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On Witten multiple zeta-functions associated with semisimple Lie algebras II

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Abstract. This is a continuation of our previous result, in which properties of multiple zeta-functions associated with simple Lie algebras of A_r type have been studied. In the present paper we consider more general situation, and discuss the Lie theoretic background structure of our theory. We show a recursive structure in the family of zeta-functions of sets of roots, which can be explained by the order relation among roots. We also point out that the recursive structure can be described in terms of Dynkin diagrams. Then we prove several analytic properties of zeta-functions associated with simple Lie algebras of B_r, C_r, and D_r types.

1. Introduction.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $s = \sigma + it$ a complex variable, and define

$$\zeta_W(s;\mathfrak{g}) = \sum_{\varphi} (\dim \varphi)^{-s}, \tag{1.1}$$

where the summation runs over all finite dimensional irreducible representations φ of $\mathfrak{g}$. Special values of this Dirichlet series were first studied by Witten [**22**] in connection with quantum gauge theory, and Zagier [**23**] called (1.1) as the Witten zeta-function associated with $\mathfrak{g}$. Some evaluation formulas of $\zeta_W(s;\mathfrak{g})$ at positive even integral arguments were given by Mordell [**18**], Zagier [**23**] and Gunnells and Sczech [**5**].

A more explicit expression of (1.1) can be obtained by using Weyl's dimension formula. Let r be the rank of $\mathfrak{g}$. Denote by $\Delta = \Delta(\mathfrak{g})$ the set of all roots of $\mathfrak{g}$, by $\Delta_+ = \Delta_+(\mathfrak{g})$ the set of all positive roots of $\mathfrak{g}$, and by $\Psi = \Psi(\mathfrak{g}) = \{\alpha_1, \ldots, \alpha_r\}$ the fundamental system of Δ. For any $\alpha \in \Delta$, we denote by $\alpha^\vee$ the associated coroot. Let $\lambda_1, \ldots, \lambda_r$ be the fundamental weights satisfying $\langle \alpha_i^\vee, \lambda_j \rangle = \lambda_j(\alpha_i^\vee) = \delta_{ij}$ (Kronecker's delta).

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Let $\boldsymbol{N}$ be the set of positive integers, $\boldsymbol{N}_0$ the set of non-negative integers, and $\boldsymbol{C}$ the set of complex numbers. Any dominant weight can be written as

$$\lambda = n_1\lambda_1 + \cdots + n_r\lambda_r \qquad (n_1, \ldots, n_r \in \boldsymbol{N}_0), \tag{1.2}$$

and especially the lowest strongly dominant form is $\rho = \lambda_1 + \cdots + \lambda_r$. Let d_λ be the dimension of the representation space corresponding to the dominant weight λ. By Weyl's dimension formula (see, for example, Section 3.8 of Samelson [**19**]), we have

$$\begin{aligned} d_\lambda &= \prod_{\alpha\in\Delta_+} \frac{\langle \alpha^\vee, \lambda+\rho\rangle}{\langle \alpha^\vee, \rho\rangle} \\ &= \prod_{\alpha\in\Delta_+} \frac{\langle \alpha^\vee, (n_1+1)\lambda_1 + \cdots + (n_r+1)\lambda_r\rangle}{\langle \alpha^\vee, \lambda_1 + \cdots + \lambda_r\rangle}. \end{aligned} \tag{1.3}$$

Hence, putting $m_j = n_j + 1$, we have

$$\begin{aligned} \zeta_W(s;\mathfrak{g}) &= \sum_\lambda \prod_{\alpha\in\Delta_+} \left(\frac{\langle \alpha^\vee, m_1\lambda_1 + \cdots + m_r\lambda_r\rangle}{\langle \alpha^\vee, \lambda_1 + \cdots + \lambda_r\rangle}\right)^{-s} \\ &= K(\mathfrak{g})^s \sum_{m_1=1}^\infty \cdots \sum_{m_r=1}^\infty \prod_{\alpha\in\Delta_+} \langle \alpha^\vee, m_1\lambda_1 + \cdots + m_r\lambda_r\rangle^{-s}, \end{aligned} \tag{1.4}$$

where the sum on the second member of the above runs over all dominant weights of the form (1.2), and

$$K(\mathfrak{g}) = \prod_{\alpha\in\Delta_+} \langle \alpha^\vee, \lambda_1 + \cdots + \lambda_r\rangle. \tag{1.5}$$

The analytic behaviour of $\zeta_W(s;\mathfrak{g})$ is determined essentially by the multiple series part of the right-hand side of (1.4). To analyze this multiple series closely, it is more flexible to introduce the following multi-variable version of the series:

$$\zeta_r(\mathbf{s};\mathfrak{g}) = \sum_{m_1=1}^\infty \cdots \sum_{m_r=1}^\infty \prod_{\alpha\in\Delta_+} \langle \alpha^\vee, m_1\lambda_1 + \cdots + m_r\lambda_r\rangle^{-s_\alpha}, \tag{1.6}$$

where $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+} \in \boldsymbol{C}^n$. (Here $n = |\Delta_+|$ is the number of positive roots of $\mathfrak{g}$.)

In particular, the recursive structure (which will be discussed in Sections 3 to 5) cannot be described without the above multi-variable version. We note that the relation

$$\zeta_W(s;\mathfrak{g}) = K(\mathfrak{g})^s \zeta_r(s,\dots,s;\mathfrak{g}) \tag{1.7}$$

holds.

When $\mathfrak{g} = \mathfrak{sl}(3)$, the series (1.6) is nothing but the classical Tornheim double series (see (2.5)). For more general $\mathfrak{g}$, (1.6) was first introduced by the second-named author [**15**] in the case $\mathfrak{g} = \mathfrak{so}(5)$, and then, for any A_r type Lie algebra $\mathfrak{sl}(r+1)$ ($r \in \boldsymbol{N}$) by the second and the third-named authors [**17**].

In [**17**], we have studied the analytic properties, such as the analytic continuation, location of singularities, recursive formulas, and functional relations for $\zeta_r(\mathbf{s};\mathfrak{g})$ when $\mathfrak{g}$ is of A_r type. In the present paper we will discuss in a more general framework, and especially discuss the Lie theoretic background structure lying in the theory of $\zeta_r(\mathbf{s};\mathfrak{g})$. Series (1.6) is convergent absolutely when $\Re s_\alpha > 1$ for any $\alpha \in \Delta_+$. Hence hereafter, except for the final section, we assume this condition. Also hereafter we frequently use the notation $\zeta_r(\mathbf{s};X_r)$, $K(X_r)$ etc. if $\mathfrak{g}$ is of X_r type (where $X = A$, B, C or D). The empty product is to be understood as 1.

In Section 2, we will prepare explicit expressions of $\zeta_r(\mathbf{s};\mathfrak{g})$ when $\mathfrak{g}$ is of B_r, C_r, or D_r type. The main body of the present paper is Sections 3 to 5, in which we will discuss the recursive structure in the family of those zeta-functions which can be described in terms of Mellin-Barnes integrals (Theorems 3.1, 4.2, 4.3). For this purpose we will introduce the notion of *multiple zeta-functions of root sets*, which includes (1.6) as special cases. To state the recursive structure we will use the notation

$$\zeta(\cdot;X) \to \zeta(\cdot;Y), \tag{1.8}$$

which implies that the zeta-function of a root set X can be expressed as an integral of Mellin-Barnes type whose integrand includes the zeta-function of another root set Y. We will observe that this recursive structure of those zeta-functions corresponds to an inclusion relation among certain sets of roots (Remarks 3.2, 4.1, 4.4). Moreover we will show that this correspondence can be explained in terms of Dynkin diagrams. The most general statement will be embodied in Theorems 5.3 and 5.4.

An important application of the recursive structure (1.8) is that it allows a detailed study of analytic properties (such as the meromorphic continuation and the determination of singularities) of $\zeta(\cdot;X)$ if the corresponding information is available for $\zeta(\cdot;Y)$. As an example of this principle, in the final section we will study the meromorphic continuation and location of singularities of zeta-functions of C_2, B_3 and C_3. Functional relations will be studied in subsequent papers ([**9**], [**10**]).

A part of the results proved in the present paper and [**10**] has been announced in [**7**], [**8**].

The case of exceptional algebras can be treated similarly, but we will devote a separate paper to this matter (see [**11**]).

2. Explicit forms.

First of all we note that if $\mathfrak{g}$ is the direct sum of two Lie algebras $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, then

$$\zeta_W(s;\mathfrak{g}) = \zeta_W(s;\mathfrak{g}_1)\zeta_W(s;\mathfrak{g}_2). \tag{2.1}$$

In fact, it is well known that any irreducible representation φ of $\mathfrak{g}$ is equivalent to the tensor product of two irreducible representations φ_1 of $\mathfrak{g}_1$ and φ_2 of $\mathfrak{g}_2$, and conversely if φ_i is an irreducible representation of $\mathfrak{g}_i$ $(i = 1, 2)$ then $\varphi_1 \otimes \varphi_2$ is an irreducible representation of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ (Section 3.4 of [**19**]). Hence

$$\begin{aligned}\zeta_W(s;\mathfrak{g}) &= \sum_{\varphi_1}\sum_{\varphi_2}(\dim \varphi_1 \otimes \varphi_2)^{-s} \\ &= \sum_{\varphi_1}(\dim \varphi_1)^{-s}\sum_{\varphi_2}(\dim \varphi_2)^{-s} = \zeta_W(s;\mathfrak{g}_1)\zeta_W(s;\mathfrak{g}_2).\end{aligned}$$

Therefore without loss of generality we may restrict our consideration to the case of simple $\mathfrak{g}$.

Hereafter we assume that $\mathfrak{g}$ is simple. First consider the A_r type of algebra $\mathfrak{g} = \mathfrak{sl}(r+1)$ $(r \in \boldsymbol{N})$. Let ε_j be the j-th coordinate function which assigns to each vector its j-th coordinate. It is known that the fundamental system is $\{\alpha_1, \ldots, \alpha_r\}$ with $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$ $(1 \le j \le r)$, and positive roots are

$$\varepsilon_i - \varepsilon_j = \sum_{i \le k < j} \alpha_k \qquad (1 \le i < j \le r+1) \tag{2.2}$$

(see, for example, the list at the end of Bourbaki [**2**]). The corresponding coroots are

$$(\varepsilon_i - \varepsilon_j)^\vee = e_i - e_j = \sum_{i \le k < j} \alpha_k^\vee \qquad (1 \le i < j \le r+1),$$

where e_j is the standard j-th coordinate vector. In the A_r case we have $\langle\alpha^\vee,\lambda\rangle = \langle\alpha,\lambda\rangle$, hence we have

$$\begin{aligned}&\big\langle(\varepsilon_i-\varepsilon_j)^\vee, m_1\lambda_1+\cdots+m_r\lambda_r\big\rangle\\ &\quad=\langle\varepsilon_i-\varepsilon_j, m_1\lambda_1+\cdots+m_r\lambda_r\rangle\\ &\quad=\sum_{i\le k<j}\langle\alpha_k, m_1\lambda_1+\cdots+m_r\lambda_r\rangle = m_i+\cdots+m_{j-1}.\end{aligned}$$

This implies

$$\zeta_r(\mathbf{s};A_r)=\sum_{m_1=1}^\infty\cdots\sum_{m_r=1}^\infty\prod_{1\le i<j\le r+1}(m_i+\cdots+m_{j-1})^{-s_{ij}}, \tag{2.3}$$

where $\mathbf{s}=\mathbf{s}(A_r)=(s_{ij})\in\boldsymbol{C}^{r(r+1)/2}$, and

$$K(A_r)=\prod_{1\le i<j\le r+1}(j-i). \tag{2.4}$$

This (2.3) coincides with (1.5) of [**17**]. In particular, $\zeta_1(s;\mathfrak{sl}(2))$ is the Riemann zeta-function $\zeta(s)$, and $\zeta_2(\mathbf{s};\mathfrak{sl}(3))$ coincides with the Tornheim double series

$$\zeta_{MT,2}(s_1,s_2,s_3)=\sum_{m_1=1}^\infty\sum_{m_2=1}^\infty m_1^{-s_1}m_2^{-s_2}(m_1+m_2)^{-s_3}, \tag{2.5}$$

with $s_1=s_{12}$, $s_2=s_{23}$, and $s_3=s_{13}$. This series was first introduced by Tornheim [**20**] when s_1, s_2 and s_3 are positive integers, and then, independently, Mordell [**18**] studied the further special case $s_1=s_2=s_3\in\boldsymbol{N}$. The analytic behaviour of (2.5) as a function of three complex variables is discussed in second-named author's paper [**12**].

Similarly as above, we can find explicit forms of $\zeta_r(\mathbf{s};\mathfrak{g})$ for other simple Lie algebras. For our later purpose, here we prepare the explicit forms of $\zeta_r(\mathbf{s};\mathfrak{g})$ in the case when $\mathfrak{g}$ is of B_r, C_r and D_r type.

