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The generalized Lefschetz number of homeomorphisms on punctured disks

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Abstract. We compute the generalized Lefschetz number of orientation-preserving self-homeomorphisms of a compact punctured disk, using the fact that homotopy classes of these homeomorphisms can be identified with braids. This result is applied to study Nielsen-Thurston canonical homeomorphisms on a punctured disk. We determine, for a certain class of braids, the rotation number of the corresponding canonical homeomorphisms on the outer boundary circle. As a consequence of this result on the rotation number, it is shown that the canonical homeomorphisms corresponding to some braids are pseudo-Anosov with associated foliations having no interior singularities.

1. Introduction.

The generalized Lefschetz number is one of the central notions in Nielsen fixed point theory. The classical Lefschetz number $L(f)$ is a well-known homotopy invariant for proving the existence of a fixed point of a continuous self-map $f : X \to X$ on a connected, finite cell complex X. It coincides with the fixed point index of the whole set $\mathrm{Fix}(f)$ of fixed points, and hence the non-vanishing of this number implies that f has a fixed point.

The generalized Lefschetz number $\mathscr{L}(f)$ is a refinement of the Lefschetz number obtained by decomposing the fixed point set $\mathrm{Fix}(f)$ into finitely many equivalence classes called fixed point classes. On the fundamental group $\pi_1(X)$, an equivalence relation, called the Reidemeister equivalence, is defined using the induced action f_π of f. An equivalence class under this relation is called a Reidemeister class. Then, a Reidemeister class is assigned to each fixed point, and the set of fixed points to which a given Reidemeister class α is assigned is called the fixed point class determined by α. The compactness of X implies that there are only finitely many Reidemeister classes determining non-empty fixed point classes. This fact enables us to define the generalized Lefschetz number $\mathscr{L}(f)$ as the formal sum of the Reidemeister classes with each class being indexed by fixed

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point index of the corresponding fixed point class. Hence, $\mathscr{L}(f)$ is not an integer, but an element of the free abelian group $\boldsymbol{Z}\mathscr{R}(f_\pi)$ generated by the set $\mathscr{R}(f_\pi)$ of Reidemeister classes. The non-vanishing of the coefficient of a Reidemeister class α in $\mathscr{L}(f)$ implies the existence of a fixed point with α assigned. Thus, by computing the generalized Lefschetz number, we can prove the existence of a fixed point corresponding to each term in $\mathscr{L}(f)$. The generalized Lefschetz number is a homotopy invariant, and the classical one $L(f)$ is obtained from $\mathscr{L}(f)$ by summing up the coefficients. See e.g. [**6**], [**14**], [**16**] for general references of Nielsen fixed point theory.

Practically, the generalized Lefschetz number is useful in studying fixed points only in the case where it is computable. Unfortunately, it is very difficult to compute it from the definition in general. The Reidemeister trace formula [**21**], [**24**], [**13**] provides a method to compute it. The classical Lefschetz number $L(f)$ is known to satisfy the following trace formula: If f is a cellular map, $L(f)$ is equal to the alternating sum of the traces of the action of f on the chain groups of X. Analogously, the generalized Lefschetz number $\mathscr{L}(f)$ satisfies the Reidemeister trace formula: $\mathscr{L}(f)$ is equal to the alternating sum of the Reidemeister traces, which are the traces of the action of a lift $\tilde{f}$ on the chain groups of the universal cover of X. Despite the existence of this formula, however, it is still difficult to make a detailed computation, particularly in the case of fundamental group being infinite and non-abelian. In this case, the author does not know any example of concrete computations carried out on large classes of maps.

In this paper, we compute the generalized Lefschetz number for orientation-preserving self-homeomorphisms f of a compact punctured disk which preserve the outer boundary circle (Theorem 1). Such homeomorphisms are of great importance in the topological study of 2-dimensional dynamical systems, for they include the homeomorphisms which are obtained from orientation-preserving disk homeomorphisms by the blow-up construction at a finite, interior invariant set (see e.g. [**5**, Section 1.6]). We should note that our computation is not complete in the sense that the problem of distinguishing Reidemeister classes is left unsettled. This means that we shall obtain an element in the group ring $\boldsymbol{Z}\pi_1(X)$ which is mapped to the generalized Lefschetz number under the projection from $\boldsymbol{Z}\pi_1(X)$ to $\boldsymbol{Z}\mathscr{R}(f_\pi)$. Thus, our result may be thought of as giving an “upper bound” of the generalized Lefschetz number. Our computation utilizes the fact that the homotopy class (or equivalently the isotopy class) of f can be identified with a braid. We show that a braid is designated by a finite sequence of positive integers, and we shall compute the generalized Lefschetz number directly from this sequence. For surfaces with boundary, Fadell and Husseini showed in [**7**] that the computation of the Reidemeister trace is reduced to that in the Fox free differential calculus on free groups. Our result is obtained by carrying out this computation. In [**19**], the author computed the image of the generalized Lefschetz number $\mathscr{L}(f)$ under the projection from $\boldsymbol{Z}\mathscr{R}(f_\pi)$ to the ring $\boldsymbol{Z}[t, t^{-1}]$ of integer polynomials in the variable t and its inverse. The present result improves the computation there.

It is a natural question whether our method is applicable in the general case where f may not preserve the outer boundary circle. In this case, f is thought of as an orientation-preserving homeomorphism on a punctured sphere, and its homotopy class is identified with a braid on a sphere. Our method is based on the fact that a braid on a plane is designated by a finite sequence of positive integers. At this moment, the author does not know a similar fact on a sphere, and cannot give an answer to the question.

On surfaces with boundary, Wagner [**23**] exploited an algorithm to compute the generalized Lefschetz number for a continuous map whose action on the fundamental group satisfies an algebraic condition. This condition is satisfied by most of continuous maps, but not by homeomorphisms. Therefore, the Wagner's algorithm is not applicable to our case.

We give two applications of our result in Section 4. The Nielsen-Thurston classification theory of isotopy classes of surface homeomorphisms provides a representative φ, called a canonical homeomorphism in each isotopy class of surface homeomorphisms. Canonical homeomorphisms play an essential role in the study of dynamics of surface homeomorphisms, because it has the "simplest" dynamical complexity among the homeomorphisms in its isotopy class. For instance, all the periodic points of φ persist under homotopy. We apply our result on $\mathscr{L}(f)$ to study periodic points of canonical homeomorphisms on a punctured disk: We determine, for a certain class of braids, the rotation number of the corresponding canonical homeomorphisms on the outer boundary circle (Proposition 2).

The second application concerns the problem of determining the type of the canonical homeomorphism in a given isotopy class. There is an algorithm to solve this problem due to Bestvina and Handel [**2**]. Similar algorithms for the disk case were given in [**9**],[**18**]. Also, different algorithms were given in [**1**],[**11**]. Our theorem provides an algebraic approach to this problem. We show that our result on the rotation number on the outer boundary circle implies that the canonical homeomorphisms corresponding to some families of braids are pseudo-Anosov with associated foliations having no interior singularities (Proposition 3).

In the last section, as a by-product of an argument in the proof of Proposition 2, we give a lower and an upper bound for the Nielsen number $N(f)$ for the class of braids treated in Proposition 2.

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2. Generalized Lefschetz number.

We recall the definition of the generalized Lefschetz number. Let X be a connected finite cell complex, and $f : X \to X$ a continuous map. Let $\mathrm{Fix}(f)$ be the fixed point set of f. Choose a base point x_0 of X, and let π denote the fundamental group $\pi_1(X, x_0)$ of X.

Given a homomorphism $\psi : \pi \to \pi$, two elements λ_1, λ_2 of π are said to be *Reidemeister equivalent* with respect to ψ (or ψ-*conjugate*) if there is a $\lambda \in \pi$ such that

$$\lambda_2 = \psi(\lambda)\lambda_1\lambda^{-1}.$$

An equivalence class under this equivalence relation is called a *Reidemeister class*. Let $\mathscr{R}(\psi)$ denote the set of Reidemeister classes, and $\boldsymbol{Z}\mathscr{R}(\psi)$ the free abelian group generated by the elements of $\mathscr{R}(\psi)$.

Choose a path τ from x_0 to $f(x_0)$. This is called a *base path*. Let $f_\pi : \pi \to \pi$ denote the composition of $f_* : \pi_1(X, x_0) \to \pi_1(X, f(x_0))$ with the isomorphism $\tau_*^{-1} : \pi_1(X, f(x_0)) \to \pi_1(X, x_0)$ induced by τ^{-1}. We shall consider Reidemeister classes with respect to f_π. For $x \in \mathrm{Fix}(f)$, take a path l from the base point x_0 to x. Then, it is easy to see that the Reidemeister class represented by $[\tau(f \circ l)l^{-1}] \in \pi$ is independent of the choice of l. This class is denoted by $R(x)$ and is called the Reidemeister class (or a coordinate) of x. For a Reidemeister class $\alpha \in \mathscr{R}(f_\pi)$, let $\mathrm{Fix}_\alpha(f) = \{x \in \mathrm{Fix}(f) \mid R(x) = \alpha\}$. This set is called the *fixed point class* of f determined by α. We then have the decomposition

$$\mathrm{Fix}(f) = \bigcup_{\alpha \in \mathscr{R}(f_\pi)} \mathrm{Fix}_\alpha(f).$$

The compactness of X implies that $\mathrm{Fix}_\alpha(f)$ is empty except for finitely many α. For an isolated set S of fixed points of f, let $\mathrm{ind}(S)$ denote the fixed point index of S with respect to f.

Definition 1. The generalized Lefschetz number $\mathscr{L}(f)$ of f is defined by

$$\mathscr{L}(f) = \sum_{\alpha \in \mathscr{R}(f_\pi)} \mathrm{ind}(\mathrm{Fix}_\alpha(f))\,\alpha \in \boldsymbol{Z}\mathscr{R}(f_\pi).$$

The generalized Lefschetz number is a homotopy invariant in the following sense: Let $g : X \to X$ be a continuous map homotopic to f through a homotopy $\{h_t\}_{0\leq t\leq 1}$. As a base path for g, take the composite of τ with the path $h_t(x_0)$ $(0 \leq t \leq 1)$ so that we have $f_\pi = g_\pi$. Then, the Nielsen fixed point theory asserts the equality $\mathscr{L}(f) = \mathscr{L}(g)$.

Let $\tilde{X}$ be the universal covering space of X. For integers q, let $C_q(\tilde{X})$ be the q-chain group of $\tilde{X}$. The action of π on $\tilde{X}$ induces an action of the group ring $\boldsymbol{Z}\pi$ on $C_q(\tilde{X})$. Then, $C_q(\tilde{X})$ becomes a finitely generated free $\boldsymbol{Z}\pi$-module. If f is a cellular map, its lift $\tilde{f}$ induces the twisted-module homomorphism $\tilde{f}_{\sharp q} : C_q(\tilde{X}) \to C_q(\tilde{X})$. Then, a trace $\operatorname{tr} \tilde{f}_{\sharp q}$ is defined as an element of $\boldsymbol{Z}\mathscr{R}(f_\pi)$. The Reidemeister trace formula [**21**], [**24**], [**13**] asserts that

$$\mathscr{L}(f) = \sum_{q\geq 0} (-1)^q \operatorname{tr} \tilde{f}_{\sharp q}.$$

Note that the classical Lefschetz number is equal to the sum of the coefficients in $\mathscr{L}(f)$, and the Nielsen number $N(f)$ is the number of Reidemeister classes with non-zero coefficients in $\mathscr{L}(f)$.

