J. Math. Soc. Japan
Vol. 61, No. 4 (2009) pp. 1243–1260
doi: 10.2969/jmsj/06141243

On a construction of the twistor spaces of Joyce metrics, II

By Nobuhiro Honda

(Received Sep. 29, 2008)

Abstract. In this note, we explicitly construct the twistor spaces of some Joyce metrics on the connected sum of arbitrary number of complex projective planes. Unlike our former construction for the case of four complex projective planes, the present construction mainly utilizes minitwistor spaces, and partially follows the method and construction given in [**5**] and [**6**].

1. Introduction.

In a recent paper [**6**], we have given a systematic method for obtaining numerous Moishezon twistor spaces admitting $\boldsymbol{C}^*$-actions. There, a key geometric object was *minitwistor spaces* associated to the twistor spaces of Joyce metrics [**8**]. More precisely, for an arbitrary Joyce metric on $n\boldsymbol{CP}^2$ and an arbitrary $U(1)$-subgroup of the torus that fixes a torus-invariant 2-sphere in $n\boldsymbol{CP}^2$, we concretely find a linear system on the twistor space (of Joyce metrics) whose associated meromorphic map can be regarded as a quotient map of the $\boldsymbol{C}^*$-action corresponding to the $U(1)$-action. The quotient spaces, which are necessarily rational surfaces, are called the minitwistor spaces. We explicitly determined defining equations of these minitwistor spaces in projective spaces, and realized projective models of the twistor spaces as conic bundles over (the minimal resolution of) the minitwistor spaces. Also, they played a main role in the study of equivariant deformations of the twistor spaces.

As we remarked in [**6**, Section 3.1], when we try to obtain the actual twistor spaces from these projective models by means of blowing-ups and downs, we face a difficulty which comes from a complexity of the base locus of the above linear system. In this note, we find a particular case for which we can give an explicit sequence of blowing-ups that eliminates the base locus completely, and consequently obtain an explicit construction of the twistor spaces of Joyce metrics on $n\boldsymbol{CP}^2$ for arbitrary $n \geq 4$. Although the number of torus-actions we

---

2000 *Mathematics Subject Classification.* Primary 32L25; Secondary 32G05, 32G07, 53A30, 53C25.

Key Words and Phrases. twistor space, minitwistor space, Moishezon manifold, conic bundle, bimeromorphic transformation, self-dual metric.

This work was partially supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

can take is only one for each n, this seems to be the first construction which works for arbitrary n. When $n = 4$, the present result gives a new construction of the twistor spaces of Joyce metrics on $4\boldsymbol{CP}^2$ whose K-action is Type I in the terminology of [**4**].

We remark that the present method is a modification of the construction in [**5**], where we constructed Moishezon twistor spaces with $\boldsymbol{C}^*$-action whose minitwistor space are the same as the present ones. In other words, the twistor spaces and their construction in this paper can be obtained as a limit of the twistor spaces and their construction in [**5**]. However, in the present case we require much more complicated operations than those in [**5**], as is displayed in the figures.

For a related work, we mention that in [**7**], we have given projective models of the twistor spaces for arbitrary Joyce metrics on $n\boldsymbol{CP}^2$.

Notation. If Z is a twistor space, F always denotes the canonical square root of the anticanonical line bundle of Z. The degree of a divisor on Z means its intersection number with twistor lines. The 2-dimensional Lie group $U(1) \times U(1)$ and its complexification $\boldsymbol{C}^* \times \boldsymbol{C}^*$ are denoted by K and G respectively. If a Lie group G is acting on Z holomorphically and D is a G-invariant divisor, G naturally acts on the vector space $H^0(Z, [D])$. Then $H^0(Z, [D])^G$ means the subspace of all G-invariant sections. If V is a non-zero vector subspace in $H^0(Z, [D])$, $|V|$ implies a linear system whose members are zero divisors of $s \in V$. Z^G means the set of G-fixed points. A $(-1,-1)$-curve in a threefold means a smooth rational curve whose normal bundle is isomorphic to $\mathscr{O}(-1)^{\oplus 2}$. The Hirzebruch surface $\boldsymbol{P}(\mathscr{O}(k) \oplus \mathscr{O})$ is denoted by Σ_k.

2. Specifying a K-action and construction of projective models.

Joyce metrics on $n\boldsymbol{CP}^2$ are determined by an effective K-action on $n\boldsymbol{CP}^2$ and a set of $(n+2)$ real numbers. We first specify the former K-action we shall consider. We start from an affine plane $\boldsymbol{C}^2 = \{(z, w)\}$ equipped with a K-action given by $(z, w) \mapsto (sz, tw)$ for $(s, t) \in K$. Let $n \geq 4$ be any integer. Then we blow-up $\boldsymbol{C}^2$ $(n-1)$ times, with the blown-up points always on the unique K-fixed point on (the proper transform of) z-axis. The resulting surface has a unique (-1)-curve. Among two K-fixed points on this curve, we blow-up the one which is not on the proper transform of z-axis. Then the K-action on the resulting surface has $(n+1)$ fixed points. By taking a natural one-point compactification and reversing the orientation, we obtain $n\boldsymbol{CP}^2$ equipped with an effective K-action. This is the K-action we shall consider. Note that if we take another K-fixed point in the final blow-up, the K-action on $n\boldsymbol{CP}^2$ contains a $U(1)$-subgroup acting semi-freely.

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_{n+2}$ be the set of real numbers and g the Joyce metric on $n\boldsymbol{CP}^2$ which has the above K-action as an automorphism group and which has $\{\lambda_1, \cdots, \lambda_{n+2}\}$ as its conformal invariant. Let Z be the twistor space of g. Since g is different from LeBrun metrics, we have $\dim |F| = 1$. Let $S \in |F|$ be a smooth member. S is a toric surface whose structure is uniquely determined by the K-action. Let C be the unique anticanonical curve on S. C consists of $2(n+2)$ rational curves $C_1, C_2, \cdots, C_{n+2}, \overline{C}_1, \overline{C}_2, \cdots, \overline{C}_{n+2}$ which form a cycle arranged in this order. Here $\overline{C}_i$ are the images of C_i under the real structure of Z. By the above blow-ups, we can suppose that

$$C_1^2 = 1-n,\ C_2^2 = -2,\ C_3^2 = -1,\ C_4^2 = -3,\ C_5^2 = \cdots = C_{n+1}^2 = -2,\ C_{n+2}^2 = -1, \quad (1)$$

for the self-intersection numbers in S. We note that when $n = 4$, this K-action coincides with the one called Type I in [**4**]. By the twistor fibration $Z \to n\boldsymbol{CP}^2$, C_1 and $\overline{C}_1$ are mapped to the closure of z-axis, and C_{n+2} and $\overline{C}_{n+2}$ are mapped to the closure of w-axis.