Let $\mathfrak{g}=\mathfrak{so}(2r+1)$ ($r\in\boldsymbol{N}$), that is, B_r type. The list of Bourbaki [**2**] shows that the fundamental system is

$$\Psi(B_r)=\{\alpha_j=\alpha_j(B_r)=\varepsilon_j-\varepsilon_{j+1}\ (1\le j\le r-1),\ \alpha_r=\alpha_r(B_r)=\varepsilon_r\}, \tag{2.6}$$

and the list of positive roots is

$$\begin{cases} \varepsilon_i = \sum_{i\le k\le r} \alpha_k & (1\le i\le r), \\ \varepsilon_i - \varepsilon_j = \sum_{i\le k<j} \alpha_k & (1\le i<j\le r), \\ \varepsilon_i + \varepsilon_j = \sum_{i\le k<j} \alpha_k + 2\sum_{j\le k\le r} \alpha_k & (1\le i<j\le r). \end{cases} \tag{2.7}$$

The fundamental coroots are $\alpha_1^\vee = e_1 - e_2, \ldots, \alpha_{r-1}^\vee = e_{r-1} - e_r$ and $\alpha_r^\vee = 2e_r$. Positive coroots are $2e_i$ $(1\le i\le r)$, $e_i \pm e_j$ $(1\le i<j\le r)$. Hence the list (2.7) can be modified to the following list of positive coroots:

$$\begin{cases} 2e_i = (\varepsilon_i)^\vee = 2\sum_{i\le k<r} \alpha_k^\vee + \alpha_r^\vee & (1\le i\le r), \\ e_i - e_j = (\varepsilon_i - \varepsilon_j)^\vee = \sum_{i\le k<j} \alpha_k^\vee & (1\le i<j\le r), \\ e_i + e_j = (\varepsilon_i + \varepsilon_j)^\vee = \sum_{i\le k<j} \alpha_k^\vee + 2\sum_{j\le k<r} \alpha_k^\vee + \alpha_r^\vee & (1\le i<j\le r). \end{cases} \tag{2.8}$$

Therefore we have

PROPOSITION 2.1.

$$\begin{aligned} \zeta_r(\mathbf{s}; B_r) = \sum_{m_1=1}^\infty \cdots \sum_{m_r=1}^\infty \prod_{1\le i\le r} \bigl(2(m_i + \cdots + m_{r-1}) + m_r\bigr)^{-s_i} \\ \times \prod_{1\le i<j\le r} (m_i + \cdots + m_{j-1})^{-s_{ij}^-} \\ \times \prod_{1\le i<j\le r} \bigl(m_i + \cdots + m_{j-1} + 2(m_j + \cdots + m_{r-1}) + m_r\bigr)^{-s_{ij}^+} \end{aligned} \tag{2.9}$$

where $\mathbf{s} = \mathbf{s}(B_r) = ((s_i), (s_{ij}^-), (s_{ij}^+)) \in \boldsymbol{C}^{r^2}$, *and*

$$K(B_r) = \prod_{1\le i\le r} (2r - 2i + 1) \prod_{1\le i<j\le r} (j-i)(2r - i - j + 1). \tag{2.10}$$

For example, we have $\zeta_1(s; \mathfrak{so}(3)) = \zeta(s)$,

$$\zeta_2(\mathbf{s};\mathfrak{so}(5)) = \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}(2m_1+m_2)^{-s_1}m_2^{-s_2}m_1^{-s_{12}^-}(m_1+m_2)^{-s_{12}^+} \tag{2.11}$$

which was introduced as (4.3) in [**15**], and

$$\begin{aligned}\zeta_3(\mathbf{s};\mathfrak{so}(7)) = &\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty}(2m_1+2m_2+m_3)^{-s_1}(2m_2+m_3)^{-s_2}m_3^{-s_3}\\ &\times m_1^{-s_{12}^-}(m_1+2m_2+m_3)^{-s_{12}^+}(m_1+m_2)^{-s_{13}^-}\\ &\times (m_1+m_2+m_3)^{-s_{13}^+}m_2^{-s_{23}^-}(m_2+m_3)^{-s_{23}^+}.\end{aligned} \tag{2.12}$$

In the case $\mathfrak{g} = \mathfrak{sp}(r)$ (C_r type), the fundamental system is

$$\Psi(C_r) = \big\{\alpha_j = \alpha_j(C_r) = \varepsilon_j - \varepsilon_{j+1}\ (1 \le j \le r-1),\ \alpha_r = \alpha_r(C_r) = 2\varepsilon_r\big\}, \tag{2.13}$$

positive roots are $2\varepsilon_i$ $(1 \le i \le r)$ and $\varepsilon_i \pm \varepsilon_j$ $(1 \le i < j \le r)$, the fundamental coroots are $\alpha_1^\vee = e_1 - e_2, \ldots, \alpha_{r-1}^\vee = e_{r-1} - e_r$ and $\alpha_r^\vee = e_r$, hence the list of positive coroots is

$$\begin{cases} e_i = \displaystyle\sum_{i\le k\le r}\alpha_k^\vee & (1 \le i \le r),\\ e_i - e_j = \displaystyle\sum_{i\le k<j}\alpha_k^\vee & (1 \le i < j \le r),\\ e_i + e_j = \displaystyle\sum_{i\le k<j}\alpha_k^\vee + 2\sum_{j\le k\le r}\alpha_k^\vee & (1 \le i < j \le r).\end{cases} \tag{2.14}$$

Therefore we have

Proposition 2.2.

$$\begin{aligned}\zeta_r(\mathbf{s};C_r) = &\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\prod_{1\le i\le r}(m_i+\cdots+m_r)^{-s_i}\\ &\times\prod_{1\le i<j\le r}(m_i+\cdots+m_{j-1})^{-s_{ij}^-}\\ &\times\prod_{1\le i<j\le r}\big(m_i+\cdots+m_{j-1}+2(m_j+\cdots+m_r)\big)^{-s_{ij}^+},\end{aligned} \tag{2.15}$$

where $\mathbf{s} = \mathbf{s}(C_r) = ((s_i), (s_{ij}^-), (s_{ij}^+)) \in \boldsymbol{C}^{r^2}$, *and*

$$K(C_r) = \prod_{1\le i\le r}(r-i+1)\prod_{1\le i<j\le r}(j-i)(2r-i-j+2). \tag{2.16}$$

We see that

$$\zeta_2(\mathbf{s};\mathfrak{sp}(2)) = \sum_{m_1=1}^\infty\sum_{m_2=1}^\infty (m_1+m_2)^{-s_1}m_2^{-s_2}m_1^{-s_{12}^-}(m_1+2m_2)^{-s_{12}^+} \tag{2.17}$$

which is equal to $\zeta_2(s_{12}^+, s_{12}^-, s_2, s_1; \mathfrak{so}(5))$, if we change m_1 and m_2 in the above. This is the natural consequence of the isomorphism $B_2 \simeq C_2$. The reverse order of variables reflects the fact that the direction of arrows of Dynkin diagrams of B_2 and C_2 are opposite. The case $r = 3$ gives

$$\begin{aligned}\zeta_3(\mathbf{s};\mathfrak{sp}(3)) = \sum_{m_1=1}^\infty\sum_{m_2=1}^\infty\sum_{m_3=1}^\infty &(m_1+m_2+m_3)^{-s_1}(m_2+m_3)^{-s_2}m_3^{-s_3}\\ &\times m_1^{-s_{12}^-}(m_1+2m_2+2m_3)^{-s_{12}^+}(m_1+m_2)^{-s_{13}^-}\\ &\times(m_1+m_2+2m_3)^{-s_{13}^+}m_2^{-s_{23}^-}(m_2+2m_3)^{-s_{23}^+}.\end{aligned} \tag{2.18}$$

Lastly, in the case $\mathfrak{g} = \mathfrak{so}(2r)$ (D_r type), the fundamental system is

$$\Psi(D_r) = \bigl\{\alpha_j = \alpha_j(D_r) = \varepsilon_j - \varepsilon_{j+1}\ (1\le j\le r-1),\ \alpha_r = \alpha_r(D_r) = \varepsilon_{r-1}+\varepsilon_r\bigr\}, \tag{2.19}$$

positive roots are $\varepsilon_i \pm \varepsilon_j$ $(1 \le i < j \le r)$, the fundamental coroots are $\alpha_1^\vee = e_1 - e_2, \ldots, \alpha_{r-1}^\vee = e_{r-1} - e_r$ and $\alpha_r^\vee = e_{r-1} + e_r$, hence the list of positive coroots is

$$\begin{cases} e_i + e_r = \displaystyle\sum_{i\le k\le r-2}\alpha_k^\vee + \alpha_r^\vee & (1\le i<r),\\ e_i - e_j = \displaystyle\sum_{i\le k<j}\alpha_k^\vee & (1\le i<j\le r),\\ e_i + e_j = \displaystyle\sum_{i\le k<j}\alpha_k^\vee + 2\sum_{j\le k\le r-2}\alpha_k^\vee + \alpha_{r-1}^\vee + \alpha_r^\vee & (1\le i<j<r).\end{cases} \tag{2.20}$$

From this list we have

PROPOSITION 2.3.

$$\zeta_r(\mathbf{s};D_r)=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\prod_{1\le i<r}\left((m_i+\cdots+m_{r-2})+m_r\right)^{-s_{ir}^{+}}$$
$$\times\prod_{1\le i<j\le r}(m_i+\cdots+m_{j-1})^{-s_{ij}^{-}}$$
$$\times\prod_{1\le i<j<r}\left(m_i+\cdots+m_{j-1}+2(m_j+\cdots+m_{r-2})+m_{r-1}+m_r\right)^{-s_{ij}^{+}} \tag{2.21}$$

for $r\ge 2$, *where* $\mathbf{s}=\mathbf{s}(D_r)=((s_{ij}^{-}),(s_{ij}^{+}))\in\boldsymbol{C}^{r(r-1)}$, *and*

$$K(D_r)=\prod_{1\le i\le r}(r-i)\prod_{1\le i<j\le r}(j-i)\prod_{1\le i<j<r}(2r-i-j). \tag{2.22}$$

We find that

$$\zeta_2\big((s_1,s_{12}^{-});\mathfrak{so}(4)\big)=\zeta(s_1)\zeta(s_{12}^{-}).$$

Since $D_2\simeq A_1\oplus A_1$, this agrees with (2.1). Also we see that $\zeta_3(\mathbf{s};\mathfrak{so}(6))$ is equal to $\zeta_3(\mathbf{s};\mathfrak{sl}(4))$ (under the suitable renaming of variables), which agrees with the fact $D_3\simeq A_3$.

3. The recursive structure for A_r.

In several papers of the second-named author, it has been pointed out that there are recursive structures in the family of various multiple zeta-functions, which can be described by Mellin-Barnes type of integrals ([**13**], [**14**], [**15**], [**16**]).

In [**17**], it has been shown that such a recursive structure exists in the family of zeta-functions attached to Lie algebras of A_r type. In fact, Theorem 2.2 of [**17**] gives a formula which expresses $\zeta_{r+1}(\mathbf{s};\mathfrak{sl}(r+2))$ as a multiple integral involving $\zeta_r(\,\cdot\,;\mathfrak{sl}(r+1))$.

In the present paper we will study this recursive structure, not only for A_r type but also for B_r, C_r and D_r types, more closely. For this purpose, we introduce the notion of *multiple zeta-functions of root sets*. Let Δ^* be a subset of $\Delta_+=\Delta_+(\mathfrak{g})$. We call Δ^* a *root set* if the condition

($*$) for any λ_j $(1\le j\le r)$, there exists an element $\alpha\in\Delta^*$ such that $\langle\alpha,\lambda_j\rangle\neq 0$

is satisfied. Under this condition, we can define the multiple zeta-function of Δ^* by

$$\zeta_r(\mathbf{s};\Delta^*)=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\prod_{\alpha\in\Delta^*}\langle\alpha^\vee,m_1\lambda_1+\cdots+m_r\lambda_r\rangle^{-s_\alpha}, \tag{3.1}$$

where $\mathbf{s}=\mathbf{s}(\Delta^*)=(s_\alpha)_{\alpha\in\Delta^*}\in\boldsymbol{C}^{n^*}$ with $n^*=|\Delta^*|$.

We formulate our recursive structure in terms of these zeta-functions. In this section we consider the case when $\mathfrak{g}=\mathfrak{sl}(r+1)$ $(r\geq 2)$. By (2.2) we see that

$$\Delta_+=\Delta_+(A_r)=\left\{\varepsilon_i-\varepsilon_j=\sum_{i\leq k<j}\alpha_k\;\middle|\;1\leq i<j\leq r+1\right\}. \tag{3.2}$$

Hence

$$\Delta_h^*(A_r)=\{\varepsilon_1-\varepsilon_j\mid 2\leq j\leq h\}\cup\{\varepsilon_i-\varepsilon_j\mid 2\leq i<j\leq r+1\}\quad(2\leq h\leq r+1) \tag{3.3}$$

is a subset of $\Delta_+(A_r)$ satisfying condition $(*)$. Then there is the relation

$$\Delta^*(A_r)\subset\Delta_2^*(A_r)\subset\cdots\subset\Delta_r^*(A_r)\subset\Delta_{r+1}^*(A_r)=\Delta_+(A_r), \tag{3.4}$$

where

$$\Delta^*(A_r)=\{\varepsilon_i-\varepsilon_j\mid 2\leq i<j\leq r+1\}.$$

The vector $\mathbf{s}(\Delta_h^*(A_r))$ can be written as

$$\mathbf{s}\big(\Delta_h^*(A_r)\big)=(s_{12},\ldots,s_{1h},\mathbf{s}_2(A_r)), \tag{3.5}$$

where $\mathbf{s}_2(A_r)=(s_{ij})_{2\leq i<j\leq r+1}$.

We now show that this relation gives the Mellin-Barnes recursive structure in the family of zeta-functions of root sets in the A_r case. The classical Mellin-Barnes integral formula is

$$(1+\lambda)^{-s}=\frac{1}{2\pi\sqrt{-1}}\int_{(c)}\frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)}\lambda^z dz, \tag{3.6}$$

where s, λ are complex numbers with $\Re s>0$, $|\arg\lambda|<\pi$, $\lambda\neq 0$, c is real with

$-\Re s < c < 0$, and the path (c) of integration is the vertical line $\Re z = c$.

