_{L,\alpha}$ does not depend on (M,N). Moreover, the coefficients of the symbol verify,

$$\begin{aligned}\beta_0(x) &= \sqrt{\pi}\tilde{c}_0\big(x', \xi_n^{-i}(x)\big) = \sqrt{\pi}\tilde{c}_0\big(x', \xi_n^c(x')\big) + \mathscr{O}\big(\sqrt{x_n + b(x')}\big),\\ \beta_m(x) &= \mathscr{O}(1) \quad (m = 0, 1, \ldots) \quad \textit{as} \quad x_n + b(x') \to 0.\end{aligned} \tag{4.33}$$

In particular, taking $M = 2[Nk/C'_L h]$ *with* $C'_L > 0$ *large enough depending on L only, one obtains,*

$$\tilde{I}_N[\tilde{c}](x,h) = \frac{e^{-\tilde{\phi}(x)/h}}{\sqrt{\tilde{r}(x)}}\left\{\sum_{m=0}^{L+[Nk/C'_L h]} \beta_m(x)\left\{\frac{h}{\tilde{r}(x)^3}\right\}^m + \mathscr{O}\big(h^{\delta_L N} + h^L\big)\right\}, \tag{4.34}$$

uniformly for $x \in \Omega(c_2(Nk)^{2/3}, c_1(Nk)^{2/3})$ *and* $h > 0$ *small enough, and where the positive constant* δ_L *does not depend on* N *large enough.*

Proof. For $x \in \Omega(c_2(Nk)^{2/3}, c_1(Nk)^{2/3})$, setting $\eta = \xi_n - \xi_n^{-i}(x)$, we can write,

$$x_n\xi_n + \tilde{g}(x', \xi_n) = \tilde{\phi}(x) + \tilde{r}(x)\eta^2 + G(x,\eta)\eta^3$$

where $G(x,\eta) := \int_0^1((1-t)^3/2)\partial_{\xi_n}^3\tilde{g}(x', \xi_n^{-i}(x) + t\eta)dt$ is holomorphic with respect to η in $\{|\operatorname{Re}\eta| < \delta_1,\ |\operatorname{Im}\eta| < \delta_1(Nk)^{1/3}\}$, with $\delta_1 > 0$ small enough (independent of N). Then, we set,

$$\tilde{r}(x)\eta^2 + G(x,\eta)\eta^3 = \tilde{r}(x)\zeta^2,$$

so that $\hat{\eta} = \eta/\tilde{r}$, $\hat{\zeta} = \zeta/\tilde{r}$ verify,

$$\hat{\eta}\sqrt{1 + G(x, \tilde{r}(x)\hat{\eta})\hat{\eta}} = \hat{\zeta},$$

where the square root is 1 for $\hat{\eta} = 0$. This equation is solvable with respect to $\hat{\eta}$, and gives $\hat{\eta} = \hat{\eta}(x, \hat{\zeta})$ where the function $\hat{\eta}(x,\hat{\zeta})$ is smooth with respect to $x \in \Omega(c_2(Nk)^{2/3}, c_1(Nk)^{2/3})$ and holomorphic with respect to $\hat{\zeta}$ in some fixed neighborhood of 0, and $(\partial\hat{\eta}/\partial\hat{\zeta})|_{\hat{\zeta}=0} = 1$.

Changing the variables from ξ_n to ζ in (4.30), we obtain,

$$\tilde{I}_N[\tilde{c}](x,h) = h^{-1/2}e^{-\tilde{\phi}(x)/h}\int_{\Gamma_N(x,h)} e^{-\tilde{r}(x)\zeta^2/h}F\Big(x, \frac{\zeta}{\tilde{r}}, h\Big)d\zeta,$$

where the contour $\Gamma_N(x,h)$ is such that,

$$0 \in \Gamma_N(x,h); \mathrm{Re}(\tilde{r}(x)\zeta^2) \geq \delta_3 |\tilde{r}(x)| \cdot |\zeta|^2 \text{ along } \Gamma_N(x,h); \tag{4.35}$$

$$|\zeta| \geq \delta_3 |\tilde{r}(x)| \text{ at the extremities of } \Gamma_N(x,h), \tag{4.36}$$

for some positive constant δ_3, and where the symbol,

$$F\big(x,\hat{\zeta},h\big) := \tilde{c}\big(x',\xi_n^{-i}(x) + \tilde{r}(x)\hat{\eta}(x,\hat{\zeta}),h\big)\frac{\partial \hat{\eta}}{\partial \hat{\zeta}}(x,\hat{\zeta})$$

can be developed asymptotically into,

$$F\big(x,\hat{\zeta},h\big) \sim \sum_{\ell \geq 0} F_\ell\big(x,\hat{\zeta}\big)h^\ell,$$

with F_ℓ holomorphic with respect to $\hat{\zeta}$ in a fixed neighborhood of 0, and $F_0(x,0) = \tilde{c}_0(x',\xi_n^{-i}(x))$. Actually, F_ℓ also depends on N, but using Lemma 8.1 (ii) and the fact that $|\tilde{r}(x)| \sim (Nk)^{1/3}$, we easily obtain that the derivatives of F_ℓ verify,

$$\left|\partial_{\hat{\zeta}}^\beta F_\ell\right| \leq C_\ell^{1+|\beta|}\beta!,$$

for some constant $C_\ell > 0$ independent of N.

Now we set $y = \tilde{r}(x)^{1/2}\zeta$, where $\tilde{r}(x)^{1/2}$ is the branch such that $\tilde{r}(x)^{1/2} \sim e^{-i\pi/4}(\nu_1(x_n+b))^{1/4}$, and, using (4.35)–(4.36), we see that a new slight modification of the contour of integration gives,

$$\tilde{I}_N[\tilde{c}](x,h) = \frac{e^{-\tilde{\phi}(x)/h}}{\sqrt{h\tilde{r}(x)}}\left(\int_{-\delta_4\tilde{r}^{3/2}}^{\delta_4\tilde{r}^{3/2}} e^{-y^2/h}F\left(x,\frac{y}{\tilde{r}^{3/2}},h\right)dy + \mathscr{O}\big(e^{-\delta_5\tilde{r}(x)^3/h}\big)\right)$$

for some constants $\delta_4, \delta_5 > 0$. As a consequence, using again that $|\tilde{r}(x)| \sim (Nk)^{1/3}$ and writing $F = \sum_{\ell=0}^L F_\ell h^\ell + \mathscr{O}(h^{L+1})$, we obtain,

$$\begin{aligned}&\tilde{I}_N[\tilde{c}](x,h)\\&\quad= \frac{e^{-\tilde{\phi}(x)/h}}{\sqrt{h\tilde{r}(x)}}\left(\sum_{\ell=0}^L h^\ell\int_{-\delta_4\tilde{r}^{3/2}}^{\delta_4\tilde{r}^{3/2}} e^{-y^2/h}F_\ell\left(x,\frac{y}{\tilde{r}^{3/2}}\right)dy + \mathscr{O}\big(h^{\delta_6 N} + h^{L+1}\big)\right)\end{aligned}$$

with some new positive constant δ_6.

At this point, we can proceed with the usual Laplace method in order to estimate each term of the previous sum. Writing,

$$F_\ell\Big(x, \frac{y}{\tilde r^{3/2}}\Big) = \sum_{m=0}^{\infty} F_{\ell,m}(x)\frac{y^m}{\tilde r^{3m/2}} = \sum_{m=0}^{M} F_{\ell,m}(x)\frac{y^m}{\tilde r^{3m/2}} + S_{M,\ell},$$

with $|F_{\ell,m}(x)|\leq C_\ell^{m+1}$ and $|S_{M,\ell}|\leq 2C_\ell^{M+2}|y/\tilde r^{3/2}|^{M+1}$, we obtain,

$$\begin{aligned}
&\tilde I_N[\tilde c](x,h)\\
&= \frac{e^{-\tilde\phi(x)/h}}{\sqrt{h\tilde r(x)}}\Big(\sum_{\ell=0}^{L}\sum_{m=0}^{M} F_{\ell,m}(x)h^\ell\int_{-\delta_4\tilde r^{3/2}}^{\delta_4\tilde r^{3/2}} e^{-y^2/h}\Big(\frac{y}{\tilde r^{3/2}}\Big)^m dy + R^{(1)}_{L,M,N}(x)\Big)\\
&= \frac{e^{-\tilde\phi(x)/h}}{\sqrt{h\tilde r(x)}}\Big(\sum_{\ell=0}^{L}\sum_{m=0}^{M} F_{\ell,m}(x)h^\ell\int_{-\infty}^{+\infty} e^{-y^2/h}\Big(\frac{y}{\tilde r^{3/2}}\Big)^m dy\\
&\qquad\qquad + R^{(1)}_{L,M,N}(x) + R^{(2)}_{L,M,N}(x)\Big)\\
&= \frac{e^{-\tilde\phi(x)/h}}{\sqrt{h\tilde r(x)}}\Big(\sum_{\ell=0}^{L}\sum_{m=0}^{[M/2]} \Gamma\Big(m+\frac12\Big)F_{\ell,2m}(x)\frac{h^{m+\ell+1/2}}{\tilde r^{3m}} + R^{(1)}_{L,M,N}(x) + R^{(2)}_{L,M,N}(x)\Big),
\end{aligned} \tag{4.37}$$

with,

$$\big|R^{(1)}_{L,M,N}(x)\big| \leq C_N h^{\delta_6 N} + C_L h^{L+1} + C_L^{M+1}\Big(\frac{h}{Nk}\Big)^{M/2} M^{M/2};$$

$$\big|R^{(2)}_{L,M,N}(x)\big| \leq C_N h^{\delta_6 N}\sum_{m=0}^{M} C_L^{m+1}\Big(\frac{h}{Nk}\Big)^{m/2} m^{m/2},$$

for some positive constant δ_6 independent of (L,M,N), some positive constant C_N independent of (L,M), and some positive constant C_L independent of (M,N). Similar estimates hold true for all the derivatives with respect to x of $R^{(1)}_{L,M,N}$ and $R^{(2)}_{L,M,N}$.

Hence, we obtain (4.31) if we set,

$$\beta_m(x) := \sum_{j+\ell=m}\Gamma\Big(j+\frac12\Big)F_{\ell,2j}(x)\tilde r(x)^{3\ell},$$

and,

$$R_{L,M,N}(x) := h^{-1/2}\Bigg(R^{(1)}_{L,M,N}(x) + R^{(2)}_{L,M,N}(x) - \sum_{\substack{\ell+m\leq L+[M/2]\\ \ell>L \text{ or } m>[M/2]}} \Gamma\left(m+\frac{1}{2}\right) F_{\ell,2m}(x)\frac{h^{m+\ell+1/2}}{\tilde{r}^{3m}}\Bigg).$$

In particular, (4.32) with $\alpha = 0$ is verified, as well as the estimates on $\beta_m(x)$ and, moreover,

$$\beta_0(x) = \Gamma\left(\frac{1}{2}\right)F_0(x,0) = \sqrt{\pi}\tilde{c}_0\big(x', \xi_n^{-i}(x)\big).$$

The estimate (4.32) for all α is obtained in the same way.

Finally, substituting $M = 2[Nk/C'_L h]$ into (4.32) with $\alpha = 0$, (4.34) follows by taking $C'_L > 4C^2_{L,0}$, since in that case, we have,

$$\begin{aligned}C^{M+1}_{L,0}\left(\frac{hM}{Nk}\right)^{M/2} &\leq C_{L,0}\left(\frac{2C^2_{L,0}}{C'_L}\right)^{M/2} \leq C_{L,0}2^{-[Nk/C'_L h]}\\ &\leq 2C_{L,0}2^{-Nk/C'_L h} = 2C_{L,0}h^{\delta_L N},\end{aligned}$$

with $\delta_L := (\ln 2)/C'_L$, and,

$$\begin{aligned}\sum_{m=0}^{M} C^{m+1}_{L,0}\left(\frac{hm}{Nk}\right)^{m/2} &\leq \sum_{m=0}^{M} C^{m+1}_{L,0}\left(\frac{2}{C'_L}\right)^{m/2}\\ &\leq C_{L,0}\sum_{m=0}^{\infty} 2^{-m/2} = \frac{\sqrt{2}}{\sqrt{2}-1}C_{L,0}.\end{aligned}$$

□

Let us observe that the principal symbol (denoted by $\tilde{a}_0(x)$) of the asymptotic expansion of $\tilde{I}_N[\tilde{c}]$ is,

$$\tilde{a}_0(x) = \frac{\beta_0(x)}{\sqrt{\tilde{r}(x)}} = \sqrt{\frac{\pi}{\tilde{r}(x)}}\tilde{c}_0\big(x', \xi_n^{-i}(x)\big),$$

and it behaves like,

$$\tilde{a}_0(x) = \frac{\sqrt{\pi}c_0(x', \xi^c_n(x'))}{\nu_1(x', \xi^c_n(x'))^{1/4}} z^{-1/2}(1 + \mathscr{O}(z)), \tag{4.38}$$

as $z = -i\sqrt{x_n + b(x')} \to 0$. Recall that the principal term of $I[c]$ for $x \in \tilde{\Omega}_-$ should coincide with $a_0(x)$ (see (4.14)), that is,

$$a_0(x) = \sqrt{\frac{\pi}{r(x)}}\, c_0\big(x', \xi_n^+(x)\big),$$

and it has the same behavior (4.38) as $\tilde{a}_0(x)$, as $z = \sqrt{-x_n - b(x')} \to 0$.

4.3. Global WKB solution near $\partial \ddot{O}$.

The previous study shows that, for any point x^1 of Γ and for any $N \geq 1$, the WKB solution $w = h^{-n/4} e^{-S/h} I[c]$ can be extended in a neighborhood of x^1 of the form,

$$\Omega_N(x^1) := \bigcup_{-t_0 < t < (Nk)^{2/3}} \exp t\nabla f(\omega(x_1)),$$

where $\omega(x_1)$ is a fixed small enough neighborhood of x^1 in the caustic set $\mathscr{C}$, and f is such that $f = 0$ is an equation of $\mathscr{C}$ near x^1, with $\{f > 0\} \cap \ddot{O}^C \neq \emptyset$. Therefore, by using a standard partition of unity in a neighborhood $\omega(\Gamma)$ of Γ in $\mathscr{C}$, we obtain an extension w_N of w in an open set of the form,

$$\Omega_N := \bigcup_{-t_0 < t < (Nk)^{2/3}} \exp tX(\omega(\Gamma)),$$

where X is any vector-field transverse to $\mathscr{C}$ near Γ and directed towards $\ddot{O}^C$ (for instance, one can take $X = -\nabla V$).

Moreover, by Assumption (A4) and (4.9) (see also [**HeSj1**, Remarque 10.4]), we see that, if $x \in \mathscr{C}$ is such that $\mathrm{dist}(\Gamma, x) \sim (Nk)^{1/2}$, then $\mathrm{Re}\, \tilde{\phi}(x) \sim Nk$, and thus $w_N(x) = \mathscr{O}(h^{-n/4} e^{-(S + \mathrm{Re}\, \tilde{\phi})/h}) = \mathscr{O}(h^{\delta N} e^{-S/h})$ for some constant $\delta > 0$. We also observe that, on $\mathscr{C}$, we have $d(x_0, x) - S = \mathrm{Re}\, \tilde{\phi}(x) \sim \mathrm{dist}(\Gamma, x)^2$. As a consequence (thanks to (4.28)), if we set,

$$\gamma(\mathscr{C}) := \{x \in \mathscr{C}\, ;\, S + Nk \leq d(x_0, x) \leq S + 2Nk\},$$

then, we also have,

$$w_N = \mathscr{O}\big(h^{\delta N} e^{-S/h}\big) \ \text{ on } \ \gamma_N^+(\mathscr{C}) := \bigcup_{0 \leq t < (Nk)^{2/3}} \exp tX(\gamma(\mathscr{C})).$$

We extend our WKB solution w_N in the domain

$$M_\eta := \{x \in \ddot{O}; \mathrm{dist}(x, \partial\ddot{O}) > \eta\},$$

for small $\eta > 0$, by patching it (in the standard way by using a partition of unity) with the eigenfunction corresponding to the first eigenvalue of the Dirichlet problem of P on M_η. Since they coincide modulo $\mathscr{O}(h^\infty e^{-d(x_0,x)/h})$ locally uniformly in $\Omega \cap M_\eta$ (see [**HeSj2**]), the extension (that we always denote by w_N) still satisfies

$$(P - \rho(h))w_N(x, h) = \mathscr{O}\big(h^\infty e^{-(S+\mathrm{Re}\,\tilde{\phi}(x))/h}\big)$$

locally uniformly in $\Omega \cup M_\eta$.

Now, for $\varepsilon, t_0 > 0$ small enough, we set,

$$\omega(\varepsilon) := \{x \in \mathscr{C}; d(x_0, x) \le S + \varepsilon\};$$
$$\omega^+(\varepsilon, t_0) := \bigcup_{0 \le t < t_0} \exp tX(\omega(\varepsilon));$$
$$\Omega(\varepsilon, t_0) := \{x \in \Omega \cup M_\eta; d(x_0, x) < S + \varepsilon\} \cup \omega^+(\varepsilon, t_0).$$

Then $\Omega(\varepsilon, t_0)$ is an open set including $\overline{B_d(x_0, S)}$ for sufficiently small η, and the previous discussion shows that w_N is well defined and C^∞ on $\Omega(2Nk, (Nk)^{2/3})$, with,

$$w_N = \mathscr{O}\big(h^{\delta N} e^{-S/h}\big) \text{ on } \Omega(2Nk, (Nk)^{2/3}) \backslash \Omega(Nk, (Nk)^{2/3}), \tag{4.39}$$

(see [**HeSj2**] for this estimate in $M_\eta \backslash \Omega$), and moreover, by construction, it satisfies

$$(P - \rho(h))w_N(x, h) = \mathscr{O}\big(h^{\delta N} e^{-(S+\mathrm{Re}\,\tilde{\phi}(x))/h}\big) \tag{4.40}$$

on $\Omega(2Nk, (Nk)^{2/3})$, for some constant $\delta > 0$.