Next let $G_1 \subset G$ be the isotropy subgroup of C_1 and $S \to \boldsymbol{CP}^1$ the (holomorphic) quotient map of the G_1-action. The latter has exactly two reducible fibers and they are explicitly given by

$$f = C_2 + 2C_3 + \sum_{4 \le i \le n+2} C_i \quad (2)$$

and its conjugation. In particular, $G_1 \simeq \boldsymbol{C}^*$ acts on C_i by weight one for $i \neq 3$ (and $i \neq 1$) and by weight 2 on C_3. Thus the sequence $(k_2, k_3, k_4, \cdots, k_{n+2})$ obtained by arranging the coefficients of the reducible fiber f is given by $(1, 2, 1, \cdots, 1)$. Hence the number m defined in [**6**, Definition 2.2] is computed to be 2. By [**6**, Proposition 2.5], the system $|2F| = |-K|$ contains a member Y and $\overline{Y}$ which are not in the subsystem $|V_2|$ composed of a pencil $|F|$. By the formula (9) in [**6**], they are explicitly given by

$$Y = S_1^+ + S_2^+ + S_3^- + S_{n+2}^- \quad (3)$$

and its conjugation, where $S_i^+ + S_i^-$ $(1 \le i \le n+2)$ are reducible members of $|F|$ having the property that $L_i := S_i^+ \cap S_i^-$ is the G-invariant twistor line through $C_i \cap C_{i+1}$ for $1 \le i \le n+1$ and $C_{n+2} \cap \overline{C}_1$ for $i = n+2$. Here, S_i^+ and S_i^- are distinguished by imposing $C_1 \subset S_i^-$. The 2-dimensional system $|V_2|$ and Y and $\overline{Y}$ generate a 4-dimensional subsystem of $|2F|$ and it coincides with $|2F|^{G_1}$ by Proposition 2.11 in [**6**]. Let $\Phi_2^{G_1} : Z \to \boldsymbol{CP}^4$ be the associated meromorphic map and $\mathscr{T} = \Phi_2^{G_1}(Z)$ the image surface which is a minitwistor space of Z with respect to G_1 ([**6**, Definition 2.9]). Then we have the following commutative diagram of meromorphic maps

$$\begin{array}{ccc} Z & \xrightarrow{\Phi_2^{G_1}} & \boldsymbol{P}^\vee H^0(Z, 2F)^{G_1} \\ {\scriptstyle\Psi_2}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ \Lambda_2 & \longrightarrow & \boldsymbol{P}^\vee V_2, \end{array} \tag{4}$$

where Ψ_2 is the meromorphic map onto the conic $\Lambda_2 \simeq \boldsymbol{CP}^1$ in $\boldsymbol{CP}^2$ associated to $|V_2|$, π_2 is the linear projection induced from $V_2 \subset H^0(2F)^{G_1}$ and $\Lambda_2 \to \boldsymbol{P}^\vee V_2$ is an embedding as a conic. The restriction of π_2 to $\mathscr{T}$ is still denoted by π_2. We can suppose that the conformal invariant $\{\lambda_1, \cdots, \lambda_{n+2}\}$ satisfies $\lambda_i = \Psi_2(S_i^+) = \Psi_2(S_i^-)$. By Proposition 2.12 and 2.14 of [**6**], we have the following.

PROPOSITION 2.1. *The minitwistor space $\mathscr{T}$ satisfies the following.* (i) *The indeterminacy locus of the projection $\pi_2 : \mathscr{T}_2 \to \Lambda_2$ consists of two points.* (ii) *These two points coincide with the singular locus of the surface $\mathscr{T}$.* (iii) *π_2 has reducible fibers precisely over the 4 points λ_i, $i = 1, 2, 3, n+2$, and all of them consist of two lines.*

Let $\tilde{\mathscr{T}}$ be the minimal resolution of $\mathscr{T}$, Γ and $\overline{\Gamma}$ the exceptional curves, and $\tilde{\pi}_2$ the composition $\tilde{\mathscr{T}} \to \mathscr{T} \to \Lambda_2$. Γ and $\overline{\Gamma}$ are sections of $\tilde{\pi}_2$. As an abstract complex surface, $\tilde{\mathscr{T}}$ is obtained as 4-points blow-up of $\Sigma_2 = \boldsymbol{P}(\mathscr{O}(-2) \oplus \mathscr{O})$ over Λ_2, where the 4 points are lying on the $(+2)$-section $\boldsymbol{P}(\mathscr{O}(-2))$ and over the 4 points λ_i with $i = 1, 2, 3, n+2$.

The structure of $\tilde{\mathscr{T}}$ is as in Figure 1, where the irreducible components s_i^+ and s_i^- of reducible fibers are named after the fact that they are the images of S_i^+ and S_i^- under $\Phi_2^{G_1}$ respectively (cf. [**6**, Section 3.1]), and f_i, $4 \le i \le n+1$, are the images of S_i^+ and S_i^- under the same map. In the following we explicitly give a $\boldsymbol{CP}^2$-bundle $\boldsymbol{P}(\mathscr{E}) \to \tilde{\mathscr{T}}$ and a conic bundle $X \to \tilde{\mathscr{T}}$ in $\boldsymbol{P}(\mathscr{E})$. For this, we define two holomorphic line bundles $\mathscr{N}^\vee$ and $\overline{\mathscr{N}}^\vee$ by

$$\mathscr{N}^\vee = \mathscr{O}(\overline{\Gamma} + (3-n)s_{n+2}^+ + s_2^- + (n-2)\mathfrak{f}) \tag{5}$$

and

$$\overline{\mathscr{N}}^\vee = \mathscr{O}(\Gamma + (3-n)s_{n+2}^- + s_2^+ + (n-2)\mathfrak{f}) \tag{6}$$

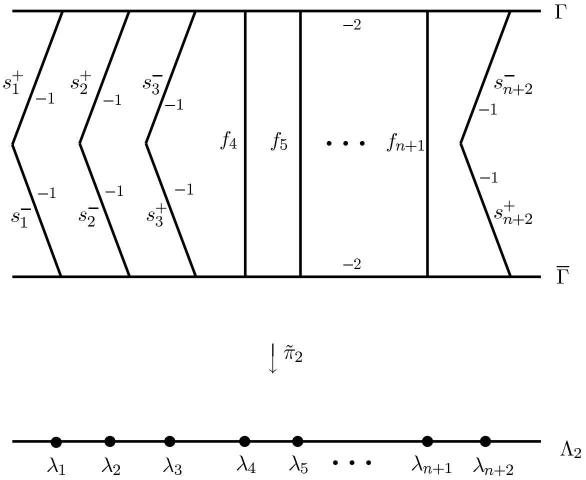

Figure 1. The structure of the resolved minitwistor space $\tilde{\mathscr{T}}$.

