

## Concrete classification and centralizers of certain $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ -actions

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**Abstract.** We introduce a new class of actions of the group  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  on finite von Neumann algebras and call them twisted Bernoulli shift actions. We classify these actions up to conjugacy and give an explicit description of their centralizers. We also distinguish many of those actions on the AFD  $\text{II}_1$  factor in view of outer conjugacy.

### 1. Introduction.

We consider the classification of  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ -actions on finite von Neumann algebras in this paper. Mainly, we concentrate on the case that the finite von Neumann algebra is the AFD factor of type  $\text{II}_1$  or non-atomic abelian.

There are two difficulties for analyzing discrete group actions on operator algebras. The first is that we do not have various ways to construct actions. The second is that we can not analyze them by concrete calculation in most cases. To give many examples of actions which admit concrete analysis, we introduce a class of trace preserving  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ -actions on finite von Neumann algebras and call them twisted Bernoulli shift actions. We classify those actions up to conjugacy and study them up to outer conjugacy.

An action  $\beta(H, \mu, \chi)$  in the class is defined for a triplet  $(H, \mu, \chi)$ , where  $H$  is an abelian countable discrete group,  $\mu$  is a normalized scalar 2-cocycle of  $H$  and  $\chi$  is a character of  $H$ . We obtain the action by restricting the so-called generalized Bernoulli shift action to a subalgebra  $N(H, \mu)$  and “twisting” it by the character  $\chi$ . The process of restriction has a vital role in concrete analysis of these actions.

A  $*$ -isomorphism which gives conjugacy between two twisted Bernoulli shift actions  $\beta(H_a, \mu_a, \chi_a)$  and  $\beta(H_b, \mu_b, \chi_b)$  must be induced from an isomorphism between the two abelian groups  $H_a$  and  $H_b$ . We prove this by concrete calculation (Section 4). It turns out that there exist continuously many, non-conjugate  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ -actions on the AFD factor of type  $\text{II}_1$  (Section 5). By using the

same technique, we describe the centralizers of all twisted Bernoulli shift actions. Here we should mention that the present work was motivated by the previous ones [Ch], [NPS], where similar studies were carried out in the case of  $SL(n, \mathbf{Z})$ .

In Section 6, we distinguish many twisted Bernoulli shift actions in view of outer conjugacy. The classification for actions of discrete amenable groups on the AFD factor of type  $\text{II}_1$  was given by Ocneanu [Oc]. Outer actions of countable amenable groups are outer conjugate. In the contrast to this, V. F. R. Jones [Jon] proved that any discrete non-amenable group has at least two non outer conjugate actions on the AFD factor of type  $\text{II}_1$ . S. Popa ([Po3], [Po4], [PoSa], etc.) used the malleability/deformation arguments for the Bernoulli shift actions to study (weak) 1-cocycles for the actions. For some of twisted Bernoulli shift actions, which we introduce in this paper, it is shown that (weak) 1-cocycles are represented in simple forms under some assumption on the (weak) 1-cocycles. We prove that there exist continuously many twisted Bernoulli shift actions which are mutually non outer conjugate. This strengthens the result mentioned above due to Jones in the  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  cases.

## 2. Preparations.

### 2.1. Functions det and gcd.

For the definition of twisted Bernoulli shift actions in Section 3, we define two  $\mathbf{Z}$ -valued functions det and gcd. The function det is given by the following equation:

$$\det\left(\left(\begin{pmatrix} q \\ r \end{pmatrix}, \begin{pmatrix} q_0 \\ r_0 \end{pmatrix}\right)\right) = qr_0 - rq_0, \quad \left(\begin{pmatrix} q \\ r \end{pmatrix}, \begin{pmatrix} q_0 \\ r_0 \end{pmatrix}\right) \in \mathbf{Z}^2.$$

The value of the function gcd at  $k \in \mathbf{Z}^2$  is the greatest common divisor of the two entries. For  $0 \in \mathbf{Z}^2$ , let the value of gcd be 0.

LEMMA 2.1.

(1) *The action of  $SL(2, \mathbf{Z})$  on  $\mathbf{Z}^2$  preserves the functions det and gcd, that is,*

$$\begin{aligned} \det(k, k_0) &= \det(\gamma \cdot k, \gamma \cdot k_0), \\ \gcd(k) &= \gcd(\gamma \cdot k), \quad k, k_0 \in \mathbf{Z}^2, \gamma \in SL(2, \mathbf{Z}). \end{aligned}$$

(2) *The following equation holds true:*

$$\det(k, k_0) = \gcd(k) + \gcd(k_0) - \gcd(k + k_0) \pmod{2}, \quad k, k_0 \in \mathbf{Z}^2.$$

PROOF. The claim (1) is a well-known fact, so we prove the claim (2). For the function gcd, we get

$$\gcd\left(\begin{pmatrix} q \\ r \end{pmatrix}\right) = \begin{cases} 1 \pmod{2}, & \text{(either } q \text{ or } r \text{ is odd),} \\ 0 \pmod{2}, & \text{(both } q \text{ and } r \text{ are even).} \end{cases}$$

Since the action of  $SL(2, \mathbf{Z})$  on  $\mathbf{Z}^2$  preserves the functions det and gcd, it suffices to show the desired equation against the following four pairs:

$$(k, k_0) = \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right), \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right), \\ \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right), \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \pmod{2}. \quad \square$$

**2.2. Scalar 2-cocycles for abelian groups.**

We fix some notations for countable abelian groups and their scalar 2-cocycles. For the rest of this paper, let  $H$  be an abelian countable discrete group and suppose that any scalar 2-cocycle  $\mu: H \times H \rightarrow \mathbf{T} = \{z \in \mathbf{C} \mid |z| = 1\}$  is normalized, that is,  $\mu(g, 0) = 1 = \mu(0, g)$  for  $g \in H$ . We denote by  $\mu^*$  the 2-cocycle for  $H$  given by  $\mu^*(g, h) = \overline{\mu(h, g)}$ ,  $g, h \in H$ . Let  $\mu^*\mu$  be the function on  $H \times H$  defined by

$$\mu^*\mu(g, h) = \overline{\mu(h, g)} \mu(g, h), \quad g, h \in H.$$

This is a bi-character, that is,  $\mu^*\mu(g, \cdot)$  and  $\mu^*\mu(\cdot, h)$  are characters of  $H$ . By using this function, we can describe the cohomology class of  $\mu$ . See [OPT] for the proof of the following Proposition:

PROPOSITION 2.2. *Two scalar 2-cocycles  $\mu_1$  and  $\mu_2$  of  $H$  are cohomologous if and only if  $\mu_1^*\mu_1 = \mu_2^*\mu_2$ .*

Let  $\mathbf{C}_\mu(H)$  be the twisted group algebra of  $H$  with respect to the 2-cocycle  $\mu$ . We denote by  $\{u_h \mid h \in H\}$  the standard basis for  $\mathbf{C}_\mu(H)$  as  $\mathbf{C}$ -linear space. We recall that the  $\mathbf{C}$ -algebra  $\mathbf{C}_\mu(H)$  has a structure of  $*$ -algebra defined by

$$u_g u_h = \mu(g, h) u_{g+h}, \quad u_g^* = \overline{\mu(g, -g)} u_{-g}, \quad g, h \in H.$$

Let  $\tilde{\mu}$  be the  $\mathbf{T}$ -valued function on  $\bigoplus_{\mathbf{Z}^2} H \times \bigoplus_{\mathbf{Z}^2} H$  defined by

$$\tilde{\mu}(\lambda_1, \lambda_2) = \prod_{k \in \mathbf{Z}^2} \mu(\lambda_1(k), \lambda_2(k)), \quad \lambda_1, \lambda_2 \in \bigoplus_{\mathbf{Z}^2} H. \quad (\text{Eq1})$$

The function  $\tilde{\mu}$  is a normalized scalar 2-cocycle for  $\bigoplus_{\mathbf{Z}^2} H$ . Let  $\Lambda(H)$  be the abelian group defined by

$$\Lambda(H) = \left\{ \lambda : \mathbf{Z}^2 \rightarrow H \mid \text{finitely supported and } \sum_{k \in \mathbf{Z}^2} \lambda(k) = 0 \right\}.$$

Its additive rule is defined by pointwise addition.

### 2.3. Definition of a $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ -action on $\Lambda(H)$ .

The group  $SL(2, \mathbf{Z})$  acts on  $\mathbf{Z}^2$  as matrix-multiplication and the group  $\mathbf{Z}^2$  also does on  $\mathbf{Z}^2$  by addition. These two actions define the action of  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  on  $\mathbf{Z}^2$  which is explicitly described as

$$\left( \begin{pmatrix} q \\ r \end{pmatrix}, \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) \cdot \begin{pmatrix} q_0 \\ r_0 \end{pmatrix} = \begin{pmatrix} q + xq_0 + yr_0 \\ r + zq_0 + wr_0 \end{pmatrix},$$

for all

$$\left( \begin{pmatrix} q \\ r \end{pmatrix}, \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z}), \quad \begin{pmatrix} q_0 \\ r_0 \end{pmatrix} \in \mathbf{Z}^2.$$

We define an action of  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  on  $\bigoplus_{\mathbf{Z}^2} H$  as

$$(\gamma \cdot \lambda)(k) = \lambda(\gamma^{-1} \cdot k), \quad k \in \mathbf{Z}^2,$$

for  $\gamma \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  and  $\lambda \in \bigoplus_{\mathbf{Z}^2} H$ .

### 2.4. On the relative property (T) of Kazhdan.

We recall the definition of the relative property (T) of Kazhdan for a pair of discrete groups.

**DEFINITION 2.3.** Let  $G \subset \Gamma$  be an inclusion of discrete groups. We say that the pair  $(\Gamma, G)$  has the relative property (T) if the following condition holds: There exist a finite subset  $F$  of  $\Gamma$  and  $\delta > 0$  such that if  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$  is a unitary representation of  $\Gamma$  on a Hilbert space  $\mathcal{H}$  with a unit vector  $\xi \in \mathcal{H}$  satisfying  $\|\pi(g)\xi - \xi\| < \delta$  for  $g \in F$ , then there exists a non-zero vector  $\eta \in \mathcal{H}$  such that  $\pi(h)\eta = \eta$  for  $h \in G$ .

Instead of this original definition, we use the following condition.

PROPOSITION 2.4 ([Jol]). *Let  $G \subset \Gamma$  be an inclusion of discrete groups. The pair  $(\Gamma, G)$  has the relative property (T) if and only if the following condition holds: For any  $\epsilon > 0$ , there exist a finite subset  $F$  of  $\Gamma$  and  $\delta > 0$  such that if  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$  is a unitary representation of  $\Gamma$  on a Hilbert space  $\mathcal{H}$  with a unit vector  $\xi \in \mathcal{H}$  satisfying  $\|\pi(g)\xi - \xi\| < \delta$  for  $g \in F$ , then  $\|\pi(h)\xi - \xi\| < \epsilon$  for  $h \in G$ .*

The pair  $(\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z}), \mathbf{Z}^2)$  is a typical example of group with the relative property (T). See [Bu] or [Sh] for the proof.

### 2.5. Weakly mixing actions.

An action of a countable discrete group  $G$  on a von Neumann algebra  $N$  is said to be *ergodic* if any  $G$ -invariant element of  $N$  is a scalar multiple of 1. The *weak mixing property* is a stronger notion of ergodicity.

DEFINITION 2.5. Let  $N$  be a von Neumann algebra with a faithful normal state  $\phi$ . A state preserving action  $(\rho_g)_{g \in G}$  of a countable discrete group  $G$  on  $N$  is said to be weakly mixing if for every finite subset  $\{a_1, a_2, \dots, a_n\} \subset N$  and  $\epsilon > 0$ , there exists  $g \in G$  such that  $|\phi(a_i \rho_g(a_j)) - \phi(a_i)\phi(a_j)| < \epsilon$ ,  $i, j = 1, \dots, n$ .

The following is a basic characterization of the weak mixing property. Between two von Neumann algebra  $N$  and  $M$ ,  $N \otimes M$  stands for the tensor product von Neumann algebra.

PROPOSITION 2.6 (Proposition D.2 in [Vaes]). *Let  $N$  be a finite von Neumann algebra and  $\text{tr}$  be a faithful normal tracial state. Let a countable discrete group  $G$  act on  $N$  by trace preserving automorphisms  $(\rho_g)_{g \in G}$ . The following statements are equivalent:*

- (1) *The action  $(\rho_g)$  is weakly mixing.*
- (2) *The only finite-dimensional invariant subspace of  $N$  is  $\mathbf{C}1$ .*
- (3) *For any action  $(\alpha_g)$  of  $G$  on a finite von Neumann algebra  $(M, \tau)$ , we have  $(N \otimes M)^{\rho \otimes \alpha} = 1 \otimes M^\alpha$ , where  $(N \otimes M)^{\rho \otimes \alpha}$  and  $M^\alpha$  are the fixed point subalgebras.*

### 2.6. A remark on group von Neumann algebras.

Let  $\Gamma$  be a discrete group and let  $\mu$  be a scalar 2-cocycle of  $\Gamma$ . A group  $\Gamma$  acts on the Hilbert space  $\ell^2\Gamma$  by the following two ways;

$$u_\gamma(\delta_g) = \mu(\gamma, g)\delta_{\gamma g}, \quad \rho_\gamma(\delta_g) = \mu(g, \gamma^{-1})\delta_{g\gamma^{-1}}, \quad \gamma, g \in \Gamma.$$

These two representations commute with each other. The von Neumann algebra  $L_\mu(\Gamma)$  generated by the image of  $u$  is called the group von Neumann algebra of  $\Gamma$  twisted by  $\mu$ . The normal state  $\langle \cdot, \delta_e, \delta_e \rangle$  is a trace on  $L_\mu(\Gamma)$ . The vector  $\delta_e$  is separating for  $L_\mu(\Gamma)$ . For any element  $a \in L_\mu(\Gamma)$ , we define the square summable function  $a(\cdot)$  on  $\Gamma$  by  $a\delta_e = \sum a(g)\delta_g$ . The function  $a(\cdot)$  is called the *Fourier coefficient* of  $a$ . We write  $a = \sum_{g \in \Gamma} a(g)u_g$  and call this the Fourier expansion of  $a$ . The Fourier expansion of  $a^*$  is given by  $a^* = \sum_{g \in \Gamma} \overline{\mu(g, g^{-1})a(g^{-1})}u_g$ , since the Fourier coefficient  $a^*(g) = \langle a^*\delta_e, \delta_g \rangle$  is described as

$$\langle \delta_e, a\rho_{g^{-1}}\delta_e \rangle = \left\langle \rho_{g^{-1}}^*\delta_e, \sum_g a(g)\delta_g \right\rangle = \overline{\mu(g, g^{-1})a(g^{-1})}.$$

Here we used the equation  $\rho_{g^{-1}}^* = \overline{\mu(g, g^{-1})}\rho_g$ , which is verified by direct computation. For two elements  $a, b$ , the Fourier coefficient of  $ab$  is given by

$$ab(\gamma) = \langle b\delta_e, \rho_{\gamma^{-1}}a^*\delta_e \rangle = \sum_g \overline{a^*(g)\mu(g, \gamma)}b(g\gamma) = \sum_g \mu(g^{-1}, g\gamma)a(g^{-1})b(g\gamma).$$

This equation allows us to calculate the Fourier coefficient algebraically, that is,

$$ab = \sum_\gamma \left( \sum_g \mu(g^{-1}, g\gamma)a(g^{-1})b(g\gamma) \right) u_\gamma = \sum_\gamma \sum_{gh=\gamma} a(g)b(h)u_gu_h.$$

For a subgroup  $\Lambda \subset \Gamma$ , the subalgebra  $\{u_\lambda \mid \lambda \in \Lambda\}'' \subset L_\mu(\Gamma)$  is isomorphic to  $L_\mu(\Lambda)$ . We sometimes identify them. An element  $a \in L_\mu(\Gamma)$  is in the subalgebra  $L_\mu(\Lambda)$  if and only if the Fourier expansion  $a(\cdot): \Gamma \rightarrow \mathcal{C}$  is supported on  $\Lambda$ , since the conditional expectation  $E$  from  $L_\mu(\Gamma)$  onto  $L_\mu(\Lambda)$  preserving  $\langle \cdot, \delta_e, \delta_e \rangle$  is described as  $E(a) = \sum_{\lambda \in \Lambda} a(\lambda)u_\lambda$ .