3. Computation on punctured disks.

We shall fix an integer n with $n \geq 3$. Let D_n be a compact n-punctured disk, namely, it is a compact surface obtained from a closed disk D by removing the interiors of n disjoint closed disks $D(1), \ldots, D(n)$ contained in the interior of D. D_n has $n+1$ boundary circles. One of these is ∂D called the *outer boundary circle* of D_n, and the others $\partial D(1), \ldots, \partial D(n)$ are called the inner boundary circles of D_n. Let $\mathrm{Homeo}_+(D_n, \partial D)$ denote the set of orientation-preserving homeomorphisms $f : D_n \to D_n$ which preserve the outer boundary circle ∂D setwise. In this paper, we shall compute the generalized Lefschetz number $\mathscr{L}(f)$ for any $f \in \mathrm{Homeo}_+(D_n, \partial D)$ up to distinguishing Reidemeister classes.

An isotopy class of such homeomorphisms can be identified with a braid: Let $\mathrm{Iso}_+(D_n, \partial D)$ be the group of isotopy classes of homeomorphisms in $\mathrm{Homeo}_+(D_n, \partial D)$. Let B_n denote the n-braid group. Then, we have an isomorphism of groups

$$\mathrm{Iso}_+(D_n, \partial D) \to B_n/Z_n,$$

where Z_n is the center of B_n. This identification is defined in the following way: Choose an isotopy $\{f_t : D \to D\}_{0\leq t\leq 1}$ such that $f_0 = \mathrm{id}$ and that f_1 coincides with f on D_n. The existence of such an isotopy is guaranteed using the well-known Alexander's trick. Then, the subset $\bigcup_{0\leq t\leq 1}(f_t(D(1) \cup \cdots \cup D(n)) \times \{t\})$ of $D \times$

$[0,1]$ consists of disjoint n tubes. If we regard each tube as a string, we obtain an n-braid. We denote this braid by $\beta(f)$, and call it the *braid of* f. The element of the quotient group B_n/Z_n represented by $\beta(f)$ does not depend on the choice of an isotopy $\{f_t\}$. Thus we obtain a map $\mathrm{Iso}_+(D_n, \partial D) \to B_n/Z_n$. It is known that this map becomes an isomorphism.

We can assume that the centers of the sub-disks $D(1), \dots, D(n)$ lie on a line in that order, hence so do the initial points of the braid $\beta(f)$. For $i = 1, \dots, n-1$, we denote by σ_i the i-th elementary braid, in which the i-th string overcrosses the $(i+1)$-th string once and all other strings go straight from the top to the bottom. The braid group B_n admits a presentation with generators $\sigma_1, \dots, \sigma_{n-1}$ and defining relations (see e.g. [**3**]):

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{if } |i-j| \geq 2,\ 1 \leq i,j \leq n-1, \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && 1 \leq i \leq n-2. \end{aligned}$$

Define $\rho \in B_n$ by $\rho = \sigma_{n-1}\cdots\sigma_2\sigma_1$. Let θ be the *full-twist* n-braid defined by $\theta = (\sigma_1\sigma_2\cdots\sigma_{n-1})^n$. θ is a generator of the center Z_n. In particular, it commutes with every braid. Note that ρ^n is equal to θ, since $\rho = \Delta(\sigma_1\cdots\sigma_{n-1})\Delta^{-1}$, where Δ is a half-twist braid $(\sigma_1\sigma_2\cdots\sigma_{n-1})\cdots(\sigma_1\sigma_2)\sigma_1$.

For a positive integer i, let $\beta(i) = \sigma_1^i\rho \in B_n$. Let d be a positive integer. Given a sequence $I = (i_1, \dots, i_d)$ of positive integers, define an n-braid $\beta(I)$ by

$$\beta(I) = \beta(i_1)\cdots\beta(i_d) = \sigma_1^{i_1}\rho\cdots\sigma_1^{i_d}\rho.$$

The following proposition has been proved in [**19**]. We give here a slightly simplified proof.

PROPOSITION 1. *Any braid is conjugate to a braid of the form $\theta^\mu\beta(I)$, where μ is an integer and I is a finite sequence of positive integers.*

PROOF. By the defining relations of B_n, it is easy to see that for $i = 1, \dots, n-2$,

$$\sigma_i\rho = \rho\sigma_{i+1}.$$

This implies that

$$\sigma_i = \rho^{1-i}\sigma_1\rho^{i-1} \tag{1}$$

for any i. Also, we have

$$(\sigma_1\rho)^{n-1} = \theta. \tag{2}$$

For any i, we have by (1), (2)

$$\sigma_i^{-1} = \rho^{1-i}\sigma_1^{-1}\rho^{i-1} = \rho^{2-i}(\sigma_1\rho)^{-1}\rho^{i-1} = \theta^{-1}\rho^{2-i}(\sigma_1\rho)^{n-2}\rho^{i-1}.$$

This and (1) imply that σ_i's and σ_i^{-1}'s can be written as a product of σ_1, ρ, ρ^{-1}, and θ^{-1}, and hence any braid β is conjugate to a braid of the form $\theta^\mu\sigma_1^{k_1}\rho^{l_1}\cdots\sigma_1^{k_s}\rho^{l_s}$, where $\mu \le 0$, $k_1,\dots,k_s > 0$ and $l_1,\dots,l_s \in \boldsymbol{Z}$. We can rewrite it in the form where all the exponents of ρ are equal to 1. In fact, since $\rho^{-1} = \theta^{-1}(\sigma_1\rho)^{n-2}\sigma_1$ by (2), we have $\rho^j = \rho^{nk}\rho^{-l} = \theta^{k-l}((\sigma_1\rho)^{n-2}\sigma_1)^l$ for any integer j, where k is an integer and $0 \le l < n$ with $j = kn - l$. □

Note that the arguments in the proof also give a procedure how to find μ, I and γ with $\beta = \gamma^{-1}\theta^\mu\beta(I)\gamma$ for a given $\beta \in B_n$.

EXAMPLE 1. Let $n = 3$ and consider $\sigma_1\sigma_2^{-1}$. Since $\sigma_2^{-1} = \theta^{-1}(\sigma_1\rho)\rho$, we have $\sigma_1\sigma_2^{-1} = \theta^{-1}\sigma_1^2\rho^2$. ρ^2 is equal to $(\sigma_1\rho)\sigma_1$, since $kn - l = 2$ for $k = l = 1$. Therefore, $\sigma_1\sigma_2^{-1} = \theta^{-1}\sigma_1^3\rho\sigma_1 = \sigma_1^{-1}\theta^{-1}\beta(4)\sigma_1$. Hence, $\mu = -1, I = (4)$, and $\gamma = \sigma_1$.

REMARK 1. μ and I in this proposition are not unique. For instance, we have by (2)

$$\beta(i, \underbrace{1,\dots,1}_{n-2}, j) = \sigma_1^{i-1}(\sigma_1\rho)^{n-1}\sigma_1^j\rho = \theta\beta(i+j-1).$$

Also, in B_3, since (2) implies $\sigma_1\rho\sigma_1 = \theta\rho^{-1}$ and hence $(\sigma_1\rho\sigma_1)^2 = \theta^2\rho^{-2} = \theta\rho$, we have for $i, j \ge 2$

$$\beta(i, 2, j) = \sigma_1^{i-1}(\sigma_1\rho\sigma_1)^2\sigma_1^{j-1}\rho = \theta\beta(i-1, j-1).$$

The purpose of this paper is to compute the generalized Lefschetz number $\mathscr{L}(f)$ in terms of μ, I, and γ given in Proposition 1. Let d be a positive integer, and $\boldsymbol{Z}_d$ the set $\{1,\dots,d\}$ of integers mod$d$. To state our main result, it is necessary to introduce the notion of a partition of $\boldsymbol{Z}_d$.

DEFINITION 2.
(i) For integers $1 \le p, q \le d$, define a sequence $[p, q]$ of consecutive integers mod d by

$$[p,q] = \begin{cases} (p,\dots,q) & \text{if } p \le q \\ (p,\dots,d,1,\dots,q) & \text{if } p > q. \end{cases}$$

This sequence is called a *block* in $\boldsymbol{Z}_d$, and the number of integers contained in it is called its length. For a block B, let $\underline{B}$ denote its underlying set, the set of integers contained in B.

(ii) A set $\{B_1,\dots,B_s\}$ of blocks in $\boldsymbol{Z}_d$ is a *partition* of $\boldsymbol{Z}_d$ if

(a) the underlying sets $\underline{B_1},\dots,\underline{B_s}$ are mutually disjoint and $\underline{B_1}\cup\cdots\cup\underline{B_s} = \boldsymbol{Z}_d$, and
(b) each of $B_1,\dots,B_s$ has length less than or equal to $n-1$.

Note that a partition contains at most one block of type $[p,q]$ with $p>q$.

(iii) Let $\mathscr{P}(d)$ denote the set of partitions of $\boldsymbol{Z}_d$.

EXAMPLE 2. Assume $n \ge 5$. Consider the case of $d=4$. In this case, any block has length less than or equal to $n-1$. Therefore, $\mathscr{P}(4)$ consists of the following fifteen partitions:

$$\{(1),(2),(3),(4)\},$$
$$\{(1,2),(3),(4)\},\quad \{(1),(2,3),(4)\},\quad \{(1),(2),(3,4)\},\quad \{(2),(3),(4,1)\},$$
$$\{(1,2),(3,4)\},\quad \{(2,3),(4,1)\},$$
$$\{(1,2,3),(4)\},\quad \{(1),(2,3,4)\},\quad \{(2),(3,4,1)\},\quad \{(3),(4,1,2)\},$$
$$\{(1,2,3,4)\},\quad \{(2,3,4,1)\},\quad \{(3,4,1,2)\},\quad \{(4,1,2,3)\}.$$

The fundamental group $\pi = \pi_1(D_n, x_0)$ is identified with a free group F_n of rank n. We shall define an action of B_n on F_n. Let e be the unit element of F_n. Assume that $x_0 \in \partial D$. Let $\xi_1,\dots,\xi_n$ be the standard generators of $\pi = F_n$ which are defined in the following way: We can assume that D is the disk in the plane $\boldsymbol{R}^2$ with center $(0,0)$ and radius 2, $x_0 = (0,2)$, and for $i=1,\dots,n$, the sub-disk $D(i)$ has radius $1/2(n+1)$ and center $(-1+(2/(n+1))\,i,\,0)$. Then, the element ξ_i is represented by a loop which traces a straight line connecting x_0 to a point in $\partial D(i)$, encircles $\partial D(i)$ once in the anti-clockwise direction, and retraces the line back to x_0. An action of the braid group B_n on F_n is defined by putting $\sigma_i(\xi_j) = \xi_i\xi_{i+1}\xi_i^{-1}$, ξ_i, or ξ_j according to whether $j=i, j=i+1$, or $j\ne i,i+1$. Thus, any braid β can be thought of as an automorphism of F_n (see [**3**, Corollary 1.8.3]).

In the case of $\beta(f)$, the corresponding automorphism of F_n can be described geometrically by using the isotopy $\{f_t\}$. Let $Z = \bigcup_{0\le t\le 1}(f_t(D_n)\times\{t\})$. Z is a solid cylinder with n disjoint open tubes removed. Define a vertical path v in Z by $v(t) = (x_0, t)$. For $\epsilon = 0, 1$, let $i_\epsilon : D_n \to Z$ be the inclusion map defined by $i_\epsilon(x) = (x, \epsilon)$. Given an element $w \in \pi$, choose a loop l based at x_0 representing w. Then, it is easy to see that the loop $v^{-1}(i_0 \circ l)v$ in Z is homotopic to $i_1 \circ l'$ in Z for some loop l' in D_n. Then the image $\beta(f)(w)$ coincides with the element of π represented by l'.

As a base path τ, we shall take τ given by $\tau(t) = f_t(x_0)$. Then, for any loop l in D_n based at x_0, the loop $v^{-1}(i_0 \circ l)v$ is homotopic to $i_1 \circ (\tau(f \circ l)\tau^{-1})$ in Z. Therefore, $\beta(f)(w) \in \pi$ is represented by the loop $\tau(f \circ l)\tau^{-1}$, and hence it is equal to $f_\pi(w)$. Thus, we have shown that

$$f_\pi = \beta(f) : F_n \to F_n. \tag{3}$$

For $w \in F_n$, we shall use the symbol w^β to denote its image under the automorphism β.