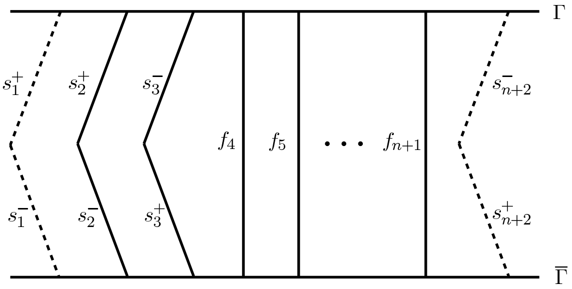

Figure 2. The discriminant curve of the conic bundle $X \to \tilde{\mathscr{T}}$.

where $\mathfrak{f}$ denotes the fiber class of $\tilde{\pi}_2$. Then we define a rank-3 bundle over $\tilde{\mathscr{T}}$ by

$$\mathscr{E} := \mathscr{N}^\vee \oplus \overline{\mathscr{N}}^\vee \oplus \mathscr{O}. \tag{7}$$

$\mathscr{E}$ is equipped with the natural real structure induced from that on $\tilde{\mathscr{T}}$. We readily have

$$\det \mathscr{E} \simeq \mathscr{O}(\Gamma + \overline{\Gamma} + n\mathfrak{f}) \tag{8}$$

which is also equipped with the natural real structure. As a real member of a linear system $|\det \mathscr{E}|$ we choose

$$\Gamma + \overline{\Gamma} + (s_2^+ + s_2^-) + (s_3^+ + s_3^-) + \sum_{4 \le i \le n+1} f_i. \tag{9}$$

(See Figure 2.) Let $P \in H^0(\tilde{\mathscr{T}}, \det \mathscr{E})$ be a real section that defines this curve. Then we define a conic bundle $p : X \to \tilde{\mathscr{T}}$ by

$$xy = Pz^2 \tag{10}$$

where $(x, y, z) \in \mathscr{E}$. The discriminant curve of p is exactly (9). Obviously, the inverse images of irreducible components of the discriminant curve consist of two irreducible components. For any singular point of the discriminant curve (9), there exists a unique ordinary double point over there of the 3-dimensional space X (given by $(x, y, z) = (0, 0, 1)$). There are no other singularities of X. This is the projective model we start with. We note that X is determined only by the conformal invariant $\{\lambda_1, \cdots, \lambda_{n+2}\}$.

The surface $\tilde{\mathscr{T}}$ has an obvious effective $\boldsymbol{C}^*$-action, which fixes Γ and $\overline{\Gamma}$. Hence $\boldsymbol{P}(\mathscr{E})$ admits an effective G-action as a combination of the $\boldsymbol{C}^*$-action on $\tilde{\mathscr{T}}$ and a $\boldsymbol{C}^*$-action on $\boldsymbol{P}(\mathscr{E})$ defined by $(x, y, z) \mapsto (tx, t^{-1}y, z)$ for $t \in \boldsymbol{C}^*$. p has two G-invariant distinguished sections

$$E_1 := \{x = z = 0\} \text{ and } \overline{E}_1 = \{y = z = 0\}. \tag{11}$$

Also, the two irreducible components of $p^{-1}(\Gamma)$ are G-invariant. We name these as E_2 and E_4, where E_2 is the one intersecting E_1. Then the conjugate divisors $\overline{E}_2$ and $\overline{E}_4$ are irreducible components of $p^{-1}(\overline{\Gamma})$ intersecting $\overline{E}_1$ and E_1 respectively. Thus we obtain 6 irreducible divisors $E_i, \overline{E}_i$ for $i = 1, 2, 4$. For each of these divisors, a $\boldsymbol{C}^*$-subgroup of G is acting trivially. (E_i will be contracted to C_i of the G-invariant cycle C in Z through our construction.)

Let $q : X \to \Lambda_2$ be the composition of the two morphisms $p : X \to \tilde{\mathscr{T}}$ and $\tilde{\pi}_2 : \tilde{\mathscr{T}} \to \Lambda_2$. Any fiber of q is G-invariant. (q will correspond to Ψ_2.) If $\lambda \neq \lambda_i$ for $1 \le i \le n+2$, $q^{-1}(\lambda)$ is a smooth toric surface. The intersection $q^{-1}(\lambda) \cap (E_1 + E_2 + E_4 + \overline{E}_1 + \overline{E}_2 + \overline{E}_4)$ is the unique G-invariant anticanonical cycle on $q^{-1}(\lambda)$. If $\lambda = \lambda_i$ for some $1 \le i \le n+2$, $q^{-1}(\lambda)$ consists of 2 or 4 irreducible components; if $i = 2, 3$, it consists of 4 components and otherwise 2 components. Note that $q^{-1}(\lambda_i)$ are mutually isomorphic for $4 \le i \le n+1$. We also note that all irreducible components of $q^{-1}(\lambda_i)$ are smooth toric surfaces, and their structure can be readily determined by our explicit description. These are illustrated as in (a) of Figures 4–11. There, dotted points (appearing in (a) of Figures 6, 7, 9, 10, 11) are precisely the singular locus of the threefold X. Namely, X has singularities at points where four G-invariant smooth divisors meet, and all of them are ordinary double points. In this way we obtain a projective fiber space over $\Lambda_2 \simeq \boldsymbol{CP}^1$ whose fibers are toric surfaces.

3. Construction of the twistor spaces.

In this section, starting from the projective 3-fold X given in the previous section, we construct the twistor spaces of Joyce metrics whose K-action is the one we specified in the beginning of Section 2, by applying a number of blowing-ups and downs. All these operations are given in such a way that they preserve the G-action and the real structure.

Broadly speaking, there are two kinds of operations we are going to apply. One is a blowing-up along a section of $q : X \to \Lambda_2$. These operations of course affect every fibers of p. The other is a blowing-up or down inside a singular fiber of p, which does not affect other fibers. (In particular, we do not make a blow-up or down inside a smooth fiber of q.) In the following we first give a sequence of blow-ups along sections of q (1° below), and next give sequences of blowing-ups and downs for each reducible fibers of q (2°–8° below). Any of the latter sequences involve the former sequence as a subsequence.

1° Blowing-ups along G-invariant sections of q.

First we blow-up $E_2 \cap E_4$ and $\overline{E}_2 \cap \overline{E}_4$. Let E_3 and $\overline{E}_3$ be the exceptional divisors respectively. Next we blow-up $E_1 \cap \overline{E}_4$ and $\overline{E}_1 \cap E_4$, and let E_5 and $\overline{E}_5$ be the exceptional divisors respectively. Next we blow-up $E_1 \cap \overline{E}_5$ and $\overline{E}_1 \cap E_5$, and let E_6 and $\overline{E}_6$ be the exceptional divisors respectively. Repeat these blow-ups until obtaining the exceptional divisors E_{n+2} and $\overline{E}_{n+2}$. Under these blow-ups, smooth fibers of q are transformed as in Figure 3. By looking the self-intersection numbers of the irreducible components of the anti-canonical cycle, the last toric surface is isomorphic to the surface $S = \Psi_2^{-1}(\lambda)$ in Section 2 which is a smooth member of the pencil $|F|$.

2° Operations for the fibers over $\boldsymbol{\lambda_{n+2}}$ and $\boldsymbol{\lambda_1}$.

For these two reducible fibers, we do not make a blowing-up or down inside the fibers, with only exception in the following contractions. Namely after applying all the blow-ups in 1°, the two fibers are transformed as in Figures 4 and 5 respectively to become (reducible) toric surfaces (c). Then we contract bold $(-1,-1)$-curves inside the fibers which are denoted by the bold lines. Then the images of the $(-1,-1)$-curves become ordinary double points of the threefold represented by the dotted points in (d). After these operations, the fibers become isomorphic to the reducible members $\Psi_2^{-1}(\lambda_{n+2})$ and $\Psi_2^{-1}(\lambda_1)$ in $|F|$ respectively.