As a consequence, if we take a cut-off function χ_N such that $\mathrm{Supp}\,\chi_N \subset \Omega(2Nk, (Nk)^{2/3})$, $\chi_N = 1$ on $\Omega(Nk, (1/2)(Nk)^{2/3})$, $\partial^\alpha \chi_N = \mathscr{O}(k^{-N_\alpha})$ (for all $\alpha \in \boldsymbol{N}^n$, and some $N_\alpha \ge 0$ independent of N), then, setting,

$$\tilde{w}_N := \chi_N w_N,$$

we see that $\tilde{w}_N$ is C^∞ on $\{\mathrm{dist}(x, \ddot{O}) < \delta_0 (Nk)^{2/3}\}$ for some fixed $\delta_0 > 0$, and it verifies,

$$(P - \rho)\tilde{w}_N = \mathscr{O}\big(h^{\delta N} e^{-\mathrm{Re}\,\tilde{\phi}/h}\big) \text{ in } \{\mathrm{dist}(x, \ddot{O}) < \delta_0 (Nk)^{2/3}\}.$$

Summing up (and slightly changing the notations by writing w_N instead of $\tilde{w}_{2^{3/2}N/\delta_0^{3/2}}$), we have proved,

Proposition 4.6. *For any large enough N, there exists a smooth function $w_N(x,h) \in C^\infty(\ddot{O}_N)$, with $\ddot{O}_N := \{\mathrm{dist}(x,\ddot{O}) < 2(Nk)^{2/3}\}$, verifying the following properties*:

(i) *There exists a constant $\delta > 0$, independent of N, such that uniformly in $\ddot{O}_N$, and for all $\alpha \in \boldsymbol{Z}_+^n$, one has,*

$$\partial^\alpha w_N(x,h) = \mathscr{O}\big(h^{-m_\alpha} e^{-(S+\mathrm{Re}\,\tilde{\phi}(x))/h}\big);$$
$$(P-\rho(h))w_N(x,h) = \mathscr{O}\big(h^{\delta N} e^{-(S+\mathrm{Re}\,\tilde{\phi}(x))/h}\big),$$

for some $m_\alpha \geq 0$, and where $\tilde{\phi}$ is defined by $\tilde{\phi}(x) = d(x_0,x) - S$ for $x \in \Omega \cup M_\eta$, and by (4.18) for $x \in \omega^+(2Nk,(Nk)^{2/3})$.

(ii) *In any compact subset of Ω, for any $M \in \boldsymbol{N}$, one has,*

$$w_N(x,h) = h^{-n/4} e^{-(S+\tilde{\phi}(x))/h} \left(\sum_{j=0}^{M} a_j(x) h^j + \mathscr{O}(h^{M+1}) \right),$$

as $h \to 0$, where $a_j(x)$ are extensions of those given in (2.3), and a_0 is elliptic.

(iii) *In $\{(Nk)^{2/3} < \mathrm{dist}(x,\ddot{O}) < 2(Nk)^{2/3}\} \cap \omega^+(Nk,(1/2)(Nk)^{2/3})$, for any large enough L, there exist $C'_L > 0$ and $\delta_L > 0$ independent of N such that*

$$w_N(x,h) = h^{-n/4} e^{-(S+\tilde{\phi}(x))/h} \left(\sum_{j=0}^{L+[Nk/C'_L h]} \tilde{a}_j(x) h^j + \mathscr{O}\big(h^{\delta_L N} + h^L\big) \right), \tag{4.41}$$

as $h \to 0$, with $\tilde{a}_j$ (independent of h) of the form,

$$\tilde{a}_j(x) = (\mathrm{dist}(x,\mathscr{C}))^{-3j/2-1/4} \tilde{\beta}_j(x, \mathrm{dist}(x,\mathscr{C})), \tag{4.42}$$

where $\tilde{\beta}_j$ is smooth near $\Gamma \times \{0\}$ for all j, and in particular $\tilde{\beta}_0$ is elliptic near $\Gamma \times \{0\}$.

5. Comparison in the island.

In this section, we compare the WKB solution w_N with the true resonant state u inside $\ddot{O}$ near a point of interaction x^1. More precisely, we obtain an estimate on the difference up to a distance of order $(Nk)^{2/3}$ of x^1.

We use the same notations as in Section 4. Let $x \in \tilde{\Omega}_-$ be a point sufficiently close to x^1. Using the representation formula (4.12) for ϕ (see also [**HeSj1**, Formula (10.22)]), we see that,

$$\phi(x) \geq \phi(x', -b(x')) + (x_n + b(x'))\xi_n^c(x') - C_1|x_n + b(x')|^{3/2}, \tag{5.1}$$

for some positive constant C_1. Moreover, thanks to Assumption (A4), we already know that $\phi(x', -b(x')) = \phi|_{\mathscr{C}}(x', -b(x')) \geq \delta|x'|^2$ with $\delta > 0$ constant, and thus, using (4.4), we see that $\phi(x', -b(x')) + (x_n + b(x'))\xi_n^c(x') \geq 0$ near x^1. As a consequence, we deduce from (5.1),

$$d(x_0, x) \geq S - C_1|x_n + b(x')|^{3/2}. \tag{5.2}$$

In particular, if $x \in \Omega(-(Nk)^{2/3}, 0)$, $k = h\log(1/h)$, we have,

$$e^{-s(x)/h} = \mathscr{O}\big(h^{-C_1N}e^{-S/h}\big).$$

The aim of this section is to show a local a priori estimate near a point of interaction x^1 (Proposition 5.1), and then, as a direct consequence, a global a priori estimate in a neighborhood of $\partial\ddot{O}$ (Proposition 5.2).

Proposition 5.1. *There exists $N_2 \in \boldsymbol{Z}$ and $C > 0$, such that, for any $N > 0$, one has,*

$$\|u(x,h) - w_{CN}(x,h)\|_{H^1(\Omega(-(Nk)^{2/3},0))} = \mathscr{O}\big(h^{-N_2}e^{-S/h}\big),$$

uniformly as $h \to 0$.

Proof. Recall (Theorem 2.2) that there exists N_0 such that

$$\big\|e^{s(x)/h}u(x,h)\big\|_{H^1(\tilde{\Omega}_-)} = \mathscr{O}(h^{-N_0}).$$

The WKB solution w_{CN} also satisfies the same estimate (see Proposition 4.6), and hence so does the difference,

$$\left\|e^{s(x)/h}(u(x,h)-w_{CN}(x,h))\right\|_{H^1(\tilde\Omega_-)}=\mathscr{O}(h^{-N_0}).$$

In particular,

$$\|u(x,h)-w_{CN}(x,h)\|_{H^1(\tilde\Omega_-\cap\{d(x_0,x)\geq S-2k\})}=\mathscr{O}\big(h^{-N_0'}e^{-S/h}\big),$$

for some other constant N_0'.

Now, we set,

$$\Omega_1=\Omega_1(h):=B_d(x_0,S-k)\cap\tilde\Omega_-.$$

Since every point of Ω_1 can be connected to x_0 by a smooth minimal geodesic (with respect to the Agmon distance), the arguments of the previous section show that the WKB solution $w_{CN}(x,h)$ is well defined in all of Ω_1 (we use its integral representation when x becomes too close to a point of interaction). Moreover, it is not difficult to construct $\chi_h\in C_0^\infty(\Omega_1)$, such that $\chi_h=1$ on $\{d(x_0,x)\leq S-2k\}$, $0\leq\chi_h\leq 1$ everywhere, and, for all $\alpha\in\boldsymbol{N}^n$,

$$\partial^\alpha\chi_h=\mathscr{O}(h^{-N_\alpha}),$$

for some constant $N_\alpha\geq 0$. Then, we set,

$$\hat w:=\chi_h(x)w_{CN}(x,h),$$

and, for $N\geq 1$ arbitrarily large,

$$\begin{aligned}\phi_N(x)=\min\big(d(x_0,x)+C_1Nk+k(S-d(x_0,x))^{1/3},&\\ S+(1-k^{1/3})(S-d(x_0,x))\big).&\end{aligned}$$

On $\Omega(-(Nk)^{2/3},0)$, by (5.2) we have $d(x_0,x)\geq S-C_1Nk$. Therefore, $\phi_N(x)\geq S$ there, and it suffices to show that there exists N_0 such that, for any $N\geq 1$,

$$\left\|e^{\phi_N/h}(\chi_hu-\hat w)\right\|_{H^1(\Omega_1)}=\mathscr{O}(h^{-N_0}).$$

We prove it by using Agmon estimates (see Lemma 8.2 in Appendix). At first, we observe that, by construction (and since $\phi_N\leq d(x_0,x)+(C_1N+S^{1/3})k$), we have (uniformly in Ω_1),

$$
\begin{aligned}
(P-\rho(h))\hat{w} &= [P,\chi_h]w_{CN} + \mathscr{O}\big(h^{\delta CN}e^{-d(x_0,x)/h}\big)\\
&= \mathscr{O}\big(1_{\operatorname{Supp}\nabla\chi_h}h^{-M_1}e^{-S/h}\big) + \mathscr{O}\big(h^{\delta CN-C_1N-S^{1/3}}e^{-\phi_N/h}\big), \qquad (5.3)
\end{aligned}
$$

for some $M_1 \geq 0$ constant. Moreover, using (2.7),

$$
\big\|e^{\phi_N/h}(P-\rho(h))\chi_h u\big\|_{L^2} = \big\|e^{\phi_N/h}[P,\chi_h]u\big\|_{L^2} = \mathscr{O}\big(h^{-M_1'}e^{(F_N-S)/h}\big), \qquad (5.4)
$$

for some other constant $M_1' \geq 0$, and with,

$$
F_N := \sup_{\operatorname{Supp}\nabla\chi_h} \phi_N.
$$

Since $S - d(x_0,x) \leq 2k$ on $\operatorname{Supp}\nabla\chi_h$, we have $F_N \leq S + 2(1-k^{1/3})k \leq S + 2k$, and thus, we deduce from (5.4),

$$
\big\|e^{\phi_N/h}(P-\rho(h))\chi_h u\big\|_{L^2} = \mathscr{O}\big(h^{-M_1'-2}\big). \qquad (5.5)
$$

Setting,

$$
u_h' := \chi_h u - \hat{w},
$$

and choosing C such that $\delta C \geq C_1$, we obtain from (5.3)–(5.5),

$$
\big\|e^{\phi_N/h}(P-\rho(h))u_h'\big\|_{L^2} = \mathscr{O}(h^{-M_2}), \qquad (5.6)
$$

for some constant $M_2 \geq 0$, independent of N.

We also observe, that, on $\Omega_1^- := \Omega_1 \cap \{d(x_0,x) + C_1Nk + k(S-d(x_0,x))^{1/3} < S + (1-k^{1/3})(S-d(x_0,x))\}$, we have,

$$
\nabla\phi_N = \left(1 - \frac{k}{3(S-d(x_0,x))^{2/3}}\right)\nabla d(x_0,x),
$$

and, on $\Omega_1^+ := \Omega_1 \cap \{d(x_0,x) + C_1Nk + k(S-d(x_0,x))^{1/3} > S + (1-k^{1/3})(S - d(x_0,x))\}$,

$$
\nabla\phi_N = -(1-k^{1/3})\nabla d(x_0,x).
$$

Since $k(S-d(x_0,x))^{-2/3} \leq k^{1/3} \ll 1$, for h sufficiently small we easily deduce,

$$V - \operatorname{Re}\rho - |\nabla\phi_N|^2 \geq \frac{k}{3(S - d(x_0,x))^{2/3}}(V - E_0) - (\operatorname{Re}\rho - E_0), \text{ on } \Omega_1^-;$$

and,

$$V - \operatorname{Re}\rho - |\nabla\phi_N|^2 \geq k^{1/3}(V - E_0) - (\operatorname{Re}\rho - E_0), \text{ on } \Omega_1^+.$$

Now, since $\nabla V \neq 0$ on $\partial\ddot{O}$, a quick examination of the Hamilton curves of $q = \xi^2 - V(x)$ starting from $\partial\ddot{O} \times \{0\}$, shows that, for $x \in \ddot{O}$ close enough to $\partial\ddot{O}$, one has $d(x, \partial\ddot{O}) = \mathscr{O}((V(x) - E_0)^{3/2})$. Therefore, by the triangle inequality, we deduce,

$$d(x_0, x) \geq S - C_2(V(x) - E_0)^{3/2}, \tag{5.7}$$

where $C_2 > 0$ is a constant, and the inequality is actually valid in all of $\ddot{O}$ except for some fixed small enough neighborhood U_0 of x_0 (since $V - E_0 > 0$ on $\ddot{O}\backslash\{x_0\}$).

In particular, U_0 can be assumed to be disjoint from Ω_1^+, and then (5.7) shows that $V(x) - E_0 \geq (k/C_2)^{2/3}$ on Ω_1^+. Therefore, observing also that $|\rho - E_0| \leq C_3 h$ with $C_3 > 0$ constant, on this set, we obtain,

$$V - \operatorname{Re}\rho - |\nabla\phi_N|^2 \geq \frac{k}{C_2^{2/3}} - C_3 h \geq \frac{k}{2C_2^{2/3}}, \tag{5.8}$$

for $h > 0$ small enough.

Moreover, by (5.7), on $\ddot{O}\backslash U_0$, we also have,

$$\frac{V - E_0}{3(S - d(x_0,x))^{2/3}} \geq \frac{1}{3C_2^{2/3}},$$

and thus, if $x \in \Omega_1^- \backslash U_0$,

$$V - \operatorname{Re}\rho - |\nabla\phi_N|^2 \geq \frac{k}{3C_2^{2/3}} - C_3 h \geq \frac{k}{4C_2^{2/3}}, \tag{5.9}$$

for $h > 0$ small enough. On the other hand, by (2.5) and the results of [**HeSj2**], we know that $w_{CN}(x,h)$ is a good approximation of $u(x,h)$ on U_0, in the sense that,

$$\left\| e^{d(x_0,x)/h} u_h' \right\|_{L^2(U_0)} = \mathscr{O}(h^\infty).$$

Since $e^{\phi_N/h} = \mathscr{O}(h^{-C_1 N - S^{1/3}} e^{d(x_0,x)/h})$, we deduce,

$$\left\| e^{\phi_N/h} u'_h \right\|_{L^2(U_0)} = \mathscr{O}(h^\infty). \tag{5.10}$$

Now, we apply the identity (3.14) with $u'_h, \phi_N, \operatorname{Re}\rho$ instead of v_h, ϕ, λ_h. Using (5.6), (5.8), (5.9), and (5.10), this permits us to obtain,

$$h^2 \left\| \nabla(e^{\phi_N/h} u'_h) \right\|^2 + k \left\| e^{\phi_N/h} u'_h \right\|^2 = \mathscr{O}\big(h^\infty + h^{-M_2} \| e^{\phi_N/h} u'_h \|\big).$$

In particular,

$$\left\| e^{\phi_N/h} u'_h \right\| = \mathscr{O}(h^{-(M_2+1)}),$$

and thus, also,

$$\left\| \nabla(e^{\phi_N/h} u'_h) \right\| = \mathscr{O}(h^{-(M_2+3/2)}),$$

and the result follows. □

Now, we estimate $u - w_{CN}$ globally in an N, h-dependent small neighborhood $U_N := \{x; \operatorname{dist}(x, \partial\ddot{O}) < 2(Nk)^{2/3}\}$ of the boundary of the island. We show

PROPOSITION 5.2. *There exist $N_2 \in \boldsymbol{Z}$ and $C > 0$ such that, for any N, one has as $h \to 0$*

$$\|u - w_{CN}\|_{H^1(U_N)} = \mathscr{O}\big(h^{-N_2} e^{-S/h}\big). \tag{5.11}$$

PROOF. Let $U_{N,1}$ be the neighborhood of Γ in U_N defined by,

$$U_{N,1} = U_N \cap \Omega(Nk, t_0), \tag{5.12}$$

with $t_0 > 0$ small enough. We may assume,

$$U_{N,1} \subset \bigcup_{x^1 \in \Gamma} \Omega^1_{x^1}\big(-(Nk)^{2/3}, (Nk)^{2/3}\big),$$

where $\Omega^1_{x^1}(-(Nk)^{2/3}, (Nk)^{2/3})$ is the neighborhood of each $x^1 \in \Gamma$ defined by (4.29).

Then Proposition 5.1, (2.7) and an estimate of $\tilde{I}[\tilde{c}]$ in the sea mean that there exists N_2 such that

$$\|u - w_{CN}\|_{H^1(U_{N,1})} = \mathscr{O}\big(h^{-N_2}e^{-S/h}\big). \tag{5.13}$$

It follows from (2.7) and the fact $(U_N \backslash U_{N,1}) \cap B_d(x_0, S) = \emptyset$ that

$$\|u\|_{H^1(U_N \backslash U_{N,1})} = \mathscr{O}\big(h^{-N_2}e^{-S/h}\big),$$

and since w_{CN} vanishes in $U_N \backslash U_{N,1}$, we obtain (5.11). □

6. Comparison in the sea.

In this section, we give a more precise estimate on $v_N = e^{S/h}(u - w_{CN})$ ($C > 0$ being as in Proposition 5.1), in the neighborhood U_N of $\partial \ddot{O}$. We will show

Proposition 6.1. *For any $L > 0$ and for any $\alpha \in \boldsymbol{Z}_+^n$, there exists $N_{L,\alpha} \geq 1$ such that, for any $N \geq N_{L,\alpha}$, one has,*

$$\partial_x^\alpha v_N(x, h) = \mathscr{O}(h^L) \quad \text{as } h \to 0, \tag{6.1}$$

uniformly in U_N.

Let $\hat{x}$ be an arbitrary point on $\partial \ddot{O}$. In the sequel, all the estimates we give are locally uniform with respect to $\hat{x} \in \partial \ddot{O}$ (and thus, indeed, globally uniform since $\partial \ddot{O}$ is compact).

Here again, we choose Euclidian coordinates x as in Section 4.1 but centered at $\hat{x}$ such that $T_{\hat{x}}(\partial \ddot{O})$ is given by $x_n = 0$, and $\partial/\partial x_n$ is the exterior normal of $\ddot{O}$ at this point.

Consider h-dependent neighborhoods of $\hat{x}$, of the form

$$\hat{\omega}_N(h) = \big\{x; |x_n - \hat{x}_n| < (Nk)^{2/3}; |x' - \hat{x}'| < (Nk)^{1/2}\big\}, \tag{6.2}$$

where $k = h \log(1/h)$.