### 3. Definition of twisted Bernoulli shift actions.

In this section, we introduce twisted Bernoulli shift actions of  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  on finite von Neumann algebras. The action is defined for a triplet  $i = (H, \mu, \chi)$ , where  $H \neq \{0\}$  is an abelian countable discrete group,  $\mu$  is a normalized scalar 2-cocycle of  $H$  and  $\chi$  is a character of  $H$ . The finite von Neumann algebra, on which the group  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  acts, is defined by the pair  $(H, \mu)$ .

We introduce a group structure on the set  $\Gamma_0 = \widehat{H} \times \mathbf{Z}^2 \times SL(2, \mathbf{Z})$  as

$$(c_1, k, \gamma_1)(c_2, l, \gamma_2) = \left( c_1c_2\chi^{\det(k, \gamma_1 \cdot l)}, k + \gamma_1 \cdot l, \gamma_1\gamma_2 \right)$$

for any  $c_1, c_2 \in \widehat{H}, k, l \in \mathbf{Z}^2, \gamma_1, \gamma_2 \in SL(2, \mathbf{Z})$ . The associativity is verified by Lemma 2.1 (1). It turns out that the subsets  $\widehat{H} = \widehat{H} \times \{0\} \times \{e\}$  and  $G_0 = \widehat{H} \times \mathbf{Z}^2 \times \{e\}$  are subgroups in  $\Gamma_0$ . It is easy to see that  $G_0$  is a normal subgroup of  $\Gamma_0$  and that  $\widehat{H}$  is a normal subgroup of  $G_0$  and  $\Gamma_0$ . We get a normal inclusion of groups  $G_0/\widehat{H} \subset \Gamma_0/\widehat{H}$  and this is isomorphic to  $\mathbf{Z}^2 \subset \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ .

Before stating the definition of the twisted Bernoulli shift action, we define a  $\Gamma_0$ -action  $\rho$  on the von Neumann algebra  $L_{\widehat{\mu}}(\bigoplus_{\mathbf{Z}^2} H)$ . We denote by  $u(\lambda) \in L_{\widehat{\mu}}(\bigoplus_{\mathbf{Z}^2} H)$  the unitary corresponding to  $\lambda \in \bigoplus_{\mathbf{Z}^2} H$ . We define a faithful normal trace  $\text{tr}$  of  $L_{\widehat{\mu}}(\bigoplus_{\mathbf{Z}^2} H)$  in the usual way. For  $c \in \widehat{H}, k \in \mathbf{Z}^2, \gamma \in SL(2, \mathbf{Z})$ , let  $\rho(c), \rho(k), \rho(\gamma)$  be the linear transformations on  $\mathcal{C}_{\widehat{\mu}}(\bigoplus_{\mathbf{Z}^2} H)$  given by,

$$\begin{aligned}\rho(c)(u(\lambda)) &= \left( \prod_{l \in \mathbf{Z}^2} c(\lambda(l)) \right) u(\lambda), \\ \rho(k)(u(\lambda)) &= \left( \prod_{m \in \mathbf{Z}^2} \chi(\lambda(m))^{\det(k, m)} \right) u(k \cdot \lambda), \\ \rho(\gamma)(u(\lambda)) &= u(\gamma \cdot \lambda), \quad \lambda \in \Lambda(H).\end{aligned}$$

These maps are compatible with the multiplication rule and the  $*$ -operation of  $\mathcal{C}_{\widehat{\mu}}(\bigoplus_{\mathbf{Z}^2} H)$ . Since these maps preserve the trace, they extend to  $*$ -automorphisms on  $L_{\widehat{\mu}}(\bigoplus_{\mathbf{Z}^2} H)$ . It is immediate to see that  $\rho(c)$  commutes with  $\rho(k)$  and  $\rho(\gamma)$ . For  $k, l \in \mathbf{Z}^2$ , we have the following relation:

$$\begin{aligned}\rho(k) \circ \rho(l)(u(\lambda)) &= \prod_{m \in \mathbf{Z}^2} \chi(\lambda(m))^{\det(l, m)} \rho(k)(u(l \cdot \lambda)) \\ &= \prod_{m \in \mathbf{Z}^2} \chi(\lambda(m))^{\det(l, m)} \prod_{m \in \mathbf{Z}^2} \chi((l \cdot \lambda)(m))^{\det(k, m)} u(k \cdot (l \cdot \lambda)) \\ &= \prod_{m \in \mathbf{Z}^2} \chi(\lambda(m))^{\det(l, m)} \chi(\lambda(m))^{\det(k, m+l)} u((k+l) \cdot \lambda).\end{aligned}$$

By  $\det(l, m) + \det(k, m+l) = \det(k, l) + \det(k+l, m)$ , this equals to

$$\begin{aligned}\left( \prod_{m \in \mathbf{Z}^2} \chi(\lambda(m)) \right)^{\det(k, l)} \prod_{m \in \mathbf{Z}^2} \chi(\lambda(m))^{\det(k+l, m)} u((k+l) \cdot \lambda) \\ = \rho(\chi^{\det(k, l)}) \circ \rho(k+l)(u(\lambda)).\end{aligned}$$

Since  $\det$  is  $SL(2, \mathbf{Z})$ -invariant (Lemma 2.1 (1)), for  $k \in \mathbf{Z}^2, \gamma \in SL(2, \mathbf{Z})$ , we get

$$\begin{aligned}
\rho(\gamma \cdot k) \circ \rho(\gamma)(u(\lambda)) &= \rho(\gamma \cdot k)(u(\gamma \cdot \lambda)) \\
&= \prod_{l \in \mathbf{Z}^2} \chi((\gamma \cdot \lambda)(l))^{\det(\gamma \cdot k, l)} u((\gamma \cdot k) \cdot (\gamma \cdot \lambda)) \\
&= \prod_{l \in \mathbf{Z}^2} \chi(\lambda(l))^{\det(\gamma \cdot k, \gamma \cdot l)} u(\gamma \cdot (k \cdot \lambda)) \\
&= \prod_{l \in \mathbf{Z}^2} \chi(\lambda(l))^{\det(k, l)} \rho(\gamma)(u(k \cdot \lambda)) \\
&= \rho(\gamma) \circ \rho(k)(u(\lambda)), \quad \lambda \in \Lambda(H).
\end{aligned}$$

By the above two equations,  $\rho$  satisfies the following formula:

$$\begin{aligned}
&(\rho(c_1) \circ \rho(k) \circ \rho(\gamma_1)) \circ (\rho(c_2) \circ \rho(l) \circ \rho(\gamma_2)) \\
&= \rho(c_1) \circ \rho(c_2) \circ \rho(k) \circ \rho(\gamma_1) \circ \rho(l) \circ \rho(\gamma_2) \\
&= \rho(c_1) \circ \rho(c_2) \circ \rho(k) \circ \rho(\gamma_1 \cdot l) \circ \rho(\gamma_1) \circ \rho(\gamma_2) \\
&= \rho(c_1) \circ \rho(c_2) \circ \rho(\chi^{\det(k, \gamma_1 \cdot l)}) \circ \rho(k + (\gamma_1 \cdot l)) \circ \rho(\gamma_1 \gamma_2) \\
&= \rho(c_1 c_2 \chi^{\det(k, \gamma_1 \cdot l)}) \circ \rho(k + (\gamma_1 \cdot l)) \circ \rho(\gamma_1 \gamma_2).
\end{aligned}$$

With  $\rho(c, k, \gamma) = \rho(c) \circ \rho(k) \circ \rho(\gamma)$ ,  $\rho$  gives a  $\Gamma_0$ -action on  $L_{\tilde{\mu}}(\bigoplus_{\mathbf{Z}^2} H)$ .

We define the finite von Neumann algebra  $N(H, \mu)$  as the group von Neumann algebra  $L_{\tilde{\mu}}(\Lambda(H))$ . By using Fourier coefficients, we can prove that  $N(H, \mu)$  is the fixed point algebra under the  $\widehat{H}$ -action  $\rho(\widehat{H}, 0, e)$  on  $L_{\tilde{\mu}}(\bigoplus_{\mathbf{Z}^2} H)$ . We get a  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ -action on  $N(H, \mu)$  by

$$\beta(k, \gamma)(x) = \rho(1, k, \gamma)(x), \quad k \in \mathbf{Z}^2, \gamma \in SL(2, \mathbf{Z}), x \in N(H, \mu).$$

This is the definition of the twisted Bernoulli shift action  $\beta = \beta(H, \mu, \chi)$  on  $N(H, \mu)$ .

We obtained the actions  $\beta(H, \mu, \chi)$  not only by twisting generalized Bernoulli shift actions but also restricting to subalgebras  $N(H, \mu) \subset L_{\tilde{\mu}}(\bigoplus_{\mathbf{Z}^2} H) = \bigotimes_{\mathbf{Z}^2} L_{\mu}(H)$ . This restriction allows us to classify the actions up to conjugacy in the next section. In order to give a variety of the actions, we twisted the shift actions by the character  $\chi$  of the abelian group  $H$ .

**REMARK 3.1.** The action  $\beta|_{\mathbf{Z}^2} = \beta(H, \mu, \chi)|_{\mathbf{Z}^2}$  has the weak mixing property. In definition 2.5, we may assume that the Fourier coefficients of  $a_i \in N = N(H, \mu)$  are finitely supported ( $i = 1, 2, \dots, n$ ), by approximating in the  $L^2$ -norm. Then for appropriate  $k \in \mathbf{Z}^2$ , we get  $\text{tr}(a_i \beta(k)(a_j)) = \text{tr}(a_i) \text{tr}(a_j)$ ,  $i, j = 1, 2, \dots, n$ .

**4. Classification up to conjugacy.**

In this section, we classify the twisted Bernoulli shift actions  $\{\beta(H, \mu, \chi)\}$  up to conjugacy (Theorem 4.1). We prove that an isomorphism which gives conjugacy between two twisted Bernoulli shift actions is of a very special form. In fact, it comes from an isomorphism on the level of base groups  $H$ . We also determine the centralizer of the  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ -action  $\beta(H, \mu, \chi)$  on  $N(H, \mu)$  (Theorem 4.4).

We fix some notations for the proofs. We define  $0, e_1, e_2 \in \mathbf{Z}^2$  as

$$0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let  $\xi$  be the element of  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  satisfying

$$\xi \cdot 0 = e_1, \quad \xi \cdot e_1 = e_2, \quad \xi \cdot e_2 = 0.$$

The elements  $\xi$  and  $\xi^2$  are explicitly described as

$$\begin{cases} \xi &= \left( e_1, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \right), \\ \xi^2 &= \left( e_2, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \right). \end{cases}$$

The order of  $\xi$  is 3. Let  $\eta, \delta \in SL(2, \mathbf{Z})$  be given by  $\eta = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \delta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ .

Let  $D$  be the subset of all elements of  $\mathbf{Z}^2$  fixed under the action of  $\delta$ , that is,

$$D = \left\{ \begin{pmatrix} n \\ 0 \end{pmatrix} \mid n \in \mathbf{Z} \right\}.$$

Then we get

$$\xi \cdot D = \left\{ \begin{pmatrix} 1-n \\ n \end{pmatrix} \mid n \in \mathbf{Z} \right\}, \quad \xi^2 \cdot D = \left\{ \begin{pmatrix} 0 \\ 1-n \end{pmatrix} \mid n \in \mathbf{Z} \right\}.$$

We define the subgroup  $\Lambda_D(H)$  of  $\Lambda(H)$  by

$$\Lambda_D(H) = \{\lambda \in \Lambda(H) \mid \lambda : \mathbf{Z}^2 \rightarrow H \text{ is supported on } D\}.$$

Let  $(H_a, \mu_a, \chi_a)$  and  $(H_b, \mu_b, \chi_b)$  be triplets of countable abelian groups, their normalized 2-cocycles and characters. For  $h \in H_a$ , we define  $\lambda_h \in \Lambda_D(H_a)$  as

$$\lambda_h(k) = \begin{cases} h & (k = e_1), \\ -h & (k = 0), \\ 0 & (k \neq e_1, 0). \end{cases}$$

For  $g \in H_b$ , we define  $\sigma_g \in \Lambda_D(H_b)$  as

$$\sigma_g(k) = \begin{cases} g & (k = e_1), \\ -g & (k = 0), \\ 0 & (k \neq e_1, 0). \end{cases}$$

We denote by  $v(\sigma) \in N(H_b, \mu_b)$  the unitary corresponding to  $\sigma \in \Lambda(H_b)$ .

**THEOREM 4.1.** *If  $\pi : N(H_a, \mu_a) \rightarrow N(H_b, \mu_b)$  is a \*-isomorphism giving conjugacy between  $\beta_a = \beta(H_a, \mu_a, \chi_a)$  and  $\beta_b = \beta(H_b, \mu_b, \chi_b)$ , then there exists a group isomorphism  $\phi = \phi_\pi : H_a \rightarrow H_b$  satisfying*

- (1)  $\pi(u(\lambda)) = v(\phi \circ \lambda) \bmod \mathbf{T}$  for  $\lambda \in \Lambda(H_a)$ ,
- (2) the 2-cocycles  $\mu_a(\cdot, \cdot)$  and  $\mu_b(\phi(\cdot), \phi(\cdot))$  of  $H_a$  are cohomologous,
- (3)  $\chi_a^2 = (\chi_b \circ \phi)^2$ .