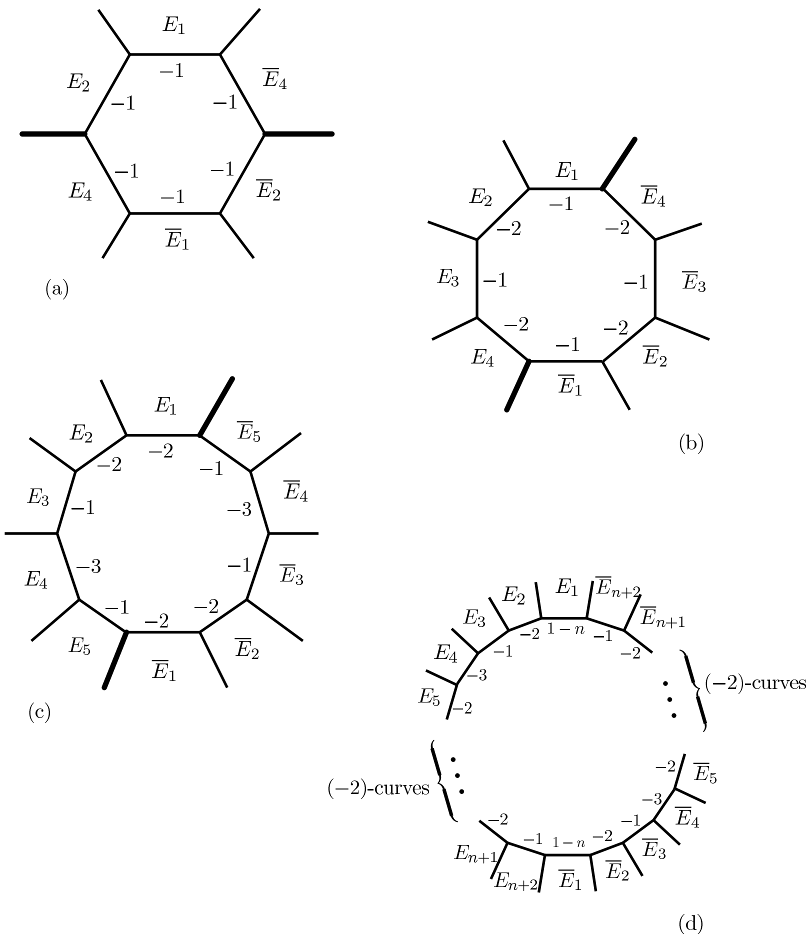

Figure 3. Changes of smooth fibers.

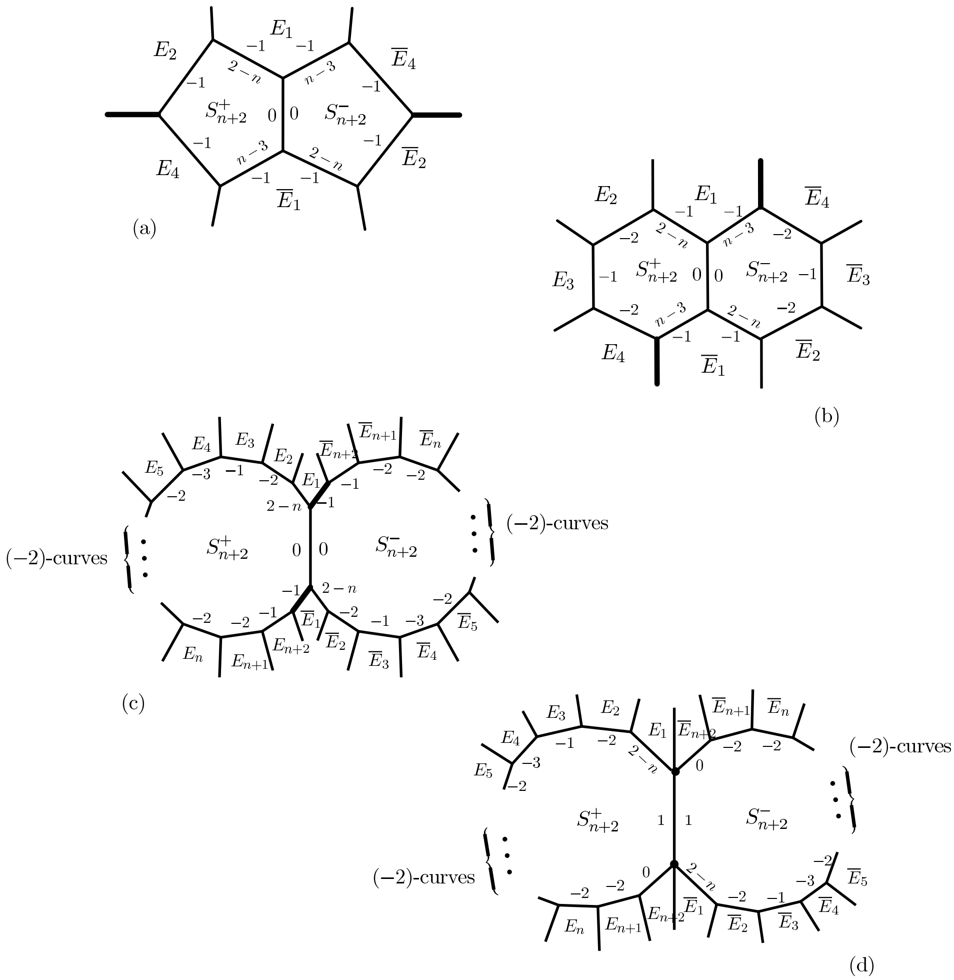

Figure 4. Changes of fibers over λ_{n+2}.