Let $(x(t), \xi(t)) = \exp tH_p(0, 0)$ be the Hamilton flow passing by the origin at time 0 i.e.

$$\begin{cases} \dfrac{dx}{dt} = 2\xi, & x(0) = 0; \\[2mm] \dfrac{d\xi}{dt} = -\dfrac{\partial V}{\partial x}, & \xi(0) = 0. \end{cases} \tag{6.3}$$

Recall that, by the non-trapping condition (A3), there exists, at any point $\hat{x}$ on $\partial\ddot{O}$, a positive constant $C_0 = C_0(\hat{x})$ such that the potential V is written in the form (4.2). Hence $-(\partial V/\partial x) = (0, \ldots, 0, C_0) + \mathscr{O}(|x|)$, and the flow is tangent to the ξ_n-axis. As $t \to 0$, one has

$$\begin{aligned} x_n(t) &= C_0 t^2 + \mathscr{O}(t^3), \quad & \xi_n(t) &= C_0 t + \mathscr{O}(t^2), \\ x'(t) &= \mathscr{O}(t^4), & \xi'(t) &= \mathscr{O}(t^3). \end{aligned} \tag{6.4}$$

When $t \to \pm\infty$, on the other hand, $|x(t)| \to \infty$ by Assumption (A3), and $\xi(t) \to \xi_\infty^\pm$ for some $\xi_\infty^\pm \in \boldsymbol{R}^n$ satisfying $|\xi_\infty^\pm|^2 = E_0$ by (A1). That is, as $t \to \pm\infty$,

$$x(t) = 2\xi_\infty^\pm t + o(|t|), \quad \xi(t) = \xi_\infty^\pm + o(1), \quad |\xi_\infty^\pm| = \sqrt{E_0}. \tag{6.5}$$

In particular,

$$\begin{cases} x(t) \cdot \xi(t) = 2E_0 t + o(|t|) \\ |x(t)| \cdot |\xi(t)| = 2E_0 |t| + o(|t|) \end{cases} \quad \text{as } t \to \pm\infty. \tag{6.6}$$

6.1. Propagation in the incoming region.

Here, we study the microlocal estimate of $e^{S/h}u$ and $e^{S/h}w_{CN}$ independently along the incoming Hamilton flow $\bigcup_{t<0}(x(t), \xi(t))$. For the estimate on $e^{S/h}u$, we use the fact that u is *outgoing* at infinity and the propagation of *frequency set*. For that of $e^{S/h}w_{CN}$, we use the result of Section 4.

6.1.1. Microlocal estimate of $e^{S/h}u$.

We first study $\tilde{u} = e^{S/h}u$. Using the Bargmann-FBI transform T_μ of (8.5), we plan to prove that, for some convenient $\mu > 0$, $T_\mu \tilde{u}(x, \xi, h)$ is exponentially small for (x, ξ) close enough to $(x(-t), \xi(-t))$, $t > 0$ sufficiently large. Actually, we prove something slightly better, namely,

Lemma 6.2. *For any $S_1 > 0$, there exist $t_1 > 0$ and $\mu > 0$, such that, for all $t \geq t_1$, one has,*

$$T_\mu u(x, \xi, h) = \mathscr{O}(e^{-S_1/h}),$$

uniformly for (x, ξ) in a neighborhood of $(x(-t), \xi(-t))$.

Proof. Let $F(x)$ be the function used to define the distorted operator P_θ (see (2.1)), and let $\chi \in C^\infty(\boldsymbol{R}_+)$ verifying $\chi(|x|) = 0$ on $\pi_x(\operatorname{Supp}\psi_0) \cup \{F(x) \neq x\}$, $\chi = 1$ on $[R, +\infty)$ for $R > 1$ large enough, $\chi' \geq 0$ everywhere.

For $\delta > 0$ small enough, we consider the distortion,

$$G_\delta(x) := x + i\delta\chi(|x|)x,$$

and the corresponding distortion operator $\tilde{U}_\delta$, formally given by,

$$\tilde{U}_\delta\phi(x) := (\det G_\delta)^{1/2}\phi(G_\delta(x)).$$

Then, the distorted operator $P_{\theta,\delta} := \tilde{U}_\delta P_\theta \tilde{U}_\delta^{-1}$ is well defined, and its principal symbol $p_{\theta,\delta}$ verifies,

$$\begin{aligned} &p_{\theta,\delta}(x,\xi) = p_\theta(x,\xi) \text{ if } x \in \pi_x(\operatorname{Supp}\psi_0) \cup \{F(x) \neq x\};\\ &p_{\theta,\delta}(x,\xi) = (1+i\theta)^{-2}(dG_\delta(x)^{-1}\xi)^2 + V((1+i\theta)G_\delta(x))\\ &\qquad\qquad\qquad\qquad \text{if } x \notin \pi_x(\operatorname{Supp}\psi_0) \cup \{F(x) \neq x\}. \end{aligned}$$

Next, we observe that,

$$dG_\delta(x) = (1 + i\delta\chi(|x|))I + i\delta A(x),$$

with,

$$A(x) = \frac{\chi'(|x|)}{|x|}(x_j x_k)_{1\leq j,k\leq n}.$$

In particular, one has $\langle A(x)y,y\rangle = (\chi'(|x|))/(|x|)\langle x,y\rangle^2 \geq 0$ for all $y \in \boldsymbol{R}^n$, and thus, it is not difficult to deduce that $\operatorname{Im} {}^t dG_\delta(x)^{-1} dG_\delta(x)^{-1} \leq 0$ for $\delta > 0$ small enough. As a consequence, we see that, for $x \notin \pi_x(\operatorname{Supp}\psi_0) \cup \{F(x) \neq x\}$, one has,

$$\operatorname{Im} p_{\theta,\delta}(x,\xi) \leq -\theta\xi^2 + \mathscr{O}\big(\langle x\rangle^{-\delta_1}\big),$$

and thus, if $F(x)$ and ψ_0 have been conveniently constructed, and using (3.1), we obtain,

$$-\operatorname{Im} p_{\theta,\delta}\big(x - t\partial_x\psi_0 - it\partial_\xi\psi_0, \xi - t\partial_\xi\psi_0 + it\partial_x\psi_0\big) \geq \frac{k}{C},$$

for some constant $C > 0$, and for (x,ξ) such that $|\operatorname{Re} p_{\theta,\delta}(x,\xi) + W(x) - E_0| \leq \langle\xi\rangle^2/C$. As for $\tilde{P}_\theta$ (see Section 2), this implies that $(P_{\theta,\delta} + W - \rho)^{-1}$ is well defined and has a norm $\mathscr{O}(k^{-1})$ on H_t.

On the other hand, we also know that $u_{\theta,\delta} := \tilde{U}_\delta u_\theta = \tilde{U}_\delta U_{i\theta} u$ is well defined, and is in $L^2(\boldsymbol{R}^n)$ (see, e.g., [**HeMa**]). Thus, we can write,

$$(P_{\theta,\delta} + W - \rho)u_{\theta,\delta} = W u_{\theta,\delta} = W u,$$

that is, $u_{\theta,\delta} = (P_{\theta,\delta} + W - \rho)^{-1} W u$, and thus,

$$\|u_{\theta,\delta}\|_{L^2(\boldsymbol{R}^n)} = \mathscr{O}(h^{-M})\|u_{\theta,\delta}\|_t = \mathscr{O}(h^{-M}k^{-1}), \tag{6.7}$$

for some $M > 0$ constant, independent of $\delta > 0$ small enough.

Making in the expression of $T_\mu u$ the change of contour of integration,

$$\boldsymbol{R}^n \ni y \mapsto G_{\theta,\delta}(y) := G_\delta(y) + i\theta F(G_\delta(y)),$$

we obtain,

$$T_\mu u(x,\xi) = c_\mu \int_{\boldsymbol{R}^n} e^{i(x-G_{\theta,\delta}(y))\xi/h - \mu(x-G_{\theta,\delta}(y))^2/2h} u_{\theta,\delta}(y) \det dG_{\theta,\delta}(y) dy,$$

and thus, using (6.7) and the Cauchy-Schwarz inequality,

$$T_\mu u(x,\xi) = \mathscr{O}(h^{-M_1})\left[\int e^{\{2\,\mathrm{Im}\,G_{\theta,\delta}(y)\xi + \mu(\mathrm{Im}\,G_{\theta,\delta}(y))^2 - \mu(x-\mathrm{Re}\,G_{\theta,\delta}(y))^2\}/h} dy\right]^{1/2} \tag{6.8}$$

for some $M_1 > 0$ constant, independent of δ.

Now, let $R_1 \gg 1$ be some fixed number arbitrarily large, and take $t > 0$ sufficiently large to have $|x(-t)| \geq R + 2R_1$. Then, for (x,ξ) close enough to $(x(-t),\xi(-t))$, and setting $\tilde{\delta} := \delta + \theta$, we deduce from (6.8),

$$\begin{aligned} T_\mu u(x,\xi) = \mathscr{O}(h^{-M_1})&\left[\int_{|y-x|\leq|x|/2} e^{2\tilde{\delta}y\xi/h + \mu\tilde{\delta}^2 y^2/h - \mu(x-y+\theta\delta y)^2/h} dy\right]^{1/2} \\ &+ \mathscr{O}\big(h^{-M_1} e^{-\mu x^2/16h}\big) \\ &\qquad \times \left[\int_{|y-x|\geq|x|/2} e^{2\tilde{\delta}y\xi/h + \mu\tilde{\delta}^2 y^2/h - \mu(x-\mathrm{Re}\,G_{\theta,\delta}(y))^2/2h} dy\right]^{1/2}, \end{aligned}$$

and thus, for δ/μ small enough, (and since $|y-x| \geq |x|/2$ implies $|y-x| \geq |y|/4$, and we have $\operatorname{Re} G_{\theta,\delta}(y) = y + \mathscr{O}(\theta\delta|y|)$, and $|\xi|$ remains uniformly bounded),

$$T_\mu u(x,\xi) = \mathscr{O}(h^{-M_1})\left[\int_{|y-x|\leq|x|/2} e^{2\tilde\delta y\xi/h+\mu\tilde\delta^2 y^2/h}dy\right]^{1/2} + \mathscr{O}\big(e^{-\mu R_1^2/4h}\big). \quad (6.9)$$

Moreover, if $|y-x|\leq |x|/2$, we have $y\cdot\xi\leq x\cdot\xi+|x|\cdot|\xi|/2$, and thus, by (6.6), and for (x,ξ) close enough to $(x(-t),\xi(-t))$, we obtain (possibly by taking t larger),

$$y\cdot\xi\leq -\frac{E_0 t}{2}.$$

In the same way, using (6.5), we also obtain,

$$|y|\leq\frac{3|x|}{2}\leq 4\sqrt{E_0}t,$$

and thus, inserting these estimates into (6.9), we find,

$$T_\mu u(x,\xi) = \mathscr{O}\big(h^{-M_1}e^{-E_0\tilde\delta t(1-16\mu\tilde\delta t)/h}\big) + \mathscr{O}\big(e^{-\mu R_1^2/4h}\big).$$

In particular, for any $S_1>0$, if we first fix $\mu>0$ such that $E_0>64\mu S_1$, then R_1 such that $\mu R_1^2/4\geq S_1$, then $t\gg 1$ such that $|x(-t)|\geq R+2R_1$, and finally $\delta := (32\mu t)^{-1}-\theta = (32\mu t)^{-1}-k$, we obtain,

$$\begin{aligned} T_\mu u(x,\xi) &= \mathscr{O}\big(h^{-M_1}e^{-E_0\tilde\delta t/2h}+e^{-S_1/h}\big)\\ &= \mathscr{O}\big(h^{-M_1}e^{-E_0/64\mu h}+e^{-S_1/h}\big)\\ &= \mathscr{O}\big(e^{-S_1/h}\big),\end{aligned}$$

uniformly for (x,ξ) close enough to $(x(-t),\xi(-t))$ and $h>0$ small enough. □

In particular, taking $S_1>S$, we obtain,

$$T_\mu(e^{S/h}u) = \mathscr{O}(h^\infty) \text{ near } (x(-t_1),\xi(-t_1)),$$

and therefore,

$$(x(-t_1),\xi(-t_1))\notin FS(e^{S/h}u),$$

where $FS(e^{S/h}u)$ stands for the *frequency set* of $e^{S/h}u$ (see, e.g., [**GuSt**], [**Ma1**]). Moreover, by (2.7), we know that, for any $K \subset \boldsymbol{R}^n \backslash B_d(x_0, S)$ compact, $\|e^{S/h}u\|_{H^1(K)} = \mathscr{O}(h^{-N_K})$ for some $N_K > 0$ constant. Since $\boldsymbol{R}^n \backslash B_d(x_0, S)$ is a neighborhood of $\{x(-t); t > 0\}$, and $(P - \rho)(e^{S/h}u) = 0$ in $\mathscr{D}'(\boldsymbol{R}^n \backslash \overline{B_d(x_0, S)})$, the standard result of propagation of the frequency set for solutions of real-principal type partial differential equations (see, e.g., [**Ma1**, Chapter 4, Exercise 7]) can be applied above this set, and tells us,

$$(x(-t), \xi(-t)) \notin FS(e^{S/h}u) \text{ for all } t > 0. \tag{6.10}$$

In particular, for any $\mu > 0$ fixed (independent of h), for any $t > 0$, and for any $K \subset \boldsymbol{R}^n \backslash B_d(x_0, S)$ compact containing $(x(-t), \xi(-t))$ in its interior, we have (denoting by 1_K the characteristic function of K),

$$T_\mu(1_K e^{S/h}u) = \mathscr{O}(h^\infty) \text{ uniformly near } (x(-t), \xi(-t)). \tag{6.11}$$

Now, we set,

$$\boldsymbol{T}_N u(x, \xi) := \int e^{i(x-y)\xi/h - \mu_n(x_n - y_n)^2/2h - (x'-y')^2/2h} u(y) dy, \tag{6.12}$$

where $\mu_n := (Nk)^{-1/3}$.

Lemma 6.3. *For any $t > 0$, for any $K \subset \boldsymbol{R}^n \backslash B_d(x_0, S)$ compact containing $x(-t)$ in its interior, and for any $N \geq 1$, we have,*

$$\boldsymbol{T}_N(1_K e^{S/h}u) = \mathscr{O}(h^\infty) \textit{ uniformly near } (x(-t), \xi(-t)).$$

Proof. We write,

$$\boldsymbol{T}_N(1_K e^{S/h}u) = (\boldsymbol{T}_N T_1^*) T_1(1_K e^{S/h}u),$$

and a straightforward computation shows that the distribution kernel K_N of $\boldsymbol{T}_N T_1^*$ verifies (see, e.g., [**Ma1**, proof of Proposition 3.2.5]),

$$|K_N(x, \xi; z, \zeta)| = \alpha e^{-(\mu_n/(1+\mu_n))(x_n - z_n)^2/2h - (1/(1+\mu_n))(\xi_n - \zeta_n)^2/2h} \times e^{-(x'-z')^2/4h - (\xi'-\zeta')^2/4h},$$

with $\alpha = \mathscr{O}(h^{-n})$. Then, the result easily follows from the obvious observation that, for any fixed $\delta > 0$, one has,

$$e^{-(\mu_n/(1+\mu_n))\delta/h} + e^{-(1/(1+\mu_n))\delta/h} = \mathscr{O}(h^\infty). \qquad \square$$

Now, we fix once for all a compact set $K_1 = K \backslash B_d(x_0, S)$, where $K \Subset \boldsymbol{R}^n$ is a compact neighborhood of the closure of $\ddot{O}$.

Lemma 6.4. *There exists $\delta_0 > 0$ such that, for any $\delta \in (0, \delta_0]$, for all $N \geq 1$ large enough, and for $t_N := \delta^{-1}(Nk)^{1/3}$, one has,*

$$\boldsymbol{T}_N(1_{K_1} e^{S/h} u) = \mathscr{O}(h^{\delta N}) \text{ uniformly in } \mathscr{W}(t_N, h),$$

where,

$$\mathscr{W}(t_N, h) := \left\{ \begin{array}{c} |x_n - x_n(-t_N)| \leq \delta(Nk)^{2/3}, \ \ |\xi_n - \xi_n(-t_N)| \leq \delta(Nk)^{1/3}, \\ |x' - x'(-t_N)| \leq \delta(Nk)^{1/3}, \quad |\xi' - \xi'(-t_N)| \leq \delta(Nk)^{1/3} \end{array} \right\}.$$

Proof. At first, we cut off the function $e^{S/h}u$ by setting

$$u_1 := \chi_+ e^{S/h} u,$$

where $\chi_+ \in C^\infty(\boldsymbol{R}^n)$, $\operatorname{Supp} \chi_+ \subset \{k^{2/3} \leq \operatorname{dist}(x, \ddot{O}) \leq 2\}$, $\chi_+ = 1$ on $\{2k^{2/3} \leq \operatorname{dist}(x, \ddot{O}) \leq 3/2\}$, and $\partial^\alpha \chi_+ = \mathscr{O}(k^{-2|\alpha|/3})$ for all $\alpha \in \boldsymbol{Z}_+^n$. In particular, by (2.7), we have $\|(P-\rho)u_1\|_{L^2} = \|[P, \chi_+]e^{S/h}u\|_{L^2} = \mathscr{O}(h^{-N_1})$ for some $N_1 \geq 0$ constant, and, if $\psi = \psi(x, \xi) \in C_0^\infty(\boldsymbol{R}^{2n})$ is such that

$$\pi_x \operatorname{Supp} \psi \subset \{3(Nk)^{2/3} \leq \operatorname{dist}(x, \ddot{O}) \leq 1\}, \quad \sup |\psi| \leq 2, \tag{6.13}$$

then, for any $M \geq 1$, we have,

$$h^{-M\psi} \boldsymbol{T}_N (P-\rho) u_1(x, \xi) = h^{-M\psi} \boldsymbol{T}_N [P, \chi_+] u_1(x, \xi)$$

and, since $|x_n - y_n| \geq (Nk)^{2/3}$ for $x \in \pi_x \operatorname{Supp} \psi$ and $y \in \operatorname{Supp}[P, \chi_+]u_1$, we easily obtain,

$$\left\| h^{-M\psi} \boldsymbol{T}_N (P-\rho) u_1 \right\|_{L^2} = \mathscr{O}\big(h^{-N_1} + h^{-2M} e^{-\mu_n (Nk)^{4/3}/2h}\big),$$

that is,

$$\left\| h^{-M\psi} \boldsymbol{T}_N (P-\rho) u_1 \right\|_{L^2} = \mathscr{O}(h^{-N_1} + h^{-2M+N/2}).$$

In particular, for $M \leq N/4$, this gives,

$$\left\| h^{-M\psi} \boldsymbol{T}_N (P-\rho) u_1 \right\|_{L^2} = \mathscr{O}(h^{-N_1}). \tag{6.14}$$

Now, in order to specify the function ψ we are going to work with, we first make a symplectic change of variables near $(0,0) \in \boldsymbol{R}^{2n}$:

$$\begin{cases} y' = x', \quad y_n = x_n - \dfrac{1}{C_0}\xi_n^2, \\ \eta' = \xi', \quad \eta_n = \xi_n. \end{cases} \tag{6.15}$$

In this new coordinates, we have

$$p = \eta'^2 - C_0 y_n + W\Big(y', y_n + \frac{1}{C_0}\eta_n^2\Big), \tag{6.16}$$

$$H_p = 2\eta' \frac{\partial}{\partial y'} + C_0 \frac{\partial}{\partial \eta_n} - \nabla W \frac{\partial}{\partial \eta} + \frac{2}{C_0}\eta_n \partial_{x_n} W \frac{\partial}{\partial y_n}. \tag{6.17}$$

Now, we fix $t_0 > 0$ small enough, and we consider the function,

$$\psi(y,\eta) := f(\eta_n)\chi\Big(\frac{|\eta'|}{|\eta_n|}\Big)\chi\Big(\frac{|y_n|}{|\eta_n|^2}\Big)\chi\Big(\frac{|y'|}{|\eta_n|}\Big),$$

where $\chi \in C_0^\infty(\boldsymbol{R}_+;[0,1])$ is such that,

$$\operatorname{Supp}\chi \subset [0,a], \quad \chi = 1 \text{ on } \Big[0,\frac{a}{2}\Big], \quad -\frac{4}{a} \leq \chi' \leq 0 \text{ on } \boldsymbol{R}_+,$$

for some constant $a > 0$ small enough, and $f \in C_0^\infty(\boldsymbol{R};[0, 1+t_0+\varepsilon'])$ is defined in the following way,

$$f(s) = \chi_0(s) f_1(s) := \chi_0(s)\Big(-Cs - 2C_1^2 \int_{s(Nk)^{-1/3}}^{-C_1} \frac{dt}{t^3}\Big), \tag{6.18}$$

where $C, C_1 > 0$ are large enough constants, and χ_0 is a cut-off function such that,

$\chi_0 \in C_0^\infty\big([-t_0-\varepsilon, -C_1(Nk)^{1/3}];[0,1]\big)$;

$\chi_0 = 1$ on $[-t_0, -2C_1(Nk)^{1/3}]$;

$\chi_0' \leq 0$ on $[-2C_1(Nk)^{1/3}, -C_1(Nk)^{1/3}]$;

for all $\ell \geq 0$, $\chi_0^{(\ell)} = \mathscr{O}((Nk)^{-|\ell|/3})$ on $[-2C_1(Nk)^{1/3}, -C_1(Nk)^{1/3}]$;

for all $\ell \geq 0$, $\chi_0^{(\ell)} = \mathscr{O}(1)$ on $[-t_0 - \varepsilon, -t_0]$.