In our computation, we shall not use the standard generators, but use the generators $a_1, \ldots, a_n$ for F_n defined by $a_i = \xi_1 \cdots \xi_i$. Then, the action of σ_i on F_n is written in a slightly simpler way as

$$a_j^{\sigma_i} = \begin{cases} a_{i+1}a_i^{-1}a_{i-1} & \text{if } j = i, \\ a_j & \text{if } j \ne i, \end{cases}$$

where we put $a_0 = e$. Note that $a_n^\beta = a_n$ for any braid β, since $a_n^{\sigma_i} = a_n$ for any i.

Let $\boldsymbol{Z}F_n$ be the group ring of F_n over $\boldsymbol{Z}$. For $\beta \in B_n$, the automorphism β of F_n induces the ring automorphism of $\boldsymbol{Z}F_n$, which will be denoted by the same letter β. For $\eta \in \boldsymbol{Z}F_n$ and $\beta \in B_n$, let $\eta^\beta \in \boldsymbol{Z}F_n$ denote the image of η under β.

Let $I = (i_1, \ldots, i_d)$ be a sequence of positive integers. We shall introduce a map $W_I : \mathscr{P}(d) \to \boldsymbol{Z}F_n$ which is necessary to state the main result. First, for integers $j \ge 0$, define $c_j \in F_n$ and $g_j \in \boldsymbol{Z}F_n$ by

$$c_j = \begin{cases} a_2^{j/2} & \text{if } j \text{ is even}, \\ a_1 a_2^{(j-1)/2} & \text{if } j \text{ is odd}, \end{cases}$$
$$g_j = (-1)^{j+1} c_j.$$

For $1 \le l \le d$, let $\beta_l(I) = \beta(i_l, \ldots, i_d) \in B_n$. Note that $\beta_1(I) = \beta(I)$. Suppose a block $B = [p, q]$ in $\boldsymbol{Z}_d$ is given. Denote its length by $|B|$. Define the braids

$\alpha(B), \omega(B)$ by

$$\alpha(B) = \beta_p(I), \quad \omega(B) = \begin{cases} \beta_q(I) & \text{if } p \leq q, \\ \beta_q(I)\beta(I)^{-1} & \text{if } p > q, \end{cases}$$

and define $W_I(B) \in \boldsymbol{Z}F_n$ as follows:

$$W_I(B) = \begin{cases} (g_0 + \cdots + g_{i_p-2})^{\alpha(B)} a_{|B|+1}^{\omega(B)} & \text{if } |B| < n-1,\ i_p \geq 2, \\ 0 & \text{if } |B| < n-1,\ i_p = 1, \\ g_{i_p}^{\alpha(B)} a_{n-1}^{\omega(B)} & \text{if } |B| = n-1. \end{cases}$$

Then, the map $W_I : \mathscr{P}(d) \to \boldsymbol{Z}F_n$ is defined as follows: Let $\mathscr{B} \in \mathscr{P}(d)$. Let $B_1 = [p_1, q_1], \ldots, B_s = [p_s, q_s]$ be the blocks in $\mathscr{B}$. We can assume that $1 \leq p_1 < p_2 < \cdots < p_s \leq d$ by rearranging the blocks if necessary. Then, define $W_I(\mathscr{B})$ by

$$W_I(\mathscr{B}) = W_I(B_1) \cdots W_I(B_s).$$

Let $\Phi_\beta : \boldsymbol{Z}F_n \to \boldsymbol{Z}\mathscr{R}(\beta)$ denote the surjective homomorphism induced by the projection $F_n = \pi \to \mathscr{R}(\beta)$. By the definition of the Reidemeister equivalence, we have

$$\Phi_\beta(w) = \Phi_\beta(w^\beta) \quad \text{for any } w \in F_n. \tag{4}$$

More generally, we have

$$\Phi_\beta(w'w) = \Phi_\beta(w^\beta w') \quad \text{for any } w, w' \in F_n. \tag{5}$$

Recall that $\beta(f)$ can be written as $\gamma^{-1}\theta^\mu\beta(I)\gamma$ for some $\mu \in \boldsymbol{Z}, \gamma \in B_n$, and some sequence I of positive integers. Our main result is the following:

Theorem 1. *Suppose $f : D_n \to D_n$ is an orientation-preserving homeomorphism which preserves the outer boundary circle setwise. We choose an isotopy $\{f_t\} : D \to D$ such that $f_0 = \mathrm{id}$ and f_1 coincides with f on D_n. As a base path for f, take the path τ defined by $\tau(t) = f_t(x_0)$. Suppose $\beta(f) = \gamma^{-1}\theta^\mu\beta(I)\gamma$, where μ is an integer, $\gamma \in B_n$, and I is a sequence of positive integers with length d. Then*

$$\mathscr{L}(f) = -\Phi_{\beta(f)}\left(a_n^\mu \sum_{\mathscr{B} \in \mathscr{P}(d)} W_I(\mathscr{B})^\gamma \right) \in \boldsymbol{Z}\mathscr{R}(\beta(f)).$$

EXAMPLE 3. (a) Let $\beta(f) = \beta(i)$, where $i \geq 2$. In this case, $\mu = 0$, $\gamma = e$, $I = (i)$, and $d = 1$. The partition $\{(1)\}$ is the only element of $\mathscr{P}(d) = \mathscr{P}(1)$, and $\alpha((1)) = \omega((1)) = \beta(i)$. Therefore, by the above theorem and (4), we have

$$\begin{aligned}\mathscr{L}(f) &= -\Phi_{\beta(f)}(W_{(i)}(\{(1)\})) = -\Phi_{\beta(i)}((g_0 + \cdots + g_{i-2})^{\beta(i)} a_2^{\beta(i)}) \\ &= -\Phi_{\beta(i)}(g_2 + \cdots + g_i) \\ &= \Phi_{\beta(i)}(c_2) - \Phi_{\beta(i)}(c_3) + \cdots + (-1)^i \Phi_{\beta(i)}(c_i).\end{aligned}$$

(b) Let $\beta(f) = \beta(i_1, i_2)$, where $i_1, i_2 \geq 2$. In this case, $\mu = 0$, $\gamma = e$, $I = (i_1, i_2)$, and $d = 2$. $\mathscr{P}(d) = \mathscr{P}(2)$ consists of the three partitions $\mathscr{B}_1 = \{(1), (2)\}$, $\mathscr{B}_2 = \{(1, 2)\}$, and $\mathscr{B}_3 = \{(2, 1)\}$. Therefore, we have $\mathscr{L}(f) = -\Phi_{\beta(f)}(W_I(\mathscr{B}_1) + W_I(\mathscr{B}_2) + W_I(\mathscr{B}_3))$, where

$$W_I(\mathscr{B}_1) = \sum_{j=2}^{i_1} \sum_{k=2}^{i_2} g_j^{\beta(I)} g_k^{\beta(i_2)},$$

$$W_I(\mathscr{B}_2) = \begin{cases} \left(\displaystyle\sum_{j=0}^{i_1-2} g_j^{\beta(I)} \right) a_3^{\beta(i_2)} & \text{if } n \geq 4, \\ g_{i_1}^{\beta(I)} a_2^{\beta(i_2)} & \text{if } n = 3, \end{cases}$$

$$W_I(\mathscr{B}_3) = \begin{cases} \left(\displaystyle\sum_{k=0}^{i_2-2} g_k^{\beta(i_2)} \right) a_3 & \text{if } n \geq 4, \\ g_{i_2}^{\beta(i_2)} a_2 & \text{if } n = 3. \end{cases}$$

As a consequence of our theorem, we can give an upper bound for the Nielsen number $N(f)$. For $\eta \in \boldsymbol{Z}F_n$, let $\nu(\eta)$ denote the number of elements of F_n with non-zero coefficient in η. Then, for a block $B = [p, q]$, we have

$$\nu(W_I(B)) = \begin{cases} i_p - 1 & \text{if } |B| < n - 1, \\ 1 & \text{if } |B| = n - 1. \end{cases}$$

For a partition $\mathscr{B} = \{B_1, \ldots, B_s\}$, let $\nu_I(\mathscr{B}) = \nu(W_I(B_1)) \cdots \nu(W_I(B_s))$. Then, we have

COROLLARY 1. *Under the same hypothesis of Theorem 1, we have* $N(f) \leq \sum_{\mathscr{B} \in \mathscr{P}(d)} \nu_I(\mathscr{B})$.

PROOF. It follows from Theorem 1 that

$$N(f)=\nu(\mathscr{L}(f))\leq\nu\left(\sum_{\mathscr{B}\in\mathscr{P}(d)}W_I(\mathscr{B})\right)\leq\sum_{\mathscr{B}\in\mathscr{P}(d)}\nu(W_I(\mathscr{B})).$$

Since $\nu(W_I(\mathscr{B}))\leq\prod_{r=1}^{s}\nu(W_I(B_r))=\nu_I(\mathscr{B})$, this gives the proof. □

Example 4. Consider the braids treated in Example 3.

(a) Let $\beta(f)=\beta(i)$, $i\geq 2$. Then, the partition $\{(1)\}$ is the only element of $\mathscr{P}(1)$, and hence $\sum_{\mathscr{B}\in\mathscr{P}(d)}\nu_I(\mathscr{B})=\nu_{(i)}(\{(1)\})=i-1$. Therefore, by Corollary 1, we have $N(f)\leq i-1$. In fact, the equality $N(f)=i-1$ holds for this braid. This is proved by using the ring homomorphism $\mathscr{T}:\boldsymbol{Z}F_n\to\boldsymbol{Z}[t,t^{-1}]$ defined by $\mathscr{T}(\xi_j)=t$ for $j=1,\dots,n$. Then, $\mathscr{T}$ induces the homomorphism $\mathscr{T}:\boldsymbol{Z}\mathscr{R}(\beta(f))\to\boldsymbol{Z}[t,t^{-1}]$, and the Reidemeister classes $\Phi_{\beta(i)}(c_2),\dots,\Phi_{\beta(i)}(c_i)$ in $\mathscr{L}(f)$ are sent to mutually different elements $t^2,\dots,t^i$ by $\mathscr{T}$. Hence they are different elements in $\mathscr{R}(\beta(i))$ and we obtain $N(f)=i-1$.

(b) Let $\beta(f)=\beta(i_1,i_2)$, where $i_1,i_2\geq 2$. Consider first the case of $n\geq 4$. Let $\mathscr{B}_1,\mathscr{B}_2,\mathscr{B}_3$ be the partitions as in Example 3(b). Then, $\nu_I(\mathscr{B}_j)=(i_1-1)(i_2-1),i_1-1,i_2-1$ for $j=1,2,3$ respectively, and so we have $\sum_{\mathscr{B}\in\mathscr{P}(d)}\nu_I(\mathscr{B})=(i_1-1)(i_2-1)+(i_1-1)+(i_2-1)=i_1i_2-1$. Hence, it follows from Corollary 1 that $N(f)\leq i_1i_2-1$. Consider next the case of $n=3$. Then, Corollary 1 implies $N(f)\leq(i_1-1)(i_2-1)+2$. If $i_1,i_2\geq 3$, the sharper estimate $N(f)\leq(i_1-1)(i_2-1)-2$ holds, because the images of $W_I(\mathscr{B}_2)=-g_{i_1}^{\beta(I)}g_2^{\beta(i_2)}$ and $W_I(\mathscr{B}_3)=-g_{i_2}^{\beta(i_2)}g_2$ under $\Phi_{\beta(f)}$ cancel by the images of two terms in $W_I(\mathscr{B}_1)$. For a class of braids including this example, we shall give a sharper estimation than Corollary 1 in Section 8.