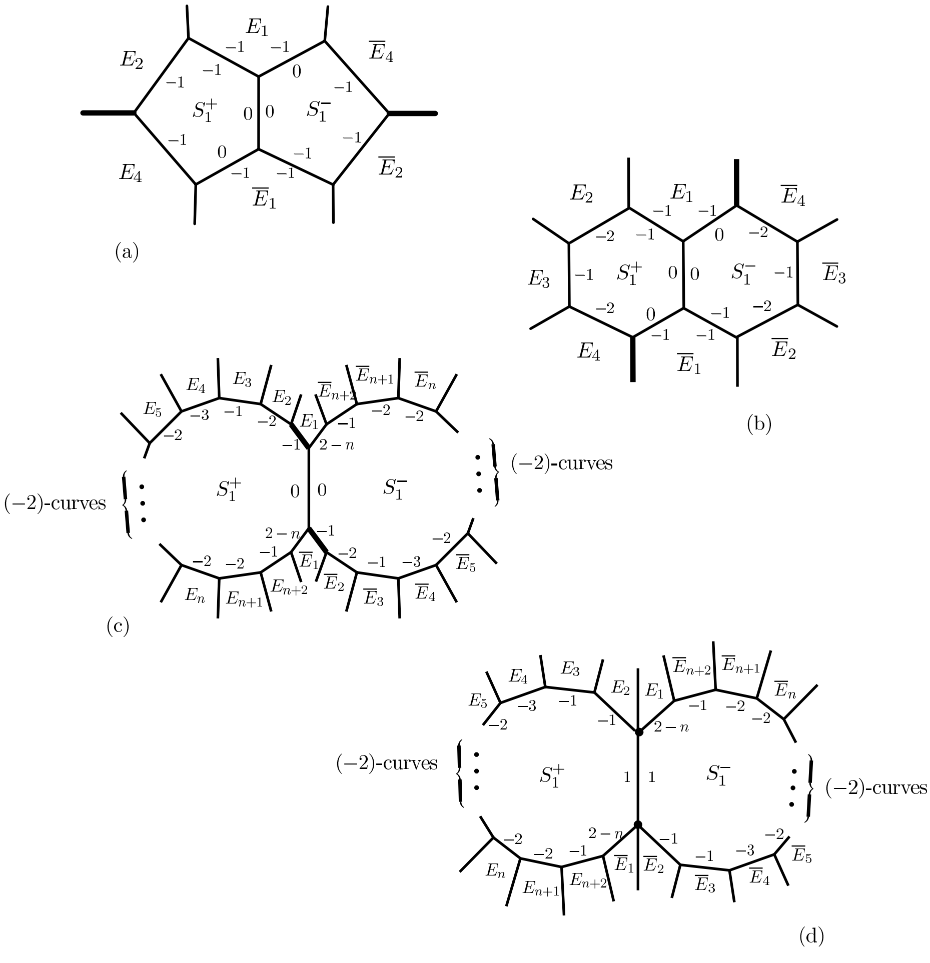

Figure 5. Changes of fibers over λ_1.

3° Operations for the fiber over $\boldsymbol{\lambda_2}$.

The fiber $q^{-1}(\lambda_2)$ consists of two Σ_1's (named S_2^+ and S_2^-) and two Σ_0's as in Figure 6 (a), and contains 3 ordinary double points of X. We take small resolutions that do not change two Σ_0's to obtain the situation displayed in (b) in the figure. Then we can blow-down two Σ_0's along both of the projections. We blow-down these in such a way that the divisors E_2 and $\overline{E}_2$ are not changed as in (c) in the figure. Next we apply all the blow-ups in 1° to obtain the (reducible) toric surface (e) in the figure. Finally we contract the two bold $(-1,-1)$-curves. The resulting surface is isomorphic to $\Psi_2^{-1}(\lambda_2) \in |F|$ in Z.

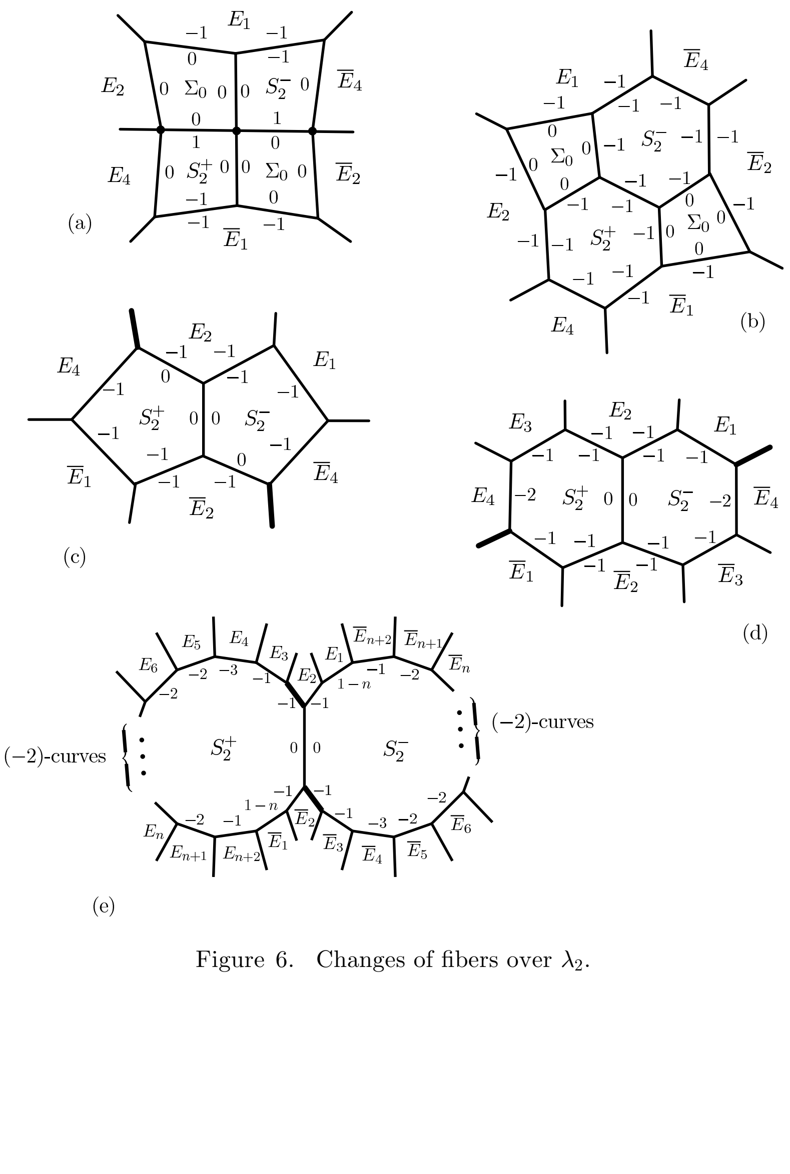

Figure 6. Changes of fibers over λ_2.

4° Operations for the fiber over λ_3.

The fiber $q^{-1}(\lambda_3)$ requires complicated operations. First, noting that $q^{-1}(\lambda_3)$ is isomorphic to $q^{-1}(\lambda_2)$, we apply the same small resolutions to obtain the situation displayed in Figure 7 (b). Next we insert the blow-ups of $E_2 \cap E_4, \overline{E}_2 \cap \overline{E}_4, E_1 \cap \overline{E}_4$ and $\overline{E}_1 \cap E_4$ of 1° to obtain (d). We subsequently blow-up the two bold curves in (d) to obtain Figure 8 (e). Then we contract the two bold curves that are $(-1,-1)$-curves in the threefold to obtain (f). The two dotted points are the resulting ordinary double points. We can interpret the operations from (c) to (f) as 'inserting two exceptional divisors (E_5 and one Σ_0) and their conjugations' and they can be replaced by a single operation of blowing-up along reducible