This ψ satisfies the condition (6.13), since on the support of ψ, one has

$$\begin{aligned} E_0 - V(x) &= \eta^2 + C_0 y_n - W\left(y', y_n + \frac{1}{C_0}\eta_n^2\right) \\ &\geq \eta_n^2 - aC_0\eta_n^2 + \mathscr{O}\big(a^2\eta_n^2 + \eta_n^4\big) \\ &\geq \frac{1}{2}\eta_n^2 \geq \frac{C_1^2}{2}(Nk)^{2/3} \end{aligned}$$

for sufficiently small a. In particular, (6.14) is valid with such a ψ.

Moreover, one has

$$\partial_{y'}^{\alpha'}\partial_{y_n}^{\alpha_n}\partial_\eta^\beta \psi = \mathscr{O}\big((Nk)^{-(|\alpha'|+2|\alpha_n|+|\beta|)/3}\big). \tag{6.19}$$

Therefore, we see that ψ satisfies the conditions (8.6)–(8.7) in the (y', η')-coordinates, with $\rho = 0$, and the same conditions in the (y_n, η_n)-coordinates, with $\rho = -1/3$.

Then, by a straightforward generalization of Proposition 8.3, and by using (8.9) and (6.14), we obtain,

$$\begin{aligned} &k\big\langle (MH_p\psi + q_{M\psi})h^{-M\psi}\boldsymbol{T}_N u_1, h^{-M\psi}\boldsymbol{T}_N u_1\big\rangle \\ &\quad = \mathscr{O}(h)\big\|\langle\eta\rangle h^{-M\psi}\boldsymbol{T}_N u_1\big\|^2 + \mathscr{O}(h^{-N_1})\big\|h^{-M\psi}\boldsymbol{T}_N u_1\big\|, \end{aligned} \tag{6.20}$$

where $M \leq N/4$ and $N_1 \geq 1$ is independent of N, and $q_{M\psi}$ is defined in (8.9).

In the sequel, we use the notations,

$$\begin{aligned} I_1 &:= \operatorname{Supp}\psi \cap \big\{\eta_n \in [-t_0 - \varepsilon', -t_0 + \varepsilon']\big\}; \\ I_2 &:= \operatorname{Supp}\psi \cap \big\{\eta_n \in [-t_0 + \varepsilon', -2C_1(Nk)^{1/3}]\big\}; \\ I_3 &:= \operatorname{Supp}\psi \cap \big\{\eta_n \in [-2C_1(Nk)^{1/3}, -C_1(Nk)^{1/3}]\big\}. \end{aligned}$$

Let us now estimate $|H_p\psi|$ from below on $I_2 \cup I_3$. First, observe that one has

$$|f(\eta_n)| \leq 2\chi_0(\eta_n) \tag{6.21}$$

if $t_0 \leq 1/C$, and in particular on $I_2 \cup I_3$, where $\chi_0' \leq 0$, one has

$$|f'(\eta_n)| \geq \left(C + \frac{2C_1^2(Nk)^{2/3}}{|\eta_n|^3}\right)\chi_0(\eta_n) \geq C\chi_0(\eta_n). \tag{6.22}$$

Using these estimates, and the expression (6.17), we can easily show

LEMMA 6.5. *For $t_0 < C_0/2$*

$$|H_p\psi| \geq |f'(\eta_n)|\left(C_0\chi_1\chi_2\chi_3 - \mathscr{O}\left(\frac{1}{C}\right)\right), \tag{6.23}$$

where

$$\chi_1 = \chi\left(\frac{|\eta'|}{|\eta_n|}\right), \quad \chi_2 = \chi\left(\frac{|y_n|}{|\eta_n|^2}\right), \quad \chi_3 = \chi\left(\frac{|y'|}{|\eta_n|}\right).$$

PROOF. One can estimate $|C_0(\partial/\partial\eta_n)\psi|$ from below by

$$\left|C_0\frac{\partial}{\partial\eta_n}\psi\right| \geq C_0|f'(\eta_n)|\chi_1\chi_2\chi_3.$$

One can also estimate $2|\eta'(\partial/\partial y')\psi|$, $|\nabla W(\partial/\partial\eta')\psi|$, $|\eta_n|\,|\nabla_{x_n}W(\partial/\partial y_n)\psi|$ from above by $|f'|$ times a constant of $\mathscr{O}(1/C)$. For example,

$$2\left|\eta'\frac{\partial}{\partial y'}\psi\right| \leq f\chi_1\chi_2\frac{|\eta'|}{|\eta_n|}|\chi_3'| \leq 4f\chi_1\chi_3 \leq \frac{8}{C}\chi_1\chi_2|f'|,$$

using the facts $|\eta'|/|\eta_n| \leq a$, $|\chi_3'| \leq 4/a$, $f \leq 2f'/C$. On the other hand, for $|\nabla W(\partial/\partial\eta_n)\psi|$, one has

$$|y|\left|\frac{\partial}{\partial\eta_n}\psi\right| \leq a|\eta_n|\left|\frac{\partial}{\partial\eta_n}\psi\right| \leq |t_0|\left|\frac{\partial}{\partial\eta_n}\psi\right|,$$

which is smaller than $|(\partial/\partial\eta_n)\psi|$ for sufficiently small t_0. □

Now, let $C_2 > C_1$ be another large enough constant. We set,

$$\Omega_1 := (I_2 \cup I_3) \cap \left\{ \chi_0 \geq \frac{1}{C_2}; \chi_1 \geq \frac{1}{C_2}; \chi_2 \geq \frac{1}{C_2}; \chi_3 \geq \frac{1}{C_2} \right\};$$

$$\Omega_2 := (I_2 \cup I_3) \backslash \Omega_1.$$

Then, by construction, on Ω_2 we have by (6.21),

$$\psi \leq \frac{1}{C_2} \sup f \leq \frac{2}{C_2}. \tag{6.24}$$

Moreover, in a neighborhood of $X_N := (0; 0, -3C_1(Nk)^{1/3})$, of the form,

$$\mathscr{W}_N = \left\{ |\eta'| \leq \delta|\eta_n|,\, |y_n| \leq \delta|\eta_n|^2,\, |y'| \leq \delta|\eta_n|,\, |(Nk)^{-1/3}\eta_n + 3C_1| < \delta \right\},$$

(where $\delta > 0$ is a small enough constant), we see that,

$$\psi \geq \frac{1}{2}\psi(X_N) = 3CC_1(Nk)^{1/3} - 2C_1^2 \int_{-3C_1}^{-C_1} \frac{dt}{t^3} \geq \frac{8}{9} =: r_0, \tag{6.25}$$

and, by (6.24), we can fix C_2 large enough, in such a way that,

$$\sup_{\Omega_2} \psi \leq \frac{1}{2} r_0. \tag{6.26}$$

On the other hand, by (6.22), (6.23), on Ω_1, we have,

$$|H_p\psi| \geq \frac{C_0}{2C_2^3}|f'| \geq \frac{C_0}{2C_2^4}\left(C + \frac{2C_1^2(Nk)^{2/3}}{|\eta_n|^3} \right). \tag{6.27}$$

Using the expression of $q_{M\psi}$ deduced from Proposition 8.3, and the fact that here, $p(x,\xi) = \xi^2 + V(x)$, we have

Lemma 6.6. *As $h \to 0$,*

$$q_{M\psi} = -2kM^2 \mu_n \partial_{x_n}\psi \partial_{\xi_n}\psi + \mathscr{O}\left(\frac{1}{\ln \frac{1}{h}} \right). \tag{6.28}$$

In particular, on Ω_1, for M/N sufficiently small, one has

$$|q_{M\psi}| \leq \frac{M}{2}|H_p\psi|. \tag{6.29}$$

PROOF. Taking into account that

$$\partial_{x'}\psi = \mathscr{O}(k^{-1/3}), \quad \partial_{x_n}\psi = \mathscr{O}(k^{-2/3}),$$
$$\partial_{\xi'}\psi = \mathscr{O}(k^{-1/3}), \quad \partial_{\xi_n}\psi = \mathscr{O}(k^{-1/3}),$$

and hence that

$$\partial_{z'_\mu}\psi = \mathscr{O}(k^{-1/3}), \quad \mu_n^{-1}\partial_{z_{\mu n}}\psi = \mathscr{O}(k^{-1/3}),$$

we see that for $p = \xi^2 + V(x)$,

$$\operatorname{Im} p\big(x - 2k\mu^{-1}\partial_{z_\mu}\psi, \xi + ik\partial_{z_\mu}\psi\big) = kH_p\psi - 2k^2\mu_n\partial_{x_n}\psi\partial_{\xi_n}\psi + \mathscr{O}(k^{4/3}),$$

and that

$$\begin{aligned} &h\partial_{z_\mu}\left[\frac{1}{\mu}\frac{\partial p}{\partial \operatorname{Re} x} - i\frac{\partial p}{\partial \operatorname{Re}\xi}\right]\big(x - 2k\mu^{-1}\partial_{z_\mu}\psi, \xi + ik\partial_{z_\mu}\psi\big) \\ &\quad = h\partial_{z_\mu}\left[\frac{1}{\mu}\frac{\partial p}{\partial \operatorname{Re} x} - i\frac{\partial p}{\partial \operatorname{Re}\xi}\right](x,\xi) + \mathscr{O}(h). \end{aligned}$$

Since $\partial_{z_\mu}[(1/\mu)(\partial p/\partial \operatorname{Re} x) - i(\partial p/\partial \operatorname{Re}\xi)](x,\xi)$ is real, we obtain (6.28).

The estimate (6.29) follows from (6.28) and (6.27), because using the estimate

$$\partial_{\eta_n}\psi = \mathscr{O}(|\eta_n|^{-1}),$$

one sees that

$$kM^2\mu_n\partial_{x_n}\psi\partial_{\xi_n}\psi = \tilde{C}\frac{M}{N}\cdot M\frac{(Nk)^{2/3}}{|\eta_n|^3},$$

for some constant $\tilde{C}$ independent of M, N. □

Thus, still by (6.27),

$$|MH_p\psi + q_{M\psi}| \geq \frac{MCC_0}{4C_2^4} \text{ on } \Omega_1. \tag{6.30}$$

Now, we turn back to (6.20), that we rewrite as,

$$\begin{aligned}&\big\langle (MH_p\psi + q_{M\psi})h^{-M\psi}\boldsymbol{T}_N u_1, h^{-M\psi}\boldsymbol{T}_N u_1\big\rangle_{\Omega_1}\\&\quad = -\big\langle (MH_p\psi + q_{M\psi})h^{-M\psi}\boldsymbol{T}_N u_1, h^{-M\psi}\boldsymbol{T}_N u_1\big\rangle_{I_1\cup\Omega_2}\\&\qquad + \mathscr{O}\big(hk^{-1}\|\langle\eta\rangle h^{-M\psi}\boldsymbol{T}_N u_1\|^2 + h^{-N_1}k^{-1}\|h^{-M\psi}\boldsymbol{T}_N u_1\|\big),\end{aligned}$$

and thus, since $MH_p\psi + q_{M\psi} = \mathscr{O}(h^{-N_3})$ for some $N_3 \geq 1$ constant, and η is bounded on $\operatorname{Supp}\psi$,

$$\begin{aligned}&\big\langle (MH_p\psi + q_{M\psi})h^{-M\psi}\boldsymbol{T}_N u_1, h^{-M\psi}\boldsymbol{T}_N u_1\big\rangle_{\Omega_1}\\&\quad = \mathscr{O}\big(h^{-N_3}\|h^{-M\psi}\boldsymbol{T}_N u_1\|^2_{I_1\cup\Omega_2} + hk^{-1}\|h^{-M\psi}\boldsymbol{T}_N u_1\|^2_{\Omega_1}\big)\\&\qquad + \mathscr{O}\big(hk^{-1}\|\langle\eta\rangle\boldsymbol{T}_N u_1\|^2 + h^{-N_1}k^{-1}\|h^{-M\psi}\boldsymbol{T}_N u_1\|\big).\end{aligned}$$

Using (6.30), we deduce,

$$\begin{aligned}\big\|h^{-M\psi}\boldsymbol{T}_N u_1\big\|^2_{\Omega_1} &= \mathscr{O}\big(h^{-N_3}\|h^{-M\psi}\boldsymbol{T}_N u_1\|^2_{I_1\cup\Omega_2} + hk^{-1}\|h^{-M\psi}\boldsymbol{T}_N u_1\|^2_{\Omega_1}\big)\\&\quad + \mathscr{O}\big(hk^{-1}\|\langle\eta\rangle\boldsymbol{T}_N u_1\|^2 + h^{-N_1}k^{-1}\|h^{-M\psi}\boldsymbol{T}_N u_1\|\big),\end{aligned}$$

and thus, since $hk^{-1} = |\ln h|^{-1} \to 0$ as $h \to 0_+$,

$$\begin{aligned}\big\|h^{-M\psi}\boldsymbol{T}_N u_1\big\|^2_{\Omega_1} &= \mathscr{O}\big(h^{-N_3}\|h^{-M\psi}\boldsymbol{T}_N u_1\|^2_{I_1\cup\Omega_2}\big)\\&\quad + \mathscr{O}\big(hk^{-1}\|\langle\eta\rangle\boldsymbol{T}_N u_1\|^2 + h^{-N_1}k^{-1}\|h^{-M\psi}\boldsymbol{T}_N u_1\|\big),\end{aligned}$$

uniformly for $h > 0$ small enough. Therefore, setting $N_4 = \max(N_3, N_1 + 1)$, we obtain,

$$\begin{aligned}&\big\|h^{-M\psi}\boldsymbol{T}_N u_1\big\|^2_{\Omega_1} - \tilde{C}h^{-N_4}\big\|h^{-M\psi}\boldsymbol{T}_N u_1\big\|_{\Omega_1}\\&\qquad \leq \tilde{C}h^{-N_4}\big(\|h^{-M\psi}\boldsymbol{T}_N u_1\|^2_{I_1\cup\Omega_2} + \|\langle\xi\rangle\boldsymbol{T}_N u_1\|^2\big),\end{aligned}$$

for some positive constant $\tilde{C}$, and thus,

$$\big\|h^{-M\psi}\boldsymbol{T}_N u_1\big\|_{\Omega_1} \leq C'h^{-N_4}\big(1 + \|h^{-M\psi}\boldsymbol{T}_N u_1\|_{I_1\cup\Omega_2} + \|\langle\xi\rangle\boldsymbol{T}_N u_1\|\big),$$

for some other constant $C' > 0$.