Remark 2. The image of $-\mathscr{L}(f)$ under $\mathscr{T}$ coincides with the trace of the reduced Burau matrix $\mathrm{Bur}(\beta(f))$ of the braid $\beta(f)$ (cf. [**12**]). This trace was computed in [**19**] using the same expression of braids as in Proposition 1. Given a square matrix A of size ν with entries in a commutative ring R, let $\mathrm{PM}(A;k)$ be the sum of principal minors of A of order k if $1\leq k\leq\nu$ and zero otherwise. Then, we have the equality

$$\operatorname{tr}A^d=\sum_{\mathscr{B}\in\mathscr{P}(d)}(-1)^{d+\sharp\mathscr{B}}\mathrm{PM}(A;|B_1|)\cdots\mathrm{PM}(A;|B_s|)$$

for any positive integer d, where $\mathscr{B}=\{B_1,\dots,B_s\}$. Applying this to the case of $A=\mathrm{Bur}(\beta(i))$, we have for $I=(i,\dots,i)\in\boldsymbol{N}^d$ that

$$\operatorname{tr}\mathrm{Bur}(\beta(I))=\operatorname{tr}\,\mathrm{Bur}(\beta(i))^d=\sum_{\mathscr{B}\in\mathscr{P}(d)}(-1)^{d+\sharp\mathscr{B}}P(i;|B_1|)\cdots P(i;|B_s|),$$

where $P(i;k)$ denotes $\mathrm{PM}(\mathrm{Bur}(\beta(i));k)$ for any k. Theorem 1 in [**19**] generalizes this equality to an arbitrary sequence $I \in \boldsymbol{N}^d$ as follows:

$$\operatorname{tr}\mathrm{Bur}(\beta(I)) = \sum_{\mathscr{B}\in\mathscr{P}(d)} (-1)^{d+\sharp\mathscr{B}} P(i_{p_1};|B_1|)\cdots P(i_{p_s};|B_s|),$$

where p_r is the initial element of B_r for $1 \le r \le s$. Our main result, Theorem 1 above, gives a refinement of this equality.

REMARK 3. In our setting, Reidemeister classes can be visualized by the method of Jiang [**15**] using the mapping torus. For $t \in [0,1]$, let $[t]$ denote the corresponding point in the circle $S^1 = \boldsymbol{R}/\boldsymbol{Z}$. Define a subspace T of $D \times S^1$ by $T = \bigcup_{0\le t\le 1}(f_t(D_n) \times \{[t]\})$, which is homeomorphic to the mapping torus of f. Then the set of Reidemeister classes is in one-to-one correspondence with the set of free homotopy classes of loops in T. The Reidemeister class $R(x)$ of $x \in \mathrm{Fix}(f)$ corresponds to the free homotopy class of the loop $(f_t(x),[t])$ $(0 \le t \le 1)$ under this identification.

4. Nielsen-Thurston classification of surface homeomorphisms.

We shall apply the theorem in the previous section to study periodic points of Nielsen-Thurston canonical homeomorphisms on a punctured disk, and also to the classification problem of homeomorphisms into isotopy classes. We recall briefly the Nielsen-Thurston classification theory of surface homeomorphisms ([**8**],[**22**]). Let M be a compact surface. A homeomorphism $\varphi : M \to M$ is said to be of *finite order* if some of its iterates is equal to the identity map. The map φ is said to be *pseudo-Anosov*, if the following conditions are satisfied:

(i) There exists a pair of transverse foliations on M, carrying measures which are uniformly expanded and contracted by φ respectively.

(ii) These foliations have finitely many singularities which coincide in the interior $\mathrm{Int}\, M$ and alternate on the boundary ∂M. Any singularity is p-pronged for some integer $p \ge 3$ if it is in the interior of M, and it is 3-pronged if it is in ∂M. (We consider segments of the boundary to be prongs.)

φ is said to be *reducible* if there exists a finite collection of disjoint annuli in M such that φ maps their union A to itself, and that each connected component N of $M - A$, called a *component* of φ, has negative Euler characteristic and for any iterate φ^m mapping N to itself, its restriction to N is either of finite order or pseudo-Anosov. The Nielsen-Thurston classification theory states that every homeomorphism $f : M \to M$ is isotopic to a homeomorphism $\varphi : M \to M$ which is of finite order, pseudo-Anosov, or reducible. The homeomorphism φ is called a *canonical homeomorphism* in the isotopy class of f. In the case where φ is irreducible, i.e., of finite order or pseudo-Anosov, the surface M is called the component of φ.

One of the common features of canonical homeomorphisms φ is that they have periodic points on every boundary circle C. In fact, in the case where C is contained in a pseudo-Anosov component, the singularities of associated foliations on it are periodic points. Also, in the case where C is contained in a finite-order component N, all the points in N are periodic. Since the restriction of φ to C is an orientation-preserving homeomorphism of a circle, the periodic points in C have the same least period. We shall consider the problem of determining the period of periodic points and the rotation number on C in the case where $M = D_n$ and $C = \partial D$. The reason why we choose the outer boundary circle as the subject is that this is the easiest case to deal with by using the generalized Lefschetz number. The result we shall obtain will be applied to classify homeomorphisms up to isotopy.

Let φ be an orientation-preserving canonical homeomorphism on D_n preserving ∂D setwise. We denote by $m(\varphi)$ the least period of periodic points on ∂D. Let N_φ be the component of φ containing ∂D. Choose an isotopy $\varphi_t : D \to D$ such that $\varphi_0 = \mathrm{id}$ and that φ_1 coincides with φ on D_n. Assume the base point x_0 is in ∂D. Define a base path τ for φ by $\tau(t) = \varphi_t(x_0)$. Note that τ is contained in ∂D. For every positive integer m, define a base path τ_m for φ^m by $\tau_m = \tau(\varphi \circ \tau) \cdots (\varphi^{m-1} \circ \tau)$. Choose a periodic point x on ∂D. Since x_0 and x are contained in ∂D, we can choose a path l connecting these points contained in ∂D. Then, the loop $\tau_{m(\varphi)}(\varphi^{m(\varphi)} \circ l)l^{-1}$ is contained in ∂D and hence it represents an element $a_n^{\nu(\varphi)} \in F_n$ for some integer $\nu(\varphi)$. Note that $\nu(\varphi)$ does not depend on the choice of the periodic point x and the path l. It depends, however, on the choice of an isotopy φ_t, but is uniquely determined modulo $m(\varphi)$. The number $\nu(\varphi)/m(\varphi)$ modulo $\boldsymbol{Z}$ is equal to the rotation number of φ on ∂D.

The following lemma shows that, in the case where $m(\varphi)$ and $\nu(\varphi)$ are relatively prime, the problem of determining these numbers is reduced to the computation of the generalized Lefschetz number.

Lemma 1. *Let m and ν be integers with $m > 0$. Assume that they are relatively prime, and that the coefficient of $\Phi_{\beta(\varphi^m)}(a_n^\nu) \in \mathscr{R}(\beta(\varphi^m))$ in $\mathscr{L}(\varphi^m)$ is non-zero. Then, $m = m(\varphi)$ and $\nu = \nu(\varphi)$.*

Proof. Since the coefficient of $\Phi_{\beta(\varphi^m)}(a_n^\nu)$ in $\mathscr{L}(\varphi^m)$ is non-zero, φ^m has a fixed point x with Reidemeister class $\Phi_{\beta(\varphi^m)}(a_n^\nu)$. Then, we can take a path l from the base point x_0 to x so that $\tau_m(\varphi^m \circ l)l^{-1}$ represents a_n^ν. We shall show that x is φ^m-related to ∂D, namely there exists a path connecting a point in ∂D with x which is homotopic to its image under φ^m via a homotopy of paths such that each path in the homotopy connects a point in ∂D with x. Choose a loop λ contained in ∂D based at x_0 which represents a_n^ν. Let $\eta = \tau_m^{-1}\lambda$ and for $0 \leq u \leq 1$, let $\eta_u(t) = \eta(ut + 1 - u)$ and $l_u = \eta_u l$. Then, $\{l_u\}$ gives a homotopy of paths such that $l_u(0) \in \partial D, l_u(1) = x$, and l_0 and $l_1 = \tau_m^{-1}\lambda l$ are homotopic to l and $\varphi^m \circ l$ fixing end points respectively. Thus, we have shown that l is a desired path to prove x being φ^m-related to ∂D. (This is proved in a different way by Guaschi [**10**, Proposition 14(b)].) Then, it follows from Jiang and Guo [**17**, Lemma 3.4] that $x \in \partial D$ if $\varphi|_{N_\varphi}$ is pseudo-Anosov, and there exists a path in $\mathrm{Fix}(\varphi^m) \cap N_\varphi$ connecting x to ∂D if $\varphi|_{N_\varphi}$ is of finite order. Hence, $m = qm(\varphi)$ for some positive integer q. Moreover, since a_n is fixed under $\varphi_\pi = \beta(\varphi)$, we have $\nu = q\nu(\varphi)$. Since m and ν are relatively prime, q must be one. Thus the proof is completed. □

Let LCM denote the least common multiple for positive integers. Using this lemma, Theorem 1 can be applied to obtain

PROPOSITION 2. *Suppose the braid $\beta(\varphi)$ is conjugate to $\theta^\mu\beta(I)$, where $I = (i_1, \ldots, i_d)$ is a sequence of positive integers. Assume either that $n \geq 4$ and $i_1, \ldots, i_d \geq 2$, or that $n = 3$ and $i_1, \ldots, i_d \geq 3$. Then*

$$m(\varphi) = \frac{\mathrm{LCM}\{d, n-2\}}{d}, \quad \nu(\varphi) = m(\varphi)\mu + \frac{\mathrm{LCM}\{d, n-2\}}{n-2}.$$

This proposition will be proved in Section 7 using some lemmas on the computation of the generalized Lefschetz number given in Section 6. When $n = 3$, this result cannot be extended to the case of $i_1, \ldots, i_d \geq 2$. For instance, $\beta(2) \in B_3$ is conjugate to ρ^2, and so φ is of finite-order and $m(\varphi) = 3$. This is not equal to $\mathrm{LCM}\{d, n-2\}/d = 1$.

This proposition has a consequence on the classification problem of canonical homeomorphisms on a punctured disk. Boyland [**4**] proved that if n is prime and $\beta(\varphi)$ is cyclic, that is, the permutation on the punctures induced by $\beta(\varphi)$ is cyclic, then φ is irreducible. He also proved that if φ is irreducible, $\beta(\varphi)$ is cyclic, and the exponent sum of $\beta(\varphi)$ is not divisible by $n-1$, then φ is pseudo-Anosov. In particular, if n is prime, $\beta(I)$ is cyclic, and $i_1 + \cdots + i_d$ is not divisible by $n-1$, then φ is pseudo-Anosov. Matsuoka [**19**] has proved that, under the assumption of Proposition 2, the canonical homeomorphism φ with braid $\beta(I)$ contains a pseudo-Anosov component, except only for the case where $n \geq 4, I = (2, \ldots, 2)$ and $n = 3, I = (3, \ldots, 3)$. This result was proved by using the computation of the reduction $\mathscr{T}(\mathscr{L}(f))$ mentioned in Remark 2. Our main theorem on the computation of the unreduced number $\mathscr{L}(f)$ can be applied to improve this result. In fact, as a consequence of Proposition 2, we have the following proposition.