*Conversely, given a group isomorphism  $\phi : H_a \rightarrow H_b$  satisfying (2) and (3), there exists a \*-isomorphism  $\pi = \pi_\phi : N(H_a, \mu_a) \rightarrow N(H_b, \mu_b)$  which satisfies condition (1) and gives conjugacy between  $\beta_a, \beta_b$ .*

We note that by Proposition 2.2 the condition (2) for  $\phi$  is equivalent to

$$(2)' \quad \mu_a^* \mu_a(g, h) = \mu_b^* \mu_b(\phi(g), \phi(h)), \quad g, h \in H_a.$$

**PROOF FOR THE FIRST HALF OF THEOREM 4.1.** Suppose that there exists a (not necessarily trace preserving) \*-isomorphism  $\pi$  from  $N(H_a, \mu_a)$  onto  $N(H_b, \mu_b)$  such that  $\pi \circ \beta_a(\gamma) = \beta_b(\gamma) \circ \pi$ ,  $\gamma \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ .

We prove that for every  $h \in H_a$  there exists  $\phi(h) \in H_b$  satisfying  $\pi(u(\lambda_h)) = v(\sigma_{\phi(h)}) \bmod \mathbf{T}$ . Let  $U_h$  denote the unitary in  $N(H_b, \mu_b)$

$$U_h = \pi\left(\overline{\mu_a(h, -h)} u(\lambda_h)\right), \quad h \in H_a.$$

We identify  $N(H, \mu)$  with the subalgebra of the infinite tensor product  $\bigotimes_{\mathbf{Z}^2} L_\mu(H)$ , which is canonically isomorphic to  $L_{\tilde{\mu}}(\bigoplus_{\mathbf{Z}^2} H)$ . The preimage  $\pi^{-1}(U_h)$  can be written as  $u_h^* \otimes u_h$ . Here  $u_h$  is the unitary corresponding to  $h \in H_a$  and placed on  $1 \in \mathbf{Z}^2$  and the unitary  $u_h^*$  is placed on  $0 \in \mathbf{Z}^2$ . We describe  $U_h$  as the Fourier expansion  $U_h = \sum_{\sigma \in \Lambda(H_b)} c(\sigma)v(\sigma)$ . Since  $e_1$  and  $0$  are fixed under the action of  $\delta$ , one has

$$\beta_b(\delta)^n(U_h) = \pi \circ \beta_a(\delta)^n(\pi^{-1}(U_h)) = U_h.$$

It follows that the Fourier expansion  $U_h = \sum_{\sigma \in \Lambda(H_b)} c(\sigma)v(\sigma)$  must satisfy that  $c(\sigma) = c(\delta^{-n} \cdot \sigma)$  for every  $\sigma \in \Lambda(H_b)$  and  $n \in \mathbf{Z}$ . For  $\sigma \in \Lambda(H_b) \setminus \Lambda_D(H_b)$ , the orbit of  $\sigma$  under the action of  $\delta^{-1}$  is an infinite set, since the support  $\text{supp}(\sigma) \subset \mathbf{Z}^2$  is not included in  $D$ . It turns out that  $c(\sigma) = 0$  for all  $\sigma \in \Lambda(H_b) \setminus \Lambda_D(H_b)$  due to  $\sum_{\sigma \in \Lambda(H_b)} |c(\sigma)|^2 = 1 < +\infty$ , so that  $U_h = \sum_{\sigma \in \Lambda_D(H_b)} c(\sigma)v(\sigma)$ .

The unitary  $\chi_a(h)U_h^*$  is also fixed under the action of  $\delta$  and can be written as

$$\begin{aligned} \chi_a(h)U_h^* &= \pi\left(\chi_a(h)\overline{\mu_a(h, -h)}u(-\lambda_h)\right) \\ &= \pi\left(\chi_a(h)\overline{\mu_a(h, -h)}u(\xi \cdot \lambda_h)\right)\pi\left(\overline{\mu_a(h, -h)}u(\xi^2 \cdot \lambda_h)\right) \\ &= \beta_b(\xi)(U_h)\beta_b(\xi^2)(U_h). \end{aligned}$$

Letting  $ne_1 = (n, 0)^T \in \mathbf{Z}^2$ , we get

$$\begin{aligned} \beta_b(\xi)(U_h) &= \beta_b(\xi)\left(\sum c(\sigma)v(\sigma)\right) = \sum_{\sigma \in \Lambda_D(H_b)} c(\sigma)v(\xi \cdot \sigma) \prod_{n \in \mathbf{Z}} \chi_b(\sigma(ne_1))^n, \\ \beta_b(\xi^2)(U_h) &= \beta_b(\xi^2)\left(\sum c(\sigma)v(\sigma)\right) = \sum_{\sigma \in \Lambda_D(H_b)} c(\sigma)v(\xi^2 \cdot \sigma). \end{aligned}$$

Since Fourier expansion admits algebraical calculation as in subsection 2.6, the expansion of  $\chi_a(h)U_h^*$  is

$$\begin{aligned} \chi_a(h)U_h^* &= \beta_b(\xi)(U_h)\beta_b(\xi^2)(U_h) \\ &= \sum_{\sigma_1, \sigma_2 \in \Lambda_D(H_b)} c(\sigma_1)c(\sigma_2)v(\xi \cdot \sigma_1)v(\xi^2 \cdot \sigma_2) \prod_{n \in \mathbf{Z}} \chi_b(\sigma_1(ne_1))^n \\ &= \sum_{\sigma_1, \sigma_2 \in \Lambda_D(H_b)} c(\sigma_1)c(\sigma_2)\tilde{\mu}_b(\xi \cdot \sigma_1, \xi^2 \cdot \sigma_2) \prod_{n \in \mathbf{Z}} \chi_b(\sigma_1(ne_1))^n v(\xi \cdot \sigma_1 + \xi^2 \cdot \sigma_2). \end{aligned}$$

The map  $\Lambda_D(H_b) \times \Lambda_D(H_b) \ni (\sigma_1, \sigma_2) \mapsto \xi \cdot \sigma_1 + \xi^2 \cdot \sigma_2 \in \Lambda(H_b)$  is injective. Indeed,  $\sigma_1$  is uniquely determined by  $\xi \cdot \sigma_1 + \xi^2 \cdot \sigma_2$ , since  $\sigma_1(k) = (\xi \cdot \sigma_1 + \xi^2 \cdot \sigma_2)(\xi \cdot k)$ ,  $k \in D \setminus \{e_1\}$  and  $\sigma_1(e_1) = -\sum_{k \in D \setminus \{e_1\}} \sigma_1(k)$ . Here we used the condition  $\sum \sigma_1(k) = 0$ . The element  $\sigma_2$  is also determined by  $\xi \cdot \sigma_1 + \xi^2 \cdot \sigma_2$ . Thus the index  $(\sigma_1, \sigma_2)$  uniquely determines  $\xi \cdot \sigma_1 + \xi^2 \cdot \sigma_2$ .

We take arbitrary elements  $\sigma_1, \sigma_2 \in \Lambda_D(H_b)$  and suppose that  $c(\sigma_1) \neq 0$ ,  $c(\sigma_2) \neq 0$ . Since the unitary  $\chi_a(h)U_h^*$  is invariant under the action of  $\delta$  and the coefficient of  $\xi \cdot \sigma_1 + \xi^2 \cdot \sigma_2$  is not zero,  $\xi \cdot \sigma_1 + \xi^2 \cdot \sigma_2$  is supported on  $D$ . It follows that the elements  $\sigma_1$  and  $\sigma_2$  can be written as  $\sigma_1 = \sigma_{\phi(h)} = \sigma_2$ , by some  $\phi(h) \in H_b$ . Indeed, since the subsets  $D \setminus \{0, e_1\}$ ,  $\xi D \setminus \{e_1, e_2\}$  and  $\xi^2 D \setminus \{e_2, 0\}$  are mutually disjoint, the element  $\xi \cdot \sigma_1$  must be supported on  $\{e_1, e_2\}$  and the element  $\xi^2 \cdot \sigma_2$  must be supported on  $\{e_2, 0\}$ . By the assumption  $\sum_{k \in \mathbf{Z}^2} \sigma_i(k) = 0$  ( $i = 1, 2$ ),  $\sigma_i$  can be written as  $\sigma_{\phi(h_i)}$ . Then using the fact that  $(\xi \cdot \sigma_1 + \xi^2 \cdot \sigma_2)(e_2) = \sigma_1(e_1) + \sigma_2(0) = 0$ , we get that  $\sigma_1 = \sigma_{\phi(h)} = \sigma_2$  for some  $\phi(h) \in H_b$ . This means that there exists only one  $\sigma \in \Lambda(H_b)$  such that  $c(\sigma) \neq 0$  and that it is of the form  $\sigma = \sigma_{\phi(h)}$ . Then the unitary  $U_h$  satisfies  $U_h = \pi(u(\lambda_h)) = v(\sigma_{\phi(h)}) \pmod{\mathbf{T}}$ .

We claim that the map  $\phi = \phi_\pi : H_a \rightarrow H_b$  is a group isomorphism. For all  $h_1, h_2 \in H_a$ , we get

$$\begin{aligned} \pi(u(\lambda_{h_1+h_2})) &= \pi(u(\lambda_{h_1})) \pi(u(\lambda_{h_2})) = v(\sigma_{\phi(h_1)}) v(\sigma_{\phi(h_2)}) \\ &= v(\sigma_{\phi(h_1)} + \sigma_{\phi(h_2)}) = v(\sigma_{\phi(h_1)+\phi(h_2)}) \pmod{\mathbf{T}}. \end{aligned}$$

On the other hand, we get  $\pi(u(\lambda_{h_1+h_2})) = v(\sigma_{\phi(h_1+h_2)}) \pmod{\mathbf{T}}$ . Since  $\{v(\sigma)\}$  are linearly independent, we get  $\sigma_{\phi(h_1+h_2)} = \sigma_{\phi(h_1)+\phi(h_2)}$ , and hence

$$\phi(h_1 + h_2) = \phi(h_1) + \phi(h_2).$$

This means that the map  $\phi$  is a group homomorphism. The bijectivity of the  $*$ -isomorphism  $\pi$  leads to that of the group homomorphism  $\phi = \phi_\pi$ . Since  $\{\gamma \cdot \lambda_h \mid \gamma \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z}), h \in H_a\} \subset \Lambda(H_a)$  generates  $\Lambda(H_a)$ , we get  $\pi(u(\lambda)) = v(\phi \circ \lambda) \pmod{\mathbf{T}}$  for  $\lambda \in \Lambda(H_a)$ .

We prove that the group isomorphism  $\phi = \phi_\pi$  satisfies the conditions (2) and (3) in the theorem. For all  $h \in H_a$ , there exists  $c(h) \in \mathbf{T}$  satisfying

$$U_h = \pi\left(\overline{\mu_a(h, -h)} u(\lambda_h)\right) = c(h) \overline{\mu_b(\phi(h), -\phi(h))} v(\sigma_{\phi(h)}).$$

Since  $(e_1, \eta) \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  acts on  $\mathbf{Z}^2$  as  $(e_1, \eta) \cdot e_1 = 0$ ,  $(e_1, \eta) \cdot 0 = e_1$ , we get

$$\begin{aligned} U_h \beta_b(e_1, \eta)(U_h) &= \pi\left(\overline{\mu_a(h, -h)} u(\lambda_h) \overline{\mu_a(h, -h)} u(-\lambda_h)\right) \\ &= \overline{\mu_a(h, -h)}^2 \tilde{\mu}_a(\lambda_h, -\lambda_h) = 1, \end{aligned}$$

where the function  $\tilde{\mu}_a$  is defined by the equation (Eq1) in subsection 2.2. The following equation also holds:

$$\begin{aligned} U_h \beta_b(e_1, \eta)(U_h) &= c(h) \overline{\mu_b(\phi(h), -\phi(h))} v(\sigma_{\phi(h)}) c(h) \overline{\mu_b(\phi(h), -\phi(h))} v(\sigma_{-\phi(h)}) \\ &= c(h)^2 \overline{\mu_b(\phi(h), -\phi(h))}^2 \tilde{\mu}_b(\sigma_{\phi(h)}, -\sigma_{\phi(h)}) = c(h)^2. \end{aligned}$$

Thus we have  $c(h) \in \{1, -1\}$  for  $h \in H_a$ .

Since  $\xi \cdot e_1 = e_2$ ,  $\xi \cdot e_2 = 0$  and  $\xi \cdot 0 = e_1$ , we have

$$\begin{aligned} U_h \beta_b(\xi)(U_h) \beta_b(\xi^2)(U_h) &= \pi\left(\overline{\mu_a(h, -h)} u(\lambda_h)\right) \pi\left(\chi_a(h) \overline{\mu_a(h, -h)} u(\xi \cdot \lambda_h)\right) \pi\left(\overline{\mu_a(h, -h)} u(\xi^2 \cdot \lambda_h)\right) \\ &= \chi_a(h). \end{aligned}$$

On the other hand, we have the following:

$$\begin{aligned} U_h \beta_b(\xi)(U_h) \beta_b(\xi^2)(U_h) &= c(h) \overline{\mu_b(\phi(h), -\phi(h))} v(\sigma_{\phi(h)}) \\ &\quad c(h) \chi_b(\phi(h)) \overline{\mu_b(\phi(h), -\phi(h))} v(\xi \cdot \sigma_{\phi(h)}) \\ &\quad c(h) \overline{\mu_b(\phi(h), -\phi(h))} v(\xi^2 \cdot \sigma_{\phi(h)}) \\ &= c(h)^3 \chi_b(\phi(h)) = c(h) \chi_b(\phi(h)). \end{aligned}$$

It follows that

$$c(h) = \overline{\chi_b(\phi(h))} \chi_a(h) \tag{Eq2}$$

and  $\chi_b(\phi(h))^2 = \chi_a(h)^2$ , for all  $h \in H_a$ .