In particular, since $\mathscr{W}_N \subset \Omega_1$ and $\|\langle\xi\rangle\boldsymbol{T}_N u_1\| = \|u_1\|_{H^1} = \mathscr{O}(1)$,

$$\begin{aligned}\big\|h^{-M\psi}\boldsymbol{T}_N u_1\big\|_{\mathscr{W}_N} &= \mathscr{O}(h^{-N_4})\big(1+\|h^{-M\psi}\boldsymbol{T}_N u_1\|_{I_1\cup\Omega_2}\big)\\ &= \mathscr{O}(h^{-N_4})\big(1+\|h^{-M\psi}\boldsymbol{T}_N u_1\|_{\Omega_2}+h^{-2M}\|\boldsymbol{T}_N u_1\|_{I_1}\big).\end{aligned}$$

Then, using Lemma 6.3, we see that $\|\boldsymbol{T}_N u_1\|_{I_1} = \mathscr{O}(h^\infty)$ if a has been taken sufficiently small, and thus, using (6.25)–(6.26), we obtain,

$$h^{-Mr_0}\|\boldsymbol{T}_N u_1\|_{\mathscr{W}_N} = \mathscr{O}\big(h^{-N_4-(M/2)r_0}\big),$$

that is,

$$\|\boldsymbol{T}_N u_1\|_{\mathscr{W}_N} = \mathscr{O}\big(h^{-N_4+(M/2)r_0}\big), \tag{6.31}$$

where the estimate is valid for N large enough, M/N small enough, and is uniform with respect to $h>0$ small enough. This completes the proof. □

6.1.2. Microlocal estimate of $e^{S/h}w_{CN}$.

Now, we study the microlocal behavior of the WKB solution w_{CN} in $\mathscr{W}(t_N,h)$. For $N\geq 1$ arbitrary, we denote by χ_N a cut-off function of the type,

$$\chi_N(x) := \chi_0\bigg(\frac{|x_n-\hat{x}_n|}{(Nk)^{2/3}}\bigg)\chi_0\bigg(\frac{|x'-\hat{x}'|}{(Nk)^{1/2}}\bigg), \tag{6.32}$$

where the function $\chi_0 \in C_0^\infty(\boldsymbol{R}_+;[0,1])$ verifies $\chi_0 = 1$ in a sufficiently large neighborhood of 0, and is fixed in such a way that $\chi_N(x) = 1$ in $\{|x_n-\hat{x}_n| \leq |x_n(-t_N)-\hat{x}_n| + 2\delta(Nk)^{2/3}; |x'-\hat{x}'| \leq |x'(-t_N)-\hat{x}'| + 2\delta(Nk)^{1/2}\}$ (here, t_N and δ are those of Lemma 6.4). Then, setting,

$$\tilde{w}_N := 1_{B_d(x_0,S)^C}\chi_N w_{CN},$$

(with $C>0$ fixed large enough, as in Proposition 5.1), we have,

LEMMA 6.7. *For any $L\in\boldsymbol{N}$ large enough, there exists $\delta_L>0$ such that, for any $\delta\in(0,\delta_L]$, for all $N\geq 1$ large enough, and for $t_N := \delta^{-1}(Nk)^{1/3}$, one has,*

$$\boldsymbol{T}_N\big(e^{S/h}\tilde{w}_N\big) = \mathscr{O}\big(h^{\delta N}+h^L\big) \text{ uniformly in } \mathscr{W}(t_N,h).$$

PROOF. Let $\chi^1(r)\in C_0^\infty(\boldsymbol{R}^+;[0,1])$ be a cut-off function such that $\chi^1 = 1$ for $0\leq r\leq 2\delta$ and $\chi^1 = 0$ for $3\delta\leq r$ and set,

$$\chi_N^1(x) := \chi_1\Big(\frac{|x_n - x_n(-t_N)|}{(Nk)^{2/3}}\Big)\chi_1\Big(\frac{|x' - x'(-t_N)|}{(Nk)^{1/2}}\Big).$$

We write,

$$\boldsymbol{T}_N\big(e^{S/h}\tilde{w}_N\big) = \boldsymbol{T}_N\big(e^{S/h}\chi_N^1\tilde{w}_N\big) + \boldsymbol{T}_N\big(e^{S/h}(1-\chi_N^1)\tilde{w}_N\big) =: I_1 + I_2.$$

First we study I_2. We have $|x_n - y_n| \geq \delta(Nk)^{2/3}$ or $|x' - y'| \geq \delta(Nk)^{1/2}$ if $|x_n - x_n(-t_N)| \leq \delta(Nk)^{2/3}$, and $|x' - x'(-t_N)| \leq \delta(Nk)^{1/2}$, and $y \in \operatorname{Supp}(1-\chi_N^1)$. Hence, there we have

$$e^{-\mu_n(x_n-y_n)^2/2h-(x'-y')^2/2h} \leq h^{\delta^2 N/2}.$$

With the estimate of $e^{S/h}w_N$ in Proposition 4.6 (i), we deduce,

$$|I_2(x,\xi;h)| = \mathscr{O}(h^{\delta^2 N/4}),$$

uniformly for $(x,\xi) \in \mathscr{W}(t_N,h)$.

Next we study I_1. Since $\operatorname{Supp}\chi_N^1 \subset \Omega(c_2(Nk)^{2/3}, c_1(Nk)^{2/3})$ for some $c_1 > c_2 > 0$, we can use the WKB expansion (4.41), that we prefer to write in the coordinates $z' = y'$, $z_n = y_n + b(y')$ as in (4.34). Using also (4.24) and (4.27), we obtain,

$$e^{S/h}w_N(y;h) = h^{-n/4}e^{-\tilde{\phi}(z',z_n)/h}A(z',z_n;h) + \mathscr{O}\big(h^{\delta_L N} + h^L\big),$$

where

$$\tilde{\phi}(z',z_n) = a(z') - b(z')\xi_n^c(z') + \xi_n^c(z')z_n - i\tilde{\nu}\big(z', -iz_n^{1/2}\big)z_n^{3/2},$$

$$A(z',z_n;h) = \sum_{j=0}^{L+[Nk/c'_L h]} \frac{\tilde{f}_j(z',-iz_n^{1/2})}{(-iz_n^{1/2})^{1/2+3j}}h^j,$$

with $\tilde{\nu}(z',-iz_n^{1/2})$ and $\tilde{f}_j(z',-iz_n^{1/2})$ holomorphic with respect to $z_n^{1/2}$ for $|z_n| < c_1(Nk)^{2/3}$, and $\tilde{\nu}(0,0) = (2/3)\sqrt{C_0}$ (see (4.24), (4.25), (4.5)). In particular, on $\operatorname{Supp}\chi_N^1$, we have,

$$|A(z',z_n;h)| = \mathscr{O}((Nk)^{-1/6}). \tag{6.33}$$

Now, I_1 is written as,

$$I_1(x,\xi,h) = h^{-n/4}\int_{\boldsymbol{R}^n} e^{i\psi(x,z,\xi)/h} d(x,z,h)dz + \mathscr{O}\big(h^{\delta_L N} + h^L\big),$$

with

$$\begin{aligned}\psi(x,z,\xi) &= (x'-z')\cdot\xi' + (x_n - z_n + b(z'))\xi_n + i\tilde{\phi}(z',z_n)\\ &\quad + \frac{i(x'-z')^2}{2} + \frac{i\mu_n(x_n-z_n+b(z'))^2}{2};\\ d(x,z;h) &= \chi_N^1(z', z_n - b(z'))A(z',z_n;h).\end{aligned}$$

By the change of scale,

$$\begin{aligned} x' &= (Nk)^{1/2}\tilde{x}'; \quad x_n = (Nk)^{2/3}\tilde{x}_n;\\ z' &= (Nk)^{1/2}\tilde{z}'; \quad z_n = (Nk)^{2/3}\tilde{z}_n;\\ \xi' &= (Nk)^{1/2}\tilde{\xi}'; \quad \xi_n = (Nk)^{1/3}\tilde{\xi}_n,\end{aligned} \tag{6.34}$$

and setting

$$\tilde{h} := (Nk)^{-1}h \quad (\ll 1), \tag{6.35}$$

I_1 becomes,

$$I_1(x,\xi,h) = (Nk)^{(n+1)/3+(n-1)\varepsilon}h^{-n/4}\int e^{i\tilde{\psi}(\tilde{x},\tilde{z},\tilde{\xi})/\tilde{h}}\tilde{d}(\tilde{x},\tilde{z},\tilde{h})d\tilde{z} + \mathscr{O}\big(h^{\delta_L N} + h^L\big),$$

where

$$\tilde{\psi} = (\tilde{x}-\tilde{z})\cdot\tilde{\xi} + \frac{2}{3}\sqrt{C_0}\tilde{z}_n^{3/2} + \frac{i(\tilde{x}-\tilde{z})^2}{2} + \tilde{a}(\tilde{z}') + \mathscr{O}((Nk)^{1/3}),$$

with $\tilde{a}(\tilde{z}') = \mathscr{O}(|\tilde{z}'|^2)$ real-valued, and $\tilde{d}(\tilde{x},\tilde{z};\tilde{h})$ is a smooth function in $\tilde{z}$ supported in

$$\big\{|\tilde{z}_n - C_0\delta^{-2} + \mathscr{O}((Nk)^{1/3})| < 3\delta\big\} \cap \big\{|\tilde{z}' + \mathscr{O}((Nk)^{5/6})| \le 3\delta\big\}, \tag{6.36}$$

(recall from (6.4) and (4.4) that $x(-t_N) = (\mathscr{O}((Nk)^{4/3}), C_0\delta^{-2}(Nk)^{2/3} + \mathscr{O}(Nk))$ and $a(z'), b(z'), \xi_n^c(z')$ are real-valued functions that are $\mathscr{O}(|z'|^2)$). Moreover, $\tilde{d}$ satisfies the same estimate as (6.33), and it is holomorphic with respect to $\tilde{z}_n$ in a

(h-independent) small neighborhood of $\tilde{z}_n = \tilde{x}_n$.

Then it suffices to show,

$$\mathrm{Re}\,\frac{\partial\tilde{\psi}}{\partial\tilde{z}_n} \geq \tilde{\delta}, \tag{6.37}$$

for some positive constant $\tilde{\delta}$ independent of h and N. Indeed, in that case, a standard modification of the integration path with respect to $\tilde{z}_n$ around $\tilde{z}_n = \tilde{x}_n$ to the upper complex plane with small enough radius, shows that $I_1 = \mathscr{O}(e^{-\delta/\tilde{h}})$ as $\tilde{h} \to 0$, with another constant $\delta > 0$, and this means that $I_1 = \mathscr{O}(h^{\delta N})$ as $h \to 0$.

The fact that (6.37) holds for $\tilde{z}$ in the support of $\tilde{d}$ (where (6.36) holds) and $(x,\xi) \in \mathscr{W}(t_N, h)$ follows from,

$$\mathrm{Re}\,\frac{\partial\tilde{\psi}}{\partial\tilde{z}_n} = -\tilde{\xi}_n + \sqrt{C_0}\,\tilde{z}_n^{1/2} + \mathscr{O}((Nk)^{1/3}),$$

and the estimates (6.36) and $-\tilde{\xi}_n \geq C_0\delta^{-1} - \delta$ implies

$$-\tilde{\xi}_n + \sqrt{C_0}\,\tilde{z}_n^{1/2} \geq 2C_0\delta^{-1} - 5\delta.$$

This completes the proof. □

6.2. Propagation up to the outgoing region.

Now, for $N \geq 1$ large enough, we set,

$$v_N := \chi_N v = \chi_N e^{S/h}(u - w_{CN}),$$

where χ_N is as in (6.32), and $C > 0$ must be fixed large enough as in Proposition 5.1. Lemmas 6.4 and 6.7 imply that for any L large enough, there exists $\delta_L > 0$ such that for any $N \geq L/\delta_L$,

$$\boldsymbol{T}_N(v_N) = \mathscr{O}(h^{\delta N} + h^L) \text{ uniformly in } \mathscr{W}(t_N, h), \tag{6.38}$$

where $\delta > 0$ is a fixed small enough constant independent of N, L, and $t_N = \delta^{-1}(Nk)^{1/3}$. Moreover, since $\mathrm{Re}\,\tilde{\phi} \geq -C_1 Nk$ on $\mathrm{Supp}\,\chi_N$ (for some $C_1 > 0$ constant), we deduce from Proposition 4.6 that, if $C > 0$ has been chosen sufficiently large, then,

$$(P - \rho(h))v_N = [P, \chi_N]e^{S/h}(u - w_{CN}) + \mathscr{O}(h^{\delta N}).$$

Now, we introduce the (N,h)-dependent distance $\tilde{d}_N$, associated with the metric,

$$\frac{|dx'|^2}{Nk} + \frac{dx_n^2}{(Nk)^{4/3}}.$$

Then, using Proposition 5.1, we see that if

$$\tilde{d}_N(x, \operatorname{Supp} \nabla \chi_N) \geq \varepsilon \tag{6.39}$$

for some fixed $\varepsilon > 0$, thanks to the Gaussian factor in the definition of $\boldsymbol{T}_N$ (see (6.12) for the definition), we have,

$$\boldsymbol{T}_N(P - \rho(h))v_N(x,\xi) = \mathscr{O}(h^{-N_5+\varepsilon^2 N/2}),$$

for some $N_5 > 0$ independent of N, and thus,

$$\boldsymbol{T}_N(P - \rho(h))v_N(x,\xi) = \mathscr{O}(h^{\varepsilon^2 N/4}), \tag{6.40}$$

for all N large enough, and uniformly with respect to $h > 0$ small enough and $(x,\xi) \in \boldsymbol{R}^{2n}$ verifying (6.39).

Still working in the same coordinates (for which $\hat{x} = 0$), we consider the (h,N)-dependent change of variables,

$$x = (x', x_n) \mapsto \tilde{x} = (\tilde{x}', \tilde{x}_n) := \big((Nk)^{-1/2}x', (Nk)^{-2/3}x_n\big),$$

and the corresponding unitary operator U_N on $L^2(\boldsymbol{R}^n)$. Under this change, the function v_N is transformed into,

$$\tilde{v}_N(\tilde{x}) := U_N v_N(\tilde{x}) = (Nk)^{n/4+1/12} v_N\big((Nk)^{1/2}\tilde{x}', (Nk)^{2/3}\tilde{x}_n\big),$$

and one can check,

$$T\tilde{v}_N\big(\tilde{x}, \tilde{\xi}; \tilde{h}\big) = c_1 (Nk)^{-n/4-1/12} \boldsymbol{T}_N v_N\big(A_N(\tilde{x},\tilde{\xi}); h\big), \tag{6.41}$$

where $T = T_1$ is the standard FBI transform defined in (8.5) with $c_1 = 2^{-n/2}(\pi h)^{-3n/4}$, and,

$$A_N\big(\tilde{x},\tilde{\xi}\big) := \big((Nk)^{1/2}\tilde{x}', (Nk)^{2/3}\tilde{x}_n; (Nk)^{1/2}\tilde{\xi}', (Nk)^{1/3}\tilde{\xi}_n\big);$$

$$\tilde{h} := \frac{h}{Nk} = \left(N \ln \frac{1}{h}\right)^{-1}.$$

Then, setting,

$$(Nk)^{-2/3}(P-\rho) =: \tilde{P} = -(Nk)^{1/3}\tilde{h}^2\Delta_{\tilde{x}'} - \tilde{h}^2\partial^2_{\tilde{x}_n} + \tilde{V}(\tilde{x}),$$

with,

$$\begin{aligned}\tilde{V}(\tilde{x},\tilde{h}) &:= (Nk)^{-2/3}V\big((Nk)^{1/2}\tilde{x}', (Nk)^{2/3}\tilde{x}_n\big) - (Nk)^{-2/3}\rho \\ &= -C_0\tilde{x}_n + (Nk)^{-2/3}\big[E_0 - \rho(h) + W\big((Nk)^{1/2}\tilde{x}', (Nk)^{2/3}\tilde{x}_n\big)\big],\end{aligned}$$

we deduce from (6.41) that (6.40) becomes,

$$T\tilde{P}\tilde{v}_N\big(\tilde{x},\tilde{\xi};\tilde{h}\big) = \mathscr{O}\big(c_1(Nk)^{-(n+3)/4}e^{-\varepsilon^2/4\tilde{h}}\big) = \mathscr{O}\big(e^{-\varepsilon^2/6\tilde{h}}\big), \tag{6.42}$$

for any $N \geq 1$ large enough, and uniformly with respect to $h > 0$ small enough and $(\tilde{x},\tilde{\xi}) \in \boldsymbol{R}^{2n}$ verifying (6.39). (Here, we have used the fact that $Nk = h/\tilde{h} = \tilde{h}^{-1}e^{-1/(N\tilde{h})}$.)

Moreover, setting,

$$\tilde{p}\big(\tilde{x},\tilde{\xi}\big) := (Nk)^{1/3}|\tilde{\xi}'|^2 + \tilde{\xi}_n^2 + \tilde{V}\big(\tilde{x},\tilde{h}\big) = (Nk)^{-2/3}p \circ A_N\big(\tilde{x},\tilde{\xi}\big),$$

a direct computation shows that, for all $\tilde{t} \in \boldsymbol{R}$, one has,

$$\exp \tilde{t}H_{\tilde{p}} = A_N^{-1} \circ \big(\exp(Nk)^{1/3}\tilde{t}H_p\big) \circ A_N.$$

As a consequence, still using (6.41), we see that (6.38) can be rewritten as,

$$\begin{aligned}T\tilde{v}_N\big(\tilde{x},\tilde{\xi};\tilde{h}\big) &= c_1(Nk)^{-n/4-1/12}\mathscr{O}\big(e^{-\delta/\tilde{h}} + e^{-\delta_L/\tilde{h}}\big) \\ &= \mathscr{O}\big(e^{-\delta_L'/2\tilde{h}}\big), \quad \delta_L' = \min(\delta,\delta_L)\end{aligned} \tag{6.43}$$

uniformly in the tubular domain

$$\tilde{\mathscr{W}}(\tilde{h}) := \left\{\begin{matrix} |\tilde{x}_n - \tilde{x}_n(-\delta^{-1})| \leq \delta, & \big|\tilde{\xi}_n - \tilde{\xi}_n(-\delta^{-1})\big| \leq \delta, \\ |\tilde{x}' - \tilde{x}'(-\delta^{-1})| \leq \delta(Nk)^{-1/6}, & \big|\tilde{\xi}' - \tilde{\xi}'(-\delta^{-1})\big| \leq \delta(Nk)^{-1/6} \end{matrix}\right\}, \tag{6.44}$$

where $(\tilde{x}(\tilde{t}),\tilde{\xi}(\tilde{t}))=\exp\tilde{t}H_{\tilde{p}}(0,0)$.

Moreover, using (2.7), Proposition 5.1, and the properties of w_{CN} in the sea, we see that there exists $N_1\geq 0$, such that,

$$\|\tilde{v}_N\|_{H^1}=\mathscr{O}(h^{-N_1})=\mathscr{O}\big(e^{N_1/(N\tilde{h})}\big). \tag{6.45}$$

In particular, for any $\varepsilon>0$, one has $\|\tilde{v}_N\|_{H^1}=\mathscr{O}(e^{\varepsilon/\tilde{h}})$ when N is large enough.