Proposition 3. *Assume $n \geq 5$. Let I be a sequence of integers $i_1, \dots, i_d \geq 2$ which are all odd or all even. Assume that $n-2$ and d are relatively prime. Then, the canonical homeomorphism φ with braid $\beta(I)$ is pseudo-Anosov. Moreover, the foliations associated to φ have no interior singularities.*

Proof. Since $n-2$ and d are assumed to be relatively prime, we have $m(\varphi) = n-2$ and $\nu(\varphi) = d$ by Proposition 2. This implies that the periodic points on ∂D have period $n-2$ and rotation number $d/(n-2)$. Let μ be the permutation on the inner boundary circles of D_n induced by φ. Assume $\varphi|_{N_\varphi}$ were of finite-order. Then, $\varphi|_{N_\varphi}$ is topologically conjugate to the rigid rotation on the unit disk by angle $2\pi d/(n-2)$ restricted to the exterior of an appropriate set of punctures. Hence, there exist $n-2$ boundary circles $C_1, \dots, C_{n-2}$ of N_φ cyclically permuted by φ. If none of $C_1, \dots, C_{n-2}$ is a boundary circle of D_n, each of them surrounds at least two boundary circles of D_n. Therefore, there must exist at least $2(n-2)$ boundary circles of D_n. Since $n \geq 5$, this number exceeds n, which is impossible. Therefore, some of $C_1, \dots, C_{n-2}$ is a boundary circle of D_n, and so are all of $C_1, \dots, C_{n-2}$, since they are cyclically permuted by φ. Therefore, μ has a cycle with length $n-2$. We shall show that this contradicts to an assumption of the proposition. In the case where $i_1, \dots, i_d$ are all even, μ is equal to the permutation induced by ρ^d, and hence it is the d-th power of a cyclic permutation on n circles. Hence, $n-2$ must divide n, which is a contradiction since $n \geq 5$. Also, in the case where $i_1, \dots, i_d$ are all odd, μ fixes one of the inner boundary circles and on the other $n-1$ inner boundary circles, μ is the permutation induced by $(\sigma_1\rho)^d$, which is the d-th power of a cyclic permutation. Thus, $n-2$ must divide $n-1$, which is a contradiction. Therefore, $\varphi|_{N_\varphi}$ is not of finite order, and hence it must be pseudo-Anosov.

Let c be the number of inner boundary circles of N_φ. Choose one of the foliations on N_φ and let $\mathscr{S}$ denote the set of its singularities. Denote by $p(x)$ the number of prongs at a singularity x. Then we have the following Euler-Poincaré formula (see e.g. [**8**], p.75):

$$\sum_{x \in \mathscr{S}} (2 - p(x)) = 2\chi(N_\varphi) = 2(1-c). \tag{6}$$

Since the singularities on ∂D are periodic points with least period $n-2$, there exist at least $n-2$ singularities on ∂D. Also, each inner boundary circle of N_φ

contains at least one singularity. Hence $\sharp(\mathscr{S} \cap \partial N_\varphi) \geq n-2+c$. Therefore, since $2-p(x) = -1$ for every singularity on ∂N_φ and $n \geq c$, we have by (6)

$$\begin{aligned}\sum_{x \in \mathscr{S} \cap \operatorname{Int} N_\varphi} (2-p(x)) &= \sum_{x \in \mathscr{S}} (2-p(x)) - \sum_{x \in \mathscr{S} \cap \partial N_\varphi} (2-p(x)) \\ &= 2(1-c) - (-\sharp(\mathscr{S} \cap \partial N_\varphi)) \geq n-c \geq 0.\end{aligned}$$

This implies that there are no interior singularities on N_φ, since $2-p(x) < 0$ for any $x \in \mathscr{S} \cap \operatorname{Int} N_\varphi$, and also that $0 \geq n-c$. Hence $c = n$, and so $N_\varphi = D_n$ and φ is pseudo-Anosov. $\square$

The above proposition cannot be extended to the case of $n = 3, 4$. For instance, $\beta(2) \in B_3$ is conjugate to ρ^2, which corresponds to a finite-order homeomorphism. Also, $\beta(2) \in B_4$ is conjugate to $\rho\sigma_3\sigma_2$, which corresponds to a reducible homeomorphism having only finite-order components.

5. Proof of Theorem 1.

For surfaces with boundary, Fadell and Husseini showed in [**7**] that the computation of the generalized Lefschetz number is reduced to that in the Fox free differential calculus on free groups. The Fox partial derivative operator $\partial/\partial a_j : \boldsymbol{Z}F_n \to \boldsymbol{Z}F_n$, $j = 1, \ldots, n$, is defined by the following rules (see [**3**],[**20**]):

(i) $\dfrac{\partial}{\partial a_j}(\eta_1 + \eta_2) = \dfrac{\partial \eta_1}{\partial a_j} + \dfrac{\partial \eta_2}{\partial a_j}, \quad \eta_1, \eta_2 \in \boldsymbol{Z}F_n,$

(ii) $\dfrac{\partial}{\partial a_j}(w_1 w_2) = \dfrac{\partial w_1}{\partial a_j} + w_1 \dfrac{\partial w_2}{\partial a_j}, \quad w_1, w_2 \in F_n,$

(iii) $\dfrac{\partial a_i}{\partial a_j} = \delta_{i,j}, \quad 1 \leq i, j \leq n,$

where $\delta_{i,j} = 1$ or 0 according to whether $i = j$ or $i \neq j$.

(iv) $\dfrac{\partial e}{\partial a_j} = 0.$

These rules imply that for $v, w \in F_n$,

$$\frac{\partial}{\partial a_j}(vwv^{-1}) = (1 - vwv^{-1})\frac{\partial v}{\partial a_j} + v\frac{\partial w}{\partial a_j}. \tag{7}$$

Given a braid $\beta \in B_n$, let $J(\beta)$ be the Jacobian matrix $(\partial a_i^\beta / \partial a_j)$. As an application of the Reidemeister trace formula, Fadell and Husseini proved that $\mathscr{L}(f) = \Phi_{\beta(f)}(1 - \operatorname{tr} J(\beta(f)))$ ([**7**, Theorem 2.3]). For a matrix A with entries in $\boldsymbol{Z}F_n$, let A^β denote the matrix obtained from A by replacing each entry with its image under β. Then, we have by (5)

$$\Phi_\beta(\operatorname{tr} A'A) = \Phi_\beta(\operatorname{tr} A^\beta A') \tag{8}$$

for any matrices A, A'. Using the chain rule for the Fox calculus, we have $J(\beta\beta') = J(\beta)^{\beta'} J(\beta')$ for any braids β, β'. Let $\beta \in B_n$. Since $a_n^\beta = a_n$, the last row of $J(\beta)$ is $(0 \cdots 01)$. Let $\bar{J}(\beta)$ denote the reduced matrix obtained from $J(\beta)$ by deleting the last column and the last row. Then, $\operatorname{tr} \bar{J}(\beta) = \operatorname{tr} J(\beta) - 1$ and hence we have

$$\mathscr{L}(f) = -\Phi_{\beta(f)}(\operatorname{tr} \bar{J}(\beta(f))). \tag{9}$$

We shall show that $\mathscr{L}(f)$ is determined essentially by $\beta(I)$. Note that

$$\bar{J}(\beta\beta') = \bar{J}(\beta)^{\beta'} \bar{J}(\beta'). \tag{10}$$

Note also that since $a_i^\theta = a_n a_i a_n^{-1}$ for any i, we have $\bar{J}(\theta) = a_n I_{n-1}$, where I_{n-1} is the identity matrix. Therefore, we have that

$$\bar{J}(\theta^\mu \beta) = \bar{J}(\theta^\mu)^\beta \bar{J}(\beta) = a_n^\mu \bar{J}(\beta). \tag{11}$$

We have by (10) that

$$\bar{J}(\gamma\beta(f)) = \bar{J}(\theta^\mu \beta(I)\gamma) = \bar{J}(\theta^\mu \beta(I))^\gamma \bar{J}(\gamma).$$

Also, we have $\bar{J}(\gamma)\bar{J}(\gamma^{-1})^\gamma = \bar{J}(\gamma\gamma^{-1})^\gamma = I_{n-1}$. Therefore, using (8), we have

$$\begin{aligned}
\Phi_{\beta(f)}(\operatorname{tr} \bar{J}(\beta(f))) &= \Phi_{\beta(f)}(\operatorname{tr} \bar{J}(\gamma^{-1})^{\gamma\beta(f)} \bar{J}(\gamma\beta(f))) \\
&= \Phi_{\beta(f)}(\operatorname{tr} \bar{J}(\gamma\beta(f)) \bar{J}(\gamma^{-1})^\gamma) \\
&= \Phi_{\beta(f)}(\operatorname{tr} \bar{J}(\theta^\mu \beta(I))^\gamma \bar{J}(\gamma) \bar{J}(\gamma^{-1})^\gamma) \\
&= \Phi_{\beta(f)}(\operatorname{tr} \bar{J}(\theta^\mu \beta(I))^\gamma).
\end{aligned}$$

Therefore, we have by (9) and (11) that

$$\mathscr{L}(f) = -\Phi_{\beta(f)}(a_n^\mu \operatorname{tr} \bar{J}(\beta(I))^\gamma). \tag{12}$$

Lemma 2.

(i) *Two elements w_1, w_2 of F_n are Reidemeister equivalent with respect to $\beta(I)$ if and only if $a_n^\mu w_1^\gamma, a_n^\mu w_2^\gamma \in F_n$ are Reidemeister equivalent with respect to $\beta(f)$.*

(ii) *Suppose $\eta_1, \eta_2 \in \boldsymbol{Z}F_n$. Then, $\Phi_{\beta(I)}(\eta_1) = \Phi_{\beta(I)}(\eta_2)$ if and only if $\Phi_{\beta(f)}(a_n^\mu \eta_1^\gamma) = \Phi_{\beta(f)}(a_n^\mu \eta_2^\gamma)$.*

Proof.

(i) Suppose $w_1, w_2 \in F_n$ are Reidemeister equivalent with respect to $\beta(I)$. Then, there exists an element $w \in F_n$ such that $w_2 = w^{\beta(I)} w_1 w^{-1}$. Then, since

$$w^{\gamma\beta(f)} a_n^\mu = (w^{\theta^\mu})^{\beta(I)\gamma} a_n^\mu = (a_n^\mu w a_n^{-\mu})^{\beta(I)\gamma} a_n^\mu = a_n^\mu w^{\beta(I)\gamma},$$

we have

$$a_n^\mu w_2^\gamma = a_n^\mu w^{\beta(I)\gamma} w_1^\gamma (w^{-1})^\gamma = (w^\gamma)^{\beta(f)} (a_n^\mu w_1^\gamma)(w^\gamma)^{-1},$$

which shows that $a_n^\mu w_1^\gamma$ and $a_n^\mu w_2^\gamma$ are Reidemeister equivalent with respect to $\beta(f)$.

Conversely, suppose $a_n^\mu w_1^\gamma$ and $a_n^\mu w_2^\gamma$ are Reidemeister equivalent with respect to $\beta(f)$. Then, there exists an element $u \in F_n$ such that $a_n^\mu w_2^\gamma = u^{\beta(f)} a_n^\mu w_1^\gamma u^{-1}$. Let $v = u^{\gamma^{-1}}$. Then, since $u^{\beta(f)} a_n^\mu = v^{\gamma\beta(f)} a_n^\mu = a_n^\mu v^{\beta(I)\gamma}$, we have $a_n^\mu w_2^\gamma = a_n^\mu v^{\beta(I)\gamma} w_1^\gamma (v^{-1})^\gamma$. Therefore, $w_2 = v^{\beta(I)} w_1 v^{-1}$, which shows that w_1 and w_2 are Reidemeister equivalent with respect to $\beta(I)$.

(ii) This follows easily from (i). □

By (12) and Lemma 2(ii), it is enough for the proof of Theorem 1 to show that

$$\Phi_{\beta(I)}(\operatorname{tr} \bar{J}(\beta(I))) = \Phi_{\beta(I)}\left(\sum_{\mathscr{B} \in \mathscr{P}(d)} W_I(\mathscr{B})\right).$$

To prove this equality, we shall compute the matrix $\bar{J}(\beta(I))$. First consider the case where I has length one. For positive integers m, let

$$\Gamma_m = \begin{cases} g_2 + \cdots + g_m & \text{if } m \geq 2, \\ 0 & \text{if } m = 1. \end{cases}$$

Then, we have

Lemma 3. *For positive integers m, let*

$$A_m = \begin{pmatrix} \Gamma_m & -g_m & -(\Gamma_m a_2^{-1} + g_{m-1}) & 0 & \dots & 0 \\ -a_2 & 0 & 1 & 0 & & \vdots \\ -a_3 & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & 1 \\ -a_{n-1} & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then, $\bar{J}(\beta(m)) = A_m^{\beta(m)}$.