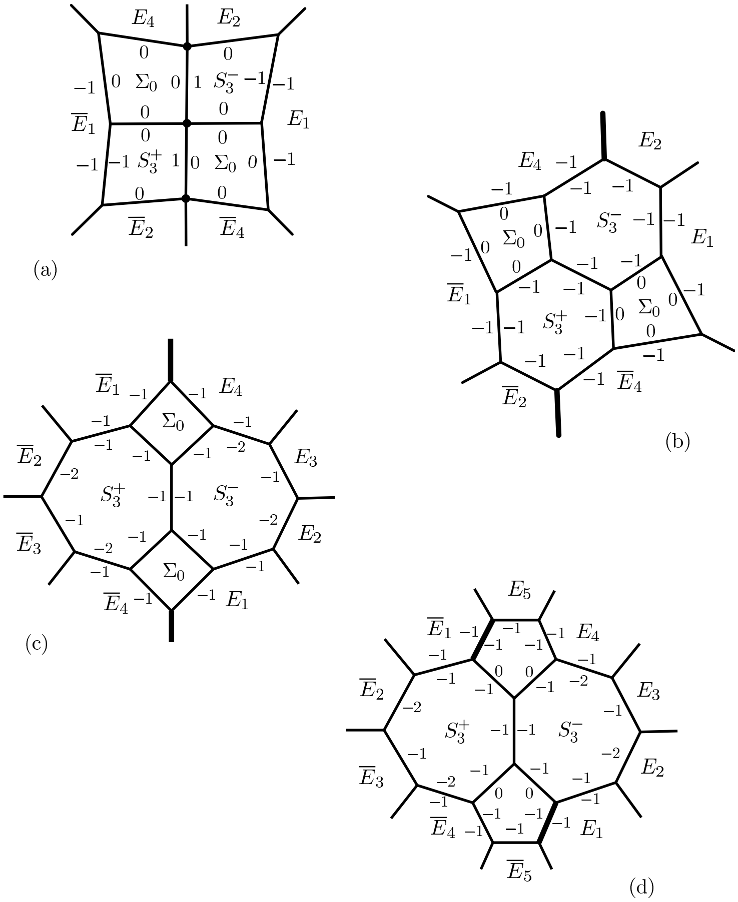

Figure 7. Changes of fibers over λ_3.

curves (which are the unions of the bold curves in (c) and (d)). We repeat this procedure until obtaining the exceptional divisors E_{n+2} and $\overline{E}_{n+2}$ as in (g). (All the dotted points are ordinary double points of the threefold.) Then all the squares in (g) are isomorphic to Σ_0 and can be simultaneously blow-down in such a way that their images are contained in the anticanonical cycles of S_4^+ and S_4^- as

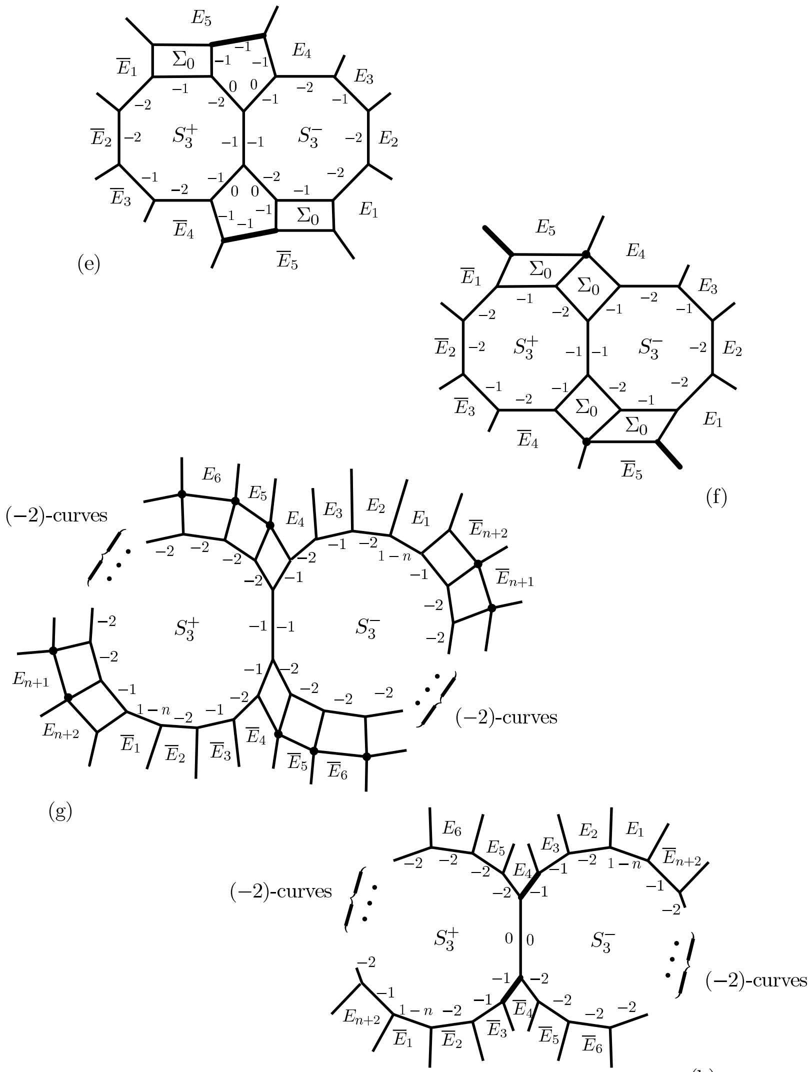

Figure 8. Changes of fibers over λ_3 (continued).

in (h). Finally, we contract the bold $(-1,-1)$-curves to ordinary double points. Then the resulting (reducible) toric surface is isomorphic to $\Psi_2^{-1}(\lambda_3) \in |F|$.

5° Operations for the fiber over $\boldsymbol{\lambda_4}$.

The fiber $q^{-1}(\lambda_4)$ consists of two irreducible components, both of which are isomorphic to Σ_1 that are mapped surjectively to $p^{-1}(\lambda_4) = f_4 \simeq \boldsymbol{CP}^1$ (Figure 9 (a)). These two components share two nodes of X as in the figure. We first take their small resolutions to obtain the situation displayed in (b). Then we subsequently blow-up G-invariant sections in the order specified in 1°, obtaining (d). Finally we contract a conjugate pair of bold $(-1,-1)$-curves. Then the resulting (reducible) toric surface is isomorphic to $\Psi_2^{-1}(\lambda_4)$.

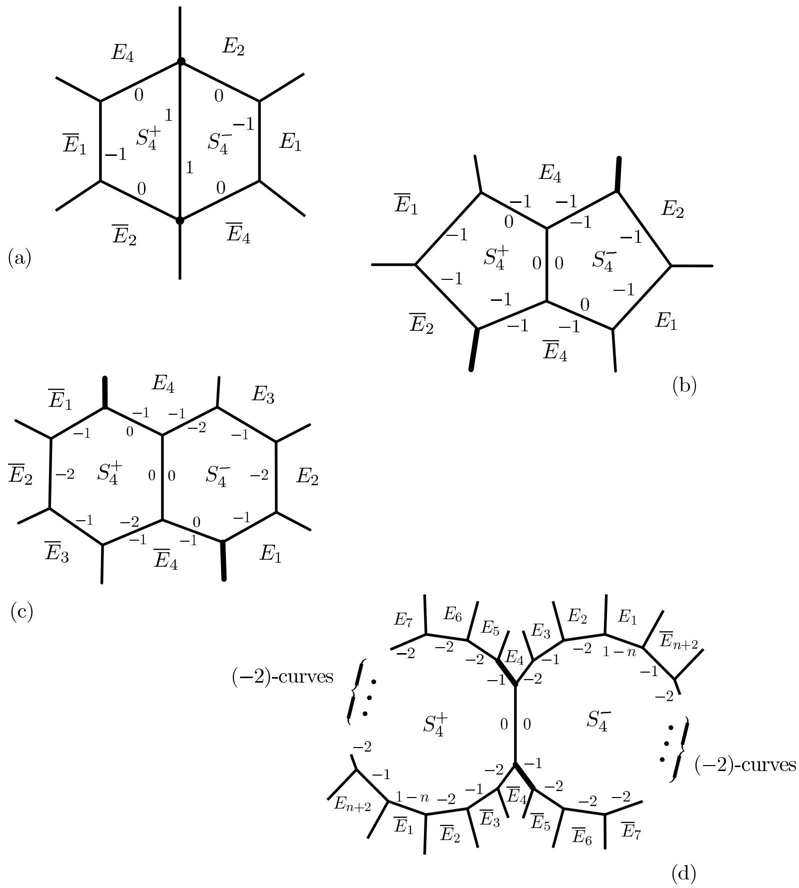

Figure 9. Changes of fibers over λ_4.