Now, we are in a situation very similar to that of the propagation of analytic singularities, except for the fact that the symbol of $\tilde{P}$ is not analytic. However, denoting by W_N a holomorphic $C(Nk)^{2/3}$-approximation of W near 0 (in the sense of Lemma 8.1, and with $C>0$ sufficiently large), and setting,

$$\begin{aligned}
\tilde{W}_N(\tilde{x}) &:= W_N\big((Nk)^{1/2}\tilde{x}',(Nk)^{2/3}\tilde{x}_n\big);\\
\tilde{V}_N(\tilde{x}) &:= -C_0\tilde{x}_n+(Nk)^{-2/3}\big[E_0-\rho(h)+\tilde{W}_N(\tilde{x})\big];\\
\tilde{P}_N &:= -(Nk)^{1/3}\tilde{h}^2\Delta_{\tilde{x}'}-\tilde{h}^2\partial_{\tilde{x}_n}^2+\tilde{V}_N(\tilde{x}),
\end{aligned}$$

we deduce from (6.42), (6.45), (and, e.g., the fact that $(Nk)^N=\tilde{h}^{-N}e^{-1/\tilde{h}}=\mathscr{O}_N(e^{-1/2\tilde{h}})$), that, for any $\varepsilon>0$ fixed small enough and for any $N\geq 1$ large enough, we have,

$$T\tilde{P}_N\tilde{v}_N\big(\tilde{x},\tilde{\xi};\tilde{h}\big)=\mathscr{O}_N\big(e^{-\varepsilon^2/6\tilde{h}}\big), \tag{6.46}$$

uniformly with respect to $\tilde{h}>0$ small enough and $(\tilde{x},\tilde{\xi})\in\boldsymbol{R}^{2n}$ verifying (6.39), which can be expresses as

$$|\tilde{x}'|<|\tilde{x}'(-\delta^{-1})|+2\delta-\varepsilon,\quad |\tilde{x}_n|<|\tilde{x}_n(-\delta^{-1})|+2\delta-\varepsilon. \tag{6.47}$$

Now, by construction, the symbol of $\tilde{P}_N$ is holomorphic in a (arbitrarily large) complex neighborhood of $(0,0)$, and since $E_0-\rho(h)=\mathscr{O}(h)$ and $\partial^\alpha W(x)=\mathscr{O}(|x|^{(2-|\alpha|)_+})$, we see that the total $\tilde{h}$-semiclassical symbol $\tilde{p}_N$ of $\tilde{P}_N$ verifies,

$$\tilde{p}_N\big(\tilde{x},\tilde{\xi}\big)=\tilde{\xi}_n^2-C_0\tilde{x}_n+(Nk)^{1/3}\big(\tilde{\xi}'\big)^2+(Nk)^{1/3}\mathscr{O}\left(|\tilde{x}|^2+\left(N\ln\frac{1}{h}\right)^{-1}\right),$$

that tends to,

$$\tilde{p}_0\big(\tilde{x},\tilde{\xi}\big):=\tilde{\xi}_n^2-C_0\tilde{x}_n,$$

as $\tilde{h}$ tends to 0. In particular, $\tilde{p}_0$ does not depend on N, and the point $\exp(-\delta^{-1}H_{\tilde{p}}(0,0))$ tends to $\exp(-\delta^{-1}H_{\tilde{p}_0}(0,0))$ as $\tilde{h}\to 0_+$.

From now on, we fix $\varepsilon>0$ small enough and the cut-off function χ_0 in such a way that $\mathrm{dist}(\pi_x(\exp\tilde{t}H_{\tilde{p}_0}(0,0)),\mathrm{Supp}\,\nabla\chi_0)\geq 2\varepsilon$ for all $\tilde{t}\in[-\delta^{-1},\delta^{-1}]$, where δ is the same as in (6.43).

Then, modifying the proof of the theorem of the propagation of analytic singularities (see, e.g., [**Sj1**, Theorem 9.1], or [**Ma1**, Theorem 4.3.7]), we can show that, in our case, the estimates (6.45), (6.43) and (6.46) imply

Proposition 6.8. *There exists a constant $\delta_1>0$ independent of L, such that, for all L large enough (and $N=L/\delta_L$), one has, for $\tilde{h}>0$ small enough,*

$$T\tilde{v}_N\big(\tilde{x},\tilde{\xi};\tilde{h}\big)=\mathscr{O}\big(e^{-\delta_1\delta_L/\tilde{h}}\big) \tag{6.48}$$

uniformly in $V(\delta_1)=\{\tilde{x};|\tilde{x}|\leq\delta_1\}\times\{\tilde{\xi};\,(Nk)^{1/6}|\tilde{\xi}'|+|\tilde{\xi}_n|\leq\delta_1\}$.

Proof. As in (6.15), we make a symplectic change of coordinates

$$\begin{cases}\tilde{y}'=\tilde{x}', & \tilde{y}_n=\tilde{x}_n-\dfrac{1}{C_0}\tilde{\xi}_n^2,\\ \tilde{\eta}'=\tilde{\xi}', & \tilde{\eta}_n=\tilde{\xi}_n,\end{cases} \tag{6.49}$$

which leads to

$$\tilde{p}_0=-C_0\tilde{y}_n,\quad H_{p_0}=C_0\frac{\partial}{\partial\tilde{\eta}_n}. \tag{6.50}$$

For positive constants a,b,c,d with $b<a$ and α,β with $\alpha<2\beta d$, we take $f\in C_0^\infty(]-a,d[;[0,\alpha])$ and $\chi\in C_0^\infty(]-c,c[;[0,1])$ such that

$$f'\leq-\beta\ \text{on}\ \Big[-b,\frac{d}{2}\Big],\quad f(0)=\frac{\alpha}{2},$$

$$\chi=1\ \text{on}\ \Big[-\frac{c}{4},\frac{c}{4}\Big],\quad \chi\geq\frac14\ \text{on}\ \Big[-\frac{c}{2},\frac{c}{2}\Big],\quad \chi\leq\frac14\ \text{outside}\ \Big[-\frac{c}{2},\frac{c}{2}\Big].$$

Then the weight function

$$\psi(\tilde{y},\tilde{\eta})=f(\tilde{\eta}_n)\chi(|\tilde{y}_n|)\chi\big((Nk)^{1/6}|\tilde{y}'|\big)\chi\big((Nk)^{1/6}|\tilde{\eta}'|\big)$$

satisfies

$$|H_{\tilde{p}_0}\psi| \geq \frac{C_0\beta}{64} \quad \text{in } \tilde{\Omega}_1, \tag{6.51}$$

$$|\psi| \leq \max\left(\frac{\alpha}{4}, \frac{\alpha - \beta d}{2}\right) = \frac{\alpha}{4} \quad \text{in } \tilde{\Omega}_2, \tag{6.52}$$

where $\operatorname{Supp}\psi \subset \tilde{\Omega}_1 \cup \tilde{\Omega}_2 \cup \tilde{\Omega}_3$, with

$$\begin{aligned} \tilde{\Omega}_1 &= \mathscr{V}_{c/2} \times \left[-b, \frac{d}{2}\right], \\ \tilde{\Omega}_2 &= (\mathscr{V}_c \times [-b, d]) \backslash \tilde{\Omega}_1, \\ \tilde{\Omega}_3 &= \mathscr{V}_c \times [-a, -b] \end{aligned}$$

and

$$\mathscr{V}_c := \big\{(\tilde{y}, \tilde{\eta}'); |(\tilde{y}', \tilde{\eta}')| \leq c(Nk)^{-1/6}, \ \ |\tilde{y}_n| \leq c\big\}.$$

Remark that $\tilde{\Omega}_3 \subset \tilde{\mathscr{W}}(\tilde{h})$ if a, b are suitably chosen.

A microlocal exponential estimate leads us, in our case, to

$$\theta^2 \big\|(H_{\tilde{p}_0}\psi)e^{\theta\psi/\tilde{h}} T\tilde{v}_N\big\|^2 \leq C(k^{1/3} + \theta^3)\big\|e^{\theta\psi/\tilde{h}} T\tilde{v}_N\big\|^2 + \big\|e^{\theta\psi/\tilde{h}} T\tilde{P}_N\tilde{v}_N\big\|^2,$$

for a small parameter θ and for each fixed L, $N = L/\delta_L$. Let θ_L be such small number that $C(k^{1/3}/\theta^2 + \theta) \times C_0\beta/64 < 1/2$ holds for sufficiently small h, and let denote again by δ_L the minimum of δ_L' and θ_L. Then we have, by (6.51),

$$\big\|e^{\delta_L\psi/\tilde{h}} T\tilde{v}_N\big\|^2_{L^2(\tilde{\Omega}_1)} \leq \big\|e^{\delta_L\psi/\tilde{h}} T\tilde{v}_N\big\|^2_{L^2(\tilde{\Omega}_1 \cup \tilde{\Omega}_3)} + C_L\big\|e^{\delta_L\psi/\tilde{h}} T\tilde{P}_N\tilde{v}_N\big\|^2, \tag{6.53}$$

for some constant $C_L > 0$. First using (6.52) and $|f| \leq \alpha$, one can estimate the RHS of (6.53) by

$$2e^{\alpha\delta_L\psi/4\tilde{h}}\|T\tilde{v}_N\|^2_{L^2(\tilde{\Omega}_2)} + 2e^{\alpha\delta_L\psi/\tilde{h}}\|T\tilde{v}_N\|^2_{L^2(\tilde{\Omega}_3)} + C_L e^{\alpha\delta_L\psi/\tilde{h}}\|T\tilde{v}_N\|^2,$$

and next, by using (6.45), (6.43) and (6.46) (observing that (6.47) is satisfied with $\tilde{x} = \pi_x \exp \tilde{t}H_{\tilde{p}_0}$ for all $-\delta^{-1} \leq \tilde{t} \leq 0$), by,

$$C_L'\big(e^{(\alpha/4 + N_1/L)\delta_L/\tilde{h}} + e^{(\alpha - 1/2)\delta_L/\tilde{h}} + e^{(\alpha\delta_L - \varepsilon^2/6)/\tilde{h}}\big).$$

The LHS of (6.53) is estimated from below by

$$\left\| e^{\delta_L \psi/\tilde{h}} T\tilde{v}_N \right\|^2_{L^2(\tilde{\Omega}_1)} \geq \left\| e^{\delta_L \psi/\tilde{h}} T\tilde{v}_N \right\|^2_{L^2(V(\delta_1))} \geq e^{(3/8)\alpha\delta_L/\tilde{h}} \|T\tilde{v}_N\|^2_{L^2(V(\delta_1))},$$

if δ_1 is so small that $\psi \geq 3/8$ on $V(\delta_1)$. Thus we obtain

$$\|T\tilde{v}_N\|^2_{L^2(V(\delta_1))} \leq C'_L \big(e^{(-\alpha/8 + N_1/L)\delta_L/\tilde{h}} + e^{((5/8)\alpha - 1/2)\delta_L/\tilde{h}} + e^{((5/8)\alpha\delta_L - \varepsilon^2/6)/\tilde{h}} \big).$$

This implies (6.48), if one takes α and δ_1 sufficiently small. □

On the other hand, if $|\tilde{x}| \leq \delta'_1$ for small enough δ'_1 and $\tilde{\xi} \in \boldsymbol{R}^n \backslash V_\xi(\delta_1)$ (where we set $V_\xi(\delta_1) = \{\tilde{\xi}; (Nk)^{1/6}|\tilde{\xi}'| + |\tilde{\xi}_n| \leq \delta_1\}$), then,

$$\tilde{p}_N\big(\tilde{x}, \tilde{\xi}\big) \geq c\big(|\tilde{\xi}_n|^2 + (Nk)^{1/3}|\tilde{\xi}'|^2 + 1\big)$$

for some positive constant c, and again, standard techniques of microlocal analytic singularities (see, e.g., [**Ma1**, Theorem 4.2.2]) show, for any $m \geq 0$, the existence of some $\varepsilon_m > 1$ (still independent of N), such that (possibly by shrinking a little bit δ_1),

$$\left\| \langle\tilde{\xi}\rangle^m T_1 \tilde{v}_N \right\|_{L^2(\{|\tilde{x}| \leq \delta_1\} \times V_\xi(\delta_1)^C)} = \mathscr{O}\big(e^{-\varepsilon_m/\tilde{h}}\big). \tag{6.54}$$

Gathering (6.48) and (6.54), we obtain,

$$\left\| \langle\tilde{\xi}\rangle^m T_1 \tilde{v}_N \right\|_{L^2(\{|\tilde{x}| \leq \delta_1\} \times \boldsymbol{R}^n)} = \mathscr{O}\big(e^{-\delta'_1 \delta_L/\tilde{h}}\big).$$

In particular, using the fact that,

$$\begin{aligned} &\|T_1 \tilde{v}_N\|^2_{L^2(\{|\tilde{x}| \leq \delta_1\} \times \boldsymbol{R}^n)} \\ &\quad = (2\pi\tilde{h})^n c_1^2 \int_{\{|\tilde{x}| \leq \delta_1\} \times \boldsymbol{R}^n} e^{-(\tilde{x} - \tilde{y})^2/\tilde{h}} |\tilde{v}_N(\tilde{y})|^2 d\tilde{y} d\tilde{x} \\ &\quad \geq (\pi\tilde{h})^{-n/2} \int_{|\tilde{x} - \tilde{y}| \leq \sqrt{\tilde{h}},\, |\tilde{y}| \leq (1/2)\delta'_1} e^{-(\tilde{x} - \tilde{y})^2/\tilde{h}} |\tilde{v}_N(\tilde{y})|^2 d\tilde{y} d\tilde{x} \\ &\quad \geq \frac{b_n}{e} \pi^{-n/2} \|\tilde{v}_N\|^2_{L^2(\{|\tilde{x}| \leq (1/2)\delta'_1\})}, \end{aligned}$$

(where b_n stands for the volume of the unit ball of $\boldsymbol{R}^n$), and turning back to the previous coordinates, we obtain,

$$\|v_N\|_{L^2(\{|x'|\le(1/2)\delta_1'(Nk)^{1/2},|x_n|\le(1/2)\delta_1'(Nk)^{2/3}\})} = \mathscr{O}\big(h^{\delta_1' L}\big).$$

In the same way, working with $\langle\xi\rangle^m T_1\tilde{v}_N$ instead of $T_1\tilde{v}_N$, we also find,

$$\|v_N\|_{H^m(\{|x'|\le(1/2)\delta_1'(Nk)^{1/2},|x_n|\le(1/2)\delta_1'(Nk)^{2/3}\})} = \mathscr{O}\big(h^{\delta_1' L}\big),$$

for large enough L. Since L is arbitrarily large, (6.1) holds uniformly in $\mathscr{W}(t_N, h)$ by standard Sobolev estimates, and since $\hat{x} \in \partial\ddot{O}$ was taken arbitrarily, Proposition 6.1 follows.

7. Asymptotics of the width.

We calculate the asymptotic expansion of $\operatorname{Im}\rho(h)$ using the formula (1.2) and the results of the preceding sections.

Let $W_\sigma \subset \boldsymbol{R}^n$ be an N, h-dependent open domain containing $\ddot{O}$ defined by

$$W_\sigma = \big\{x; \operatorname{dist}(x, \ddot{O}) < \sigma(Nk)^{2/3}\big\}$$

for $1 < \sigma < 2$.

The boundary ∂W_σ is in U_N for $1 < \sigma < 2$. Hence, replacing u by w_{CN} in the formula (1.2) by using Proposition 6.1, and noticing that $\|u\|_{L^2(W_N(h))} - 1$ is exponentially small, we have

$$\operatorname{Im}\rho(h) = -h^2 \operatorname{Im}\int_{W_\sigma} \frac{\partial w_{CN}}{\partial n}\overline{w_{CN}}dS + \mathscr{O}\big(h^{2L}e^{-2S/h}\big). \tag{7.1}$$

Moreover, the domain of integration ∂W_σ can be replaced by $\partial W_\sigma \cap U_{N,1}$ using the facts (4.39) and Proposition 4.6 (i).

Then we can substitute the asymptotic formula (4.41) into the integrand of (7.1); for any L large enough, there exist $\delta_L, c_L (= 1/C_L') > 0$ such that for all $N > L/\delta_L$, one has

$$h^2\frac{\partial w_{CN}}{\partial n}\overline{w_{CN}} = h^{1-n/2}e^{-2(S+\operatorname{Re}\tilde{\phi})/h} \times \left\{\sum_{j,k=0}^{L+c_L N|\ln h|}\left(\frac{\partial\tilde{\phi}}{\partial n}\tilde{a}_j\overline{\tilde{a}_k} + h\frac{\partial\tilde{a}_j}{\partial n}\overline{\tilde{a}_k}\right)h^{j+k} + \mathscr{O}(h^L)\right\}.$$

The vector field $\partial \operatorname{Im}\tilde{\phi}/\partial x \cdot \partial/\partial x$ is transversal to the caustics $\mathscr{C}$ where $\operatorname{Im}\tilde{\phi} = \operatorname{Im}\phi = 0$. Let ι be the one-to-one map which associates to a point x in $\partial W_\sigma \cap U_{N,1}$ the point $y = \iota(x)$ on $\mathscr{C}$ such that the integral curve of $\partial \operatorname{Im}\tilde{\phi}/\partial x \cdot \partial/\partial x$ emanating from y passes by x.