Proof. For all $1 \leq i \leq n-1$, we have

$$a_i^{\sigma_1^m} = \begin{cases} a_2^{\frac{m-1}{2}} a_2 a_1^{-1} a_2^{-\frac{m-1}{2}} & \text{if } i = 1,\ m \text{ is odd}, \\ a_2^{\frac{m}{2}} a_1 a_2^{-\frac{m}{2}} & \text{if } i = 1,\ m \text{ is even}, \\ a_i & \text{if } 2 \leq i \leq n-1. \end{cases}$$

Also, $a_i^{\rho} = a_{i+1} a_1^{-1}$. These imply that

$$a_i^{\beta(m)} = \begin{cases} (a_3 a_1^{-1})^{\frac{m-1}{2}} a_3 a_2^{-1} (a_3 a_1^{-1})^{-\frac{m-1}{2}} & \text{if } i = 1,\ m \text{ is odd}, \\ (a_3 a_1^{-1})^{\frac{m}{2}} a_2 a_1^{-1} (a_3 a_1^{-1})^{-\frac{m}{2}} & \text{if } i = 1,\ m \text{ is even}, \\ a_{i+1} a_1^{-1} & \text{if } 2 \leq i \leq n-1. \end{cases} \tag{13}$$

We first compute $\partial a_1^{\beta(m)} / \partial a_j$ for $j = 1, \dots, n-1$. Let $v = (a_3 a_1^{-1})^{[m/2]}$, where $[m/2]$ denotes the largest integer which does not exceed $m/2$. Since $(a_3 a_1^{-1})^r = (a_2^{\beta(m)})^r = -g_{2r}^{\beta(m)}$ for any positive r, we have

$$\frac{\partial v}{\partial a_j} = \begin{cases} -\sum_{r=1}^{[m/2]} (a_3 a_1^{-1})^r = \sum_{r=1}^{[m/2]} g_{2r}^{\beta(m)} & \text{if } j = 1, \\ \sum_{r=0}^{[m/2]-1} (a_3 a_1^{-1})^r = -\frac{\partial v}{\partial a_1} (a_2^{-1})^{\beta(m)} & \text{if } j = 3, \\ 0 & \text{otherwise}. \end{cases} \tag{14}$$

It follows from the definition of g_{2r} that

$$(1-a_1)\sum_{r=1}^{[m/2]} g_{2r} = \begin{cases} \Gamma_m & \text{if } m \text{ is odd,} \\ \Gamma_{m+1} & \text{if } m \text{ is even.} \end{cases} \tag{15}$$

Suppose m is odd. Let $w = a_3 a_2^{-1}$. Then $vwv^{-1} = a_1^{\beta(m)}$ by (13). We shall compute the right-hand side of the equality (7). We have by (14), (15)

$$(1-vwv^{-1})\frac{\partial v}{\partial a_j} = \begin{cases} \Gamma_m^{\beta(m)} & \text{if } j=1, \\ -(\Gamma_m a_2^{-1})^{\beta(m)} & \text{if } j=3, \\ 0 & \text{otherwise,} \end{cases}$$

and, since $v = (a_2^{[m/2]})^{\beta(m)}$, we have

$$v\frac{\partial w}{\partial a_j} = \begin{cases} -vw = -a_1^{\beta(m)} v = -g_m^{\beta(m)} & \text{if } j=2, \\ v = -g_{m-1}^{\beta(m)} & \text{if } j=3, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose m is even. Let $w' = a_2 a_1^{-1}$. Then $vw'v^{-1} = a_1^{\beta(m)}$ by (13). By (14), (15), we have

$$(1-vw'v^{-1})\frac{\partial v}{\partial a_j} = \begin{cases} \Gamma_{m+1}^{\beta(m)} & \text{if } j=1, \\ -(\Gamma_{m+1} a_2^{-1})^{\beta(m)} = -(\Gamma_m a_2^{-1} + g_{m-1})^{\beta(m)} & \text{if } j=3, \\ 0 & \text{otherwise.} \end{cases}$$

$$v\frac{\partial w'}{\partial a_j} = \begin{cases} -vw' = -a_1^{\beta(m)} v = -g_{m+1}^{\beta(m)} & \text{if } j=1, \\ v = -g_m^{\beta(m)} & \text{if } j=2, \\ 0 & \text{otherwise.} \end{cases}$$

These computations and the equality (7) imply that $\partial a_1^{\beta(m)}/\partial a_j$ is equal to the $(1,j)$ entry of the matrix $A_m^{\beta(m)}$ in either case of m odd or even.

The i-th row for $i \geq 2$ is obtained from the following:

$$\frac{\partial a_i^{\beta(m)}}{\partial a_j} = \frac{\partial a_{i+1} a_1^{-1}}{\partial a_j} = \begin{cases} -a_i^{\beta(m)} & \text{if } j=1, \\ 1 & \text{if } j=i+1, \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof. □

Now consider the general case of I having an arbitrary length. Fix a sequence $I = (i_1, \ldots, i_d)$ of positive integers. We shall give formulas for entries of the matrix $\bar{J}(\beta(I))$ in Lemma 5 below. To state these formulas, we need four families of elements of $\boldsymbol{Z}F_n$. The first one is α_q^l defined for integers q, l. For integers $1 \le l \le d$, denote $\beta_l(I)$ simply by β_l, and let $\beta_{d+1} = e$. Then, $\alpha_q^l \in \boldsymbol{Z}F_n$ is defined for integers q, l by

$$\alpha_q^l = \begin{cases} a_q^{\beta_l} & \text{if } 1 \le q \le n-1, 1 \le l \le d, \\ 0 & \text{otherwise.} \end{cases}$$

The second family is $W_k^l \in \boldsymbol{Z}F_n$ defined for positive integers k, l. To define these elements, we need to generalize the notion of a partition given in Definition 2 to the non-cyclic setting.

Definition 3. Suppose k, l are positive integers with $k \le l$.

(i) For integers p, q with $k \le p \le q \le l$, define a sequence $[p, q]$ of positive integers by $[p, q] = (p, \ldots, q)$. This sequence is called a *block* in $\{k, \ldots, l\}$, and the number of integers contained in it is called its *length*. For a block B, let $\underline{B}$ denote its underlying set.

(ii) A set $\{B_1, \ldots, B_s\}$ of blocks in $\{k, \ldots, l\}$ is a *partition* of $\{k, \ldots, l\}$ if $\underline{B_1}, \ldots, \underline{B_s}$ are mutually disjoint, $\underline{B_1} \cup \cdots \cup \underline{B_s} = \{k, \ldots, l\}$, and $B_1, \ldots, B_s$ have length less than or equal to $n-1$.

(iii) Let $\mathscr{P}(k, l)$ denote the set of partitions of $\{k, \ldots, l\}$.

For a subset $\mathscr{A}$ of $\mathscr{P}(d)$ or of $\mathscr{P}(k, l)$, where $1 \le k \le l \le d$, let $W_I(\mathscr{A}) = \sum_{\mathscr{B} \in \mathscr{A}} W_I(\mathscr{B})$. Then, $W_k^l \in \boldsymbol{Z}F_n$ is defined for positive integers k, l by

$$W_k^l = \begin{cases} W_I(\mathscr{P}(k, l)) & \text{if } k \le l \le d, \\ 1 & \text{if } k = l+1 \text{ and } l \le d \\ 0 & \text{otherwise.} \end{cases}$$

We prepare the next lemma, which will be used to prove Lemma 5. For $1 \le l \le d$, $1 \le \lambda \le n-1$, let $\mathscr{P}_{1,\lambda}(l)$ be the set of partitions of $\{1, \ldots, l\}$ such that the block with initial element 1 has length λ, and let $\mathscr{P}_{l,\lambda}(d)$ be the set of partitions of $\boldsymbol{Z}_d$ which contain a block with initial element l and length λ.

Lemma 4.

(i) $g_m^{\beta(m)} = -g_{m-1}^{\beta(m)} a_3 a_2^{-1}$ *for any positive integer* m.

(ii) *For $1 \le l \le d$, we have*

$$\sum_{u\ge 0}\Big(g_{i_1}^{\beta_1}\alpha_{2+u}^{2+u} + g_{i_1-1}^{\beta_1}\alpha_{3+u}^{2+u}\Big)W_{3+u}^l = W_I(\mathscr{P}_{1,n-1}(l)).$$

(iii) *For positive integers l with $d+3-n \le l \le d$, the elements*

$$\sum_{u\ge 0}\Big(\alpha_{d+2-l+u}^{1+u}W_{2+u}^{l-1}g_{i_l}^{\beta_l} + \alpha_{d+3-l+u}^{1+u}W_{2+u}^{l-1}g_{i_l-1}^{\beta_l}\Big)$$

and $W_I(\mathscr{P}_{l,n-1}(d))$ have the same $\Phi_{\beta(I)}$-image.

Proof. (i) Consider the case of m odd. Since $a_3a_1^{-1} = a_2^{\beta(m)}$, we have

$$(a_3a_1^{-1})^{(m-1)/2} = (a_2^{\beta(m)})^{(m-1)/2} = (a_2^{(m-1)/2})^{\beta(m)} = c_{m-1}^{\beta(m)} = -g_{m-1}^{\beta(m)}.$$

Therefore, we have by (13), $a_1^{\beta(m)} = -g_{m-1}^{\beta(m)}a_3a_2^{-1}(c_{m-1}^{\beta(m)})^{-1}$ and hence

$$g_m^{\beta(m)} = (a_1a_2^{(m-1)/2})^{\beta(m)} = a_1^{\beta(m)}c_{m-1}^{\beta(m)} = -g_{m-1}^{\beta(m)}a_3a_2^{-1}.$$

In the case of m even, we have by (13) the desired equality from the following:

$$\begin{aligned} g_{m-1}^{\beta(m)} &= (a_1a_2^{m/2}a_2^{-1})^{\beta(m)} \\ &= \Big[c_m^{\beta(m)}a_2a_1^{-1}(c_m^{\beta(m)})^{-1}\Big]c_m^{\beta(m)}(a_2^{-1})^{\beta(m)} \\ &= c_m^{\beta(m)}a_2a_1^{-1}(a_3a_1^{-1})^{-1} \\ &= c_m^{\beta(m)}a_2a_3^{-1} = -g_m^{\beta(m)}a_2a_3^{-1}. \end{aligned}$$

(ii) Let Σ_1 be the left-hand side of the equality (ii). Let

$$V(u) = g_{i_1}^{\beta_1}\alpha_{2+u}^{2+u}W_{3+u}^l, \quad V'(u) = g_{i_1-1}^{\beta_1}\alpha_{3+u}^{2+u}W_{3+u}^l.$$

Then, we have

$$\Sigma_1 = \sum_{u\ge 0}(V(u) + V'(u)).$$

There are three cases:

(a) $2+u > l$ or $2+u > n-1$,
(b) $2+u \le l$ and $2+u < n-1$,

(c) $2+u \leq l$ and $2+u = n-1$.