Let $3 \le h \le r+1$. The set $\Delta_h^*(A_r)$ includes the root

$$\varepsilon_1 - \varepsilon_h = \sum_{1 \le k < h} \alpha_k,$$

which produces the term $(m_1 + \cdots + m_{h-1})^{-s_{1h}}$ on the right-hand side of (3.1) for $\Delta^* = \Delta_h^*(A_r)$. We apply formula (3.6) to the above term to obtain

$$\begin{aligned}
&(m_1 + \cdots + m_{h-1})^{-s_{1h}} \\
&\quad = (m_1 + \cdots + m_{h-2})^{-s_{1h}} \left(1 + \frac{m_{h-1}}{m_1 + \cdots + m_{h-2}}\right)^{-s_{1h}} \\
&\quad = (m_1 + \cdots + m_{h-2})^{-s_{1h}} \frac{1}{2\pi\sqrt{-1}} \int_{(c_h)} \frac{\Gamma(s_{1h} + z_h)\Gamma(-z_h)}{\Gamma(s_{1h})} \\
&\qquad \times \left(\frac{m_{h-1}}{m_1 + \cdots + m_{h-2}}\right)^{z_h} dz_h,
\end{aligned} \tag{3.7}$$

where $-\Re s_{1h} < c_h < 0$. Hence

$$\begin{aligned}
&\zeta_r\bigl(\mathbf{s}(\Delta_h^*(A_r)); \Delta_h^*(A_r)\bigr) \\
&\quad = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{2 \le i < j \le r+1} (m_i + \cdots + m_{j-1})^{-s_{ij}} \\
&\qquad \times \prod_{2 \le j < h} (m_1 + \cdots + m_{j-1})^{-s_{1j}} \frac{1}{2\pi\sqrt{-1}} \int_{(c_h)} \frac{\Gamma(s_{1h} + z_h)\Gamma(-z_h)}{\Gamma(s_{1h})} \\
&\qquad \times (m_1 + \cdots + m_{h-2})^{-s_{1h} - z_h} m_{h-1}^{z_h} dz_h \\
&\quad = \frac{1}{2\pi\sqrt{-1}} \int_{(c_h)} \frac{\Gamma(s_{1h} + z_h)\Gamma(-z_h)}{\Gamma(s_{1h})} \\
&\qquad \times \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{\substack{2 \le i < j \le r+1 \\ (i,j) \ne (h-1,h)}} (m_i + \cdots + m_{j-1})^{-s_{ij}} m_{h-1}^{-s_{h-1,h} + z_h} \\
&\qquad \times \prod_{2 \le j \le h-2} (m_1 + \cdots + m_{j-1})^{-s_{1j}} (m_1 + \cdots + m_{h-2})^{-s_{1,h-1} - s_{1h} - z_h} dz_h \\
&\quad = \frac{1}{2\pi\sqrt{-1}} \int_{(c_h)} \frac{\Gamma(s_{1h} + z_h)\Gamma(-z_h)}{\Gamma(s_{1h})} \zeta_r\bigl(\mathbf{s}^*(A_r, z_h); \Delta_{h-1}^*(A_r)\bigr) dz_h,
\end{aligned} \tag{3.8}$$

where

$$\mathbf{s}^*(A_r,z_h)=\bigl(s_{12},\ldots,s_{1,h-2},s_{1,h-1}+s_{1h}+z_h,\mathbf{s}_2^*(A_r,z_h)\bigr)$$

and $\mathbf{s}_2^*(A_r,z_h)$ is almost the same as $\mathbf{s}_2(A_r)$ but $s_{h-1,h}$ is replaced by $s_{h-1,h}-z_h$. Formula (3.8) is an integral expression of $\zeta_r(\mathbf{s};\Delta_h^*(A_r))$ whose integrand includes $\zeta_r(\cdot\,;\Delta_{h-1}^*(A_r))$ $(3\le h\le r+1)$. Repeating this procedure, we find the recursive structure

$$\begin{aligned}\zeta_r(\cdot\,;A_r)=\zeta_r\bigl(\cdot\,;\Delta_{r+1}^*(A_r)\bigr)&\to\zeta_r\bigl(\cdot\,;\Delta_r^*(A_r)\bigr)\to\\ \cdots\to\zeta_r\bigl(\cdot\,;\Delta_3^*(A_r)\bigr)&\to\zeta_r\bigl(\cdot\,;\Delta_2^*(A_r)\bigr)\end{aligned}\tag{3.9}$$

(where we use the notation (1.8)).

In particular, when $h=3$, the second product on the third member of (3.8) is an empty product (hence is equal to 1), hence the sum with respect to m_1 can be separated, which produces the Riemann zeta factor. That is, we have

$$\zeta_r\bigl(\mathbf{s}^*(A_r,z_3);\Delta_2^*(A_r)\bigr)=\zeta_{r-1}\bigl(\mathbf{s}_2^*(A_r,z_3);\Delta^*(A_r)\bigr)\zeta(s_{12}+s_{13}+z_3).\tag{3.10}$$

The first factor on the right-hand side is not ζ_r but ζ_{r-1}, because this is defined by an $(r-1)$-ple sum. Hence

$$\begin{aligned}\zeta_r\bigl(\mathbf{s};\Delta_3^*(A_r)\bigr)=\frac{1}{2\pi\sqrt{-1}}&\int_{(c_3)}\frac{\Gamma(s_{13}+z_3)\Gamma(-z_3)}{\Gamma(s_{13})}\\ &\times\zeta_{r-1}\bigl(\mathbf{s}_2^*(A_r,z_3);\Delta^*(A_r)\bigr)\zeta(s_{12}+s_{13}+z_3)dz_3.\end{aligned}\tag{3.11}$$

This implies that the last step of (3.9) can be rewritten as

$$\zeta_r\bigl(\cdot\,;\Delta_3^*(A_r)\bigr)\to\zeta_{r-1}\bigl(\cdot\,;\Delta^*(A_r)\bigr),\tag{3.12}$$

by neglecting the Riemann zeta factor.

Renaming ε_i as ε_{i-1} $(2\le i\le r{+}1)$, we find that $\Delta^*(A_r)$ is equal to $\Delta_+(A_{r-1})$. Hence we now observe that $\zeta_r(\mathbf{s};A_r)$ can be expressed as an $(r-1)$-ple integral involving $\zeta_{r-1}(\cdot\,;A_{r-1})$, which is exactly the assertion of Theorem 2.2 of [**17**].

Summarizing the above argument, we obtain the following refinement of Theorem 2.2 of [**17**].

Theorem 3.1. *Between* $\zeta_r(\cdot\,;A_r)$ *and* $\zeta_{r-1}(\cdot\,;A_{r-1})$ *(for* $r\ge 2$*) there is* *the recursive relation given by* (3.9) *and* (3.12)*, which can be expressed as the Mellin-Barnes integrals* (3.8) *and* (3.11)*. This further gives the recursive relation*

$$\zeta_r(\cdot\,;A_r) \to \zeta_{r-1}(\cdot\,;A_{r-1}) \to \cdots \to \zeta_2(\cdot\,;A_2) = \zeta_{MT,2} \to \zeta. \tag{3.13}$$

REMARK 3.2. Among the coroots of A_r listed in the preceding section, the coroot $e_1 - e_{r+1}$ is the highest, and there is the order relation

$$e_1 - e_{r+1} > e_1 - e_r > \cdots > e_1 - e_2. \tag{3.14}$$

The recursive relations (3.9) and (3.12) correspond to this order relation. In fact, we see that

(i) in each step (3.8) we apply the Mellin-Barnes formula to the sum $m_1 + \cdots + m_{h-1}$ corresponding to the coroot $e_1 - e_h$,

and

(ii) the sum is divided into $m_1 + \cdots + m_{h-2}$ and m_{h-1}, where the former sum corresponds to the next coroot $e_1 - e_{h-1}$.

In the next section we will show that order relations among coroots similar to (3.14) also give the recursive structures among zeta-functions in the B_r, C_r and D_r cases.

4. The recursive structure for B_r, C_r and D_r.

In this section we consider the Mellin-Barnes recursive structure in the B_r, C_r, and D_r cases. First treat the B_r case ($r \geq 2$). From (2.7) we have

$$\Delta_+(B_r) = \{\varepsilon_i \mid 1 \leq i \leq r\} \cup \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq r\}. \tag{4.1}$$

We write the vector $\mathbf{s}(B_r)$ as

$$\mathbf{s}(B_r) = \big(s_1, \ldots, s_r, \mathbf{s}(B_r)^-, \mathbf{s}(B_r)^+\big),$$

where $\mathbf{s}(B_r)^{\pm} = (s_{ij}^{\pm})_{1 \leq i < j \leq r}$. The highest coroot is $2e_1$, and there is the order relation

$$\begin{aligned} 2e_1 > e_1 + e_2 > e_1 + e_3 > \cdots > e_1 + e_r \\ > e_1 - e_r > e_1 - e_{r-1} > \cdots > e_1 - e_3 > e_1 - e_2 \end{aligned} \tag{4.2}$$

among the coroots involving $\alpha_1^\vee$.

We give the Mellin-Barnes recursive structure according to this relation. Let

$$\Delta_h^*(B_r) = \{\varepsilon_i \mid 2 \leq i \leq r\} \cup \{\varepsilon_1 - \varepsilon_j \mid 2 \leq j \leq h\} \cup \{\varepsilon_i - \varepsilon_j \mid 2 \leq i < j \leq r\} \\ \cup \{\varepsilon_i + \varepsilon_j \mid 2 \leq i < j \leq r\} \quad (2 \leq h \leq r) \tag{4.3}$$

and

$$\Delta_h^{**}(B_r) = \{\varepsilon_i \mid 2 \leq i \leq r\} \cup \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq r\} \cup \{\varepsilon_1 + \varepsilon_j \mid h \leq j \leq r\} \\ \cup \{\varepsilon_i + \varepsilon_j \mid 2 \leq i < j \leq r\} \quad (2 \leq h \leq r). \tag{4.4}$$

Then there is the relation

$$\Delta^*(B_r) \subset \Delta_2^*(B_r) \subset \cdots \subset \Delta_r^*(B_r) \\ \subset \Delta_r^{**}(B_r) \subset \cdots \subset \Delta_2^{**}(B_r) \subset \Delta_+(B_r), \tag{4.5}$$

where

$$\Delta^*(B_r) = \{\varepsilon_i \mid 2 \leq i \leq r\} \cup \{\varepsilon_i \pm \varepsilon_j \mid 2 \leq i < j \leq r\}. \tag{4.6}$$

First, the root ε_1 (or the coroot $2e_1$) corresponds to the term

$$\big(2(m_1 + \cdots + m_{r-1}) + m_r\big)^{-s_1}$$

which is, by using (3.6), equal to

$$\frac{1}{2\pi\sqrt{-1}} \int_{(c')} \frac{\Gamma(s_1 + z')\Gamma(-z')}{\Gamma(s_1)} m_1^{z'} \big(m_1 + 2(m_2 + \cdots + m_{r-1}) + m_r\big)^{-s_1 - z'} dz'$$

with $-\Re s_1 < c' < 0$. Hence

$$\zeta_r(\mathbf{s}; B_r) = \frac{1}{2\pi\sqrt{-1}} \int_{(c')} \frac{\Gamma(s_1 + z')\Gamma(-z')}{\Gamma(s_1)} \\ \times \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{2 \leq i \leq r} \big(2(m_i + \cdots + m_{r-1}) + m_r\big)^{-s_i}$$

$$\times m_1^{z'-s_{12}^-} \prod_{\substack{1\le i<j\le r\\(i,j)\neq(1,2)}} (m_i+\cdots+m_{j-1})^{-s_{ij}^-}$$
$$\times \bigl(m_1+2(m_2+\cdots+m_{r-1})+m_r\bigr)^{-s_1-s_{12}^+-z'}$$
$$\times \prod_{\substack{1\le i<j\le r\\(i,j)\neq(1,2)}} \bigl(m_i+\cdots+m_{j-1}+2(m_j+\cdots+m_{r-1})+m_r\bigr)^{-s_{ij}^+}dz'$$
$$= \frac{1}{2\pi\sqrt{-1}}\int_{(c')}\frac{\Gamma(s_1+z')\Gamma(-z')}{\Gamma(s_1)}\zeta_r\bigl(\mathbf{s}'(B_r,z');\Delta_2^{**}(B_r)\bigr)dz', \tag{4.7}$$

where

$$\mathbf{s}'(B_r,z')=\bigl(s_2,\ldots,s_r,\mathbf{s}'(B_r,z')^-,\mathbf{s}'(B_r,z')^+\bigr),$$

$\mathbf{s}'(B_r,z')^-$ is almost the same as $\mathbf{s}(B_r)^-$ but s_{12}^- is replaced by $s_{12}^- - z'$, and $\mathbf{s}'(B_r,z')^+$ is almost the same as $\mathbf{s}(B_r)^+$ but s_{12}^+ is replaced by $s_1+s_{12}^++z'$. This implies the recursive relation

$$\zeta_r(\cdot\,;B_r)\to\zeta_r\bigl(\cdot\,;\Delta_2^{**}(B_r)\bigr). \tag{4.8}$$