We recall that the algebra  $L_{\tilde{\mu}_a}(\bigoplus_{\mathbf{Z}^2} H_a)$  is canonically identified with the infinite tensor product  $\bigotimes_{\mathbf{Z}^2} L_{\mu_a}(H_a)$ . The unitary  $\pi^{-1}(U_h) \in N(H_a, \mu_a) \subset \bigotimes_{\mathbf{Z}^2} L_{\mu_a}(H_a)$  can be written as  $1 \otimes u_h^* \otimes u_h$ , where 1 is placed on  $-e_1 \in \mathbf{Z}^2$ ,  $u_h^*$  is placed on 0 and  $u_h$  is placed on  $e_1$ . Since  $\eta \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  acts on  $\mathbf{Z}^2$  as  $\eta \cdot e_1 = -e_1$ ,  $\eta \cdot 0 = 0$ , the unitary  $\pi^{-1}(\beta_b(\eta)(U_g))$  can be written as  $u_g \otimes u_g^* \otimes 1$ . We have the following equation:

$$\begin{aligned}
& U_g \beta_b(\eta)(U_h) U_g^* \beta_b(\eta)(U_h)^* \\
&= \pi((1 \otimes u_g^* \otimes u_g)(u_h \otimes u_h^* \otimes 1)(1 \otimes u_g^* \otimes u_g)^*(u_h \otimes u_h^* \otimes 1)^*) = \mu_a^* \mu_a(g, h).
\end{aligned}$$

The unitary  $U_h$  can be written as  $c(h)(1 \otimes v_{\phi(h)}^* \otimes v_{\phi(h)}) \in N(H_b, \mu_b) \subset \bigotimes_{\mathbf{Z}^2} L_{\mu_b}(H_b)$ . Here we write  $v_{\phi(h)}$  for the unitary in  $L_{\mu_b}(H_b)$  corresponding to  $\phi(h)$ . The unitary  $\beta_b(\eta)(U_g)$  can be written as  $c(g)(v_{\phi(g)} \otimes v_{\phi(g)}^* \otimes 1)$ . Then we get

$$\begin{aligned}
& U_g \beta_b(\eta)(U_h) U_g^* \beta_b(\eta)(U_h)^* \\
&= (1 \otimes v_{\phi(g)}^* \otimes v_{\phi(g)})(v_{\phi(h)} \otimes v_{\phi(h)}^* \otimes 1)(1 \otimes v_{\phi(g)}^* \otimes v_{\phi(g)})^*(v_{\phi(h)} \otimes v_{\phi(h)}^* \otimes 1)^* \\
&= \mu_b^* \mu_b(\phi(g), \phi(h)).
\end{aligned}$$

Thus we get  $\mu_a^* \mu_a(g, h) = \mu_b^* \mu_b(\phi(g), \phi(h))$ , for all  $g, h \in H_a$ . We proved that the group isomorphism  $\phi = \phi_\pi$  satisfies the conditions (1), (2) and (3).  $\square$

From a group homomorphism which satisfies the conditions (2) and (3), we construct a  $*$ -homomorphism from  $N(H_a, \mu_a)$  to  $N(H_b, \mu_b)$  with the condition (1). In the construction, the function  $\widehat{\mu}$  on  $\Lambda(H)$  given below is useful. We fix an index for  $\mathbf{Z}^2$  as  $\mathbf{Z}^2 = \{k_0, k_1, k_2, \dots\}$  throughout the rest of this section. For a scalar 2-cocycle  $\mu$  of  $H$ , we define the function  $\widehat{\mu}$  by

$$\widehat{\mu}(\lambda) = \prod_{j=1}^n \mu \left( \sum_{i=0}^{j-1} \lambda(k_i), \lambda(k_j) \right), \quad \lambda \in \Lambda(H),$$

where  $\lambda$  is supported on  $\{k_0, k_1, k_2, \dots, k_n\}$ . This definition depends on the choice of an order on  $\mathbf{Z}^2$ . Since  $\sum_i \lambda(k_i) = 0$ , the function  $\widehat{\mu}$  is also given by the following relation in  $\mathbf{C}_\mu(H)$ :

$$\widehat{\mu}(\lambda)1 = u_{\lambda(k_0)} u_{\lambda(k_1)} u_{\lambda(k_2)} \cdots u_{\lambda(k_n)}, \quad \lambda \in \Lambda(H).$$

If  $\mu$  is a coboundary, then the definition of  $\widehat{\mu}$  does not depend on the order on  $\mathbf{Z}^2$ , since  $\mathbf{C}_\mu(H)$  is commutative.

LEMMA 4.2. *Let  $\mu_0$  be another normalized scalar 2-cocycle for  $H$ . Let  $\widetilde{\mu}_0$  be the scalar 2-cocycle on  $\Lambda(H) \times \Lambda(H)$  given in the same way as equation (Eq1) in subsection 2.2 and let  $\widehat{\mu}_0$  be the function on  $\Lambda(H)$  constructed from  $\mu_0$  in the above manner. If the scalar 2-cocycles  $\mu$  and  $\mu_0$  are cohomologous, then for all  $\lambda_1, \lambda_2 \in \Lambda(H)$ , we have the equation*

$$\widetilde{\mu}(\lambda_1, \lambda_2) \overline{\widetilde{\mu}(\lambda_1)} \overline{\widetilde{\mu}(\lambda_2)} \widehat{\mu}(\lambda_1 + \lambda_2) = \widetilde{\mu}_0(\lambda_1, \lambda_2) \overline{\widetilde{\mu}_0(\lambda_1)} \overline{\widetilde{\mu}_0(\lambda_2)} \widehat{\mu}_0(\lambda_1 + \lambda_2).$$

PROOF. We denote by  $\{\nu(g, h)\}$  the scalar 2-cocycle  $\{\overline{\mu_0(g, h)}\mu(g, h)\}$  of  $H$ . Since  $\nu$  is a 2-coboundary, there exists  $\{c(g)\}_{g \in H} \subset \mathbf{T}$  satisfying  $\nu(g, h) = b(g)b(h)\overline{b(g+h)}$ . Then the map  $\widehat{\nu}$  becomes  $\widehat{\nu}(\lambda) = \prod_i b(\lambda(k_i))$ . Since

$$\begin{aligned} \widehat{\nu}(\lambda_1) \widehat{\nu}(\lambda_2) &= \prod_i b(\lambda_1(k_i)) b(\lambda_2(k_i)), \\ \widetilde{\nu}(\lambda_1, \lambda_2) &= \prod_i b(\lambda_1(k_i)) b(\lambda_2(k_i)) \overline{b(\lambda_1(k_i) + \lambda_2(k_i))}, \\ \widehat{\nu}(\lambda_1 + \lambda_2) &= \prod_i b(\lambda_1(k_i) + \lambda_2(k_i)). \end{aligned}$$

we get  $\widehat{\nu}(\lambda_1) \widehat{\nu}(\lambda_2) = \widetilde{\nu}(\lambda_1, \lambda_2) \widehat{\nu}(\lambda_1 + \lambda_2)$ . By the definitions of  $\widetilde{\mu}$ ,  $\widetilde{\mu}_0$ ,  $\widehat{\mu}$  and  $\widehat{\mu}_0$ , the maps  $\widehat{\nu}$  and  $\widetilde{\nu}$  are given by

$$\widehat{\nu}(\lambda) = \widehat{\mu}(\lambda) \overline{\widehat{\mu}_0(\lambda)}, \quad \widetilde{\nu}(\lambda_1, \lambda_2) = \widetilde{\mu}(\lambda_1, \lambda_2) \overline{\widetilde{\mu}_0(\lambda_1, \lambda_2)}.$$

Thus the desired equality immediately follows.  $\square$

PROOF FOR THE SECOND HALF OF THEOREM 4.1. Suppose that there exists a group isomorphism  $\phi$  satisfying the conditions (2) and (3) in the theorem. We prove that there exists a  $*$ -isomorphism  $\pi = \pi_\phi$  from  $N(H_a, \mu_a)$  onto  $N(H_b, \mu_b)$  preserving the  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ -actions with the condition (1).

We define a group homomorphism  $c_\phi$  from  $H_a$  to  $\{1, -1\} \subset \mathbf{T}$  by

$$c_\phi(h) = \overline{\chi_b(\phi(h))} \chi_a(h), \quad h \in H_a.$$

Let  $\widetilde{c}_\phi$  be the group homomorphism from  $\Lambda(H_a)$  to  $\{1, -1\} \subset \mathbf{T}$  given by

$$\widetilde{c}_\phi(\lambda) = \prod_{k \in \mathbf{Z}^2} c_\phi(\lambda(k))^{\gcd(k)} = \prod_{k \in \mathbf{Z}^2} \chi_a(\lambda(k))^{\gcd(k)} \overline{\chi_b(\phi(\lambda(k)))}^{\gcd(k)}, \quad \lambda \in \Lambda(H_a).$$

We define a linear map  $\pi$  from the group algebra  $\mathbf{C}_{\widetilde{\mu}_a}(\Lambda(H_a))$  onto  $\mathbf{C}_{\widetilde{\mu}_b}(\Lambda(H_b))$  by

$$\pi\left(\overline{\widetilde{\mu}_a(\lambda)} u(\lambda)\right) = \widetilde{c}_\phi(\lambda) \overline{\widetilde{\mu}_b(\phi \circ \lambda)} v(\phi \circ \lambda), \quad \lambda \in \Lambda(H_a).$$

By direct computations, for all  $\lambda_1, \lambda_2 \in \Lambda(H_a)$ , we get

$$\begin{aligned}
& \pi\left(\overline{\widehat{\mu}_a(\lambda_1)} u(\lambda_1)\right) \pi\left(\overline{\widehat{\mu}_a(\lambda_2)} u(\lambda_2)\right) \\
&= \widetilde{c}_\phi(\lambda_1) \widetilde{c}_\phi(\lambda_2) \overline{\widehat{\mu}_b(\phi \circ \lambda_1) \widehat{\mu}_b(\phi \circ \lambda_2)} v(\phi \circ \lambda_1) v(\phi \circ \lambda_2) \\
&= \widetilde{c}_\phi(\lambda_1 + \lambda_2) \overline{\widehat{\mu}_b(\phi \circ \lambda_1) \widehat{\mu}_b(\phi \circ \lambda_2) \widehat{\mu}_b(\phi \circ \lambda_1, \phi \circ \lambda_2)} v(\phi \circ (\lambda_1 + \lambda_2)).
\end{aligned}$$

On the other hand, we have the following equation:

$$\begin{aligned}
& \pi\left(\overline{\widehat{\mu}_a(\lambda_1) \widehat{\mu}_a(\lambda_2)} u(\lambda_1) u(\lambda_2)\right) \\
&= \pi\left(\overline{\widehat{\mu}_a(\lambda_1) \widehat{\mu}_a(\lambda_2)} \widetilde{\mu}_a(\lambda_1, \lambda_2) u(\lambda_1 + \lambda_2)\right) \\
&= \overline{\widehat{\mu}_a(\lambda_1) \widehat{\mu}_a(\lambda_2)} \widetilde{\mu}_a(\lambda_1, \lambda_2) \widehat{\mu}_a(\lambda_1 + \lambda_2) \pi\left(\overline{\widehat{\mu}_a(\lambda_1 + \lambda_2)} u(\lambda_1 + \lambda_2)\right) \\
&= \widetilde{c}_\phi(\lambda_1 + \lambda_2) \overline{\widehat{\mu}_a(\lambda_1) \widehat{\mu}_a(\lambda_2) \widetilde{\mu}_a(\lambda_1, \lambda_2) \widehat{\mu}_a(\lambda_1 + \lambda_2)} \\
&\quad \overline{\widehat{\mu}_b(\phi \circ (\lambda_1 + \lambda_2))} v(\phi \circ (\lambda_1 + \lambda_2)).
\end{aligned}$$

By Lemma 4.2 and the condition (2) for the group isomorphism  $\phi$  in the theorem, we have that

$$\begin{aligned}
& \overline{\widehat{\mu}_b(\phi \circ \lambda_1) \widehat{\mu}_b(\phi \circ \lambda_2)} \widetilde{\mu}_b(\phi \circ \lambda_1, \phi \circ \lambda_2) \\
&= \overline{\widehat{\mu}_a(\lambda_1) \widehat{\mu}_a(\lambda_2)} \widetilde{\mu}_a(\lambda_1, \lambda_2) \widehat{\mu}_a(\lambda_1 + \lambda_2) \overline{\widehat{\mu}_b(\phi \circ (\lambda_1) + \phi \circ (\lambda_2))}.
\end{aligned}$$

Therefore we get  $\pi(u(\lambda_1)) \pi(u(\lambda_2)) = \pi(u(\lambda_1) u(\lambda_2))$ . The linear map  $\pi$  also preserves the  $*$ -operation. As a consequence,  $\pi$  is a  $*$ -isomorphism from  $\mathbf{C}_{\widehat{\mu}_a}^\sim(\Lambda(H_a))$  onto  $\mathbf{C}_{\widetilde{\mu}_b}^\sim(\Lambda(H_b))$  and this preserves the trace. The map  $\pi = \pi_\phi$  is extended to a normal  $*$ -isomorphism from  $N(H_a, \mu_a)$  onto  $N(H_b, \mu_b)$ .

We next prove that this  $\pi$  preserves the  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ -actions. The group homomorphism  $\widetilde{c}_\phi$  from  $\Lambda(H_a)$  to  $\{1, -1\}$  is invariant under the action of  $SL(2, \mathbf{Z})$ , by Lemma 2.1 (1). The scalar 2-cocycle  $\nu(g, h) = \mu_a(g, h) \overline{\widehat{\mu}_b(\phi(h), \phi(g))}$  satisfies  $\nu(g, h) = \nu(h, g)$  by condition (2), so the function  $\widehat{\nu}(\cdot) = \widehat{\mu}_a(\cdot) \overline{\widehat{\mu}_b(\phi \circ \cdot)}$  on  $\Lambda(H_a)$  does not depend on the order on  $\mathbf{Z}^2$  chosen before. Since

$$\begin{aligned}
\pi \circ \beta_a(\gamma)(u(\lambda)) &= \pi(u(\gamma \cdot \lambda)) \\
&= \widetilde{c}_\phi(\gamma \cdot \lambda) \overline{\widehat{\mu}_a(\gamma \cdot \lambda) \widehat{\mu}_b(\phi \circ (\gamma \cdot \lambda))} v(\phi \circ (\gamma \cdot \lambda)) \\
&= \widetilde{c}_\phi(\lambda) \overline{\widehat{\mu}_a(\lambda) \widehat{\mu}_b(\phi \circ \lambda)} v(\gamma \cdot (\phi \circ \lambda)) \\
&= \beta_b(\gamma) \left( \widetilde{c}_\phi(\lambda) \overline{\widehat{\mu}_a(\lambda) \widehat{\mu}_b(\phi \circ \lambda)} v(\phi \circ \lambda) \right) \\
&= \beta_b(\gamma) \circ \pi(u(\lambda)), \quad \gamma \in SL(2, \mathbf{Z}), \lambda \in \Lambda(H_a),
\end{aligned}$$

it turns out that the  $*$ -isomorphism  $\pi$  is compatible with the  $SL(2, \mathbf{Z})$ -actions  $\beta_a|_{SL(2, \mathbf{Z})}$  and  $\beta_b|_{SL(2, \mathbf{Z})}$ .