Consider Case (a). If $2+u > l$, then $3+u > l+1$ and so $W_{3+u}^l = 0$. Also, if $2+u > n-1$, then $\alpha_{2+u}^{2+u} = \alpha_{3+u}^{2+u} = 0$. Therefore, we have $V(u) = V'(u) = 0$. Consider Case (b). For any $i \geq 2$, $1 \leq l \leq d$, and any positive integer u with $l+u \leq d+1$ and $i+1+u \leq n-1$, we have by the equality $a_i^{\beta(m)} = a_{i+1}a_1^{-1}$ for any m that

$$(a_{i+1}a_i^{-1})^{\beta_l} = (a_{i+1}a_i^{-1})^{\beta(i_l,\dots,i_{l+u-1})\beta_{l+u}} = (a_{i+1+u}a_{i+u}^{-1})^{\beta_{l+u}}. \tag{16}$$

We have by (i) of this lemma and (16) that

$$\begin{aligned} g_{i_1}^{\beta_1} &= (g_{i_1}^{\beta(i_1)})^{\beta_2} = -(g_{i_1-1}^{\beta(i_1)}a_3a_2^{-1})^{\beta_2} \\ &= -g_{i_1-1}^{\beta_1}a_{3+u}^{\beta_{2+u}}(a_{2+u}^{\beta_{2+u}})^{-1}. \end{aligned}$$

This implies that $V(u) + V'(u) = 0$. Consider Case (c). Since $3+u = n$, we have $\alpha_{3+u}^{2+u} = 0$ and hence $V'(u) = 0$. Also, since $n-1 = 2+u \leq l \leq d$ and hence $\alpha_{2+u}^{2+u} = a_{n-1}^{\beta_{n-1}}$, we have

$$V(u) = g_{i_1}^{\beta_1}a_{n-1}^{\beta_{n-1}}W_n^l = W_I([1,n-1])W_n^l = W_I(\mathscr{P}_{1,n-1}(l)).$$

If $l \geq n-1$, putting these computations together, we have $\Sigma_1 = W_I(\mathscr{P}_{1,n-1}(l))$, and so (ii) holds. Suppose $l < n-1$. Then, $\Sigma_1 = 0$ since Case (c) does not occur, and we have $W_I(\mathscr{P}_{1,n-1}(l)) = 0$ since $\mathscr{P}_{1,n-1}(l)$ is empty. Therefore, the equality (ii) is proved.

(iii) Let l be a positive integer with $d+3-n \leq l \leq d$. Note that $d+2-l \leq n-1$. Let

$$V_l(u) = \alpha_{d+2-l+u}^{1+u}W_{2+u}^{l-1}g_{i_l}^{\beta_l}, \quad V_l'(u) = \alpha_{d+3-l+u}^{1+u}W_{2+u}^{l-1}g_{i_{l-1}}^{\beta_l},$$

and let

$$\Sigma_l = \sum_{u\geq 0}(V_l(u) + V_l'(u)).$$

There are three cases:

(a) $2+u > l$ or $d+2-l+u > n-1$,
(b) $2+u \leq l$ and $d+2-l+u < n-1$,

(c) $2+u \le l$ and $d+2-l+u = n-1$.

Consider Case (a). If $2+u > l$, then $W_{2+u}^{l-1} = 0$. Also, if $d+2-l+u > n-1$, then $\alpha_{d+2-l+u}^{1+u} = \alpha_{d+3-l+u}^{1+u} = 0$. Therefore, we have $V_l(u) = V_l'(u) = 0$. In Case (b), we have by (i) of this lemma and (16) that

$$\begin{aligned} g_{i_l}^{\beta_l} &= (g_{i_l}^{\beta(i_l)})^{\beta_{l+1}} = -g_{i_l-1}^{\beta_l}(a_3 a_2^{-1})^{\beta_{l+1}\beta_1\beta_1^{-1}} \\ &= -g_{i_l-1}^{\beta_l}(a_{3+d-l}a_{2+d-l}^{-1})^{\beta_1\beta_1^{-1}} = -g_{i_l-1}^{\beta_l}(a_{d+3-l+u}^{\beta_{1+u}}(a_{d+2-l+u}^{\beta_{1+u}})^{-1})^{\beta_1^{-1}}. \end{aligned}$$

Therefore, noting that $\beta_1 = \beta(I)$, we have by (5),

$$\begin{aligned} \Phi_{\beta(I)}(V_l(u)) &= \Phi_{\beta(I)}(W_{2+u}^{l-1} g_{i_l}^{\beta_l}(a_{d+2-l+u}^{\beta_{1+u}})^{\beta_1^{-1}}) \\ &= -\Phi_{\beta(I)}(W_{2+u}^{l-1} g_{i_l-1}^{\beta_l}(a_{d+3-l+u}^{\beta_{1+u}})^{\beta_1^{-1}}) = -\Phi_{\beta(I)}(V_l'(u)). \end{aligned}$$

Therefore, $\Phi_{\beta(I)}(V_l(u) + V_l'(u)) = 0$. In Case (c), clearly $V_l'(u) = 0$. Since $\alpha_{d+2-l+u}^{1+u} = a_{n-1}^{\beta_{1+u}}$ and the length of the block $[l, 1+u]$ is $(d-l+1)+1+u = n-1$, we have

$$\begin{aligned} \Phi_{\beta(I)}(V_l(u)) &= \Phi_{\beta(I)}(a_{n-1}^{\beta_{1+u}} W_{2+u}^{l-1} g_{i_l}^{\beta_l}) = \Phi_{\beta(I)}(W_{2+u}^{l-1} g_{i_l}^{\beta_l} a_{n-1}^{\beta_{1+u}\beta_1^{-1}}) \\ &= \Phi_{\beta(I)}(W_{2+u}^{l-1} W_I([l, 1+u])) \\ &= \Phi_{\beta(I)}(W_I(\mathscr{P}_{l,n-1}(d))). \end{aligned}$$

If $d < n-1$, Case (c) does not occur since $d+2-l+u = n-1$ implies $2+u = (n-1)-d+l > l$. Therefore, (iii) is proved by summing up these computations. $\square$

The last two families of elements of $\boldsymbol{Z}F_n$ necessary to state Lemma 5 are S_l, G_l defined for integers l as follows:

$$S_l = \begin{cases} (\Gamma_{i_l} a_2^{-1} + g_{i_l-1})^{\beta_l} & \text{if } 1 \le l \le d, \\ 0 & \text{otherwise}, \end{cases}$$

$$G_l = \begin{cases} g_{i_l}^{\beta_l} & \text{if } 1 \le l \le d, \\ -1 & \text{if } l = d+1, \\ 0 & \text{otherwise}. \end{cases}$$

Let $r_{i,j}(I)$ be the (i,j)-entry of the matrix $\bar{J}(\beta(I))$.

LEMMA 5.

$$r_{i,j}(I) = \begin{cases} -W_1^{d+2-j} S_{d+3-j} - W_1^{d+1-j} G_{d+2-j} & \text{if } i = 1, \\ \sum_{u \geq 0} \alpha_{i+u}^{1+u} \Big(W_{2+u}^{d+2-j} S_{d+3-j} + W_{2+u}^{d+1-j} G_{d+2-j} \Big) + \delta_{i,j-d} & \text{if } i \geq 2. \end{cases}$$

PROOF. We prove this lemma by induction on d. The case of $d = 1$ follows easily from Lemma 3. Assume that the lemma holds for $d-1$, and we shall prove it for d. Let $I = (i_1, \ldots, i_d)$ be a sequence of positive integers. Let $I' = (i_2, \ldots, i_d)$. Then, $r_{i,j}(I')$ is obtained from the right-hand side of this lemma by replacing α_{i+u}^{1+u} and $\delta_{i,j-d}$ with α_{i+u}^{2+u} and $\delta_{i,j-(d-1)}$ respectively, and by adding one to the subscript of each of W's. Note that by (10) and Lemma 3

$$\bar{J}(\beta(I)) = \bar{J}(\beta(i_1))^{\beta(I')} \bar{J}(\beta(I')) = A_{i_1}^{\beta_1} \bar{J}(\beta(I')). \tag{17}$$

Consider the case of $i = 1$. Let

$$M(l) = \Gamma_{i_1}^{\beta_1} W_2^l + (\Gamma_{i_1} a_2^{-1})^{\beta_1} \sum_{u \geq 0} \alpha_{3+u}^{2+u} W_{3+u}^l + W_I(\mathscr{P}_{1,n-1}(l))$$

for $l \geq 1$ and $M(l) = 0$ for $l \leq 0$. Then, we have by (17), Lemma 3, and Lemma 4(ii) that $r_{1,j}(I)$ is equal to

$$\begin{aligned}
&\Gamma_{i_1}^{\beta_1} r_{1,j}(I') - g_{i_1}^{\beta_1} r_{2,j}(I') - (\Gamma_{i_1} a_2^{-1} + g_{i_1-1})^{\beta_1} r_{3,j}(I') \\
&\quad = -\Gamma_{i_1}^{\beta_1} \Big(W_2^{d+2-j} S_{d+3-j} + W_2^{d+1-j} G_{d+2-j} \Big) \\
&\qquad - g_{i_1}^{\beta_1} \Bigg[\sum_{u \geq 0} \Big(\alpha_{2+u}^{2+u} W_{3+u}^{d+2-j} S_{d+3-j} + \alpha_{2+u}^{2+u} W_{3+u}^{d+1-j} G_{d+2-j} \Big) + \delta_{2,j-(d-1)} \Bigg] \\
&\qquad - (\Gamma_{i_1} a_2^{-1} + g_{i_1-1})^{\beta_1} \Bigg[\sum_{u \geq 0} \Big(\alpha_{3+u}^{2+u} W_{3+u}^{d+2-j} S_{d+3-j} + \alpha_{3+u}^{2+u} W_{3+u}^{d+1-j} G_{d+2-j} \Big) + \delta_{3,j-(d-1)} \Bigg] \\
&\quad = -(M(d+2-j) S_{d+3-j} + \delta_{j,d+2} S_1) - (M(d+1-j) G_{d+2-j} + \delta_{j,d+1} G_1).
\end{aligned}$$

Therefore, since $\delta_{j,d+2} S_1 = \delta_{j,d+2} S_{d+3-j}$ and $\delta_{j,d+1} G_1 = \delta_{j,d+1} G_{d+2-j}$, we have

$$r_{1,j}(I) = -(M(d+2-j) + \delta_{j,d+2}) S_{d+3-j} - (M(d+1-j) + \delta_{j,d+1}) G_{d+2-j}.$$

Since $M(l)=\sum_{\lambda=1}^{n-1} W_I(\mathscr{P}_{1,\lambda}(l))=W_1^l$ if $l\geq 1$ and $W_1^0=1$, this is equal to $-W_1^{d+2-j}S_{d+3-j}-W_1^{d+1-j}G_{d+2-j}$, which is the right-hand side of the equality of the lemma in the case of $i=1$.