Next consider $\Delta_h^{**}(B_r)$. The corresponding vector can be written as

$$\mathbf{s}\bigl(\Delta_h^{**}(B_r)\bigr)=\bigl(s_2,\ldots,s_r,\mathbf{s}(B_r)^-,s_{1h}^+,\ldots,s_{1r}^+,\mathbf{s}_2(B_r)^+\bigr),$$

where $\mathbf{s}_2(B_r)^+=(s_{ij}^+)_{2\le i<j\le r}$. When $2\le h\le r-1$, we apply (3.6) to the term

$$\bigl(m_1+\cdots+m_{h-1}+2(m_h+\cdots+m_{r-1})+m_r\bigr)^{-s_{1h}^+}$$
$$=\bigl(m_1+\cdots+m_h+2(m_{h+1}+\cdots+m_{r-1})+m_r\bigr)^{-s_{1h}^+}$$
$$\times\left(1+\frac{m_h}{m_1+\cdots+m_h+2(m_{h+1}+\cdots+m_{r-1})+m_r}\right)^{-s_{1h}^+} \tag{4.9}$$

which corresponds to the root $\varepsilon_1+\varepsilon_h$. Then

$$\zeta_r\bigl(\mathbf{s}(\Delta_h^{**}(B_r));\Delta_h^{**}(B_r)\bigr)$$
$$=\frac{1}{2\pi\sqrt{-1}}\int_{(c_{1h}^+)}\frac{\Gamma(s_{1h}^++z_{1h}^+)\Gamma(-z_{1h}^+)}{\Gamma(s_{1h}^+)}$$

$$\times \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{2\le i\le r} \bigl(2(m_i+\cdots+m_{r-1})+m_r\bigr)^{-s_i}$$
$$\times m_h^{z_{1h}^+} \prod_{1\le i<j\le r} (m_i+\cdots+m_{j-1})^{-s_{ij}^-}$$
$$\times \prod_{h<j\le r} \bigl(m_1+\cdots+m_{j-1}+2(m_j+\cdots+m_{r-1})+m_r\bigr)^{-s_{1j}^+}$$
$$\times \bigl(m_1+\cdots+m_h+2(m_{h+1}+\cdots+m_{r-1})+m_r\bigr)^{-s_{1h}^+-z_{1h}^+}$$
$$\times \prod_{2\le i<j\le r} \bigl(m_i+\cdots+m_{j-1}+2(m_j+\cdots+m_{r-1})+m_r\bigr)^{-s_{ij}^+} dz_{1h}^+, \quad (4.10)$$

where $-\Re s_{1h}^+ < c_{1h}^+ < 0$. This is equal to

$$\frac{1}{2\pi\sqrt{-1}} \int_{(c_{1h}^+)} \frac{\Gamma(s_{1h}^+ + z_{1h}^+)\Gamma(-z_{1h}^+)}{\Gamma(s_{1h}^+)} \zeta_r\bigl(\mathbf{s}^{**}(B_r, z_{1h}^+); \Delta_{h+1}^{**}(B_r)\bigr) dz_{1h}^+, \quad (4.11)$$

where

$$\mathbf{s}^{**}(B_r, z_{1h}^+) = \bigl(s_2, \ldots, s_r, \mathbf{s}^{**}(B_r, z_{1h}^+)^-,$$
$$s_{1h}^+ + s_{1,h+1}^+ + z_{1h}^+, s_{1,h+2}^+, \ldots, s_{1r}^+, \mathbf{s}_2(B_r)^+\bigr),$$

and $\mathbf{s}^{**}(B_r, z_{1h}^+)^-$ is almost the same as $\mathbf{s}(B_r)^-$ but $s_{h,h+1}^-$ is replaced by $s_{h,h+1}^- - z_{1h}^+$. When $h = r$, we apply (3.6) to

$$(m_1+\cdots+m_r)^{-s_{1r}^+} = (m_1+\cdots+m_{r-1})^{-s_{1r}^+} \left(1 + \frac{m_r}{m_1+\cdots+m_{r-1}}\right)^{-s_{1r}^+}.$$

Then we have

$$\zeta_r\bigl(\mathbf{s}(\Delta_r^{**}(B_r)); \Delta_r^{**}(B_r)\bigr)$$
$$= \frac{1}{2\pi\sqrt{-1}} \int_{(c_{1r}^+)} \frac{\Gamma(s_{1r}^+ + z_{1r}^+)\Gamma(-z_{1r}^+)}{\Gamma(s_{1r}^+)} \zeta_r\bigl(\mathbf{s}^{**}(B_r, z_{1r}^+); \Delta_r^*(B_r)\bigr) dz_{1r}^+, \quad (4.12)$$

where

$$\mathbf{s}^{**}\big(B_r, z_{1r}^+\big) = \big(s_2, \ldots, s_{r-1}, s_r - z_{1r}^+, \mathbf{s}^{**}(B_r, z_{1r}^+)^-, \mathbf{s}_2(B_r)^+\big)$$

and $\mathbf{s}^{**}(B_r, z_{1r}^+)^-$ is almost the same as $\mathbf{s}(B_r)^-$ but s_{1r}^- is replaced by $s_{1r}^- + s_{1r}^+ + z_{1r}^+$. Therefore we now find the recursive relation

$$\zeta_r\big(\cdot\,;\Delta_2^{**}(B_r)\big) \to \zeta_r\big(\cdot\,;\Delta_3^{**}(B_r)\big) \to \\ \cdots \to \zeta_r\big(\cdot\,;\Delta_r^{**}(B_r)\big) \to \zeta_r\big(\cdot\,;\Delta_r^{*}(B_r)\big). \tag{4.13}$$

Lastly we consider $\Delta_h^*(B_r)$ $(3 \le h \le r)$, with the corresponding vector

$$\mathbf{s}\big(\Delta_h^*(B_r)\big) = \big(s_2, \ldots, s_r, s_{12}^-, \ldots, s_{1h}^-, \mathbf{s}_2(B_r)^-, \mathbf{s}_2(B_r)^+\big),$$

where $\mathbf{s}_2(B_r)^- = (s_{ij}^-)_{2 \le i < j \le r}$. Apply (3.6) to the term

$$(m_1 + \cdots + m_{h-1})^{-s_{1h}^-} = (m_1 + \cdots + m_{h-2})^{-s_{1h}^-}\left(1 + \frac{m_{h-1}}{m_1 + \cdots + m_{h-2}}\right)^{-s_{1h}^-}$$

to obtain

$$\begin{aligned}
&\zeta_r\big(\mathbf{s}(\Delta_h^*(B_r)); \Delta_h^*(B_r)\big) \\
&\quad = \frac{1}{2\pi\sqrt{-1}} \int_{(c_{1h}^-)} \frac{\Gamma(s_{1h}^- + z_{1h}^-)\Gamma(-z_{1h}^-)}{\Gamma(s_{1h}^-)} \\
&\qquad \times \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{2 \le i \le r} \big(2(m_i + \cdots + m_{r-1}) + m_r\big)^{-s_i} \\
&\qquad \times \prod_{2 \le j \le h-1} (m_1 + \cdots + m_{j-1})^{-s_{1j}^-} \\
&\qquad \times (m_1 + \cdots + m_{h-2})^{-s_{1h}^- - z_{1h}^-}\, m_{h-1}^{z_{1h}^-} \prod_{2 \le i < j \le r} (m_i + \cdots + m_{j-1})^{-s_{ij}^-} \\
&\qquad \times \prod_{2 \le i < j \le r} \big(m_i + \cdots + m_{j-1} + 2(m_j + \cdots + m_{r-1}) + m_r\big)^{-s_{ij}^+} dz_{1h}^- \\
&\quad = \frac{1}{2\pi\sqrt{-1}} \int_{(c_{1h}^-)} \frac{\Gamma(s_{1h}^- + z_{1h}^-)\Gamma(-z_{1h}^-)}{\Gamma(s_{1h}^-)} \times \zeta_r\big(\mathbf{s}^*(B_r, z_{1h}^-); \Delta_{h-1}^*(B_r)\big) dz_{1h}^-,
\end{aligned} \tag{4.14}$$

where $-\Re s_{1h}^{-} < c_{1h}^{-} < 0$,

$$\mathbf{s}^*(B_r, z_{1h}^{-}) = \bigl(s_2, \ldots, s_r, s_{12}^{-}, \ldots, s_{1,h-2}^{-}, s_{1,h-1}^{-} + s_{1h}^{-} + z_{1h}^{-}, \\ \mathbf{s}_2^*(B_r, z_{1h}^{-})^-, \mathbf{s}_2(B_r)^+\bigr)$$

and $\mathbf{s}_2^*(B_r, z_{1h}^{-})^-$ is almost the same as $\mathbf{s}_2(B_r)^-$ but $s_{h-1,h}^{-}$ is replaced by $s_{h-1,h}^{-} - z_{1h}^{-}$. When $h = 3$, the sum with respect to m_1 on the right-hand side of (4.14) can be separated, which implies

$$\zeta_r\bigl(\mathbf{s}^*(B_r, z_{13}^{-}); \Delta_2^*(B_r)\bigr) = \zeta_{r-1}\bigl(\mathbf{s}_2^*(B_r, z_{13}^{-}); \Delta^*(B_r)\bigr)\zeta(s_{12}^{-} + s_{13}^{-} + z_{13}^{-}), \quad (4.15)$$

where

$$\mathbf{s}_2^*\bigl(B_r, z_{13}^{-}\bigr) = \bigl(s_2, \ldots, s_r, \mathbf{s}_2^*(B_r, z_{13}^{-})^-, \mathbf{s}_2(B_r)^+\bigr).$$

Hence we have the recursive relation

$$\zeta_r\bigl(\cdot\,; \Delta_r^*(B_r)\bigr) \to \zeta_r\bigl(\cdot\,; \Delta_{r-1}^*(B_r)\bigr) \to \\ \cdots \to \zeta_r\bigl(\cdot\,; \Delta_3^*(B_r)\bigr) \to \zeta_{r-1}\bigl(\cdot\,; \Delta^*(B_r)\bigr) \quad (4.16)$$

by neglecting the Riemann zeta factor on the right-hand side of (4.15).

Remark 4.1. The situation similar to (i), (ii) in Remark 3.2 also holds in the B_r case. In fact, the recursive relations (4.8), (4.13) and (4.16) exactly correspond to the order relation (4.2), and in each step, the term corresponding to the coroot is divided into two parts, one of which corresponds to the next coroot in the relation (4.2).

Renaming ε_i as ε_{i-1} $(2 \le i \le r)$, we see that $\Delta^*(B_r)$ coincides with $\Delta_+(B_{r-1})$. Therefore, collecting the results of the above argument, we now arrive at the following

Theorem 4.2. *Between* $\zeta_r(\cdot\,; B_r)$ *and* $\zeta_{r-1}(\cdot\,; B_{r-1})$ (*for any* $r \ge 2$) *there is the recursive relation given by* (4.8), (4.13) *and* (4.16), *which can be expressed as the Mellin-Barnes integrals* (4.7), (4.11), (4.12) *and* (4.14). *This further gives the recursive relation*

$$\zeta_r(\cdot\,; B_r) \to \zeta_{r-1}(\cdot\,; B_{r-1}) \to \cdots \to \zeta_2(\cdot\,; B_2) \to \zeta. \quad (4.17)$$

The treatment in the C_r case ($r \geq 2$) is similar. In this case

$$\Delta_+(C_r) = \{2\varepsilon_i \mid 1 \leq i \leq r\} \cup \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq r\}, \tag{4.18}$$

and the order relation is

$$\begin{aligned} & e_1 + e_2 > e_1 + e_3 > \cdots > e_1 + e_r > e_1 \\ & \quad > e_1 - e_r > e_1 - e_{r-1} > \cdots > e_1 - e_3 > e_1 - e_2. \end{aligned} \tag{4.19}$$

Define the set $\Delta_h^{**}(C_r)$ $(2 \leq h \leq r)$ by replacing $\{\varepsilon_i \mid 2 \leq i \leq r\}$ in the definition of $\Delta_h^{**}(B_r)$ by $\{2\varepsilon_i \mid 1 \leq i \leq r\}$, the sets $\Delta_h^*(C_r)$ and $\Delta^*(C_r)$ $(2 \leq h \leq r)$ by replacing $\{\varepsilon_i \mid 2 \leq i \leq r\}$ in the definitions of $\Delta_h^*(B_r)$ and $\Delta^*(B_r)$ respectively by $\{2\varepsilon_i \mid 2 \leq i \leq r\}$, and

$$\begin{aligned} \Delta^{**}(C_r) = & \{2\varepsilon_i \mid 1 \leq i \leq r\} \cup \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq r\} \\ & \cup \{\varepsilon_i + \varepsilon_j \mid 2 \leq i < j \leq r\}. \end{aligned} \tag{4.20}$$

Then we find

$$\begin{aligned} & \Delta^*(C_r) \subset \Delta_2^*(C_r) \subset \cdots \subset \Delta_r^*(C_r) \subset \Delta^{**}(C_r) \\ & \qquad \subset \Delta_r^{**}(C_r) \subset \cdots \subset \Delta_2^{**}(C_r) \subset \Delta_+(C_r). \end{aligned} \tag{4.21}$$

Guided by the same principle as in Remark 4.1, we can easily find the corresponding Mellin-Barnes recursive structure

$$\begin{aligned} & \zeta_r(\cdot\,; C_r) \to \zeta_r\big(\cdot\,; \Delta_2^{**}(C_r)\big) \to \zeta_r\big(\cdot\,; \Delta_3^{**}(C_r)\big) \to \cdots \to \zeta_r\big(\cdot\,; \Delta_r^{**}(C_r)\big) \\ & \quad \to \zeta_r\big(\cdot\,; \Delta^{**}(C_r)\big) \to \zeta_r\big(\cdot\,; \Delta_r^{*}(C_r)\big) \to \zeta_r\big(\cdot\,; \Delta_{r-1}^{*}(C_r)\big) \to \cdots \\ & \quad \to \zeta_r\big(\cdot\,; \Delta_3^{*}(C_r)\big) \to \zeta_{r-1}\big(\cdot\,; \Delta^{*}(C_r)\big). \end{aligned} \tag{4.22}$$

In the D_r case ($r \geq 3$), we have

$$\Delta_+(D_r) = \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq r\}. \tag{4.23}$$