For all  $\lambda \in \Lambda(H_a)$  and  $k \in \mathbf{Z}^2$ , we have

$$\begin{aligned} \pi \circ \beta_a(k)(u(\lambda)) &= \pi \left( \prod_{l \in \mathbf{Z}^2} \chi_a(\lambda(l))^{\det(k, l)} u(k \cdot \lambda) \right) \\ &= \prod_{l \in \mathbf{Z}^2} c_\phi((k \cdot \lambda)(l))^{\gcd(l)} \prod_{l \in \mathbf{Z}^2} \chi_a(\lambda(l))^{\det(k, l)} \widehat{\mu}_a(k \cdot \lambda) \overline{\widehat{\mu}_b(\phi \circ (k \cdot \lambda))} v(\phi \circ (k \cdot \lambda)). \end{aligned}$$

Since  $c_\phi(h)^{\det(k, l)} c_\phi(h)^{\gcd(k+l)} = c_\phi(h)^{\gcd(k)} c_\phi(h)^{\gcd(l)}$ , by Lemma 2.1 (2), the unitary  $\pi \circ \beta_a(k)(u(\lambda))$  equals to

$$\begin{aligned} & \prod_{l \in \mathbf{Z}^2} c_\phi(\lambda(l))^{\gcd(k+l)} \prod_{l \in \mathbf{Z}^2} \left( c_\phi(\lambda(l))^{\det(k, l)} \chi_b(\phi \circ \lambda(l))^{\det(k, l)} \right) \\ & \quad \widehat{\mu}_a(k \cdot \lambda) \overline{\widehat{\mu}_b(\phi \circ (k \cdot \lambda))} v(\phi \circ (k \cdot \lambda)) \\ &= \prod_{l \in \mathbf{Z}^2} c_\phi(\lambda(l))^{\gcd(k)} \prod_{l \in \mathbf{Z}^2} c_\phi(\lambda(l))^{\gcd(l)} \prod_{l \in \mathbf{Z}^2} \chi_b(\phi \circ \lambda(l))^{\det(k, l)} \\ & \quad \widehat{\mu}_a(k \cdot \lambda) \overline{\widehat{\mu}_b(\phi \circ (k \cdot \lambda))} v(k \cdot (\phi \circ \lambda)) \\ &= \widetilde{c}_\phi(\lambda) \widehat{\mu}_a(k \cdot \lambda) \overline{\widehat{\mu}_b(k \cdot (\phi \circ \lambda))} \beta_b(k)(v(\phi \circ \lambda)) \\ &= \beta_b(k) \circ \pi(u(\lambda)). \end{aligned}$$

This means that the  $*$ -isomorphism  $\pi$  preserves the  $\mathbf{Z}^2$ -actions.

We get the  $*$ -isomorphism  $\pi = \pi_\phi$  from  $N(H_a, \mu_a)$  onto  $N(H_b, \mu_b)$  giving conjugacy between  $\beta_a$  and  $\beta_b$ .  $\square$

REMARK 4.3. The proof of the first half of Theorem 4.1 shows that any isomorphism  $\pi$  giving conjugacy between  $\beta_a$  and  $\beta_b$  is of the form  $\pi_\phi$ . This means that an isomorphism which gives conjugacy between two twisted Bernoulli shift actions must be trace preserving.

This proof shows that an isomorphism giving conjugacy between the two actions  $\beta(H_a, \mu_a, \chi_a)$ ,  $\beta(H_b, \mu_b, \chi_b)$  is of a very special form derived from a group isomorphism between  $H_a$  and  $H_b$ . Taking notice of this fact, we can describe the centralizer of a twisted Bernoulli shift action. We define two topological groups before we state Theorem 4.4.

Let  $\beta$  be a trace preserving action of some group  $\Gamma$  on a separable finite von Neumann algebra  $(N, \text{tr})$ . We denote by  $\text{Aut}(N, \beta)$  the group of all automorphisms which commute with the action  $\beta$ , that is,

$$\{\alpha \in \text{Aut}(N) \mid \beta(\gamma) \circ \alpha = \alpha \circ \beta(\gamma), \quad \gamma \in \Gamma\}.$$

We regard the group  $\text{Aut}(N, \beta)$  as a topological group equipped with the pointwise-strong topology. When  $\beta$  is a twisted Bernoulli shift action on  $N$ , an automorphism  $\alpha$  commuting with  $\beta$  is necessarily trace preserving by Remark 4.3. We consider that  $\text{Aut}(N, \beta)$  is equipped with the pointwise-2-norm topology.

Let  $\text{Aut}(H, \mu, \chi)$  be the group of all automorphisms of an abelian group  $H$  which preserve its 2-cocycle  $\mu$  and character  $\chi$ , that is,

$$\{\phi \in \text{Aut}(H) \mid \mu(g, h) = \mu(\phi(g), \phi(h)), \chi(g) = \chi(\phi(g)), \quad g, h \in H\}.$$

We define the topology of  $\text{Aut}(H, \mu, \chi)$  by pointwise convergence.

**THEOREM 4.4.** *For  $\pi \in \text{Aut}(N(H, \mu), \beta(H, \mu, \chi))$ , there exists a unique element  $\phi = \phi_\pi \in \text{Aut}(H, \mu^* \mu, \chi^2)$  satisfying  $\pi(u(\lambda)) = u(\phi \circ \lambda) \pmod{\mathbf{T}}$  for  $\lambda \in \Lambda(H)$ . The map  $\pi \mapsto \phi_\pi$  gives an isomorphism between two topological groups*

$$\text{Aut}(N(H, \mu), \beta(H, \mu, \chi)) \cong \text{Aut}(H, \mu^* \mu, \chi^2).$$

**PROOF.** We use the notations in the proof of the previous theorem, letting  $H_a = H_b = H$ ,  $\mu_a = \mu_b = \mu$  and  $\chi_a = \chi_b = \chi$ . Denote  $N = N(H, \mu)$  and  $\beta = \beta(H, \mu, \chi)$ . We have already shown the first claim. Let

$$\text{Aut}(H, \mu^* \mu, \chi^2) \ni \phi \mapsto \pi_\phi \in \text{Aut}(N, \beta),$$

be the map given as in the proof of Theorem 4.1, that is,

$$\pi_\phi(\overline{\widehat{\mu}(\lambda)} u(\lambda)) = \widetilde{c}_\phi(\lambda) \overline{\widehat{\mu}(\phi \circ \lambda)} u(\phi \circ \lambda), \quad \lambda \in \Lambda(H),$$

where

$$\widetilde{c}_\phi(\lambda) = \prod_{k \in \mathbf{Z}^2} \chi(\lambda(k))^{\text{gcd}(k)} \overline{\chi(\phi \circ \lambda(k))^{\text{gcd}(k)}}.$$

It is easy to prove that  $\phi_{\pi_\phi} = \phi$  by the definition. Thus the map  $\pi \mapsto \phi_\pi$  is surjective. We prove that this map is injective. Let  $\phi$  be an element of  $\text{Aut}(H, \mu^* \mu, \chi^2)$ . Suppose that  $\pi$  is an arbitrary element of  $\text{Aut}(N, \beta)$  satisfying  $\phi = \phi_\pi$ . The set  $\{\beta(\gamma)(u(\lambda_h)) \mid h \in H, \gamma \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})\}$  generates  $N$ , so we have only to prove the uniqueness of  $c(h) \in \mathbf{T}$  satisfying

$$\pi\left(\overline{\mu(h, -h)} u(\lambda_h)\right) = c(h) \overline{\mu(\phi(h), -\phi(h))} u(\lambda_{\phi(h)}),$$

for all  $h \in H$ . In the proof of the first half of the previous theorem (equation (Eq2)), we have already shown that  $c(h) = \chi(h) \overline{\chi(\phi(h))}$ . Thus the  $*$ -isomorphism  $\pi$  is uniquely determined and the map  $\pi \mapsto \phi_\pi$  is injective.

We prove the two maps  $\phi \mapsto \pi_\phi$  and  $\pi \mapsto \phi_\pi$  are continuous. Let  $(\phi_i)$  be a net in  $\text{Aut}(H, \mu^* \mu, \chi^2)$  converging to  $\phi$ . For all  $h \in H$ , we have

$$\pi_{\phi_i}\left(\overline{\chi(h) \mu(h, -h)} u(\lambda_h)\right) = \overline{\chi(\phi_i(h)) \mu(\phi_i(h), -\phi_i(h))} u(\lambda_{\phi_i(h)}).$$

The right side of the equation converges to

$$\overline{\chi(\phi(h)) \mu(\phi(h), -\phi(h))} u(\lambda_{\phi(h)}) = \pi_\phi\left(\overline{\chi(h) \mu(h, -h)} u(\lambda_h)\right).$$

This proves that  $\pi_{\phi_i}$  converges to  $\pi_\phi$  in pointwise 2-norm topology on the generating set  $\{\beta(\gamma)(u(\lambda_h)) \mid h \in H, \gamma \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})\}$  of  $N$ . Thus  $\pi_{\phi_i}$  converges to  $\pi_\phi$  on  $N$ .

Conversely, let  $(\pi_i)$  be a net in  $\text{Aut}(N, \beta)$  converging to  $\pi$ . For all  $h \in H$ , we get  $\pi_i(u(\lambda_h)) = u(\lambda_{\phi_{\pi_i}(h)}) \bmod \mathbf{T}$ . The left side of the equation converges to  $\pi(u(\lambda_h)) = u(\lambda_{\phi_\pi(h)})$ . If  $\phi_{\pi_i}(h) \neq \phi_\pi(h)$ , then the distance between  $\mathbf{T}u(\lambda_{\phi_{\pi_i}(h)})$  and  $\mathbf{T}u(\lambda_{\phi_\pi(h)})$  is  $\sqrt{2}$  in the 2-norm. Thus  $\phi_{\pi_i}(h) = \phi_\pi(h)$  for large enough  $i$ . This means that  $(\phi_{\pi_i})$  converges to  $\phi_\pi$ .

As a consequence, the two maps  $\phi \mapsto \pi_\phi$  and  $\pi \mapsto \phi_\pi$  are continuous group homomorphisms and inverse maps of each other.  $\square$

## 5. Examples.

### 5.1. Twisted Bernoulli shift actions on $L^\infty(X)$ .

In this subsection, we consider the case of  $\mu = 1$  and  $H \neq \{0\}$ . Then the algebra  $N(H, 1)$  is abelian and has a faithful normal state, so it is isomorphic to  $L^\infty(X)$ , where  $X$  is a standard probability space. The measure of  $X$  is determined by the trace on  $N(H, 1)$ . Furthermore,  $X$  is non-atomic, since  $N(H, 1)$  is infinite dimensional and the action  $\beta(H, 1, \chi)$  is ergodic. As corollaries of Theorems 4.1 and 4.4, we get trace preserving  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ -actions on  $L^\infty(X)$  whose centralizers are isomorphic to some prescribed groups.

REMARK 5.1. The  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ -action on  $X$  defined by  $\beta = \beta(H, 1, \chi)$  is free. An automorphism  $\alpha \in \text{Aut}(L^\infty(X), \beta)$  is free or the identity map for any twisted Bernoulli shift action  $\beta$  on  $L^\infty(X)$ . This is proved as follows. We identify

$\beta(\gamma)$  ( $\gamma \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ ) and  $\alpha$  with measure preserving Borel isomorphisms on  $X$  here. Suppose that there exists a non-null Borel subset  $Y \subset X$  whose elements are fixed under  $\alpha$ . All elements in  $\tilde{Y} = \bigcup \{\beta(\gamma)(Y) \mid \gamma \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})\}$  are fixed under  $\alpha$ . By the ergodicity of  $\beta$ , the measure of  $\tilde{Y}$  is 1. Then  $\alpha$  is the identity map of  $L^\infty(X)$ .

**COROLLARY 5.2.** *For any abelian countable discrete group  $H \neq \{0\}$ , there exists a trace preserving essentially free ergodic action  $\beta$  of  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  on  $L^\infty(X)$  satisfying*

$$\text{Aut}(L^\infty(X), \beta) \cong \text{Aut}(H).$$

**PROOF.** When we define  $\beta = \beta(H, 1, 1)$ , we have the above relation by Theorem 4.4.  $\square$

In the next corollary we use the effect of twisting by a character  $\chi$ .

**COROLLARY 5.3.** *For every abelian countable discrete group  $H \neq \{0\}$ , there exist continuously many trace preserving essentially free ergodic actions  $\{\beta_c\}$  of  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  on  $L^\infty(X)$  which are mutually non-conjugate and satisfy*

$$\text{Aut}(L^\infty(X), \beta_c) \cong H \rtimes \text{Aut}(H).$$

Here the topology of  $H \rtimes \text{Aut}(H)$  is the product of the discrete topology on  $H$  and the pointwise convergence topology on  $\text{Aut}(H)$ .

**PROOF.** Let  $c \in \{e^{int} \mid t \in (0, 1/2) \setminus \mathbf{Q}\}$ . We put  $\beta_c = \beta(H \oplus \mathbf{Z}, 1, 1 \times \chi_c)$ , where the character  $\chi_c$  of  $\mathbf{Z}$  is defined as  $\chi_c(n) = c^n$ . By Theorem 4.4, we get

$$\text{Aut}(L^\infty(X), \beta_c) \cong \text{Aut}(H \oplus \mathbf{Z}, 1, 1 \times \chi_c^2).$$

Since the character  $\chi_c^2$  is injective, a group automorphism  $\alpha \in \text{Aut}(H \oplus \mathbf{Z}, 1, 1 \times \chi_c^2)$  preserves the second entry. For all  $\alpha \in \text{Aut}(H \oplus \mathbf{Z}, 1, 1 \times \chi_c^2)$ , there exist  $\phi_\alpha \in \text{Aut}(H)$  and  $h_\alpha \in H$  satisfying

$$\alpha(h, n) = (\phi_\alpha(h) + nh_\alpha, n), \quad (h, n) \in H \oplus \mathbf{Z}.$$

The map  $\text{Aut}(H \oplus \mathbf{Z}, 1, 1 \times \chi_c^2) \ni \alpha \mapsto (h_\alpha, \phi_\alpha) \in H \rtimes \text{Aut}(H)$  is a homeomorphic group isomorphism.

If  $c_1, c_2 \in \{e^{int} \mid t \in (0, 1/2) \setminus \mathbf{Q}\}$  and  $c_1 \neq c_2$ , then there exists no isomorphism from  $H \oplus \mathbf{Z}$  to  $H \oplus \mathbf{Z}$  whose pull back of the character  $1 \times \chi_{c_2}^2$  is equal to  $1 \times \chi_{c_1}^2$ . The two actions  $\beta_{c_1}$  and  $\beta_{c_2}$  are not conjugate by Theorem 4.1.  $\square$

COROLLARY 5.4. *There exist continuously many trace preserving essentially free ergodic actions  $\{\beta_c\}$  of  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  on  $L^\infty(X)$  which are mutually non-conjugate and have the trivial centralizer  $\text{Aut}(L^\infty(X), \beta_c) = \{\text{id}_{L^\infty X}\}$ .*

PROOF. Let  $\{\chi_c \mid c = e^{i\pi t}, t \in (0, 1/2)\}$  be characters of  $\mathbf{Z}$  such that  $\chi_c(m) = c^m$ . Since  $\chi_c^2(1)$  is in the upper half plane, the identity map is the only automorphism of  $\mathbf{Z}$  preserving  $\chi_c^2$ . By Theorem 4.4, we get  $\text{Aut}(\beta(\mathbf{Z}, 1, \chi_c)) = \{\text{id}\}$ .