6° Operations for the fiber over λ_5.

The fiber $q^{-1}(\lambda_5)$ is isomorphic to $q^{-1}(\lambda_4)$ (Figure 10 (a)). We make the same operations until we obtain the situation (b) in Figure 10 (which is the same as (c) of Figure 9). Next we insert the blow-up at $\overline{E}_1 \cap E_4$ and $E_1 \cap \overline{E}_4$ in 1° to obtain (c). Then the intersections $E_4 \cap S_5^+$ and $\overline{E}_4 \cap S_5^-$ become $(-1,-1)$-curves (bold curves in (c)). We flop these two curves to obtain (d). After this process, we go back to the blow-ups in 1° to obtain the situation of (e). Finally we contract $E_5 \cap S_5^+$ and $\overline{E}_5 \cap S_5^-$ which are $(-1,-1)$-curves. Then the resulting (reducible)

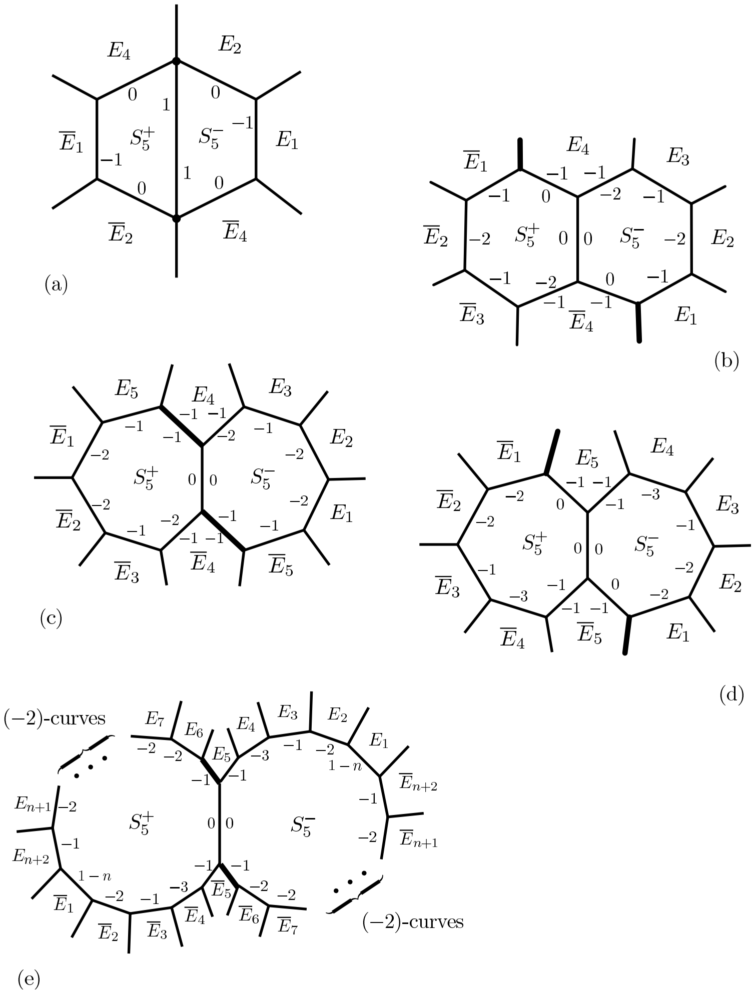

Figure 10. Changes of fibers over λ_5.

toric surface is isomorphic to $\Psi_2^{-1}(\lambda_5)$.

7° Operations for the fiber over $\boldsymbol{\lambda_6}$.

The fiber $q^{-1}(\lambda_6)$ is also isomorphic to $q^{-1}(\lambda_5)$ ((a) of Figure 11). We apply the same operation as $q^{-1}(\lambda_5)$ until we obtain the situation displayed in Figure 11 (b). Next we insert the blow-up at $\overline{E}_1 \cap E_5$ and $E_1 \cap \overline{E}_5$ in 1° to obtain (c). Then we apply flops at $E_5 \cap S_6^+$ and $\overline{E}_5 \cap S_6^-$ to obtain (d). Next we go back to the blow-

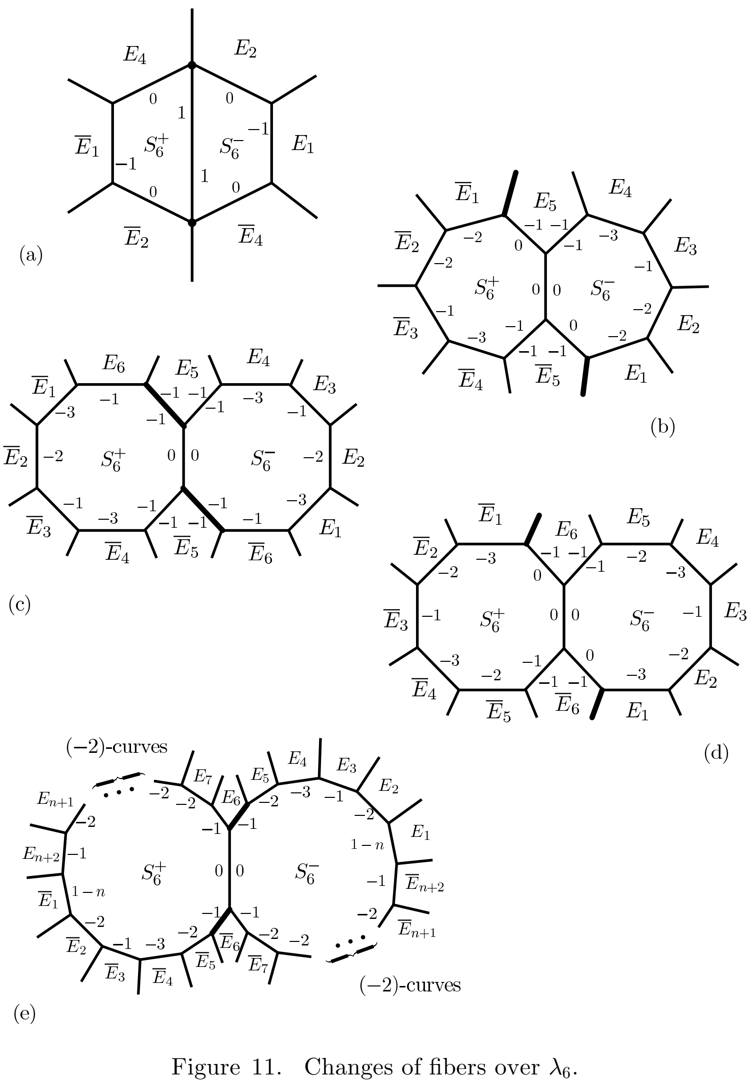

Figure 11. Changes of fibers over λ_6.

ups in 1° to obtain the situation of (e). Finally we contract $\overline{E}_6 \cap S_6^+$ and $E_6 \cap S_6^-$ which are $(-1,-1)$-curves. Then the resulting (reducible) toric surface is isomorphic to $\Psi_2^{-1}(\lambda_6)$. (So this fiber, we flop pairs of $(-1,-1)$-curves twice.)