Hence the vector $\mathbf{s}(D_r)$ is to be written as

$$\mathbf{s}(D_r) = \big(\mathbf{s}(D_r)^-, \mathbf{s}(D_r)^+\big),$$

where $\mathbf{s}(D_r)^\pm = (s_{ij}^\pm)_{1\le i<j\le r}$. Define $\mathbf{s}_2(D_r)^\pm = (s_{ij}^\pm)_{2\le i<j\le r}$,

$$\Delta_h^*(D_r) = \{\varepsilon_1 - \varepsilon_j \mid 2 \le j \le h\} \cup \{\varepsilon_i \pm \varepsilon_j \mid 2 \le i < j \le r\} \quad (2 \le h \le r), \tag{4.24}$$

$$\Delta^*(D_r) = \{\varepsilon_i \pm \varepsilon_j \mid 2 \le i < j \le r\} \text{ and}$$

$$\begin{aligned}\Delta_h^{**}(D_r) = \{&\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le r\} \cup \{\varepsilon_1 + \varepsilon_j \mid h \le j \le r\} \\ &\cup \{\varepsilon_i + \varepsilon_j \mid 2 \le i < j \le r\} \quad (2 \le h \le r).\end{aligned} \tag{4.25}$$

Then, corresponding to the order relation

$$e_1 + e_2 > e_1 + e_3 > \cdots > e_1 + e_r > e_1 - e_r > \cdots > e_1 - e_3 > e_1 - e_2, \tag{4.26}$$

we have

$$\begin{aligned}\Delta^*(D_r) &\subset \Delta_2^*(D_r) \subset \cdots \subset \Delta_r^*(D_r) \\ &\subset \Delta_r^{**}(D_r) \subset \cdots \subset \Delta_2^{**}(D_r) \subset \Delta_+(D_r).\end{aligned} \tag{4.27}$$

According to this relation, we give the Mellin-Barnes recursive structure. First, by using

$$\begin{aligned}&\big(m_1 + \cdots + m_{h-1} + 2(m_h + \cdots + m_{r-2}) + m_{r-1} + m_r\big)^{-s_{1h}^+} \\ &\quad = \big(m_1 + \cdots + m_h + 2(m_{h+1} + \cdots + m_{r-2}) + m_{r-1} + m_r\big)^{-s_{1h}^+} \\ &\qquad \times \left(1 + \frac{m_h}{m_1 + \cdots + m_h + 2(m_{h+1} + \cdots + m_{r-2}) + m_{r-1} + m_r}\right)^{-s_{1h}^+} \\ &\qquad\qquad (2 \le h \le r-2)\end{aligned}$$

and

$$\begin{aligned}&(m_1 + \cdots + m_{r-2} + m_{r-1} + m_r)^{-s_{1,r-1}^+} \\ &\quad = (m_1 + \cdots + m_{r-2} + m_r)^{-s_{1,r-1}^+}\left(1 + \frac{m_{r-1}}{m_1 + \cdots + m_{r-2} + m_r}\right)^{-s_{1,r-1}^+},\end{aligned}$$

we have

$$\zeta_r\bigl(\mathbf{s}(\Delta_h^{**}(D_r));\Delta_h^{**}(D_r)\bigr)$$
$$=\frac{1}{2\pi\sqrt{-1}}\int_{(c_{1h}^+)}\frac{\Gamma(s_{1h}^++z_{1h}^+)\Gamma(-z_{1h}^+)}{\Gamma(s_{1h}^+)}\zeta_r\bigl(\mathbf{s}^{**}(D_r,z_{1h}^+);\Delta_{h+1}^{**}(D_r)\bigr)dz_{1h}^+ \quad (4.28)$$

for $2\le h\le r-1$, where

$$\mathbf{s}^{**}\bigl(D_r,z_{1h}^+\bigr)=\bigl(\mathbf{s}^{**}(D_r,z_{1h}^+)^-,s_{1h}^++s_{1,h+1}^++z_{1h}^+,s_{1,h+2}^+,\ldots,s_{1r}^+,\mathbf{s}_2(D_r)^+\bigr)$$

and $\mathbf{s}_2^{**}(D_r,z_{1h}^+)^-$ is almost the same as $\mathbf{s}_2(D_r)^-$ but $s_{h,h+1}^-$ is replaced by $s_{h,h+1}^- - z_{1h}^+$.

Next, the term corresponding to the coroot e_1+e_r is

$$\bigl((m_1+\cdots+m_{r-2})+m_r\bigr)^{-s_{1r}^+}$$
$$=(m_1+\cdots+m_{r-2})^{-s_{1r}^+}\left(1+\frac{m_r}{m_1+\cdots+m_{r-2}}\right)^{-s_{1r}^+}. \quad (4.29)$$

Applying (3.6) to the above, we obtain

$$\zeta_r\bigl(\mathbf{s}(\Delta_r^{**}(D_r));\Delta_r^{**}(D_r)\bigr)$$
$$=\frac{1}{2\pi\sqrt{-1}}\int_{(c_{1r}^+)}\frac{\Gamma(s_{1r}^++z_{1r}^+)\Gamma(-z_{1r}^+)}{\Gamma(s_{1r}^+)}$$
$$\times\prod_{2\le i\le r-2}\bigl((m_i+\cdots+m_{r-2})+m_r\bigr)^{-s_{ir}^+}m_r^{-s_{r-1,r}^++z_{1r}^+}$$
$$\times\prod_{1\le i<j\le r}(m_i+\cdots+m_{j-1})^{-s_{ij}^-}(m_1+\cdots+m_{r-2})^{-s_{1r}^+-z_{1r}^+}$$
$$\times\prod_{2\le i<j<r}\bigl(m_i+\cdots+m_{j-1}+2(m_j+\cdots+m_{r-2})+m_{r-1}+m_r\bigr)^{-s_{ij}^+}dz_{1r}^+$$
$$=\frac{1}{2\pi\sqrt{-1}}\int_{(c_{1r}^+)}\frac{\Gamma(s_{1r}^++z_{1r}^+)\Gamma(-z_{1r}^+)}{\Gamma(s_{1r}^+)}\zeta_r\bigl(\mathbf{s}^{**}(D_r,z_{1r}^+);\Delta_r^*(D_r)\bigr)dz_{1r}^+, \quad (4.30)$$

where

$$\mathbf{s}^{**}(D_r,z_{1r}^+)=\bigl(\mathbf{s}^{**}(D_r,z_{1r}^+)^-,\mathbf{s}_2^{**}(D_r,z_{1r}^+)^+\bigr),$$

$\mathbf{s}^{**}(D_r, z_{1r}^+)^-$ is almost the same as $\mathbf{s}(D_r)^-$ but $s_{1,r-1}^-$ is replaced by $s_{1,r-1}^- + s_{1r}^+ + z_{1r}^+$, and $\mathbf{s}_2^{**}(D_r, z_{1r}^+)^+$ is almost the same as $\mathbf{s}_2(D_r)^+$ but $s_{r-1,r}^+$ is replaced by $s_{r-1,r}^+ - z_{1r}^+$.

Finally, similarly to (4.14) we obtain

$$\begin{aligned}&\zeta_r\bigl(\mathbf{s}(\Delta_h^*(D_r)); \Delta_h^*(D_r)\bigr)\\ &\quad= \frac{1}{2\pi\sqrt{-1}} \int_{(c_{1h}^-)} \frac{\Gamma(s_{1h}^- + z_{1h}^-)\Gamma(-z_{1h}^-)}{\Gamma(s_{1h}^-)} \zeta_r\bigl(\mathbf{s}^*(D_r, z_{1h}^-); \Delta_{h-1}^*(D_r)\bigr) dz_{1h}^- \\ &\qquad\qquad (3 \le h \le r), \qquad (4.31)\end{aligned}$$

where

$$\mathbf{s}^*\bigl(D_r, z_{1h}^-\bigr) = \bigl(s_{12}^-, \ldots, s_{1,h-2}^-, s_{1,h-1}^- + s_{1h}^- + z_{1h}^-, \mathbf{s}_2^*(D_r, z_{1h}^-)^-, \mathbf{s}_2(D_r)^+\bigr)$$

and $\mathbf{s}_2^*(D_r, z_{1h}^-)^-$ is almost the same as $\mathbf{s}_2(D_r)^-$ but $s_{h-1,h}^-$ is replaced by $s_{h-1,h}^- - z_{1h}^-$. When $h = 3$, we further find

$$\zeta_r\bigl(\mathbf{s}^*(D_r, z_{13}^-); \Delta_2^*(D_r)\bigr) = \zeta_{r-1}\bigl(\mathbf{s}_2^*(D_r, z_{13}^-); \Delta^*(D_r)\bigr)\zeta\bigl(s_{12}^- + s_{13}^- + z_{13}^-\bigr) \qquad (4.32)$$

with

$$\mathbf{s}_2^*\bigl(D_r, z_{13}^-\bigr) = \bigl(\mathbf{s}_2^*(D_r, z_{13}^-)^-, \mathbf{s}_2(D_r)^+\bigr).$$

Therefore, we now find the recursive structure

$$\begin{aligned}&\zeta_r(\cdot\,; D_r) \to \zeta_r\bigl(\cdot\,; \Delta_2^{**}(D_r)\bigr) \to \zeta_r\bigl(\cdot\,; \Delta_3^{**}(D_r)\bigr) \to \\ &\quad \cdots \to \zeta_r\bigl(\cdot\,; \Delta_r^{**}(D_r)\bigr) \to \zeta_r\bigl(\cdot\,; \Delta_r^*(D_r)\bigr) \to \zeta_r\bigl(\cdot\,; \Delta_{r-1}^*(D_r)\bigr) \to \\ &\quad \cdots \to \zeta_r\bigl(\cdot\,; \Delta_3^*(D_r)\bigr) \to \zeta_{r-1}\bigl(\cdot\,; \Delta^*(D_r)\bigr). \qquad (4.33)\end{aligned}$$

Summarizingly, we obtain

Theorem 4.3. *In the C_r and D_r cases, there are recursive structures* (4.22) *and* (4.33)*, which can be expressed as Mellin-Barnes integrals. These further give recursive relations*

$$\zeta_r(\cdot\,; C_r) \to \zeta_{r-1}(\cdot\,; C_{r-1}) \to \cdots \to \zeta_2(\cdot\,; C_2) \to \zeta \qquad (4.34)$$

and

$$\zeta_r(\cdot\,;D_r) \to \zeta_{r-1}(\cdot\,;D_{r-1}) \to \cdots \to \zeta_2(\cdot\,;D_2). \tag{4.35}$$

REMARK 4.4. The situation described in Remarks 3.2 and 4.1 also holds in the C_r case. On the other hand, it is to be noted that in the D_r case, the principle of type (ii) is no longer valid at the step $e_1+e_r \to e_1-e_r$. In fact, (4.29) shows that the sum $(m_1+\cdots+m_{r-2})+m_r$, corresponding to e_1+e_r, is divided into $m_1+\cdots+m_{r-2}$ and m_r, and $m_1+\cdots+m_{r-2}$ corresponds not to e_1-e_r but to e_1-e_{r-1}.

5. Recursive structures and Dynkin diagrams.

Theorems 3.1, 4.2 and 4.3, proved in the previous sections, give certain recursive structures among our zeta-functions (1.6). In this section we discuss that we can find many other recursive relations among those zeta-functions.

In order to describe the situation, it is better to introduce here the viewpoint of Dynkin diagrams. Let $\Gamma(X_r)$ be the Dynkin diagram of the root system of type X_r. Theorems 3.1, 4.2 and 4.3 give the recursive relations of the form

$$\zeta_r(\cdot\,;X_r) \to \zeta_{r-1}(\cdot\,;X_{r-1}) \tag{5.1}$$

for $X=A,B,C$ and D, by separating m_1 which corresponds to the coroot $\alpha_1^\vee$. In terms of Dynkin diagrams, this is the procedure of cutting off the leftmost edge, that is the edge joining the vertices corresponding to $\alpha_1^\vee$ and $\alpha_2^\vee$, of $\Gamma(X_r)$ (see Fig. 1).

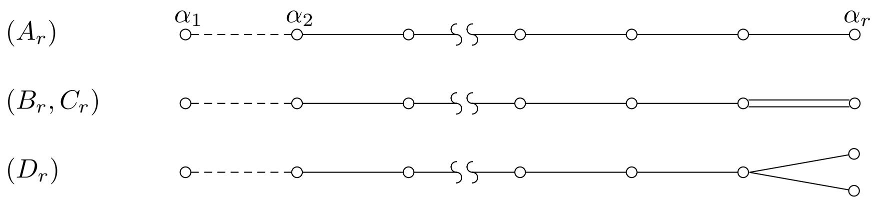


Figure 1.

However it is also possible to cut off the rightmost edge(s) of $\Gamma(X_r)$. This can be realized by separating m_r corresponding to the coroot $\alpha_r^\vee$. In the case when $X_r=B_r$, the coroots involving $\alpha_r^\vee$ are $2e_i$ $(1\le i\le r)$ and e_i+e_j $(1\le i<j\le r)$. For each i $(1\le i\le r-1)$, we can construct the Mellin-Barnes recursive relation by following

$$2e_i > e_i+e_{i+1} > e_i+e_{i+2} > \cdots > e_i+e_r > e_i-e_r, \tag{5.2}$$

similarly to the argument in the previous sections. The consequence is that we have a multiple integral expression of $\zeta_r(\mathbf{s}; B_r)$ involving $\zeta_{r-1}(\cdot\ ; A_{r-1})$ and ζ. That is, we find the recursive relation

$$\zeta_r(\cdot\,; B_r) \to \zeta_{r-1}(\cdot\,; A_{r-1}). \tag{5.3}$$

Similarly, we can obtain the relation

$$\zeta_r(\cdot\,; C_r) \to \zeta_{r-1}(\cdot\,; A_{r-1}), \tag{5.4}$$

by using

$$e_i + e_{i+1} > e_i + e_{i+2} > \cdots > e_i + e_r > e_i > e_i - e_r, \tag{5.5}$$

and

$$\zeta_r(\cdot\,; D_r) \to \zeta_{r-1}(\cdot\,; A_{r-1}), \tag{5.6}$$

by using

$$e_i + e_{i+1} > e_i + e_{i+2} > \cdots > e_i + e_r > e_i - e_{r-1} \tag{5.7}$$

(see Fig. 2).