If  $\beta(\mathbf{Z}, 1, \chi_{c_1}), \beta(\mathbf{Z}, 1, \chi_{c_2})$  are conjugate, then there exists a group isomorphism on  $\mathbf{Z}$  whose pull back of  $\chi_{c_2}^2$  is  $\chi_{c_1}^2$  by Theorem 4.1. This means  $c_1 = c_2$ . Thus the actions  $\{\beta(\mathbf{Z}, 1, \chi_c)\}$  are mutually non-conjugate.  $\square$

**5.2. Twisted Bernoulli shift actions on the AFD factor of type  $\text{II}_1$ .**

Firstly, we find a condition that the finite von Neumann algebra  $N(H, \mu)$  is the AFD factor of type  $\text{II}_1$ .

LEMMA 5.5. *For an abelian countable discrete group  $H \neq \{0\}$  and its normalized scalar 2-cocycle  $\mu$ , the following statements are equivalent:*

- (1) *The algebra  $N(H, \mu)$  is the AFD factor of type  $\text{II}_1$ .*
- (2) *The group von Neumann algebra  $L_\mu(H)$  twisted by the scalar 2-cocycle  $\mu$  is a factor (of type  $\text{II}_1$  or  $\text{I}_n$ ).*
- (3) *For all  $g \in H \setminus \{0\}$ , there exists  $h \in H$  such that  $\mu(g, h) \neq \mu(h, g)$ .*

PROOF. The amenability of the group  $\Lambda(H)$  leads the injectivity for  $N(H, \mu)$ . The injectivity for  $N(H, \mu)$  implies that  $N(H, \mu)$  is approximately finite dimensional ([Co]). We have only to show the equivalence of conditions (2), (3) and

(1)' The algebra  $N(H, \mu)$  is a factor.

By using Fourier expansion it is easy to see that condition (2) holds true if and only if for any  $g \in H \setminus \{0\}$  there exists  $h \in H$  satisfying  $u_g u_h \neq u_h u_g$ . This is equivalent to condition (3). Similarly, condition (1)' is equivalent to

(1)'' For any  $\lambda_1 \in \Lambda(H) \setminus \{0\}$ , there exists  $\lambda_2 \in \Lambda(H)$  satisfying  $\overline{\tilde{\mu}(\lambda_2, \lambda_1)} \tilde{\mu}(\lambda_1, \lambda_2) \neq 1$ .

Suppose condition (3). For any  $\lambda_1$ , choose element  $k, l \in \mathbf{Z}^2$  so that  $k \in \text{supp}(\lambda_1)$  and  $l \notin \text{supp}(\lambda_1)$ . By condition (3), there exists  $h \in H$  satisfying  $\mu^* \mu(\lambda_1(k), h) \neq 1$ . Let  $\lambda_2$  be the element in  $\Lambda(H)$  which takes  $h$  at  $k$ ,  $-h$  at  $l$  and 0 for the other places. The element  $\lambda_2$  satisfies  $\overline{\tilde{\mu}(\lambda_2, \lambda_1)} \tilde{\mu}(\lambda_1, \lambda_2) = \mu^* \mu(\lambda_1(k), h) \neq 1$ . Here we get condition (1)''. The implication from (1)'' to (3) is easily shown.  $\square$

REMARK 5.6. The twisted Bernoulli shift action  $\beta = \beta(H, \mu, \chi)$  is an outer action of  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ . Any non-trivial automorphism in  $\text{Aut}(R, \beta(H, \mu, \chi))$  is also outer. This is proved by the weak mixing property of the action  $\beta(H, \mu, \chi)$  as follows. If  $\alpha \in \text{Aut}(R, \beta(H, \mu, \chi))$  is an inner automorphism  $\text{Ad}(u)$ , then we have

$$\begin{aligned} \text{Ad}(\beta(\gamma)(u))(x) &= \beta(\gamma)(u\beta(\gamma)^{-1}(x)u^*) = \beta(\gamma) \circ \alpha \circ \beta(\gamma)^{-1}(x) \\ &= \alpha(x) = \text{Ad}(u)(x), \end{aligned}$$

for all  $x \in R$  and  $\gamma \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ . Since  $\text{Ad}(\beta(\gamma)(u)u^*) = \text{id}$ ,  $\mathbf{C}u \subset R$  is an invariant subspace of the action  $\beta$ . The only subspace invariant under the weakly mixing action  $\beta$  is  $\mathbf{C}1$  (Proposition 2.6), thus we get  $\alpha = \text{id}$ .

Using Theorems 4.1 and 4.4, we give continuously many actions of  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  on  $R$  such that there exists no commuting automorphism except for trivial one.

COROLLARY 5.7. *There exist continuously many ergodic outer actions  $\{\beta_c\}$  of  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  on the AFD factor  $R$  of type  $\text{II}_1$  which are mutually non-conjugate and have the trivial centralizer  $\text{Aut}(R, \beta_c) = \{\text{id}_R\}$ .*

PROOF. We can choose and fix a character  $\chi$  on  $\mathbf{Z}^2$  such that  $\chi^2$  is injective. Let  $\{\mu_c \mid c = e^{i\pi t}, t \in (0, 1/2) \setminus \mathbf{Q}\}$  be scalar 2-cocycles for  $\mathbf{Z}^2$  defined by

$$\mu_c \left( \begin{pmatrix} s_1 \\ t_1 \end{pmatrix}, \begin{pmatrix} s_2 \\ t_2 \end{pmatrix} \right) = c^{s_1 t_2 - t_1 s_2}, \quad s_1, t_1, s_2, t_2 \in \mathbf{Z}.$$

We put  $\beta_c = \beta(\mathbf{Z}^2, \mu_c, \chi)$ . The 2-cocycle  $\mu_c$  satisfies condition (3) in Lemma 5.5. Thus  $\beta_c$  defines a  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ -action on  $R$ . By Theorem 4.4, we get the following isomorphism between topological groups:

$$\text{Aut}(R, \beta_c) \cong \text{Aut}(\mathbf{Z}^2, \mu_c^* \mu_c, \chi^2) = \text{Aut}(\mathbf{Z}^2, \mu_{c^2}, \chi^2).$$

Since the character  $\chi^2$  of  $\mathbf{Z}^2$  is injective, so the group of the right side is  $\{\text{id}|_{\mathbf{Z}^2}\}$ . This means that the action  $\beta_c$  has trivial centralizers.

Finally, we prove that the actions  $\{\beta_c \mid c = e^{i\pi t}, t \in (0, 1/2) \setminus \mathbf{Q}\}$  are mutually non-conjugate. Suppose that actions  $\beta_{c_1}$  and  $\beta_{c_2}$  are conjugate. By Theorem 4.1, there exists a group isomorphism  $\phi$  of  $\mathbf{Z}^2$  satisfying

$$\mu_{c_1}^2(g, h) = \mu_{c_2}^2(\phi(g), \phi(h)), \quad g, h \in \mathbf{Z}^2.$$

A group isomorphism of  $\mathbf{Z}^2$  is given by an element of  $GL(2, \mathbf{Z})$ . If the automorphism  $\phi$  is given by an element of  $SL(2, \mathbf{Z})$ , we get  $c_2^2 = c_1^2$ . If  $\phi$  is given by an element of  $GL(2, \mathbf{Z}) \setminus SL(2, \mathbf{Z})$ , then we get  $c_2^2 = -c_1^2$ . Since both  $c_1$  and  $c_2$  have the form  $e^{i\pi t}$ ,  $t \in (0, 1/2)$ , we get  $c_1 = c_2$ .  $\square$

Any cyclic group of an odd order can be realized as the centralizer of a twisted Bernoulli shift actions on  $R$ .

COROLLARY 5.8. *Let  $q$  be an odd natural number  $\geq 3$  and denote by  $H_q$  the abelian group  $(\mathbf{Z}/q\mathbf{Z})^2$ . We define the 2-cocycle  $\mu_q$  and the character  $\chi_q$  on  $H_q$  as*

$$\mu_q\left(\left(\begin{matrix} s_1 \\ t_1 \end{matrix}\right), \left(\begin{matrix} s_2 \\ t_2 \end{matrix}\right)\right) = \exp(2\pi i s_1 t_2 / q), \quad \chi_q\left(\left(\begin{matrix} s_1 \\ t_1 \end{matrix}\right)\right) = \exp(2\pi i s_1 / q).$$

*Then the algebra  $N(H_q, \mu_q)$  is the AFD factor  $R$  of type  $\text{II}_1$  and the centralizer of the twisted Bernoulli shift action  $\beta_a = \beta(H_q, \mu_q, \chi_q)$  is isomorphic to  $\mathbf{Z}/q\mathbf{Z}$ .*

PROOF. By Lemma 5.5, it is shown that the algebra  $N(H_Q, \mu_Q)$  is the AFD factor of type  $\text{II}_1$ . Using Theorem 4.4, we have only to prove that  $\text{Aut}(H_q, \mu_q^* \mu_q, \chi_q^2) \cong \mathbf{Z}/q\mathbf{Z}$ .

Let  $\phi$  be in  $\text{Aut}(H_q, \mu_q^* \mu_q, \chi_q^2)$ . The automorphism  $\phi$  of  $H_q$  is given by a  $2 \times 2$  matrix  $A$  of  $\mathbf{Z}/q\mathbf{Z}$ . Since  $\phi$  preserves  $\mu_q^* \mu_q$ , the determinant of  $A$  must be 1. Since  $q$  is odd, the value of  $\chi_q^2$  determines the first entry of  $(\mathbf{Z}/q\mathbf{Z})^2$  and  $\phi$  preserves  $\chi_q^2$ . The matrix  $A$  is of the form

$$\begin{pmatrix} 1 & 0 \\ t_\phi & 1 \end{pmatrix}, \quad t_\phi \in \mathbf{Z}/q\mathbf{Z}.$$

The map  $\phi \mapsto t_\phi$  is an isomorphism. In turn, if the matrix  $A$  is of this form, it defines an element in  $\text{Aut}(H_q, \mu_q^* \mu_q, \chi_q^2)$ .  $\square$

COROLLARY 5.9. *For a set  $Q$  consisting of odd prime numbers, let  $\beta_Q$  be the tensor product  $\bigotimes_{q \in Q} \beta_q$  of the actions  $\beta_q$  on the AFD factor of type  $\text{II}_1$ . The centralizer of  $\beta_Q$  is isomorphic to  $\prod_{q \in Q} \mathbf{Z}/q\mathbf{Z}$ .*

PROOF. The action  $\beta_Q$  is the twisted Bernoulli shift action  $\beta(H_Q, \mu_Q, \chi_Q)$ , where  $H_Q$  is the abelian group  $\bigoplus_{q \in Q} H_q$  and the scalar 2-cocycle  $\mu_q$  and the character  $\chi_Q$  on  $H_Q$  are given by

$$\begin{aligned}\mu_Q((s_q), (t_q)) &= \prod_{q \in Q} \mu_q(s_q, t_q), \\ \chi_Q((s_q)) &= \prod_{q \in Q} \chi_q(s_q), \quad (s_q), (t_q) \in H_Q, s_q, t_q \in H_q.\end{aligned}$$

Using Theorem 4.4, we have only to prove

$$\text{Aut}(H_Q, \mu_Q^* \mu_Q, \chi_Q^2) \cong \prod_{q \in Q} \mathbf{Z}/q\mathbf{Z}.$$

A group automorphism  $\phi$  of  $H_Q = \bigoplus_{q \in Q} H_q$  has a form  $\phi((k_q)) = (\phi_q(k_q))$ , for some  $\{\phi_q \in \text{Aut}(H_q)\}$ . Thus we get

$$\text{Aut}(H_Q, \mu_Q^* \mu_Q, \chi_Q^2) \cong \prod_{q \in Q} \text{Aut}(H_q, \mu_q^* \mu_q, \chi_q^2).$$

Together with the previous corollary, we get the conclusion.  $\square$

REMARK 5.10. If  $Q_1 \neq Q_2$ , then the two groups  $\prod_{q \in Q_1} \mathbf{Z}/q\mathbf{Z}$  and  $\prod_{q \in Q_2} \mathbf{Z}/q\mathbf{Z}$  are not isomorphic. The continuously many outer actions  $\{\beta_Q\}$  are distinguished in view of conjugacy only by using the centralizers  $\{\text{Aut}(R, \beta_Q)\}$ .

## 6. Malleability and rigidity arguments.

In this section, we give *malleability* and *rigidity* type arguments invented by S. Popa, in order to examine weak 1-cocycles for actions. See Popa [Po2], [Po3], [Po4] and Popa-Sasyk [PoSa] for the references. S. Popa in [Po3] showed that every 1-cocycle for a Connes-Størmer Bernoulli shift by a property (T) group (or  $w$ -rigid group like  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ ) vanishes modulo scalars. As a consequence, two such actions are cocycle conjugate if and only if they are conjugate. In our case, 1-cocycles do not vanish modulo scalars but they are still in the situation that cocycle (outer) conjugacy implies conjugacy. We need the following notion to examine outer conjugacy of two group actions.

DEFINITION 6.1. Let  $\alpha$  be an action of discrete group  $\Gamma$  on a von Neumann algebra  $\mathcal{M}$ . A weak 1-cocycle for  $\alpha$  is a map  $w : \Gamma \rightarrow \mathcal{U}(M)$  satisfying

$$w_{gh} = w_g \alpha_g(w_h) \pmod{\mathbf{T}}, \quad g, h \in \Gamma.$$

The weak 1-cocycle  $w$  is called a weak 1-coboundary if there exists a unitary  $v \in \mathcal{U}(M)$  satisfying  $w_g = v \alpha_g(v)^* \pmod{\mathbf{T}}$ . Two weak 1-cocycles  $w$  and  $w'$  are said to be equivalent when  $w'_g = v w_g \alpha_g(v)^* \pmod{\mathbf{T}}$  for some  $v \in \mathcal{U}(M)$ .

Let  $N$  be a finite von Neumann algebra with a faithful normal trace. The following is directly obtained by combining Lemmas 2.4 and 2.5 in [PoSa], although these Lemmas were proved for Bernoulli shift actions on standard probability space. The following can be also regarded as a weak 1-cocycle version of Proposition 3.2 in [Po4].

PROPOSITION 6.2. *Let  $G$  be a countable discrete group. Let  $\beta$  be a trace preserving weakly mixing action of  $G$  on  $N$ . A weak 1-cocycle  $\{w_g\}_{g \in G} \subset N$  for  $\beta$  is a weak 1-coboundary if and only if there exists a non-zero element  $\tilde{x}_0 \in N \otimes N$  satisfying*

$$(w_g \otimes 1)(\beta_g \otimes \beta_g)(\tilde{x}_0)(1 \otimes w_g^*) = \tilde{x}_0, \quad g \in G.$$

The following is a weak 1-cocycle version of Proposition 3.6.3° in [Po4].