Lemma 7.1. *On $\partial W_\sigma \cap U_{N,1}$, the function $\operatorname{Re}\tilde{\phi}(x)$ reaches its (transversally non-degenerate) minimum S at $\iota^{-1}(\Gamma)$ modulo $\mathscr{O}(h^\infty)$. More precisely, one has,*

$$\operatorname{Re}\tilde{\phi}(x) \mid_{\partial W_\sigma \cap U_{N,1}} = \phi(\iota(x)) + \mathscr{O}(h^\infty).$$

Proof. This is a direct consequence of (4.9) and (4.28). □

Lemma 7.2. *Let $x \in \partial W_\sigma \cap U_{N,1}$, and $y = \iota(x) \in \mathscr{C}$. There exists a family of smooth functions $\{\beta'_m(y, \operatorname{dist}(x, \mathscr{C}))\}_{m=0}^\infty$ defined in $\mathscr{C} \times [0, 2(Nk)^{2/3})$, with $\beta'_0(y, 0) > 0$, such that, for any large L, there exist $\delta_L, c_L > 0$ such that for all $N > L/\delta_L$, one has,*

$$-h^2 \operatorname{Im}\frac{\partial w_{CN}}{\partial n}\overline{w_{CN}} = h^{1-n/2} e^{-2(S+\phi(y))/h} \times \left\{ \sum_{m=0}^{2L+2c_L N|\ln h|+1} \beta'_m(y, \delta(Nk)^{2/3}) \left(N \ln\frac{1}{h}\right)^{-m} + \mathscr{O}(h^L) \right\}.$$

Proof. We know by (4.19) and (4.42) that

$$\frac{\partial \operatorname{Im}\tilde{\phi}}{\partial n} = \mathscr{O}\big(\operatorname{dist}(x, \mathscr{C})^{1/2}\big), \quad \tilde{a}_j = \mathscr{O}\big(\operatorname{dist}(x, \mathscr{C})^{-3j/2-1/4}\big).$$

It follows that, for $j + k = m$,

$$\frac{\partial \operatorname{Im}\tilde{\phi}}{\partial n}\tilde{a}_j\overline{\tilde{a}_k} h^m \sim \operatorname{dist}(x, \mathscr{C})^{-3m/2} h^m = \delta^{-3m/2}\left(\frac{h}{Nk}\right)^m,$$

$$\frac{\partial \tilde{a}_j}{\partial n}\overline{\tilde{a}_k} h^{m+1} \sim \operatorname{dist}(x, \mathscr{C})^{-3(m+1)/2} h^{m+1} = \delta^{-3(m+1)/2}\left(\frac{h}{Nk}\right)^{m+1}.$$

In particular, the principal term $\beta'_0(y, 0)$ is positive. In fact,

$$\beta'_0(y, \delta(Nk)^{2/3}) = \frac{\partial\tilde{\phi}}{\partial n}|\tilde{a}_0|^2.$$

In local coordinates as in Section 4, $\partial/\partial n = (\mathscr{O}(|x|)\partial/\partial x', (1+\mathscr{O}(|x|))\partial/\partial x_n)$ by (4.4), and

$$-\frac{\partial \operatorname{Im}\tilde{\phi}}{\partial x'}|\tilde{a}_0|^2 = \mathscr{O}(|x'|) + \mathscr{O}(x_n + b(x')),$$

$$-\frac{\partial \operatorname{Im}\tilde{\phi}}{\partial x_n}|\tilde{a}_0|^2 = \frac{\pi c_0(x', \xi_n^c(x'))^2}{\nu_1(x', \xi_n^c(x'))} + \mathscr{O}\big(\sqrt{x_n + b(x')}\big),$$

as $x_n + b(x')$ tends to 0, by (4.38) and (4.19). □

Now expanding β'_m in Taylor series with respect to $\operatorname{dist}(x, \mathscr{C})$, we obtain

$$-\operatorname{Im}\rho = h^{1-n/2}e^{-2S/h} \sum_{m=0}^{2L+2c_L N|\ln h|+1} \left(N \ln\frac{1}{h}\right)^{-m} \sum_{j=0}^{[3L/2]+1} \delta^{j-3m/2}(Nk)^{2j/3}$$
$$\times \int_{\mathscr{C}\cap U_{N,1}} e^{-2\phi(y)/h}\beta'_{m,j}(y)dy + \mathscr{O}\big(h^{L+1-n/2}e^{-2S/h}\big)\int_{\mathscr{C}\cap U_{N,1}} e^{-2\phi(y)/h}dy.$$

To the integrals of the RHS, we apply the stationary phase method using Assumption (A4), which means that the phase $\phi(y)$ attains its transversally non-degenerate minimum on the whole submanifold Γ. For any large L, we obtain,

$$\int_{\mathscr{C}\cap U_{N,1}} e^{-2\phi(y)/h}\beta'_{m,j}(y)dy = h^{(n-1-n_\Gamma)/2}\left\{\sum_{l=0}^{L-1} d_{l,j,m}h^l + \mathscr{O}(h^L)\right\},$$

where $\{d_{l,j,m}\}$ is a family of real numbers with $d_{0,0,0} > 0$. Here $n_\Gamma = \dim\Gamma$. Hence we have

$$-h^{-(1-n_\Gamma)/2}e^{2S/h}\operatorname{Im}\rho(h)$$
$$= \sum_{(l,j,m)\in\mathscr{N}} d_{l,j,m}h^l k^{2j/3}\left(\ln\frac{1}{h}\right)^{-m}(N^{2/3}\delta)^{j-3m/2} + \mathscr{O}(h^L), \tag{7.2}$$

where

$$\mathscr{N} = \left\{(l,j,m)\in \boldsymbol{N}^3; l \le L-1,\ j \le \left[\frac{3L}{2}\right]+1,\ m \le 2(L + c_L N|\ln h|)+1\right\}.$$

Lemma 7.3. *Let $(l,j,m) \in \mathscr{N}$. The real number $d_{l,j,m}$ vanishes if $j - 3m/2 \neq 0$.*

Proof. Observing that $h \ll k^{2/3} \ll |\ln h|^{-1}$, we introduce an order relation in the set $\mathscr{N}$. We write $(l,j,m) < (l',j',m')$ if one of the following three conditions holds:

$$\text{(i) } l < l', \quad \text{(ii) } l = l' \text{ and } j < j', \quad \text{(iii) } l = l',\ j = j',\ \text{and } m < m'.$$

Suppose there exists $d_{l,j,m} \neq 0$ with $j - 3m/2 \neq 0$ and let (l_0, j_0, m_0) be the smallest among such (l,m,j)'s. Then the RHS of (7.2) becomes

$$\sum_{(l,j,m)<(l_0,j_0,m_0)} d_{l,j,m}h^{l+m} + d_{l_0,j_0,m_0}h^{l_0}k^{2j_0/3}\left(\ln\frac{1}{h}\right)^{-m_0}(N^{2/3}\delta)^{j_0-3m_0/2}$$
$$+ o\left(h^{l_0}k^{2j_0/3}\left(\ln\frac{1}{h}\right)^{-m_0}\right).$$

Here, δ is an arbitrary number varying between 1 and 2 and $d_{l,j,m}$ are independent of δ. On the other hand, the LHS of (7.2) is independent of δ. This is a contradiction and we have proved Lemma 7.3. □

Then the proof of Theorem 2.3 follows from (7.2) and this Lemma 7.3, since for $j - 3m/2 = 0$, one has

$$\left(\ln\frac{1}{h}\right)^{-m}k^{2j/3} = h^m.$$

8. Appendix.

8.1. Holomorphic δ-approximation.

Let $f = f(x)$ be a smooth function on $\boldsymbol{R}^n$ uniformly bounded together with all its derivatives. A function $\tilde{f}(x,y)$ on $\boldsymbol{R}^{2n}$ is said to be an *almost-analytic extension* of f if,

$$\tilde{f}(x,0) = f(x),$$

and, for all $\alpha \in \boldsymbol{Z}_+^{2n}$,

$$\partial^\alpha\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\tilde{f}(x,y) = \mathscr{O}(|y|^\infty), \tag{8.1}$$

as $|y| \to 0_+$, uniformly with respect to x. We can construct an almost-analytic extension (see, e.g., [**MeSj**]) by setting,

$$\tilde{f}(x,y) = \sum_{\alpha\in \boldsymbol{N}^n} \frac{(iy)^\alpha}{\alpha!}\partial^\alpha f(x)\Big(1-\chi\Big(\frac{\epsilon_\alpha}{|y|}\Big)\Big), \tag{8.2}$$

where $\chi \in C_0^\infty(\boldsymbol{R})$ is a fixed cutoff function that is equal to 1 near 0, and $(\epsilon_\alpha)_{\alpha\in\boldsymbol{N}^n}$ is a decreasing sequence of positive numbers converging to 0 sufficiently rapidly. More precisely, we choose ε_α such that, for any $\beta \le \alpha$, one has,

$$|y|\sup\left|\Big(1-\chi\Big(\frac{\varepsilon_\alpha}{|y|}\Big)\Big)\partial^{\alpha+\beta}f\right| \le \alpha!.$$

Then, the corresponding almost-analytic extension have the following elementary properties:

LEMMA 8.1. *Let f be as above.*

(i) *If $\tilde{f}(x,y)$ and $\hat{f}(x,y)$ are both almost-analytic extensions of $f(x)$, then, for any $\delta > 0$, one has*

$$\tilde{f}(x,y) - \hat{f}(x,y) = \mathscr{O}(|y|^\infty);$$
$$\sup_{|y|\le\delta} |\tilde{f}(x,y)| \le \min\{\sup|f| + 2, \sup|f| + \delta(1+\sup|\nabla f|)\}.$$

(ii) *Let $\tilde{f}$ be an almost-analytic extension of f and let $I_1,\dots,I_n \subset \boldsymbol{R}$ be bounded open intervals. Then, for any $\delta > 0$ there exists a function f_δ, holomorphic in $\Gamma_\delta := \{z\in\boldsymbol{C}^n; \mathrm{dist}(z_j, I_j) < \delta,\ j=1,\dots,n\}$, such that, for all $\alpha\in\boldsymbol{Z}_+^{2n}$, $\beta\in\boldsymbol{Z}_+^n$, and $N\ge 1$, there exists $C(\alpha,N) > 0$, such that,*

$$\sup_{x+iy\in\Gamma_\delta} \left|\partial^\alpha\big(f_\delta(x+iy) - \tilde{f}(x,y)\big)\right| \le C(\alpha,N)\delta^N; \tag{8.3}$$

$$\sup_{z\in\Gamma_\delta}\left|\partial_z^\beta f_\delta\right| \le \sup_{x+iy\in\Gamma_{2\delta}}\left|\tilde{f}(x,y)\right|\delta^{-|\beta|}\beta!, \tag{8.4}$$

uniformly as $\delta\to 0_+$. (Such a function f_δ will be called a holomorphic δ-approximation of f on $I_1\times\cdots\times I_n$.)

(iii) *Suppose $n=1$ and $f(x)$ is real valued. If $f'(x_0)\ne 0$, then any almost-analytic extension $\tilde{f}(x,y)$ is one to one in a neighborhood of $(x,y)=(x_0,0)$ and the inverse $\tilde{f}^{-1}(u,v)$ defined in a neighborhood of $(u,v)=(f(x_0),0)$ is an almost-analytic extension of $f^{-1}(u)$.*

Proof. The proof of (i) is easy and we proceed with that of (ii). We denote by $\gamma(\delta)$ the (positively oriented) n-contour,

$$\gamma(\delta) := \{\zeta \in \boldsymbol{C}^n; \mathrm{dist}(\zeta_j, I_j) = 2\delta,\, j = 1, \dots, n\},$$

and, for $z \in \gamma(\delta)$, we set,

$$f_\delta(z) = \frac{1}{(2i\pi)^n} \int_{\gamma(\delta)} \frac{\tilde{f}(\operatorname{Re}\zeta, \operatorname{Im}\zeta)}{(z_1 - \zeta_1)\cdots(z_n - \zeta_n)} d\zeta_1 \cdots d\zeta_n.$$

Then, f_δ is clearly holomorphic in Γ_δ, and since $|z_j - \zeta_j| \geq \delta$ for $\zeta \in \gamma(\delta)$ and $z \in \Gamma_\delta$, (8.4) is obtained in a standard way by differentiating under the integral-sign. Moreover, for $z = x + iy \in \Gamma_\delta$, we have,

$$f_\delta(z) - \tilde{f}(x, y) = \frac{1}{(2i\pi)^n} \int_{\gamma(\delta)} \frac{\tilde{f}(\operatorname{Re}\zeta, \operatorname{Im}\zeta) - \tilde{f}(x, y)}{(z_1 - \zeta_1)\cdots(z_n - \zeta_n)} d\zeta_1 \cdots d\zeta_n,$$

and, using the notations $\partial_z = (1/2)(\partial_x - i\partial_y)$ and $\overline{\partial}_z = (1/2)(\partial_x + i\partial_y)$, we see that,

$$\begin{aligned}
&\tilde{f}(\operatorname{Re}\zeta, \operatorname{Im}\zeta) - \tilde{f}(x, y) \\
&\quad = (\zeta - z)\int_0^1 \big(\partial_z \tilde{f}\big)(t\zeta + (1-t)z)dt + \big(\overline{\zeta - z}\big)\int_0^1 \big(\overline{\partial}_z \tilde{f}\big)(t\zeta + (1-t)z)dt,
\end{aligned}$$

where $\partial_z \tilde{f}(z)$ stands for $\partial_z \tilde{f}(x, y)$, and similarly for $\overline{\partial}_z \tilde{f}(z)$. Therefore, since $\tilde{f}$ is almost-analytic, and $|\operatorname{Im} z| + |\operatorname{Im}\zeta| = \mathscr{O}(\delta)$, we obtain,

$$f_\delta(z) - \tilde{f}(x, y) = \sum_{j=1}^n \int_{\gamma(\delta)} \frac{F_j(z, \zeta)}{\prod_{\ell \neq j}(z_\ell - \zeta_\ell)} d\zeta_1 \cdots d\zeta_n + r(x, y),$$

with $F_j(z, \zeta) := \int_0^1 (\partial_{z_j}\tilde{f})(t\zeta + (1-t)z)dt$, and $\partial^\alpha r = \mathscr{O}(\delta^\infty)$ uniformly. Thus, since $F_j(z, \zeta)\prod_{\ell \neq j}(z_\ell - \zeta_\ell)^{-1}$ depends smoothly on ζ_j in the domain $A_j := \{\zeta_j; \mathrm{dist}(\zeta_j, I_j) \leq 2\delta\}$, by the Stokes formula, we obtain,

$$f_\delta(z) - \tilde{f}(x, y) = -i\sum_{j=1}^n \int_{\gamma_j(\delta)} \frac{\overline{\partial}_{\zeta_j} F_j(z, \zeta)}{\prod_{\ell \neq j}(z_\ell - \zeta_\ell)} d\zeta_1 \cdots d\zeta_n \wedge d\overline{\zeta}_j + r(x, y),$$

with $\gamma_j(\delta) := \{\zeta_j \in A_j,\ \mathrm{dist}(\zeta_\ell, I_\ell) = 2\delta,\ \ell \neq j\}$. Then, (8.3) follows from the fact that $\overline{\partial}_{\zeta_j} F_j = \mathscr{O}(\delta^\infty)$ together with all its derivatives.

Now, we prove (iii). Let us use the coordinates $(z, \bar z) = (x+iy, x-iy)$, $(\zeta, \bar\zeta) = (u+iv, u-iv)$ and regard $\tilde f$ and $g \equiv \tilde f^{-1}$ as functions of $(z, \bar z)$ and $(\zeta, \bar\zeta)$ respectively. Then

$$\tilde f\big(g(\zeta,\bar\zeta), \overline{g(\zeta,\bar\zeta)}\big) = \zeta.$$

Differentiating by $\bar\zeta$, one gets

$$\partial_z \tilde f \bar\partial_\zeta g + \bar\partial_z \tilde f \bar\partial_\zeta \bar g = 0,$$

where $\partial_z = (1/2)(\partial/\partial x - i(\partial/\partial y))$, $\bar\partial_z = (1/2)(\partial/\partial x + i(\partial/\partial y))$ and $\partial_\zeta = (1/2)(\partial/\partial u - i(\partial/\partial v))$, $\bar\partial_\zeta = (1/2)(\partial/\partial u + i(\partial/\partial v))$.

Since $\partial_z \tilde f$ does not vanish near x_0 by assumption, we can conclude that $\bar\partial_\zeta g = \mathscr{O}(|v|^\infty)$, i.e. g is an almost-analytic extension of $f^{-1}(u)$ if $\bar\partial_z \tilde f$ is, as function of (u, v), of $\mathscr{O}(|v|^\infty)$ as $v \to 0$.

First, $\bar\partial_z \tilde f = \mathscr{O}(|y|^\infty)$ since $\tilde f$ is almost-analytic. On the other hand, since $f(x)$ is real-valued, we see from (8.2) that $v = f'(x)y + \mathscr{O}(y^3)$ as $y \to 0$, and since $f'(x_0) \neq 0$, we also see that $y = \mathscr{O}(v)$ as $v \to 0$. Hence $\bar\partial_z \tilde f = \mathscr{O}(|v|^\infty)$. □

8.2. A priori estimates.

We recall some *a priori* estimates. The first is the so-called *Agmon estimate* (see for example [**HeSj2**], [**Ma1**]):

Lemma 8.2. *For any $h > 0$, $V \in L^\infty(\boldsymbol{R}^n)$ real-valued, $E \in \boldsymbol{R}$, $f \in H^1(\boldsymbol{R}^n)$, and φ real-valued and Lipshitz on $\boldsymbol{R}^n$, one has*

$$\begin{aligned}&\mathrm{Re}\,\big\langle e^{\varphi/h}(-h^2\Delta + V - E)f, e^{\varphi/h} f\big\rangle \\ &\quad = \big\| h\nabla(e^{\varphi/h} f)\big\|^2 + \big\langle (V - E - |\nabla\varphi(x)|^2) e^{\varphi/h} f, e^{\varphi/h} f\big\rangle.\end{aligned}$$

The second is a microlocal estimate originated by one of the authors ([**Ma2**]). For $u(x,h) \in S'(\boldsymbol{R}^n)$ and $\mu > 0$, we define the so-called *FBI-Bargmann transform* T_μ by the formula,

$$T_\mu u(x,\xi,h) = c_\mu \int_{\boldsymbol{R}^n} e^{i(x-y)\cdot\xi/h - \mu(x-y)^2/2h} u(y,h)\,dy, \tag{8.5}$$

where $c_\mu = \mu^{n/4} 2^{-n/2} (\pi h)^{-3n/4}$. The operator T_μ is unitary from $L^2(\boldsymbol{R}^n)$ to $L^2(\boldsymbol{R}^{2n})$, and $e^{\xi^2/2\mu h} T_\mu u(x,\xi)$ is an entire function of $z_\mu := x - i\xi/\mu$.

PROPOSITION 8.3. *Let $m \in S_{2n}(1)$, $d \geq 0$, $p \in S_{2n}(\langle\xi\rangle^{2d})$, and denote by $p_a(x,\xi;h)$ an almost-analytic extension of p. Let also $k := h\ln(1/h)$, $\rho \in [-1/3, 1/3]$ and $\psi \in C^\infty(\boldsymbol{R}^{2n};\boldsymbol{R})$, possibly h-dependent, and verifying,*

$$\partial_x^\alpha \partial_\xi^\beta \psi = \mathscr{O}\big(k^{-(a|\alpha|+(1-a)|\beta|)}\big), \tag{8.6}$$

for any $\alpha, \beta \in \boldsymbol{Z}_+^n$, with,

$$\frac{1}{3} - \min(\rho, 0) \leq a \leq \frac{2}{3} - \max(\rho, 0). \tag{8.7}$$

Then, taking $\mu = Ck^\rho$ (with $C > 0$ constant arbitrary), for $u, v \in L^2(\boldsymbol{R}^n)$, one has,

$$\begin{aligned}\big\langle mh^{-\psi}T_\mu Pu, h^{-\psi}T_\mu v\big\rangle &= \big\langle \tilde{p}(x,\xi;h)h^{-\psi}T_\mu u, h^{-\psi}T_\mu v\big\rangle \\ &\quad + \mathscr{O}\bigg(\frac{h}{\ln\frac{1}{h}}\big\|\langle\xi\rangle^d h^{-\psi}T_\mu u\big\| \cdot \big\|\langle\xi\rangle^d h^{-\psi}T_\mu v\big\|\bigg),\end{aligned} \tag{8.8}$$

with,

$$\begin{aligned}\tilde{p}(x,\xi;h) &= m(x,\xi)p_a\big(x - 2k\mu^{-1}\partial_{z_\mu}\psi, \xi + 2ik\partial_{z_\mu}\psi\big) \\ &\quad + h\partial_{z_\mu}\bigg[m(x,\xi)\bigg(\frac{1}{\mu}\frac{\partial p_a}{\partial \operatorname{Re} x} - i\frac{\partial p_a}{\partial \operatorname{Re}\xi}\bigg)\big(x - 2k\mu^{-1}\partial_{z_\mu}\psi, \xi + 2ik\partial_{z_\mu}\psi\big)\bigg],\end{aligned}$$

where we have set $\partial_{z_\mu} = (\partial_x + i\mu\partial_\xi)/2$.