(B_r, C_r)

(D_r)

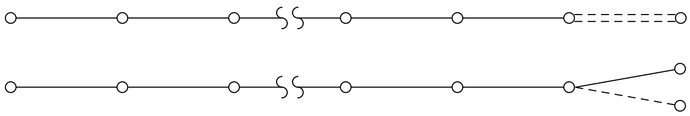


Figure 2.

In the cases of B_r and C_r, we can also cut off only one of the doubled rightmost edges of the diagram, which gives the relation

$$\zeta_r(\cdot\,; B_r) \to \zeta_r(\cdot\,; A_r), \qquad \zeta_r(\cdot\,; C_r) \to \zeta_r(\cdot\,; A_r). \tag{5.8}$$

Consider the B_r case. The sum $m_i + \cdots + m_r$, corresponding to the coroot $e_i + e_r$, exists as one of the factors of $\zeta(\mathbf{s}; A_r)$. Hence the final step of (5.2) is not necessary this time, and we obtain the first relation of (5.8) by following

$$2e_i > e_i + e_{i+1} > e_i + e_{i+2} > \cdots > e_i + e_r. \tag{5.9}$$

Similarly, by following

$$e_i + e_{i+1} > e_i + e_{i+2} > \cdots > e_i + e_r > e_i, \tag{5.10}$$

which is (5.5) without the last step, we can obtain the second relation of (5.8) (see Fig. 3).

(B_r, C_r)

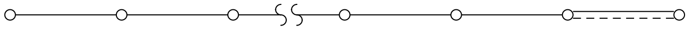


Figure 3.

In general, we can cut off any edge of the diagram and reduce a zeta-function for a root system to that for another root system. To state our assertion, we recall the definition concerning root systems and construct embeddings in a non-standard sense (that is, embeddings which do not preserve the inner products).

Let Δ be a reduced root system which may not be irreducible, Γ its Dynkin diagram and $\Psi = \{\alpha_1, \ldots, \alpha_r\}$ its fundamental system. (The j-th vertex from the left on Γ corresponds to α_j.) By $Q^\vee$ we denote the coroot lattice generated by $\Psi^\vee$. Let Γ' be a Dynkin diagram obtained by cutting off some edges from Γ. Let Δ', $\Psi' = \{\alpha'_1, \ldots, \alpha'_n\}$ and $(Q')^\vee$ be the corresponding root system, fundamental system and coroot lattice, respectively. Then we see that the map $f : (\Psi')^\vee \to \Psi^\vee$ defined by $f : (\alpha'_j)^\vee \mapsto \alpha_j^\vee$ is $\mathbb{Z}$-linearly extended to an isomorphism $f : (Q')^\vee \to Q^\vee$ as $\mathbb{Z}$-modules.

For $\beta^\vee = \sum_j c_j \alpha_j^\vee \in Q^\vee$, we denote its height by

$$\mathrm{ht}\ \beta^\vee = \sum_j c_j. \tag{5.11}$$

Lemma 5.1. $f((\Delta')_+^\vee) \subset \Delta_+^\vee$.

Proof. We show the statement by induction on their heights. We denote by $\beta^\vee \in Q^\vee$ the image $f((\beta')^\vee)$ of $(\beta')^\vee \in (\Delta')^\vee$. We first note that $\langle \alpha'_i, (\alpha'_j)^\vee \rangle \geq \langle \alpha_i, \alpha_j^\vee \rangle$ because the value of $\langle \alpha_i, \alpha_j^\vee \rangle$ is only $0, -1, -2$ or -3 for $i \neq j$ and cutting off some edges of the diagram only produces the effect of increasing the value. Hence in general, we have

$$\left\langle \alpha'_i, (\beta')^\vee \right\rangle \geq \langle \alpha_i, \beta^\vee \rangle \tag{5.12}$$

for $(\beta')^\vee \in (\Delta')_+^\vee$.

Because $(\beta')^\vee \in (\Delta')_+^\vee$ with $\mathrm{ht}\ (\beta')^\vee = 1$ implies $(\beta')^\vee = (\alpha'_i)^\vee \in (\Psi')^\vee$, for some i, it follows that $\beta^\vee = \alpha_i^\vee \in \Psi^\vee$ by definition.

Let $m \geq 1$, and assume $\beta^\vee \in \Delta_+^\vee$ for $(\beta')^\vee \in (\Delta')_+^\vee$ with ht $(\beta')^\vee \leq m$. Let $(\beta')^\vee = \sum_j c'_j(\alpha'_j)^\vee \in (\Delta')_+^\vee$ with ht $(\beta')^\vee = m+1$. It is known that there exists a decomposition $(\beta')^\vee = (\alpha')^\vee + (\alpha'_i)^\vee$ with $(\alpha')^\vee \in (\Delta')_+^\vee$, ht $(\alpha')^\vee = m$ and $(\alpha'_i)^\vee \in (\Psi')^\vee$ (see, e.g., Lemma A in Section 10.2 of [**6**] or Proposition A in Section 2.11 of [**19**]).

If $\langle \alpha'_i, (\alpha')^\vee \rangle < 0$, then $\langle \alpha_i, \alpha^\vee \rangle < 0$ by (5.12). By the assumption of induction we have $\alpha^\vee \in \Delta_+^\vee$ and the simple reflection

$$r_i(\alpha^\vee) = \alpha^\vee - \langle \alpha_i, \alpha^\vee \rangle \alpha_i^\vee$$

is also $\in \Delta_+^\vee$. Hence $\beta^\vee = \alpha^\vee + \alpha_i^\vee \in \Delta_+^\vee$ since all the roots in the $\alpha_i^\vee$-string through $\alpha^\vee$ belong to $\Delta^\vee$.

If $\langle \alpha'_i, (\alpha')^\vee \rangle \geq 0$, we consider $(\gamma')^\vee = r'_i((\beta')^\vee)$, where r'_i is the simple reflection with respect to α'_i. Then $(\gamma')^\vee \in \Delta^\vee$ because $(\beta')^\vee \in \Delta^\vee$. We see that

$$(\gamma')^\vee = r'_i\big((\alpha')^\vee + (\alpha'_i)^\vee\big) = (\alpha')^\vee - \big(\langle \alpha'_i, (\alpha')^\vee \rangle + 1\big)(\alpha'_i)^\vee$$

which implies that ht $(\gamma')^\vee \leq m-1$. Moreover we see that $(\gamma')^\vee \in \Delta_+^\vee$; in fact, since ht $(\beta')^\vee = m+1 \geq 2$, there exists a $j \neq i$ for which c'_j is positive, and so the coefficient of $(\alpha'_j)^\vee$ in $(\gamma')^\vee$ is also positive, which implies $(\gamma')^\vee \in (\Delta')_+^\vee$. Hence by the assumption of induction we have $\alpha^\vee \in \Delta_+^\vee$, $\gamma^\vee \in \Delta_+^\vee$ and

$$r_i(\gamma^\vee) = \alpha^\vee + \big(\langle \alpha'_i, (\alpha')^\vee \rangle - \langle \alpha_i, \alpha^\vee \rangle + 1\big)\alpha_i^\vee \in \Delta_+^\vee.$$

Hence $\beta^\vee = \alpha^\vee + \alpha_i^\vee \in \Delta_+^\vee$. □

Theorem 5.2. *Let Δ, Δ' be reduced root systems and Γ, Γ' be their Dynkin diagrams. Then there exists an isomorphism $f : (Q')^\vee \to Q^\vee$ such that $f((\Delta')_+^\vee) \subset \Delta_+^\vee$ and $f : (\alpha'_j)^\vee \mapsto \alpha_j^\vee$ if and only if Γ' is obtained from Γ by cutting off some edges.*

Proof. By Lemma 5.1, we have only to show $\langle \alpha'_i, (\alpha'_j)^\vee \rangle \geq \langle \alpha_i, \alpha_j^\vee \rangle$ if such f exists. Because $\langle \alpha'_i, (\alpha'_j)^\vee \rangle = \langle \alpha_i, \alpha_j^\vee \rangle = 2$ if $i = j$, we assume $i \neq j$. Since $(\Delta')_+^\vee \ni r'_i(\alpha'_j)^\vee = (\alpha'_j)^\vee - \langle \alpha'_i, (\alpha'_j)^\vee \rangle (\alpha'_i)^\vee$, we have $\Delta_+^\vee \ni f(r'_i(\alpha'_j)^\vee) = \alpha_j^\vee - \langle \alpha'_i, (\alpha'_j)^\vee \rangle \alpha_i^\vee$.

On the other hand, we have $\Delta^\vee \ni r_i \alpha_j^\vee = \alpha_j^\vee - \langle \alpha_i, \alpha_j^\vee \rangle \alpha_i^\vee$. Since the length of the $\alpha_i^\vee$-string through $\alpha_j^\vee$ is $\langle \alpha_i, \alpha_j^\vee \rangle$, it follows that the string consists of $\{\alpha_j^\vee + h\alpha_i^\vee \mid 0 \leq h \leq -\langle \alpha_i, \alpha_j^\vee \rangle\}$. Therefore we have $0 \geq \langle \alpha'_i, (\alpha'_j)^\vee \rangle \geq \langle \alpha_i, \alpha_j^\vee \rangle$. □

Since f is injective, we identify $(\Delta')_+^\vee$ and its image by f, and denote the image by the same symbol $(\Delta')_+^\vee$.

Let $(\Delta^*)^\vee = \Delta_+^\vee \setminus (\Delta')_+^\vee$ and $k = |\Delta^*|$. We fix an order

$$(\Delta^*)^\vee = \left\{\beta_1^\vee, \beta_2^\vee, \ldots, \beta_k^\vee\right\} \tag{5.13}$$

by their heights

$$\mathrm{ht}\ \beta_1^\vee \leq \mathrm{ht}\ \beta_2^\vee \leq \cdots \leq \mathrm{ht}\ \beta_k^\vee. \tag{5.14}$$

For $0 \leq j \leq k$, define

$$\left(\Delta_j^*\right)^\vee = (\Delta')_+^\vee \cup \left\{\beta_1^\vee, \ldots, \beta_j^\vee\right\} \tag{5.15}$$

so that

$$(\Delta')_+^\vee = \left(\Delta_0^*\right)^\vee \subset \left(\Delta_1^*\right)^\vee \subset \cdots \subset \left(\Delta_{k-1}^*\right)^\vee \subset \left(\Delta_k^*\right)^\vee = \Delta_+^\vee. \tag{5.16}$$

Then for $1 \leq j \leq k$ we have a decomposition $\beta_j^\vee = \alpha_l^\vee + \gamma^\vee$ with some $\alpha_l^\vee \in \Psi^\vee$ and $\gamma^\vee \in (\Delta_{j-1}^*)^\vee$ since the order is determined by their heights. Then, again by the Mellin-Barnes argument, we have

$$\begin{aligned}
\zeta_r\left(\mathbf{s}; \Delta_j^*\right) &= \sum_\lambda \prod_{\alpha^\vee \in (\Delta_j^*)^\vee} \langle \alpha^\vee, \lambda\rangle^{-s_\alpha} \\
&= \sum_\lambda \left(\prod_{\alpha^\vee \in (\Delta_{j-1}^*)^\vee} \langle \alpha^\vee, \lambda\rangle^{-s_\alpha}\right) \left\langle \alpha_l^\vee + \gamma^\vee, \lambda\right\rangle^{-s_{\beta_j}} \\
&= \sum_\lambda \prod_{\alpha^\vee \in (\Delta_{j-1}^*)^\vee \setminus \{\alpha_l^\vee, \gamma^\vee\}} \langle \alpha^\vee, \lambda\rangle^{-s_\alpha} \\
&\quad \times \frac{1}{2\pi\sqrt{-1}} \int_{(c)} \frac{\Gamma(s_{\beta_j} + z)\Gamma(-z)}{\Gamma(s_{\beta_j})} \langle \alpha_l^\vee, \lambda\rangle^{-s_{\alpha_l}+z} \langle \gamma^\vee, \lambda\rangle^{-s_{\beta_j}-s_\gamma-z} dz,
\end{aligned} \tag{5.17}$$

where Δ_j^* is the set of positive roots corresponding to $(\Delta_j^*)^\vee$. This implies the following theorem.

Theorem 5.3. *We have*

$$\zeta_r(\mathbf{s};\Delta_j^*)=\frac{1}{2\pi\sqrt{-1}}\int_{(c)}\frac{\Gamma(s_{\beta_j}+z)\Gamma(-z)}{\Gamma(s_{\beta_j})}$$
$$\times\,\zeta_r\big(\ldots,s_{\alpha_l}-z,\ldots,s_{\beta_j}+s_\gamma+z,\ldots;\Delta_{j-1}^*\big)dz, \tag{5.18}$$

and, repeating this procedure k times, we obtain the recursive relation

$$\zeta_r(\cdot\,;\Delta_+)\to\zeta_r(\cdot\,;\Delta_+'). \tag{5.19}$$

This general result includes all examples discussed above. Let $\Gamma=\Gamma(X_r)$, $X=A,B,C$ or D. When $X=B$ or C and Γ' is obtained by removing only one of the doubled rightmost edges of Γ, then Γ' is irreducible. These cases are described as (5.8).

Except for those cases, any Γ' which is obtained by cutting off edge(s) of Γ is not irreducible, hence the corresponding zeta-function $\zeta_r(\cdot\,;\Delta_+')$ is the product of two (or more) zeta-functions.

If we cut off the leftmost edge, then $\zeta_r(\cdot\,;\Delta_+')$ is the product of $\zeta_{r-1}(\cdot\,;X_{r-1})$ and the Riemann zeta-function. These cases are discussed in detail in Sections 3 and 4. The cases of cutting off the rightmost edge(s) are presented as (5.3), (5.4) and (5.6).