PROPOSITION 6.3. *Let  $\Gamma$  be a countable discrete group and  $G$  be a normal subgroup of  $\Gamma$ . The group  $\Gamma$  acts on a finite von Neumann algebra  $N$  in a trace-preserving way by  $\beta$ . Suppose that the restriction of  $\beta$  to  $G$  is weakly mixing. Let  $\{w_\gamma\}_{\gamma \in \Gamma}$  be a weak 1-cocycle for  $\beta$ . If  $w|_G$  is a weak 1-coboundary, then  $w$  is a weak 1-coboundary for the  $\Gamma$ -action.*

PROOF. Suppose that  $w|_G$  is a weak 1-coboundary, that is, there exists a unitary element  $v$  in  $N$  such that  $w_g = v\beta_g(v^*) \pmod{\mathbf{T}}$  for  $g \in G$ . It suffices to show that  $\{w'_\gamma\} = \{v^*w_\gamma\beta_\gamma(v)\}$  is in  $\mathbf{T}$  for all  $\gamma \in \Gamma$ . Take arbitrary  $\gamma \in \Gamma, g \in G$ . Write  $h = \gamma^{-1}g\gamma \in G$ . Let  $\pi_\gamma$  be the unitary on  $L^2(N)$  induced from  $\beta_\gamma$ . Since  $w'_h, w'_g \in \mathbf{T}$ , we get

$$w'_\gamma \pi_g w'^*_\gamma = (w'_\gamma \pi_\gamma)(w'_h \pi_h)(w'_\gamma \pi_\gamma)^* = w'_{\gamma h \gamma^{-1}} \pi_{\gamma h \gamma^{-1}} = \pi_g \pmod{\mathbf{T}}.$$

By applying these operators to  $\hat{1} \in \hat{N} \subset L^2(N)$ , it follows that  $w'_\gamma \beta_g(w'^*_\gamma) \in \mathbf{T}$ . Since the  $G$ -action is weakly mixing, we have  $w'_\gamma \in \mathbf{T}$ . □

By using the above propositions, we will “untwist” some weak 1-cocycles later. We require some ergodicity assumption on the weak 1-cocycles.

DEFINITION 6.4. Let  $\Gamma$  be a discrete group and  $G$  be a subgroup of  $\Gamma$ . Suppose that its restriction to  $G$  is ergodic. Let  $\beta$  be a trace preserving action of  $\Gamma$  on  $N$ . A weak 1-cocycle  $w = \{w_g\}_{g \in \Gamma}$  for  $\beta$  is said to be ergodic on  $G$ , if the action  $\beta^w$  of  $G$  is still ergodic, where  $\beta^w$  is defined by  $\beta_g^w = \text{Ad} w_g \circ \beta_g, g \in G$ .

Let  $\beta$  be a  $\Gamma$ -action on  $N$ . Suppose that the diagonal action  $\beta \otimes \beta$  on  $(N \otimes N, \text{tr} \otimes \text{tr})$  has an extension  $\tilde{\beta}$  on a finite von Neumann algebra  $(\tilde{N}, \tau)$ . The algebra

$\tilde{N}$  is not necessarily identical with  $N \otimes N$ . When the action  $\tilde{\beta}$  is ergodic on a normal subgroup  $G \subset \Gamma$ , we get the following:

**PROPOSITION 6.5.** *Let  $\{w_\gamma\}_{\gamma \in \Gamma} \subset N$  be a weak 1-cocycle for  $\beta$ . Let  $\alpha$  be a trace preserving continuous action of  $\mathbf{R}$  on  $\tilde{N}$  satisfying the following properties:*

- $\alpha_1(x \otimes 1) = 1 \otimes x$ , for all  $x \in N$ .
- $\alpha_t \circ \tilde{\beta}(\gamma)(\tilde{x}) = \tilde{\beta}(\gamma) \circ \alpha_t(\tilde{x})$ , for all  $t \in \mathbf{R}$ ,  $\gamma \in \Gamma$  and  $\tilde{x} \in \tilde{N}$ .

*Suppose that the weak 1-cocycle  $\{w_\gamma \otimes 1\} \subset \tilde{N}$  is ergodic for the  $G$ -action  $\tilde{\beta}|_G$ . If the group inclusion  $G \subset \Gamma$  has the relative property (T) of Kazhdan, then there exists a non-zero element  $\tilde{x}_0 \in \tilde{N}$  so that  $(w_g \otimes 1)\tilde{\beta}_g(\tilde{x}_0)(1 \otimes w_g^*) = \tilde{x}_0$ ,  $g \in G$ .*

This is proved in the same way for Bernoulli shift actions on the infinite tensor product of abelian von Neumann algebras ([**PoS**a], Lemma 3.5). Since we are interested in actions on the AFD  $\text{II}_1$  factor, we require the ergodicity assumption on weak 1-cocycle  $\{w_\gamma \otimes 1\}$ . For the self-containedness and in order to make it clear where the ergodicity assumption works, we write down a complete proof.

**PROOF.** For  $t \in (0, 1]$ , let  $K_t$  be the convex weak closure of

$$\{(w_g \otimes 1)\alpha_t(w_g^* \otimes 1) \mid g \in G\} \subset \tilde{N}$$

and  $\tilde{x}_t \in K_t$  be the unique element whose 2-norm is minimum in  $K_t$ . Since

$$\begin{aligned} & (w_g \otimes 1)\tilde{\beta}_g((w_{g_1} \otimes 1)\alpha_t(w_{g_1}^* \otimes 1))\alpha_t(w_g^* \otimes 1) \\ &= (w_g \beta_g(w_{g_1}) \otimes 1)\alpha_t(\beta_g(w_{g_1}^*)w_g^* \otimes 1) \\ &= (w_{gg_1} \otimes 1)\alpha_t(w_{gg_1}^* \otimes 1), \quad g, g_1 \in G, \end{aligned}$$

we have  $(w_g \otimes 1)\tilde{\beta}_g(K_t)\alpha_t(w_g^* \otimes 1) = K_t$ , for  $g \in G$ . By the uniqueness of  $\tilde{x}_t$ , we get

$$(w_g \otimes 1)\tilde{\beta}_g(\tilde{x}_t)\alpha_t(w_g^* \otimes 1) = \tilde{x}_t, \quad g \in G. \quad (\text{Eq3})$$

By the assumption, the action  $(\text{Ad}(w_g \otimes 1) \circ \tilde{\beta}_g)_{g \in G}$  is ergodic on  $\tilde{N}$ . By the calculation

$$\begin{aligned} & (w_g \otimes 1)\tilde{\beta}_g(\tilde{x}_t \tilde{x}_t^*)(w_g^* \otimes 1) \\ &= (w_g \otimes 1)\tilde{\beta}_g(\tilde{x}_t)\alpha_t(w_g^* \otimes 1)\alpha_t(w_g \otimes 1)\tilde{\beta}_g(\tilde{x}_t^*)(w_g^* \otimes 1) \\ &= \tilde{x}_t \tilde{x}_t^*, \quad g \in G, \end{aligned}$$

we get  $\tilde{x}_t \tilde{x}_t^* \in \mathbf{C}1$ . The element  $\tilde{x}_t$  is a scalar multiple of a unitary in  $\tilde{N}$ .

We shall next prove that  $\widetilde{x_{1/n}}$  is not zero for some positive integer  $n$ . The pair  $(\Gamma, G)$  has the relative property (T) of Kazhdan. By Proposition 2.4, we can find a positive number  $\delta$  and a finite subset  $F \subset \Gamma$  satisfying the following condition: If a unitary representation  $(\pi, \mathcal{H})$  of  $\Gamma$  and a unit vector  $\xi$  of  $\mathcal{H}$  satisfy  $\|\pi(\gamma)\xi - \xi\| \leq \delta$  ( $\gamma \in F$ ), then  $\|\pi(g)\xi - \xi\| \leq 1/2$  ( $g \in G$ ). By the continuity of the action  $\alpha$ , there exists  $n$  such that

$$\|(w_\gamma \otimes 1)\alpha_{1/n}(w_\gamma^* \otimes 1) - 1\|_{\text{tr},2} \leq \delta, \quad \gamma \in F.$$

The actions  $\beta$  and  $(\alpha_{1/n}^l)_{l \in \mathbf{Z}}$  on  $\widetilde{N}$  give a  $\Gamma \times \mathbf{Z}$  action on  $\widetilde{N}$ . Let  $P$  be the crossed product von Neumann algebra  $P = \widetilde{N} \rtimes (\Gamma \times \mathbf{Z})$ . Let  $(U_\gamma)_{\gamma \in \Gamma}$  and  $W$  be the implementing unitaries in  $P$  for  $\Gamma$  and  $1 \in \mathbf{Z}$  respectively. We put  $V_\gamma = (w_\gamma \otimes 1)U_\gamma$ ,  $\gamma \in \Gamma$ . We regard  $\text{Ad}V$  as a unitary representation of  $\Gamma$  on  $L^2(P)$ . Since

$$\begin{aligned} \|\text{Ad}V_\gamma(W) - W\|_{L^2(P)} &= \|(w_\gamma \otimes 1)W(w_\gamma^* \otimes 1)W^* - 1\|_{L^2(P)} \\ &= \|(w_\gamma \otimes 1)\alpha_{1/n}(w_\gamma^* \otimes 1) - 1\|_{L^2(\widetilde{N})} \leq \delta, \quad \gamma \in F, \end{aligned}$$

we have the following inequality:

$$\frac{1}{2} \geq \|\text{Ad}V_g(W) - W\|_{L^2(P)} = \|(w_g \otimes 1)\alpha_{1/n}(w_g^* \otimes 1) - 1\|_{L^2(\widetilde{N})}, \quad g \in G.$$

We get  $1/2 \geq \|\widetilde{x_{1/n}} - 1\|_{L^2(\widetilde{N})}$  and  $\widetilde{x_{1/n}} \neq 0$ .

Let  $\widetilde{u_{1/n}}$  be the unitary of  $\widetilde{N}$  given by a scalar multiple of  $\widetilde{x_{1/n}}$ . By equation (Eq3), the unitary satisfies

$$(w_g \otimes 1)\widetilde{\beta}_g(\widetilde{u_{1/n}})\alpha_{1/n}(w_g^* \otimes 1) = \widetilde{u_{1/n}}, \quad g \in G.$$

Let  $\widetilde{x_0}$  be the unitary defined by

$$\widetilde{x_0} = \widetilde{u_{1/n}}\alpha_{1/n}(\widetilde{u_{1/n}})\alpha_{2/n}(\widetilde{u_{1/n}})\alpha_{(n-1)/n}(\widetilde{u_{1/n}}).$$

By direct computations, we have the following desired equality:

$$(w_g \otimes 1)\widetilde{\beta}_g(\widetilde{x_0})(1 \otimes w_g^*) = (w_g \otimes 1)\widetilde{\beta}_g(\widetilde{x_0})\alpha_1(w_g^* \otimes 1) = \widetilde{x_0}, \quad g \in G. \quad \square$$

**THEOREM 6.6.** *Let  $\beta = \beta(H, \mu, \chi)$  be a twisted Bernoulli shift action on  $N(H, \mu)$ . Suppose that  $N(H, \mu)$  is the AFD factor of type  $\text{II}_1$  and that there exists a*

continuous  $\mathbf{R}$ -action  $(\alpha_t^{(0)})_{t \in \mathbf{R}}$  on  $L_\mu(H) \otimes L_\mu(H)$  satisfying the following properties:

- For any  $x \in L_\mu(H)$ ,  $\alpha_1^{(0)}(x \otimes 1) = 1 \otimes x$ .
- The automorphism  $\alpha_t^{(0)}$  commutes with the diagonal action of  $\widehat{H}$ .

Let  $\beta^{(1)}$  be another action of  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  on the AFD factor  $N^{(1)}$  of type  $\text{II}_1$  and suppose that its restriction to  $\mathbf{Z}^2$  is ergodic. The action  $\beta^{(1)}$  is outer conjugate to  $\beta$ , if and only if  $\beta^{(1)}$  is conjugate to  $\beta$ .

PROOF. We deduce from outer conjugacy to conjugacy in the above situation. Let  $\theta$  be a  $*$ -isomorphism from  $N^{(1)}$  onto  $N(H, \mu)$  which gives the outer conjugacy of the action  $\beta^{(1)}$  and  $\beta = \beta(H, \mu, \chi)$ . There exists a weak 1-cocycle  $\{w_\gamma\}_{\gamma \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})}$  for  $\beta$  satisfying

$$\theta \circ \beta^{(1)}(\gamma) = \text{Ad} w_\gamma \circ \beta(\gamma) \circ \theta, \quad \gamma \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z}).$$

Since the action  $\beta^{(1)}$  is ergodic on  $\mathbf{Z}^2$ , the weak 1-cocycle  $w$  is ergodic on  $\mathbf{Z}^2$ .

We use the notations  $\Gamma_0, G_0$  given in Section 3. Let  $\tilde{\rho}$  be the diagonal action  $\rho \otimes \rho$  of  $\Gamma_0$  on the tensor product algebra  $\tilde{M} = L_\mu(\bigoplus_{\mathbf{Z}^2} H) \otimes L_\mu(\bigoplus_{\mathbf{Z}^2} H)$ :

$$\tilde{\rho}(\gamma_0)(a \otimes b) = \rho(\gamma_0)(a) \otimes \rho(\gamma_0)(b).$$

The fixed point algebra  $\tilde{N} \subset \tilde{M}$  of the diagonal  $\widehat{H}$ -action contains  $N(H, \mu) \otimes N(H, \mu)$ . Since  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z}) = \Gamma_0 / \widehat{H}$ , the action  $\tilde{\rho}$  gives a  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ -action  $\tilde{\beta}$  on  $\tilde{N}$ . The action  $\tilde{\beta}$  is the extension of the diagonal action  $\beta \otimes \beta$  on  $N(H, \mu) \otimes N(H, \mu)$ . We denote by  $\alpha_t$  the action on  $\tilde{M} \cong \bigotimes_{\mathbf{Z}^2} (L_\mu(H) \otimes L_\mu(H))$  given by the infinite tensor product of the  $\mathbf{R}$ -action  $\alpha_t^{(0)}$ . By the assumption on  $\alpha_t^{(0)}$ , the  $\mathbf{R}$ -action  $\alpha_t$  commutes with the action  $\tilde{\rho}$ . It follows that the subalgebra  $\tilde{N}$  is globally invariant under  $\alpha_t$ .