8° Operations for the remaining fibers.

These are inductively given as follows. Let $5 \le i \le n$ be an integer and suppose that the operations for the previous fiber $q^{-1}(\lambda_{i-1})$ are already given. We may suppose that the number of times of flops for $q^{-1}(\lambda_{i-1})$ is $(i-5)$, up to conjugation. Noting that $q^{-1}(\lambda_i)$ is isomorphic to $q^{-1}(\lambda_{i-1})$ $(\simeq q^{-1}(\lambda_5))$, we first apply the same procedure as $q^{-1}(\lambda_{i-1})$ until just finishing the final flop. Next we insert the blow-up at $E_{i-1} \cap \overline{E}_1$ and $\overline{E}_{i-1} \cap E_1$. Then $E_{i-1} \cap S_i^+$ and $\overline{E}_{i-1} \cap S_i^-$ become $(-1,-1)$-curves. So we flop these curves. Then we go back to the blow-ups along sections of q in 1°. After this, we contract two $(-1,-1)$-curves $S_i^+ \cap \overline{E}_i$ and $S_i^- \cap E_i$ to ordinary double points. Then the resulting (reducible) toric surface is isomorphic to $\Psi_2^{-1}(\lambda_i) \in |F|$ in Z. Also, the number of flops we have applied is clearly $(i-5)+1 = i-4$. So the induction works to give operations for any $5 \le i \le n+1$.

9° Contracting the union $(\cup E_i) \cup (\cup \overline{E}_i)$.

Let $\hat{Z}'$ be the 3-fold obtained as a result of all the operations in 1°–8°, and $\hat{q} : \hat{Z}' \to \Lambda_2$ the natural projection obtained from the original projection $q : X \to \Lambda_2$. $\hat{Z}'$ is equipped with a natural G-action induced by that on X, as well as a real structure. As is already verified, any fiber $\hat{q}^{-1}(\lambda)$ is isomorphic to $\Psi_2^{-1}(\lambda) \in |F|$, where $\Psi_2 : Z \to \Lambda_2$ is the meromorphic map associated to the pencil $|F|$ on Z as before. On each reducible fibers $\hat{q}^{-1}(\lambda_i)$, $1 \le i \le n+2$, $\hat{Z}'$ has two ordinary double points. $\hat{Z}'$ contains $2(n+2)$ divisors E_i and $\overline{E}_i$ $(1 \le i \le n+2)$, all of which are G-invariant. By the explicitness of all the constructions, we can verify, after long but tedious computations, that all these divisors are isomorphic to Σ_0. We can also verify that the union $(\cup_{1\le i\le n+2} E_i) \cup (\cup_{1\le i\le n+2} \overline{E}_i)$ can be blown-down in such a way that the image becomes a cycle of rational curves which is the anticanonical curve of the images of any smooth fibers of $\hat{q}$. Let $\hat{Z}' \to Z'$ be the contraction map. In this way, starting from the projective variety X, we obtain a smooth 3-fold Z' equipped with a G-action and a real structure. This is the required twistor space as the following result shows.

THEOREM 3.1. *There exists a biholomorphic map $j : Z \to Z'$.*

PROOF. Let $\hat{Z} \to Z$ be the blowing-up of the twistor space along the cycle C. Any fibers of the natural projection $\hat{Z} \to \Lambda_2$ is biholomorphic to the corresponding fiber of $\hat{q} : \hat{Z}' \to \Lambda_2$. Therefore, by Fujiki's proof of Theorem 8.1 in

[**2**] (especially the proof of Lemmas 8.3–8.6), in order to obtain the isomorphism j, it suffices to show the existence of a smooth rational curve $\hat{L}'$ in $\hat{Z}'$ satisfying the following properties: (i) $\hat{L}'$ is disjoint from the divisor $(\cup_{1\le i\le n+2}E_i)\cup(\cup_{1\le i\le n+2}\overline{E}_i)$, (ii) the restriction of $\hat{q}$ to $\hat{L}'$ is two to one over Λ_2, and unramified at any λ_i, $1\le i\le n+2$. To find this curve, we choose and fix any $1\le i\le n+2$ and let $\hat{L}'_i\subset\hat{Z}'$ be the intersection of the two irreducible component of $\hat{q}^{-1}(\lambda_i)$. Let $L'_i\subset Z'$ be the image of $\hat{L}'_i$ by the blowing-down $\hat{Z}'\to Z'$. Then by the fact that the two irreducible components of $\hat{q}^{-1}(\lambda_i)$ intersect along $\hat{L}'_i$ transversally and the normal bundles of $\hat{L}'_i$ in the irreducible components are exactly degree one, we obtain that the normal bundle of L'_i in Z' is isomorphic to $\mathscr{O}(1)^{\oplus 2}$. Thus by deformation theory, the universal family of deformations of L'_i in Z' is parameterized by a smooth complex 4-manifold. If we choose a general member L' among this family and letting $\hat{L}'$ to be the inverse image of $\hat{L}$ by $\hat{Z}'\to Z'$, $\hat{L}'$ satisfies the properties (i) and (ii), as desired. □

References

[1] M. F. Atiyah, N. J. Hitchin and I. M. Singer, Self-duality in four-dimensional Riemannian geometry, Proc. Roy. Soc. London, Ser. A, **362** (1978), 425–461.

[2] A. Fujiki, Compact self-dual manifolds with torus actions, J. Differential Geom., **55** (2000), 229–324.

[3] N. Hitchin, Complex manifolds and Einstein's equations, Lecture Notes in Math., Springer, **970** (1982), pp. 73–99.

[4] N. Honda, On a construction of the twistor spaces of Joyce metrics, J. Algebraic Geom., **17** (2008), 709–750, available as math.DG/0603242 and math.DG/0604306.

[5] N. Honda, Explicit construction of new Moishezon twistor spaces, J. Differential Geom., **82** (2009), 411–444.

[6] N. Honda, A new series of compact minitwistor spaces and Moishezon twistor spaces over them, to appear in J. Reine Angew. Math.

[7] N. Honda, Projective models of the twistor spaces of Joyce metrics, preprint, arXiv:0805:0046.

[8] D. Joyce, Explicit construction of self-dual 4-manifolds, Duke Math. J., **77** (1995), 519–552.

[9] C. LeBrun, Explicit self-dual metrics on $\boldsymbol{CP}^2\#\cdots\#\boldsymbol{CP}^2$, J. Differential Geom., **34** (1991), 223–253.

Nobuhiro HONDA
Department of Mathematics
Graduate School of Science and Engineering
Tokyo Institute of Technology
2-12-1, O-okayama, Meguro
Tokyo 152-8551, Japan
E-mail: honda@math.titech.ac.jp