More generally, we can cut off the edge which joins two vertices corresponding to $\alpha_{\ell-1}^\vee$ and $\alpha_\ell^\vee$ ($2\le\ell\le r$ for $X=A$, $2\le\ell\le r-1$ for $X=B$ or C, and $2\le\ell\le r-2$ for $X=D$). Then (5.18) implies that $\zeta_r(\cdot\,;X_r)$ can be written as a multiple integral involving $\zeta_{\ell-1}(\cdot\,;A_{\ell-1})$ and $\zeta_{r-\ell+1}(\cdot\,;X_{r-\ell+1})$ (see Fig. 4).

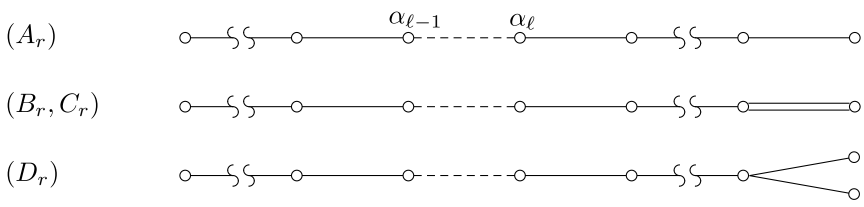
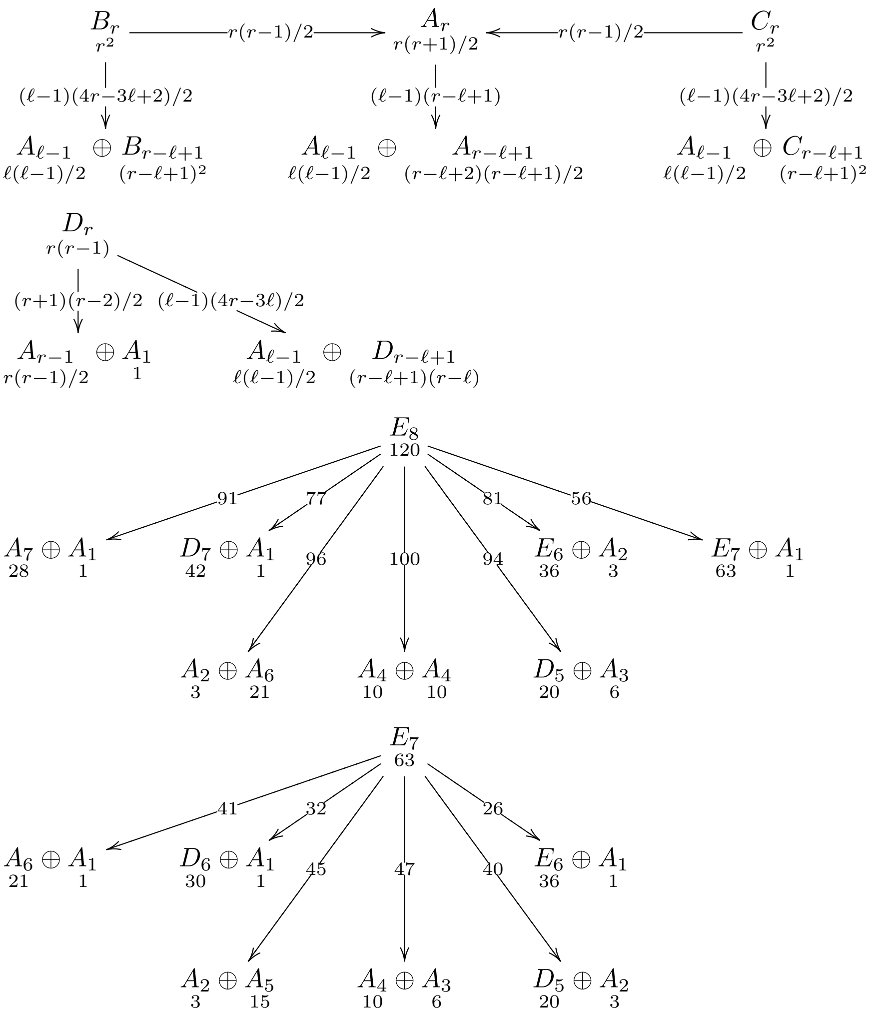


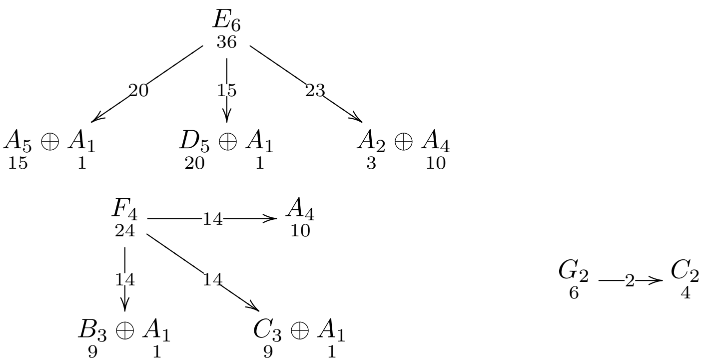
Figure 4.

We can summarize the above argument as follows.

Theorem 5.4. *By cutting off any edge of a Dynkin diagram, we find that the zeta-function of the corresponding root system can be written as a (multiple) integral, whose integrand includes zeta-functions of each connected components of the resulting Dynkin diagram.*

The following diagrams show the hierarchy of Lie algebras whose zeta-functions are connected with each other by Mellin-Barnes recursive formulas as above. The number attached to each algebra is the number of positive roots, that is, the number of variables of the associated zeta-function. The number written in the middle of each arrow is the number of iteration of integrals. The horizontal arrow implies that, in the corresponding process, the Dynkin diagram is not divided into two separate parts. Note that the following diagrams include the cases of exceptional Lie algebras.





6. The analytic continuation and the location of singularities.

The definition (1.6) of multi-variable Witten zeta-functions, or more generally, the definition (3.1) of zeta-functions of root sets, shows that the denominators of these zeta-functions are finite products of linear forms of $m_1, \ldots, m_r$. Therefore, applying Theorem 3 of [**15**], we immediately obtain the following

Theorem 6.1. *The zeta-function* $\zeta_r(\mathbf{s}; \Delta^*)$ *of any root set* Δ^* (*defined at the beginning of Section* 3) *can be continued meromorphically to the whole space* $\boldsymbol{C}^{n^*}$.

This type of result is actually a special case of Essouabri's more general result [**3**], [**4**], which was published earlier than [**15**]. However, the proof of Theorem 3 of [**15**] gives a method of obtaining more analytic information on $\zeta_r(\mathbf{s}; \Delta^*)$. For example, in [**17**], we express $\zeta_3(\mathbf{s}; A_3)$ as a double integral involving $\zeta_2(\mathbf{s}; A_2)$, and analyzing the process of integration carefully, we deduce the information on the location of singularities of $\zeta_3(\mathbf{s}; A_3)$ from that of $\zeta_2(\mathbf{s}; A_2)$. Note that in [**17**], on the line next to (4.12), $z_5 = 0, 1, \ldots, N$ is to be read as $z_5 = -s_5 - l$ $(0 \leq l \leq N)$.

The same analysis is possible for any zeta-functions of Lie algebras (or, more generally, of root sets), by going upstream the arrows in the diagrams at the end of the preceding section. This is one of the motivations of the study of recursive structures (Sections 3 to 5), though the actual procedure will become very complicated when the number of iteration of integrals becomes large.

In this section, we prove some information on the location of singularities of zeta-functions of C_2, B_3 and C_3 by this method. This is because these three zeta-functions play the leading part in the subsequent paper [**9**]. The argument is similar to that for A_3 developed in [**17**], so some details will be omitted.

First consider the zeta-function of C_2, which is (2.17). Here we change the notation of variables to write

$$\zeta_2(s_1,s_2,s_3,s_4;C_2)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^{-s_1}n^{-s_2}(m+n)^{-s_3}(m+2n)^{-s_4}. \tag{6.1}$$

Note that this can be obtained by an arrangement of (2.11). In fact, in [**15**], [**21**], [**7**], the series (6.1) is called the zeta-function of $\mathfrak{so}(5)$, that is, of B_2.

At first we assume $\Re s_j>1$ $(1\le j\le 4)$. Then we have

$$\zeta_2(s_1,s_2,s_3,s_4;C_2)=\frac{1}{2\pi\sqrt{-1}}\int_{(c)}\frac{\Gamma(s_4+z)\Gamma(-z)}{\Gamma(s_4)}\zeta_{MT,2}(s_1,s_2-z,s_3+s_4+z)dz, \tag{6.2}$$

where $-\Re s_4<c<0$ ((4.4) of [**15**]). Let L be a large positive integer, and put

$$\Phi(s_1,s_2,s_3,s_4)=(s_1+s_2+s_3+s_4-2)\prod_{\ell=0}^{L}(s_2+s_3+s_4-1+\ell).$$

Then

$$\zeta_2(s_1,s_2,s_3,s_4;C_2)=\Phi(s_1,s_2,s_3,s_4)^{-1}I, \tag{6.3}$$

where

$$I=\frac{1}{2\pi\sqrt{-1}}\int_{(c)}\frac{\Gamma(s_4+z)\Gamma(-z)}{\Gamma(s_4)}\times\zeta_{MT,2}(s_1,s_2-z,s_3+s_4+z)\Phi(s_1,s_2,s_3,s_4)dz. \tag{6.4}$$

We shift the path of integration to $\Re z=M-\varepsilon$, where M is a large positive integer and ε is a small positive number. Counting the residues at $z=m$ $(0\le m\le M-1)$, we obtain

$$\zeta_2(s_1,s_2,s_3,s_4;C_2)=\sum_{m=0}^{M-1}\binom{-s_4}{m}\zeta_{MT,2}(s_1,s_2-m,s_3+s_4+m)+\Phi(s_1,s_2,s_3,s_4)^{-1}I', \tag{6.5}$$

where I' is defined by replacing (c) in the definition of I by $(M-\varepsilon)$.

The singularities of $\zeta_{MT,2}$ have already been determined in Theorem 1 of [**12**]; the singularities of $\zeta_{MT,2}(s_1,s_2-z,s_3+s_4+z)$ are

$$\begin{cases} s_1 + s_3 + s_4 + z = 1 - k & (k \in \boldsymbol{N}_0), \\ s_2 + s_3 + s_4 = 1 - k & (k \in \boldsymbol{N}_0), \\ s_1 + s_2 + s_3 + s_4 = 2, \end{cases} \tag{6.6}$$

and all of those are true singularities. Hence the integral I' is holomorphic in the region $\mathscr{D}_{M,L}$, which is a subset of $\boldsymbol{C}^4$ defined by the conditions $\Re s_4 > -M + \varepsilon$, $\Re(s_1 + s_3 + s_4) > 1 - M + \varepsilon$, and $\Re(s_2 + s_3 + s_4) > -L$. Therefore (6.5) gives the meromorphic continuation of $\zeta_2(s_1, s_2, s_3, s_4; C_2)$ to $\mathscr{D}_{M,L}$, and candidates of singularities in this region are

$$s_1 + s_3 + s_4 = 1 - \ell \qquad (\ell \in \boldsymbol{N}_0), \tag{6.7}$$

$$s_2 + s_3 + s_4 = 1 - \ell \qquad (\ell \in \boldsymbol{N}_0), \tag{6.8}$$

and

$$s_1 + s_2 + s_3 + s_4 = 2. \tag{6.9}$$

Note that, since $\mathscr{D}_{M,L}$ tends to the whole space $\boldsymbol{C}^4$ when M and L tend to infinity, the above argument gives the meromorphic continuation of $\zeta_2(s_1, s_2, s_3, s_4; C_2)$ to $\boldsymbol{C}^4$.

Now we prove the following.

Theorem 6.2. *The list of singularities of* $\zeta_2(s_1, s_2, s_3, s_4; C_2)$ *is given by* (6.7), (6.8) *and* (6.9).

To complete the proof of this theorem, we have only to check that the candidates (6.7), (6.8) and (6.9) are indeed singularities.

Consider the case (6.7). Since I' is holomorphic in $\mathscr{D}_{M,L}$, the singularity (6.7) is coming only from the sum part on the right-hand side of (6.5). For each m satisfying $0 \le m \le \ell$, the singularity of $\zeta_{MT,2}(s_1, s_2 - m, s_3 + s_4 + m)$ of the form $s_1 + (s_3 + s_4 + m) = 1 - k$ with $k + m = \ell$ gives the singularity of the form (6.7). These singularities are not cancelled each other, as can be easily proved by the "change of variables" technique, originally introduced in Akiyama, Egami and Tanigawa [**1**] (see Section 4 of [**17**]).