In particular, if p is real-valued, one obtains,

$$\begin{aligned}&\operatorname{Im}\big\langle h^{-\psi}T_\mu Pu, h^{-\psi}T_\mu u\big\rangle \\ &\quad = k\big\langle (H_p\psi + q_\psi(x,\xi;h))h^{-\psi}T_\mu u, h^{-\psi}T_\mu u\big\rangle + \mathscr{O}(h)\big\|\langle\xi\rangle^d h^{-\psi}T_\mu u\big\|^2,\end{aligned} \tag{8.9}$$

with,

$$\begin{aligned}q_\psi(x,\xi;h) &:= h\mu\sum_{j,\ell=1}^n \big(\partial_{\xi_j}\partial_{\xi_\ell}p\big)\partial_{\xi_j}\partial_{x_\ell}\psi \\ &\quad + \sum_{\substack{\alpha\in\boldsymbol{N}^{2n} \\ 2\leq|\alpha|\leq 3}} \frac{2^{|\alpha|}k^{|\alpha|-1}}{\alpha!}\partial^\alpha p(x,\xi)\operatorname{Im}\big[(-\mu^{-1}\partial_{z_\mu}\psi, i\partial_{z_\mu}\psi)^\alpha\big].\end{aligned}$$

Proof. We follow the proof of [**Ma2**, Proposition 3.1] (see also [**BoMi**, Theorem 3]), and we do it for $d = 0$ only (the general case $d \geq 0$ can be done along the same lines and is left to the reader). We have,

$$\begin{aligned} I &:= \langle m h^{-\psi} T_\mu P u, h^{-\psi} T_\mu v \rangle \\ &= \frac{c_\mu^2}{(2\pi h)^n} \int e^{\Phi/h} m(x,\xi) p\Big(\frac{y+x'}{2}, \eta\Big) u(x') \overline{v(y')} d(x', \eta, y, y', x, \xi), \end{aligned} \tag{8.10}$$

where the integral runs over $\boldsymbol{R}^{6n}$, and where we have set,

$$\Phi = 2k\psi(x,\xi) + i(y'-y)\xi - \frac{\mu}{2}(x-y)^2 - \frac{\mu}{2}(x-y')^2 + i(y-x')\eta. \tag{8.11}$$

Then we observe that, by construction of p_a, for all $Y = (y,\eta) \in \boldsymbol{R}^{2n}$ and $X = (x - 2k\mu^{-1}\partial_{z_\mu}\psi(x,\xi), \xi + 2ik\partial_{z_\mu}\psi(x,\xi)) \in \boldsymbol{C}^{2n}$ (and setting $X_s := sY + (1-s)X$, $0 \leq s \leq 1$), we have,

$$\begin{aligned} p(Y) - p_a(X) &= \int_0^1 \Big((Y - \operatorname{Re} X) \frac{\partial p_a}{\partial \operatorname{Re} X}(X_s) - \operatorname{Im} X \frac{\partial p_a}{\partial \operatorname{Im} X}(X_s) \Big) ds \\ &= \int_0^1 \Big((Y - X) \frac{\partial p_a}{\partial \operatorname{Re} X}(X_s) + 2i \operatorname{Im} X \frac{\partial p_a}{\partial \overline{X}}(X_s) \Big) ds \\ &= (X - Y) \cdot b_1(x,\xi,Y) + r_1(x,\xi,Y), \end{aligned} \tag{8.12}$$

where b_1 and r_1 are C^∞ on $\boldsymbol{R}^{4n}$ and verify,

$$\begin{aligned} &\partial_x^\alpha \partial_\xi^\beta \partial_Y^\gamma b_1 = \mathscr{O}\big(1 + k^{\tau - a|\alpha| - (1-a)|\beta|}\big); \\ &\partial_x^\alpha \partial_\xi^\beta \partial_Y^\gamma r_1 = \mathscr{O}\big((1 + k^{\tau - a|\alpha| - (1-a)|\beta|}) |\operatorname{Im} X|^\infty\big) = \mathscr{O}(h^\infty), \end{aligned} \tag{8.13}$$

for all $\alpha, \beta \in (\boldsymbol{Z}_+)^n$, $\gamma \in (\boldsymbol{Z}_+)^{2n}$, uniformly on $\boldsymbol{R}^{4n}$, and with $\tau := \min(1-a, 1-a-\rho, a, a+\rho)$. (The last estimate comes from the fact that $|\operatorname{Im} X| = \mathscr{O}(k^{1/3})$.)

In the same way, one also has,

$$b_1(x,\xi,Y) = b_1(x,\xi,X) + B_2(x,\xi,Y)(X-Y) + r_2(x,\xi,Y), \tag{8.14}$$

with,

$$b_1(x,\xi,X) = \frac{\partial p_a}{\partial \operatorname{Re} X}(X);$$

$$\partial_x^\alpha \partial_\xi^\beta \partial_Y^\gamma B_2 = \mathscr{O}\big(1 + k^{\tau - a|\alpha| - (1-a)|\beta|}\big); \tag{8.15}$$
$$\partial_x^\alpha \partial_\xi^\beta \partial_Y^\gamma r_2 = \mathscr{O}(h^\infty).$$

Inserting (8.12) and (8.14) into (8.10), we obtain,

$$I = \big\langle m p_a(X) h^{-\psi} T_\mu u, h^{-\psi} T_\mu v \big\rangle + R_1 + R_2 + R_3 \tag{8.16}$$

with,

$$R_1 = \frac{c_\mu^2}{(2\pi h)^n} \int e^{\Phi/h} \Big(X - \Big(\frac{y+x'}{2}, \eta\Big)\Big) \frac{\partial p_a}{\partial \mathrm{Re}\, X}(X) u(x') \overline{v(y')}$$
$$\times\, m(x,\xi) d(x', \eta, y, y', x, \xi), \tag{8.17}$$

$$R_2 = \frac{c_\mu^2}{(2\pi h)^n} \int e^{\Phi/h} \Big\langle B \cdot \Big(X - \Big(\frac{y+x'}{2}, \eta\Big)\Big), X - \Big(\frac{y+x'}{2}, \eta\Big) \Big\rangle u(x')$$
$$\times\, \overline{v(y')} m(x,\xi) d(x', \eta, y, y', x, \xi), \tag{8.18}$$

where we have set,

$$B := B_2\Big(x, \xi, \frac{y+x'}{2}, \eta\Big),$$

and,

$$R_3 = \frac{c_\mu^2}{(2\pi h)^n} \int e^{\Phi/h} r\Big(x, \xi, \frac{y+x'}{2}, \eta\Big) u(x') \overline{v(y')}$$
$$\times\, m(x,\xi) d(x', \eta, y, y', x, \xi), \tag{8.19}$$

where,

$$r(x, \xi, Y) := (X - Y) \cdot r_2(x, \xi, Y) + r_1(x, \xi, Y).$$

Then, as in [**Ma2**], [**BoMi**], we observe that Φ verifies,

$$L(\Phi) = X - \Big(\frac{y+x'}{2}, \eta\Big),$$

with $L := (1/2)(-\mu^{-1}\partial_x - i\partial_\xi - i\partial_\eta, i\partial_x - \mu\partial_\xi + 2i\partial_y)$. As a consequence,

$$e^{\Phi/h}\left(X - \left(\frac{y+x'}{2}, \eta\right)\right) = hL(e^{\Phi/h}),$$

and

$$\left\langle B\cdot\left(X - \left(\frac{y+x'}{2}, \eta\right)\right), X - \left(\frac{y+x'}{2}, \eta\right)\right\rangle e^{\Phi/h} = h^2\langle B\cdot L, L\rangle e^{\Phi/h} =: h^2 A e^{\Phi/h}.$$

Thus, making an integration by parts in (8.17), we obtain,

$$R_1 = h\big\langle p_1(x,\xi)h^{-\psi}T_\mu u, h^{-\psi}T_\mu v\big\rangle, \tag{8.20}$$

with

$$\begin{aligned} p_1(x,\xi;h) &:= {}^tL\left(m(x,\xi)\frac{\partial p_a}{\partial \mathrm{Re}\, X}(X)\right) \\ &= \partial_{z_\mu}\left[m(x,\xi)\left(\frac{1}{\mu}\frac{\partial p_a}{\partial\,\mathrm{Re}\, x}(X) - i\frac{\partial p_a}{\partial\,\mathrm{Re}\,\xi}(X)\right)\right], \end{aligned}$$

(here ${}^tL$ stands for the transposed operator of L). In the same way, making two integrations by parts in (8.18), we obtain,

$$R_2 = h^2\big\langle h^{-\psi}T_{\mu,f}u, h^{-\psi}T_\mu v\big\rangle,$$

where we have set,

$$f\left(x,\xi,\frac{y+x'}{2},\eta\right) = {}^tA\cdot(m(x,\xi)), \tag{8.21}$$

and,

$$T_{\mu,f}u(x,\xi) := c_\mu\int e^{i(x-y)\xi/h-\mu(x-y)^2/2h}\mathrm{Op}_h^W(f(x,\xi,\cdot))u(y)dy.$$

In particular, by (8.15) and (8.7), we see that,

$$\partial_x^\alpha\partial_\xi^\beta\partial_Y^\gamma f = \mathscr{O}\big(k^{-1-a|\alpha|-(1-a)|\beta|}\big). \tag{8.22}$$

With the same notations, we also have,

$$R_3 = \langle h^{-\psi} T_{\mu,mr} u, h^{-\psi} T_\mu v \rangle.$$

Now, for $g \in \{f, mr\}$, we observe,

$$T_{\mu,g} u(x,\xi) = \left[T_\mu \mathrm{Op}_h^W \big(g(x', \xi', \cdot) \big) u(x,\xi) \right] \Big|_{\substack{x'=x \\ \xi'=\xi}},$$

and thus, applying a slight generalization of [**Ma1**, Proposition 3.3.1] to the case $\mu \neq 1$, we easily obtain,

$$T_{\mu,g} u(x,\xi) = \mathrm{Op}_h(\tilde{g}) T_\mu u(x,\xi), \tag{8.23}$$

where $\mathrm{Op}_h(\tilde{g})$ stands for the semiclassical pseudodifferential operator with symbol,

$$\tilde{g}\big(x, x', \xi, \xi', x^*, \xi^*\big) := g\left(x, \xi, \frac{x+x'}{2} - \xi^*, x^* \right).$$

Here, x^* and ξ^* stand for the dual variables of x and ξ, respectively. Then, writing,

$$\psi(x,\xi) - \psi(x', \xi') = (x - x')\phi_1 + (\xi - \xi')\phi_2,$$

(with $\phi_j = \phi_j(x, x', \xi, \xi'; h)$ smooth), applying Stokes formula and, in the expression of $h^{-\psi} \mathrm{Op}_h(\tilde{g}) h^\psi$, performing the change of contour,

$$\boldsymbol{R}^{2n} \ni (x^*, \xi^*) \mapsto (x^*, \xi^*) + ik(\phi_1, \phi_2),$$

we see that $h^{-\psi} \mathrm{Op}_h(\tilde{g}) h^\psi$ is an h-admissible operator with symbol,

$$g_\psi\big(x, x', \xi, \xi', x^*, \xi^*\big) := g_a\left(x, \xi, \frac{x+x'}{2} - \xi^* - ik\phi_2, x^* + ik\phi_1 \right),$$

where g_a is an almost-analytic extension of g. Moreover, by (8.6), we see that ϕ_1 and ϕ_2 verify,

$$\partial_{x,x'}^\alpha \partial_{\xi,\xi'}^\beta \phi_1 = \mathscr{O}\big(k^{-(a + a|\alpha| + (1-a)|\beta|)} \big);$$
$$\partial_{x,x'}^\alpha \partial_{\xi,\xi'}^\beta \phi_2 = \mathscr{O}\big(k^{-(1 - a + a|\alpha| + (1-a)|\beta|)} \big),$$

and thus, using (8.15), (8.22), the fact that $k \geq h$, and the Calderón-Vaillancourt theorem (see, e.g., [**Ma1**, Chapter 2, Exercise 15]), we obtain,

$$\begin{aligned} \big\| h^{-\psi}\mathrm{Op}_h(\tilde{f})h^{\psi}\big\| &= \mathscr{O}(k^{-1});\\ \big\| h^{-\psi}\mathrm{Op}_h(\widetilde{mr})h^{\psi}\big\| &= \mathscr{O}(h^{\infty}), \end{aligned}$$

and therefore,

$$\begin{aligned} R_2 &= \mathscr{O}\big(h^2k^{-1}\|h^{-\psi}T_\mu u\|\cdot\|h^{-\psi}T_\mu v\|\big);\\ R_3 &= \mathscr{O}\big(h^{\infty}\|h^{-\psi}T_\mu u\|\cdot\|h^{-\psi}T_\mu v\|\big), \end{aligned}$$

so that, by (8.16) and (8.20), (8.8) follows.

To prove (8.9), we first observe, that, by a Taylor expansion, we have,

$$\begin{aligned} &p_a\big(x-2k\mu^{-1}\partial_{z_\mu}\psi, \xi+2ik\partial_{z_\mu}\psi\big)\\ &\quad= \big(p-k\mu^{-1}\nabla_x p\nabla_x\psi - k\mu\nabla_\xi p\nabla_\xi\psi\big)(x,\xi) + ikH_p\psi(x,\xi)\\ &\qquad+\sum_{\substack{\alpha\in\boldsymbol{N}^{2n}\\ 2\leq|\alpha|\leq 3}}\frac{(2k)^{|\alpha|}}{\alpha!}\partial^\alpha p(x,\xi)\big(-\mu^{-1}\partial_{z_\mu}\psi, i\partial_{z_\mu}\psi\big)^\alpha + \mathscr{O}(k^{4/3}), \end{aligned}$$

and thus, in particular (since $k^{4/3}=\mathscr{O}(h)$),

$$\begin{aligned} &\mathrm{Im}\, p_a\big(x-2k\mu^{-1}\partial_{z_\mu}\psi, \xi+2ik\partial_{z_\mu}\psi\big)\\ &\quad= kH_p\psi(x,\xi)+\sum_{\substack{\alpha\in\boldsymbol{N}^{2n}\\ 2\leq|\alpha|\leq 3}}\frac{(2k)^{|\alpha|}}{\alpha!}\partial^\alpha p(x,\xi)\,\mathrm{Im}\left[(-\mu^{-1}\partial_{z_\mu}\psi, i\partial_{z_\mu}\psi)^\alpha\right]+\mathscr{O}(h). \end{aligned} \tag{8.24}$$

Moreover, using (8.6), we also see that,

$$\begin{aligned} &\left(\frac{1}{\mu}\frac{\partial p_a}{\partial\,\mathrm{Re}\,x} - i\frac{\partial p_a}{\partial\,\mathrm{Re}\,\xi}\right)\big(x-2k\mu^{-1}\partial_{z_\mu}\psi, \xi+2ik\partial_{z_\mu}\psi\big)\\ &\quad= \mu^{-1}\partial_x p(x,\xi) - i\partial_\xi p(x,\xi) + 2kM(x,\xi)\partial_{z_\mu}\psi(x,\xi) + \mathscr{O}(k^{2/3}), \end{aligned}$$

where M is the $n\times n$-matrix-valued function,

$$\begin{aligned} M &= -\mu^{-2}(\nabla_x \otimes \nabla_x)p + i\mu^{-1}(\nabla_\xi \otimes \nabla_x + \nabla_x \otimes \nabla_\xi)p + (\nabla_\xi \otimes \nabla_\xi)p \\ &= (\nabla_\xi \otimes \nabla_\xi)p + \mathscr{O}(k^{1/3}). \end{aligned}$$

(Here, we have used the notation $\nabla_x \otimes \nabla_\xi = (\partial_{x_j}\partial_{\xi_\ell})_{1\leq j,\ell\leq n}$.)

Since applying ∂_{z_μ} makes lose at most $k^{-2/3}$, one also easily obtains,

$$\begin{aligned} &\partial_{z_\mu}\left[\left(\frac{1}{\mu}\frac{\partial p_a}{\partial \operatorname{Re} x} - i\frac{\partial p_a}{\partial \operatorname{Re}\xi}\right)\left(x - 2k\mu^{-1}\partial_{z_\mu}\psi, \xi + 2ik\partial_{z_\mu}\psi\right)\right] \\ &\quad = \frac{\mu}{2}\Delta_\xi p + 2k\sum_{j,\ell=1}^{n}\left(\partial_{\xi_j}\partial_{\xi_\ell}p\right)\partial_{z_\mu^j}\partial_{z_\mu^\ell}\psi(x,\xi) + \mathscr{O}(1), \end{aligned}$$

and thus, if p is real-valued, one finds,

$$\begin{aligned} &\operatorname{Im}\partial_{z_\mu}\left[\left(\frac{1}{\mu}\frac{\partial p_a}{\partial \operatorname{Re} x} - i\frac{\partial p_a}{\partial \operatorname{Re}\xi}\right)\left(x - 2k\mu^{-1}\partial_{z_\mu}\psi, \xi + 2ik\partial_{z_\mu}\psi\right)\right] \\ &\quad = \frac{k}{2}\sum_{j,\ell=1}^{n}\mu\left(\partial_{\xi_j}\partial_{\xi_\ell}p\right)\left(\partial_{\xi_j}\partial_{x_\ell}\psi + \partial_{x_j}\partial_{\xi_\ell}\psi\right) + \mathscr{O}(1) \\ &\quad = k\mu\sum_{j,\ell=1}^{n}\left(\partial_{\xi_j}\partial_{\xi_\ell}p\right)\left(\partial_{x_j}\partial_{\xi_\ell}\psi\right) + \mathscr{O}(1). \end{aligned} \tag{8.25}$$

Then, (8.9) immediately follows from (8.24)–(8.25). □

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