The set of unitaries  $\{W_\gamma = w_\gamma \otimes 1\}_{\gamma \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})} \subset \tilde{N}$  is a weak 1-cocycle for  $\tilde{\beta}$ . We shall prove that this weak 1-cocycle is ergodic on  $\mathbf{Z}^2$ . Let  $a$  be an element in  $\tilde{N}$  fixed under  $\tilde{\beta}^W|_{\mathbf{Z}^2}$ . The element  $a$  can be written as  $a = \sum_{\lambda \in \bigoplus_{\mathbf{Z}^2} H} a_\lambda \otimes u(\lambda)$  in  $L^2 \tilde{M}$ , where  $a_\lambda \otimes 1 = E_{M \otimes \mathcal{C}}(a(1 \otimes u(\lambda)))^*$ . Since  $a$  is fixed under the action of  $\mathbf{Z}^2$ , we have

$$\begin{aligned}
 a &= \tilde{\beta}^W(k)(a) = \sum_{\lambda \in \bigoplus_{\mathbf{Z}^2} H} \text{Ad}w_k \circ \rho(1, k)(a_\lambda) \otimes \rho(1, k)(u(\lambda)) \\
 &= \sum_{\lambda \in \bigoplus_{\mathbf{Z}^2} H} \text{Ad}w_k \circ \rho(1, k)(a_\lambda) \otimes \prod_{l \in \mathbf{Z}^2} \chi(\lambda(l))^{\det(k, l)} u(k \cdot \lambda).
 \end{aligned}$$

Since  $\text{Ad}w_k \circ \rho(1, k)$  preserves the 2-norm, we get  $\|a_\lambda\|_2 = \|a_{k^{-1} \cdot \lambda}\|_2$ . Since  $\|a\|_2^2 = \sum \|a_\lambda\|_2^2 < \infty$  and the set  $\{a_{k^{-1} \cdot \lambda} \mid k \in \mathbf{Z}^2\}$  is infinite for  $\lambda \neq 0$ , it turns out that  $a_\lambda = 0$  for  $\lambda \neq 0$  and thus  $a \in \tilde{N} \cap (M \otimes \mathbf{C}) = N(H, \mu) \otimes \mathbf{C}$ . By the ergodicity of the  $\mathbf{Z}^2$ -action  $\{\text{Ad}w_k \circ \beta(k)\}$ , we get  $a \in \mathbf{C}$ . We conclude that the weak 1-cocycle  $\{W_\gamma\} \subset \tilde{N}$  is ergodic on  $\mathbf{Z}^2$ .

By the relative property (T) for the inclusion  $\mathbf{Z}^2 \subset \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  and Proposition 6.5, there exists a non-zero element  $\tilde{x}_0 \in \tilde{N}$  satisfying

$$(w_k \otimes 1)\tilde{\beta}(k)(\tilde{x}_0)(1 \otimes w_k^*) = \tilde{x}_0, \quad k \in \mathbf{Z}^2.$$

The element  $\tilde{x}_0$  can be written as the following Fourier expansion:

$$\tilde{x}_0 = \sum c(\lambda_1, \lambda_2) u(\lambda_1) \otimes u(\lambda_2) \in \tilde{N} \subset L_\mu\left(\bigoplus_{\mathbf{Z}^2} H\right) \otimes L_\mu\left(\bigoplus_{\mathbf{Z}^2} H\right).$$

Here  $c(\lambda_1, \lambda_2)$  is a complex number and  $(\lambda_1, \lambda_2) \in \left(\bigoplus_{\mathbf{Z}^2} H\right)^2$  runs through all pairs satisfying  $\sum_{k \in \mathbf{Z}^2} (\lambda_1(k) + \lambda_2(k)) = 0$ . Choose and fix a pair  $(\lambda_1, \lambda_2)$  satisfying

$$-\sum_{k \in \mathbf{Z}^2} \lambda_1(k) = h = \sum_{k \in \mathbf{Z}^2} \lambda_2(k), \quad c(\lambda_1, \lambda_2) \neq 0.$$

Let  $v'_h \in M$  be the unitary written as  $v'_h = u_h \otimes 1 \otimes 1 \otimes \cdots$ , where  $u_h \in L_\mu(H)$  is placed on  $0 \in \mathbf{Z}^2$ . The following unitaries  $\{w'_\gamma\} \subset N(H, \mu)$  give a weak 1-cocycle for  $\beta$ :

$$w'_{(k, \gamma_0)} = v'_h w_{(k, \gamma_0)} \rho(1, k, \gamma_0)(v'_h)^*, \quad (k, \gamma_0) \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z}).$$

Letting  $\tilde{y} = (v'_h \otimes 1)\tilde{x}_0(1 \otimes v'_h)^* \in \tilde{M}$ , we get

$$\tilde{y} = (w'_k \otimes 1)\tilde{\beta}(k)(\tilde{y})(1 \otimes w'_k)^*, \quad k \in \mathbf{Z}^2.$$

Applying the trace preserving conditional expectation  $E = E_{N(H, \mu) \otimes N(H, \mu)}$ , we get

$$\begin{aligned} E(\tilde{y}) &= (w'_k \otimes 1)E(\tilde{\beta}(k)(\tilde{y}))(1 \otimes w'_k)^* \\ &= (w'_k \otimes 1)\tilde{\beta}(k)(E(\tilde{y}))(1 \otimes w'_k)^*, \quad k \in \mathbf{Z}^2. \end{aligned}$$

Since the Fourier coefficient of  $\tilde{x}_0$  at  $(\lambda_1, \lambda_2) \in (\bigoplus_{\mathbf{Z}^2} H)^2$  is not zero, that of  $E(\tilde{y})$  at  $(\lambda_1 + \delta_{h,0}, \lambda_2 - \delta_{h,0}) \in \Lambda(H)^2$  is not zero, where  $\delta_{h,0} \in \bigoplus_{\mathbf{Z}^2} H$  is zero on  $\mathbf{Z}^2 \setminus \{0\}$  and is  $h$  on  $0 \in \mathbf{Z}^2$ . By Proposition 6.2, it follows that the weak 1-cocycle  $\{w'_{(k,e)}\}_{k \in \mathbf{Z}^2} \subset N(H, \mu)$  is a weak 1-coboundary of  $\beta|_{\mathbf{Z}^2}$ . Since the  $\mathbf{Z}^2$ -action  $\beta|_{\mathbf{Z}^2}$  is weakly mixing,  $w'$  is a weak 1-coboundary on  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ , by Proposition 6.3. In other words, there exists  $v \in N(H, \mu)$  satisfying

$$\begin{aligned} w'_\gamma &= v\beta(\gamma)(v^*) \quad \text{mod } \mathbf{T}, \\ w_\gamma &= v'_h{}^* v\rho(1, \gamma)(v^* v'_h) \quad \text{mod } \mathbf{T}, \quad \gamma \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z}). \end{aligned}$$

Noting that  $u = v^* v'_h \in M$  is a normalizer of  $N(H, \mu)$ , we get

$$\begin{aligned} (\text{Ad}(u) \circ \theta) \circ \beta^{(1)}(\gamma) &= \text{Ad}(u) \circ \text{Ad}(w_\gamma) \circ \beta(\gamma) \circ \theta \\ &= \text{Ad}(\rho(1, \gamma)(u)) \circ \beta(\gamma) \circ \theta \\ &= \rho(1, \gamma) \circ \text{Ad}(u) \circ \theta \\ &= \beta(\gamma) \circ (\text{Ad}(u) \circ \theta), \quad \gamma \in \mathbf{Z}^2 \rtimes SL(2, \mathbf{Z}). \end{aligned}$$

Thus we get the conjugacy of two  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ -actions  $\beta^{(0)}$  and  $\beta$ .  $\square$

We can always apply Theorem 6.6 if  $H$  is finite.

**COROLLARY 6.7.** *Let  $H$  be a finite abelian group and let  $\beta = \beta(H, \mu, \chi)$  be a twisted Bernoulli shift action on  $N(H, \mu)$ . Suppose that  $N(H, \mu)$  is the AFD factor of type  $\text{II}_1$ . Let  $\beta^{(1)}$  be an action of  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$  on the AFD factor  $N^{(1)}$  of type  $\text{II}_1$  and suppose that its restriction to  $\mathbf{Z}^2$  is ergodic. The action  $\beta^{(1)}$  is outer conjugate to  $\beta$ , if and only if  $\beta^{(1)}$  is conjugate to  $\beta$ .*

**PROOF.** We have only to construct an  $\mathbf{R}$ -action on  $L_\mu(H) \otimes L_\mu(H)$  satisfying the properties in Theorem 6.6. Let  $U$  be an element of  $L_\mu(H) \otimes L_\mu(H)$  defined by

$$U = \frac{1}{|H|^{1/2}} \sum_{h \in H} u_h \otimes u_h^*.$$

We note that  $\mu^* \mu(g, \cdot)$  is a character of  $H$  and that it is not identically 1 provided  $g \neq 0$  by Lemma 5.5. The element  $U$  is self-adjoint and unitary, since

$$\begin{aligned}
 U^* &= \frac{1}{|H|^{1/2}} \sum_{h \in H} u_h^* \otimes u_h = \frac{1}{|H|^{1/2}} \sum_{h \in H} \overline{\mu(h, -h)} u_{-h} \otimes \mu(h, -h) u_{-h}^* = U, \\
 U^2 &= \frac{1}{|H|} \sum_{g, h \in H} u_g u_h \otimes u_g^* u_h^* = \frac{1}{|H|} \sum_{g, h \in H} \mu^* \mu(g, h) u_{g+h} \otimes u_{g+h}^* \\
 &= \frac{1}{|H|} \sum_{g \in H} \left( \sum_{h \in H} \mu^* \mu(g, h - g) \right) u_g \otimes u_g^* = 1.
 \end{aligned}$$

The operator  $U$  is a fixed point under the action of  $\widehat{H}$ , so the projections  $P_1 = (1 + U)/2$  and  $P_{-1} = (1 - U)/2$  are also fixed points. Thus the  $\mathbf{R}$ -action  $\alpha_t^{(0)} = \text{Ad}(P_1 + \exp(i\pi t)P_{-1})$  commutes with the  $\widehat{H}$ -action. The automorphism  $\alpha_1^{(0)}$  satisfies

$$\alpha_1^{(0)}(u_g \otimes 1) = U(u_g \otimes 1)U^* = (1 \otimes u_g)UU^* = 1 \otimes u_g, \quad g \in H.$$

This verifies the first condition for  $\alpha^{(0)}$ . □

**COROLLARY 6.8.** *Let  $Q$  be a set consisting of odd prime numbers and  $\beta_Q$  be the twisted Bernoulli shift action defined in Corollary 5.9. Let  $\beta$  be a  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ -action on the AFD factor of type  $\text{II}_1$  whose restriction to  $\mathbf{Z}^2$  is ergodic. The actions  $\beta_Q$  and  $\beta$  are outer conjugate if and only if they are conjugate. In particular,  $\{\beta_Q\}$  is an uncountable family of  $\mathbf{Z}^2 \rtimes SL(2, \mathbf{Z})$ -actions which are mutually non outer conjugate.*

**PROOF.** We will use the notation given in Corollary 5.8 and 5.9. Let  $\alpha_t^{(q)}$  be the  $\mathbf{R}$ -action on  $L_{\mu_q}(H_q) \otimes L_{\mu_q}(H_q)$  constructed as in the previous corollary. We define the  $\mathbf{R}$ -action  $\alpha^{(Q)}$  on  $L_{\mu_Q}(H_Q) \otimes L_{\mu_Q}(H_Q)$  by  $\alpha_t^{(Q)}(\bigotimes_{q \in Q} x_q) = \bigotimes_{q \in Q} \alpha_t^{(q)}(x_q)$ , where  $x_q \in L_{\mu_q}(H_q) \otimes L_{\mu_q}(H_q)$  and  $x_q \neq 1$  only for finitely many  $q$ . The  $\mathbf{R}$ -action satisfies the conditions in Theorem 6.6. By Corollary 5.9,  $\{\beta_Q\}$  are mutually non conjugate and their restriction to  $\mathbf{Z}^2$  is ergodic. Thus they are mutually non outer conjugate. □

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## References

- [Bu] M. Burger, Kazhdan constants for  $SL_3(\mathbf{Z})$ , *J. Reine Angew. Math.*, **413** (1991), 36–67.
- [Ch] M. Choda, A continuum of non-conjugate property T actions of  $SL(n, \mathbf{Z})$  on the hyperfinite  $\text{II}_1$ -factor, *Math. Japon.*, **30** (1985), 133–150.
- [Co] A. Connes, Classification of injective factors. Cases  $\text{II}_1$ ,  $\text{II}_\infty$ ,  $\text{III}_\lambda$ ,  $\lambda \neq 1$ , *Ann. of Math. (2)*, **104** (1976), 73–115.
- [Jol] P. Jolissaint, On property (T) for pairs of topological groups, *Enseign. Math. (2)*, **51** (2005), 31–45.
- [Jon] V. F. R. Jones, A converse to Ocneanu’s theorem, *J. Operator Theory*, **10** (1983), 61–63.
- [Ka] Y. Kawahigashi, Cohomology of actions of discrete groups on factors of type  $\text{II}_1$ , *Pac. J. Math.*, **149** (1991), 303–317.
- [NPS] R. Nicoara, S. Popa and R. Sasyk, On  $\text{II}_1$  factors arising from 2-cocycles of  $w$ -rigid groups, *J. Funct. Anal.*, **242** (2007), 230–246.
- [Oc] A. Ocneanu, Actions of discrete amenable groups on von Neumann algebras, *Lecture Notes in Math.*, **1138**, Springer-Verlag, Berlin, 1985.
- [OPT] D. Olesen, G. K. Pedersen and M. Takesaki, Ergodic actions of compact abelian groups, *J. Operator Theory*, **3** (1980), 237–269.
- [Po1] S. Popa, On a class of type  $\text{II}_1$  factors with Betti numbers invariants, *Ann. of Math. (2)*, **163** (2006), 809–899.
- [Po2] S. Popa, Some computations of 1-cohomology groups and construction of non orbit equivalent actions, *J. Inst. Math. Jussieu*, **5** (2006), 309–332.
- [Po3] S. Popa, Some rigidity results for non-commutative Bernoulli shifts, *J. Funct. Anal.*, **230** (2006), 273–328.
- [Po4] S. Popa, Cocycle and orbit equivalence superrigidity for malleable actions of  $w$ -rigid groups, *Invent. Math.*, **170** (2007), 243–295.
- [PoSa] S. Popa and R. Sasyk, On the cohomology of Bernoulli actions, *Erg. Theory Dyn. Sys.*, **26** (2007), 1–11.
- [Sh] Y. Shalom, Bounded generation and Kazhdan’s property (T), *Inst. Hautes. Études. Sci. Publ. Math.*, **90** (1999), 145–168.
- [Vaes] S. Vaes, Rigidity results for Bernoulli actions and their von Neumann algebras (after Sorin Popa), *Séminaire Bourbaki*, exp. no. 961, *Astérisque*, **311** (2007), pp.237–294.

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