Next consider (6.9). We go back to the situation (6.3) with (6.4), which is valid when (s_1, s_2, s_3, s_4) is in the region of absolute convergence $\mathscr{D}_0$. Formula (5.3) of [**12**] is

$$\zeta_{MT,2}(s_1,s_2,s_3)=\frac{\Gamma(s_2+s_3-1)\Gamma(1-s_2)}{\Gamma(s_3)}\zeta(s_1+s_2+s_3-1)$$
$$+\sum_{m=0}^{M-1}\binom{-s_3}{m}\zeta(s_1+s_3+m)\zeta(s_2-m)$$
$$+\frac{1}{2\pi\sqrt{-1}}\int_{(M-\varepsilon)}\frac{\Gamma(s_3+z')\Gamma(-z')}{\Gamma(s_3)}\zeta(s_1+s_3+z')\zeta(s_2-z')dz', \tag{6.10}$$

where M is a large positive integer satisfying $M>\Re s_2-1+\varepsilon$. Therefore from (6.4) we have

$$I=\frac{1}{2\pi\sqrt{-1}}\int_{(c)}\frac{\Gamma(s_4+z)\Gamma(-z)}{\Gamma(s_4)}\left(S_1+\sum_{m=0}^{M-1}S_2(m)+S_3\right)\Phi(s_1,s_2,s_3,s_4)dz, \tag{6.11}$$

where

$$S_1=\frac{\Gamma(s_2+s_3+s_4-1)\Gamma(1-s_2+z)}{\Gamma(s_3+s_4+z)}\zeta(s_1+s_2+s_3+s_4-1),$$
$$S_2(m)=\binom{-s_3-s_4-z}{m}\zeta(s_1+s_3+s_4+z+m)\zeta(s_2-z-m),$$

and

$$S_3=\frac{1}{2\pi\sqrt{-1}}\int_{(M-\varepsilon)}\frac{\Gamma(s_3+s_4+z+z')\Gamma(-z')}{\Gamma(s_3+s_4+z)}$$
$$\times\zeta(s_1+s_3+s_4+z+z')\zeta(s_2-z-z')dz'.$$

Singularity (6.9) is coming from the $\Phi(s_1,s_2,s_3,s_4)^{-1}$ factor on the right-hand side of (6.3). Hence what we have to check is that I (or rather, the analytic continuation of I) does not vanish identically on the hyperplane

$$\mathscr{H}\ :\ s_1+s_2+s_3+s_4=2.$$

We first take a point $\mathbf{s}^*=(s_1^*,s_2^*,s_3^*,s_4^*)$ on the hyperplane $\mathscr{H}$, and choose a positive number a such that the point $\mathbf{s}^0=(s_1^0,s_2^0,s_3^0,s_4^0)$, where $s_j^0=s_j^*+a$, is included in $\mathscr{D}_0$. Then (6.11) is valid at $\mathbf{s}=\mathbf{s}^0$. We continue (6.11) to the point $\mathbf{s}^*$

with keeping the imaginary part of each variable s_j.

Define

$$I_n = \frac{1}{2\pi\sqrt{-1}} \int_{(c)} \frac{\Gamma(s_4+z)\Gamma(-z)}{\Gamma(s_4)} S_n \Phi(s_1,s_2,s_3,s_4) dz$$

for $n=1,3$, and also define $I_2(m)$ by replacing S_n in the above by $S_2(m)$. Consider $I_2(m)$. The poles of the integrand are $z=-s_4-\ell$, $z=\ell$, $z=1-s_1-s_3-s_4-m$, $z=s_2-1-m$ ($\ell \in \boldsymbol{N}_0$). We may choose $\mathbf{s}^*$ for which imaginary parts of these four types of poles

$$-\Im s_4,\ 0,\ 1-\Im(s_1+s_3+s_4),\ \Im s_2 - 1 \tag{6.12}$$

are all different. Then we can deform the path (c) to a new contour $\mathscr{C}$ (similarly to $\mathscr{C}$ in Section 3 of [**17**]) which does not cross the poles when the variables are moved from $\mathbf{s}^0$ to $\mathbf{s}^*$. Hence

$$I_2(m) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathscr{C}} \frac{\Gamma(s_4+z)\Gamma(-z)}{\Gamma(s_4)} S_2(m) \Phi(s_1,s_2,s_3,s_4) dz$$

around $\mathbf{s}=\mathbf{s}^*$. Since $\Phi(s_1,s_2,s_3,s_4)=0$ on $\mathscr{H}$, $I_2(m)$ vanishes identically around $\mathbf{s}^*$. Similarly we can show that I_3 vanishes identically around $\mathbf{s}^*$.

On the other hand,

$$I_1 = \frac{\Gamma(s_2+s_3+s_4-1)}{\Gamma(s_4)} \zeta(s_1+s_2+s_3+s_4-1)\Phi(s_1,s_2,s_3,s_4) J, \tag{6.13}$$

where

$$J = \frac{1}{2\pi\sqrt{-1}} \int_{(c)} \Gamma(s_4+z)\Gamma(-z) \frac{\Gamma(1-s_2+z)}{\Gamma(s_3+s_4+z)} dz.$$

We have assumed that $(s_1^0,s_2^0,s_3^0,s_4^0) \in \mathscr{D}_0$, but J is independent of s_1, hence (6.13) is valid in a wider region $\mathscr{D}_1$ which has no restriction on the value of $\Re s_1$. Now choose $s_2^{**}=2c+1$ and $s_3^{**}=0$. If $\Re s_4^{**}$ is sufficiently large and s_1^{**} is such that $s_1^{**}+2c+1+s_4^{**}=2$, then

$$\mathbf{s}^{**} = \left(s_1^{**}, 2c+1, 0, s_4^{**}\right) \in \mathscr{D}_1 \cap \mathscr{H},$$

and at this point we have

$$J = \frac{1}{2\pi\sqrt{-1}} \int_{(c)} \Gamma(-z)\Gamma(1 - s_2^{**} + z)dz$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Gamma(-c + it)|^2 dt > 0.$$

The point $\mathbf{s}^{**}$ itself does not satisfy the condition that (6.12) are all different. But from the above inequality we can find a point $\mathbf{s}^* \in \mathscr{D}_1 \cap \mathscr{H}$ near $\mathbf{s}^{**}$, where $J > 0$ and (6.12) are all different. Then around $\mathbf{s}^*$, we find that I_1 does not vanish while $I_2(m) \equiv I_3 \equiv 0$.

This implies that I does not vanish identically on $\mathscr{H}$, hence (6.9) is really singular.

The argument for (6.8) is almost the same as in the case of (6.9), so we omit it. The proof of Theorem 6.2 is complete. □

Next we consider the singularities of zeta-functions of B_3 and C_3. In the diagrams at the end of Section 5, we find two ways of arriving at B_3 (or C_3), that is, from A_3, or from $A_1 \oplus B_2$. Here we choose the way from A_3, because the number of iteration of integrals is smaller, and also, this way is along the horizontal arrow (see Remark 6.4 below).

The zeta-function of $B_3 = \mathfrak{so}(7)$ is given as (2.12) explicitly, but here we change the notation of variables as follows:

$$\begin{aligned}
&\zeta_3(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9; B_3) \\
&= \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{\infty} m_1^{-s_1} m_2^{-s_2} m_3^{-s_3} (m_1 + m_2)^{-s_4} (m_2 + m_3)^{-s_5} \\
&\quad \times (2m_2 + m_3)^{-s_6} (m_1 + m_2 + m_3)^{-s_7} (m_1 + 2m_2 + m_3)^{-s_8} \\
&\quad \times (2m_1 + 2m_2 + m_3)^{-s_9}. \qquad (6.14)
\end{aligned}$$

Similarly, we write

$$\begin{aligned}
&\zeta_3(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9; C_3) \\
&= \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{\infty} m_1^{-s_1} m_2^{-s_2} m_3^{-s_3} (m_1 + m_2)^{-s_4} (m_2 + m_3)^{-s_5} \\
&\quad \times (m_2 + 2m_3)^{-s_6} (m_1 + m_2 + m_3)^{-s_7} (m_1 + m_2 + 2m_3)^{-s_8} \\
&\quad \times (m_1 + 2m_2 + 2m_3)^{-s_9}. \qquad (6.15)
\end{aligned}$$

As for B_3, we follow the way indicated by (5.9), that is,

$$2e_1 > e_1 + e_2 > e_1 + e_3 \quad \text{and} \quad 2e_2 > e_2 + e_3.$$

Let $\Delta' = \Delta_+(B_3) \setminus \{\varepsilon_1\}$, and $\Delta'' = \Delta' \setminus \{\varepsilon_1 + \varepsilon_2\}$. Then the zeta-function of B_3 (resp. Δ', Δ'') is expressed as an integral involving the zeta-function of Δ' (resp. Δ'', A_3) in the integrand. Theorem 5.3 with $\Delta_j^* = \Delta''$ and $\Delta_{j-1}^* = A_3$ shows that

$$\begin{aligned}
&\zeta_3(s_1, s_2, s_3, s_4, s_5, s_6, s_7; \Delta'') \\
&\quad = \frac{1}{2\pi\sqrt{-1}} \int_{(c)} \frac{\Gamma(s_6 + z)\Gamma(-z)}{\Gamma(s_6)} \zeta_3(s_1, s_2 - z, s_3, s_4, s_5 + s_6 + z, s_7; A_3) dz.
\end{aligned} \tag{6.16}$$

The singularities of the zeta-function of A_3 have been completely determined in [**17**], which implies that the singularities of the zeta-function $\zeta_3(s_1, s_2 - z, s_3, s_4, s_5 + s_6 + z, s_7; A_3)$ in the integrand on the right-hand side of (6.16) are on

$$s_1 + s_4 + s_7 = 1 - \ell, \tag{6.17}$$

$$s_2 + s_4 + s_5 + s_6 + s_7 = 1 - \ell, \tag{6.18}$$

$$s_3 + (s_5 + s_6 + z) + s_7 = 1 - \ell, \tag{6.19}$$

$$s_1 + s_2 + s_4 + s_5 + s_6 + s_7 = 2 - \ell, \tag{6.20}$$

$$s_1 + s_3 + s_4 + (s_5 + s_6 + z) + s_7 = 2 - \ell, \tag{6.21}$$

$$s_2 + s_3 + s_4 + s_5 + s_6 + s_7 = 2 - \ell, \tag{6.22}$$

$$s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7 = 3, \tag{6.23}$$

where $\ell \in \boldsymbol{N}_0$. Hence the poles of the integrand with respect to z are (6.19), (6.21), and $z = -s_6 - m$ and $z = m$ ($m \in \boldsymbol{N}_0$), both of which are coming from the gamma-factors. When we shift the path of integration to $\Re z = M - \varepsilon$, the only relevant poles are $z = m$ ($0 \leq m \leq M - 1$), and we have

$$\begin{aligned}&\zeta_3(s_1,s_2,s_3,s_4,s_5,s_6,s_7;\Delta'')\\&\quad=\sum_{m=0}^{M-1}\binom{-s_6}{m}\zeta_3(s_1,s_2-m,s_3,s_4,s_5+s_6+m,s_7;A_3)\\&\quad\quad+\frac{1}{2\pi\sqrt{-1}}\int_{(M-\varepsilon)}\frac{\Gamma(s_6+z)\Gamma(-z)}{\Gamma(s_6)}\\&\quad\quad\times\zeta_3(s_1,s_2-z,s_3,s_4,s_5+s_6+z,s_7;A_3)dz. \qquad (6.24)\end{aligned}$$

The singularities of the sum part on the right-hand side of (6.24) is the same as (6.17)–(6.23), only with replacing z in (6.19) and (6.21) by m. Therefore the list of possible singularities of $\zeta_3(s_1,s_2,s_3,s_4,s_5,s_6,s_7;\Delta'')$ is (6.17), (6.18), (6.20), (6.22), (6.23) and

$$s_3+s_5+s_6+s_7=1-\ell, \qquad (6.25)$$

$$s_1+s_3+s_4+s_5+s_6+s_7=2-\ell, \qquad (6.26)$$

where $\ell\in\boldsymbol{N}_0$.

Similarly, we can express the zeta-function of Δ' as in integral involving $\zeta_3(\cdot\ ;\Delta'')$. Shifting the path to the right, and using the above data on the singularities of $\zeta_3(\cdot\ ;\Delta'')$, we obtain the following list of possible singularities of $\zeta_3(s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8;\Delta')$:

$$\begin{cases}s_1+s_4+s_7+s_8=1-\ell,\\s_3+s_5+s_6+s_7+s_8=1-\ell,\\s_2+s_4+s_5+s_6+s_7+s_8=1-\ell,\\s_1+s_2+s_4+s_5+s_6+s_7+s_8=2-\ell,\\s_1+s_3+s_4+s_5+s_6+s_7+s_8=2-\ell,\\s_2+s_3+s_4+s_5+s_6+s_7+s_8=2-\ell,\\s_1+s_2+s_3+s_4+s_5+s_6+s_7+s_8=3.\end{cases} \qquad (6.27)$$

Finally, applying the same argument to the integral expression of $\zeta(\cdot\ ;B_3)$, we obtain the B_3 part of the following theorem.

THEOREM 6.3. *The possible singularities of zeta-functions of B_3, and also of C_3, are located only on the subsets of $\boldsymbol{C}^9$ defined by one of the following equations*:

$$\begin{cases} s_1+s_4+s_7+s_8+s_9=1-\ell, \\ s_3+s_5+s_6+s_7+s_8+s_9=1-\ell, \\ s_2+s_4+s_5+s_6+s_7+s_8+s_9=1-\ell, \\ s_1+s_2+s_4+s_5+s_6+s_7+s_8+s_9=2-\ell, \\ s_1+s_3+s_4+s_5+s_6+s_7+s_8+s_9=2-\ell, \\ s_2+s_3+s_4+s_5+s_6+s_7+s_8+s_9=2-\ell, \\ s_1+s_2+s_3+s_4+s_5+s_6+s_7+s_8+s_9=3, \end{cases} \tag{6.28}$$

where $\ell \in \boldsymbol{N}_0$.

The case of C_3 can be proved similarly, by following (5.10) instead of (5.9).

It is plausible that all of the possible singularities listed above are true singularities, and probably it can be proved by the method described in Section 4 of [**17**]. However such a study would require further pages, so this time we will not discuss it.

Remark 6.4. In the proof of Theorem 6.3 for B_3, we have shifted the path of integration to the right three times, and at each time the only relevant poles are $z=m$ ($m \in \boldsymbol{N}_0$) coming from the factor $\Gamma(-z)$. This is because the arrow between A_3 and B_3 in the diagram in Section 5 is horizontal, that is, the Mellin-Barnes integral expression includes only one zeta factor in the integrand. When the arrow is not horizontal, this situation is no longer valid, and then the shift to the right is, in general, not sufficient for the continuation. A more delicate deformation of the path (such as those given in [**15**] and Section 3 of [**17**]) is necessary, and hence, the discussion of finding the possible singularities will become more complicated.

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