Consider the case of $i\geq 2$. We have

$$\begin{aligned}r_{i,j}(I)&=-a_i^{\beta_1}r_{1,j}(I')+r_{i+1,j}(I')\\&=a_i^{\beta_1}(W_2^{d+2-j}S_{d+3-j}+W_2^{d+1-j}G_{d+2-j})\\&\quad+\sum_{u\geq 0}\alpha_{i+1+u}^{2+u}(W_{3+u}^{d+2-j}S_{d+3-j}+W_{3+u}^{d+1-j}G_{d+2-j})+\delta_{i+1,j-(d-1)}.\end{aligned}$$

It is easy to show that this is equal to the right-hand side of the equality in the lemma in the case of $i\geq 2$. $\square$

We shall complete the proof of Theorem 1. Note that $S_{d+1}=S_{d+2}=0$. Let

$$L_1=\sum_{j=3}^{\nu_1}\sum_{u\geq 0}\alpha_{j+u}^{1+u}W_{2+u}^{d+2-j}(\Gamma_{i_{d+3-j}}a_2^{-1})^{\beta_{d+3-j}},$$

$$L_2=\sum_{j=3}^{\nu_2}\sum_{u\geq 0}\alpha_{j+u}^{1+u}W_{2+u}^{d+2-j}g_{i_{d+3-j}-1}^{\beta_{d+3-j}},$$

$$L_3=\sum_{j=2}^{\nu_3}\sum_{u\geq 0}\alpha_{j+u}^{1+u}W_{2+u}^{d+1-j}g_{i_{d+2-j}}^{\beta_{d+2-j}},$$

where $\nu_1=\nu_2=\min\{n-1,d+1\},\nu_3=\min\{n-1,d\}$. Then, by Lemma 5,

$$\operatorname{tr}(\bar{J}(\beta(I)))=W_1^d+L_1+L_2+L_3.$$

Let $d_1=d+3-\nu_1=\max\{d+4-n,2\},d_2=d+2-\nu_3=\max\{d+3-n,2\}$. In L_2, we can change ν_2 to $\bar{\nu}_2=\min\{n,d+1\}$, since $\alpha_{n+u}^{1+u}=0$. Then, $d+3-\bar{\nu}_2=d_2$. Putting $l=d+3-j$ in L_1 and L_2, and putting $l=d+2-j$ in L_3, we have

$$L_1=\sum_{l=d_1}^{d}\sum_{u\geq 0}\alpha_{d+3-l+u}^{1+u}W_{2+u}^{l-1}(\Gamma_{i_l}a_2^{-1})^{\beta_l},$$

$$L_2+L_3=\sum_{l=d_2}^{d}\sum_{u\geq 0}\Big(\alpha_{d+3-l+u}^{1+u}W_{2+u}^{l-1}g_{i_l-1}^{\beta_l}+\alpha_{d+2-l+u}^{1+u}W_{2+u}^{l-1}g_{i_l}^{\beta_l}\Big).$$

Let

$$Q_1 = \bigcup_{l=d_1}^{d} \bigcup_{\lambda=d+2-l}^{n-2} \mathscr{P}_{l,\lambda}(d), \quad Q_2 = \bigcup_{l=d_2}^{d} \mathscr{P}_{l,n-1}(d).$$

Since $\Phi_{\beta(I)}(\alpha_{d+3-l+u}^{1+u} W_{2+u}^{l-1}(\Gamma_{i_l} a_2^{-1})^{\beta_l})$ is equal to

$$\Phi_{\beta(I)}(W_{2+u}^{l-1} W_I([l, 1+u])) = \Phi_{\beta(I)}(W_I(\mathscr{P}_{l,d+2-l+u}(d)))$$

if $d+2-l+u \leq n-2$, and equal to zero otherwise since $\alpha_m^{1+u} = 0$ for $m \geq n$, we have

$$\Phi_{\beta(I)}(L_1) = \sum_{l=d_1}^{d} \sum_{\lambda=d+2-l}^{n-2} \Phi_{\beta(I)}(W_I(\mathscr{P}_{l,\lambda}(d))) = \Phi_{\beta(I)}(W_I(Q_1)).$$

Lemma 4(iii) implies that

$$\Phi_{\beta(I)}(L_2 + L_3) = \sum_{l=d_2}^{d} \Phi_{\beta(I)}(W_I(\mathscr{P}_{l,n-1}(d))) = \Phi_{\beta(I)}(W_I(Q_2)).$$

Since $W_1^d = W_I(\mathscr{P}(1,d))$ and $\mathscr{P}(d) = \mathscr{P}(1,d) \cup Q_1 \cup Q_2$, these equalities prove that $\Phi_{\beta(I)}(\mathrm{tr}(\bar{J}(\beta(I)))$ is equal to $\Phi_{\beta(I)}(W_I(\mathscr{P}(d)))$. Thus the proof of the theorem is completed by (12) and Lemma 2(ii).

6. Reduction of the formula.

This section makes preparations for the proof of Proposition 2. We shall show that, under the assumption of Proposition 2, the element $\sum_{\mathscr{B} \in \mathscr{P}(d)} W_I(\mathscr{B})$ of $\boldsymbol{Z}F_n$ in the formula of Theorem 1 can be reduced so that Reidemeister equivalent elements of F_n have the same coefficient. Hence, no cancellation occurs when the reduced one is projected on $\boldsymbol{Z}\mathscr{R}(\beta(f))$, which enables us to apply Lemma 1 to the problem.

Consider first the case where $n \geq 4$ and $i_1, \ldots, i_d \geq 2$. Let $\mathscr{P}'(d)$ be the set of partitions $\mathscr{B} = \{B_1, \ldots, B_s\}$ of $\boldsymbol{Z}_d$ such that $(|B_j|, |B_{j+1}|) \neq (1, n-2)$ for any $1 \leq j \leq s$, where $B_{s+1} = B_1$. For partitions $\mathscr{B} \in \mathscr{P}'(d)$, we shall define elements $W_I'(\mathscr{B})$ of $\boldsymbol{Z}F_n$ which satisfy that the sums $\sum_{\mathscr{B} \in \mathscr{P}(d)} W_I(\mathscr{B})$ and $\sum_{\mathscr{B} \in \mathscr{P}'(d)} W_I'(\mathscr{B})$ have the same image under the projection $\Phi_{\beta(I)}$. Suppose $B = [p, q]$ is a block. If $|B| < n-1$, let $S_B(I)$ denote the set of integers J with $0 \leq J \leq i_p - 2$, and let $\lambda_B(J) = c_J^{\alpha(B)} a_{|B|+1}^{\omega(B)} \in F_n$ for any $J \in S_B(I)$. If $|B| = n-1$, let $S_B(I)$ denote the set of $(j, j') \in \boldsymbol{Z}^2$ such that $2 \le j \le i_p$, $0 \le j' \le i_{p'} - 2$, $(j, j') \neq (i_p, 0)$, and let $\lambda_B(J) = c_j^{\alpha(B)} c_{j'}^{\alpha'(B)} a_{n-1}^{\omega(B)} \in F_n$ for $J = (j, j') \in S_B(I)$, where $\alpha'(B) \in B_n$ is defined by

$$\alpha'(B) = \begin{cases} \beta_{p+1} & \text{if } p \le d-1, \\ e & \text{if } p = d. \end{cases}$$

For a partition $\mathscr{B} = \{B_1, \dots, B_s\}$, let

$$S_{\mathscr{B}}(I) = S_{B_1}(I) \times \cdots \times S_{B_s}(I).$$

For an element $\mathscr{J} = (J_1, \dots, J_s)$ of $S_{\mathscr{B}}(I)$, define $\lambda_{\mathscr{B}}(\mathscr{J}) \in F_n$ by $\lambda_{\mathscr{B}}(\mathscr{J}) = \lambda_{B_1}(J_1) \cdots \lambda_{B_s}(J_s)$. For a block $B = [p, q]$, define $W_I'(B) \in \boldsymbol{Z}F_n$ by

$$W_I'(B) = \begin{cases} W_I(B) = \displaystyle\sum_{J \in S_B(I)} g_J^{\alpha(B)} a_{|B|+1}^{\omega(B)} & \text{if } |B| < n-1, \\ \displaystyle\sum_{(j,j') \in S_B(I)} g_j^{\alpha(B)} g_{j'}^{\alpha'(B)} a_{n-1}^{\omega(B)} & \text{if } |B| = n-1. \end{cases}$$

Then, $W_I'(\mathscr{B}) \in \boldsymbol{Z}F_n$ is defined for $\mathscr{B} \in \mathscr{P}'(d)$ by $W_I'(\mathscr{B}) = W_I'(B_1) \cdots W_I'(B_s)$, where $\mathscr{B} = \{B_1, \dots, B_s\}$ with $1 \le p_1 < \cdots < p_s \le d$.

For $w \in F_n$, define an integer $e(w)$ as the exponent sum of w with respect to the standard generators $\xi_1, \dots, \xi_n$. Note that $e(w)$ can be defined also by $\mathscr{T}(w) = t^{e(w)}$, where $\mathscr{T}$ is the ring homomorphism introduced in Example 4.

LEMMA 6. *Let $n \ge 4$. Assume $\beta(f) = \gamma^{-1}\theta^{\mu}\beta(I)\gamma$, where $\mu \in \boldsymbol{Z}, \gamma \in B_n$, $I = (i_1, \dots, i_d)$ with $i_1, \dots, i_d \ge 2$. Then, we have*

(i)

$$\mathscr{L}(f) = -\Phi_{\beta(f)}\left(a_n^{\mu} \sum_{\mathscr{B} \in \mathscr{P}'(d)} W_I'(\mathscr{B})^{\gamma} \right).$$

(ii) *For $\mathscr{B} \in \mathscr{P}'(d)$, we have*

$$W_I'(\mathscr{B}) = \sum_{\mathscr{J} \in S_{\mathscr{B}}(I)} (-1)^{d + e(\lambda_{\mathscr{B}}(\mathscr{J}))} \lambda_{\mathscr{B}}(\mathscr{J}).$$

(iii) *For any* $\mathscr{B} \in \mathscr{P}'(d)$ *and any* $\mathscr{J} \in S_{\mathscr{B}}(I)$, *the coefficient of* $\Phi_{\beta(f)}(a_n^{\mu}\lambda_{\mathscr{B}}(\mathscr{J})^{\gamma})$ *in* $\mathscr{L}(f)$ *is non-zero.*

Proof.

(i) For a partition $\mathscr{B}$, let $K(\mathscr{B})$ be the set of integers $k \in \boldsymbol{Z}_d$ such that either $[k, k+n-2]$ is a block in $\mathscr{B}$, or both (k) and $[k+1, k+n-2]$ are blocks in $\mathscr{B}$, where integers are taken modulo d. Let $\mathscr{K}(d)$ be the set of subsets K of $\boldsymbol{Z}_d$ such that, if K is written as $K = \{k_1, \ldots, k_t\}$, where $1 \leq k_1 < \cdots < k_t \leq d$, then $k_{r+1} - k_r \geq n-1$ for any $1 \leq r \leq t$, where we put $k_{t+1} = k_1 + d$. We assume that the empty set is contained in $\mathscr{K}(d)$. Note that a subset K of $\boldsymbol{Z}_d$ is contained in $\mathscr{K}(d)$ if and only if there is a partition $\mathscr{B}$ with $K(\mathscr{B}) = K$. For $K \in \mathscr{K}(d)$, let $\mathscr{P}_K$ be the set of partitions $\mathscr{B}$ with $K(\mathscr{B}) = K$.

Assume $d \geq n-1$. Let $K = \{k_1, \ldots, k_t\} \in \mathscr{K}(d)$, where $k_1 < \cdots < k_t$. For $1 \leq r \leq t$, let $B(r) = [k_r, k_r + n - 2]$ and let $X_r = W_I(B(r)) \in \boldsymbol{Z}F_n$. Also, define $Y_r \in \boldsymbol{Z}F_n$ by

$$Y_r = \begin{cases} W_I((d))W_I([1, n-2])^{\beta_1^{-1}} & \text{if } r = t, k_t = d, \\ W_I((k_r))W_I([k_r+1, k_r+n-2]) & \text{otherwise.} \end{cases}$$

For $1 \leq k, l \leq d$ with $k \leq l+1$, define $Z(k,l) \in \boldsymbol{Z}F_n$ by $Z(k,l) = W_I(\mathscr{P}(k,l))$ if $k \leq l$ and $Z(k,l) = e$ if $k = l+1$. For r with $1 \leq r < t$, let $Z_r = Z(k_r + n - 1, k_{r+1} - 1)$. If $k_t + n - 2 < d$, let $Z_0 = Z(1, k_1 - 1)$ and $Z_t = Z(k_t + n - 1, d)$. If $k_t + n - 2 \geq d$, let $Z_0 = Z(k_t + n - 1 - d, k_1 - 1)$ and $Z_t = e$. Let

$$\Lambda'_K = \prod_{r=1}^{t-1}(X_r + Y_r)Z_r, \quad \Lambda_K = Z_0\Lambda'_K(X_t + Y_t)Z_t.$$

In the case of $k_t < d$, it is easy to see that $\sum_{\mathscr{B}\in\mathscr{P}_K} W_I(\mathscr{B}) = \Lambda_K$. In the case of $k_t = d$, we have $\sum_{\mathscr{B}\in\mathscr{P}_K} W_I(\mathscr{B}) = Z_0\Lambda'_K X_t + W_I([1, n-2])Z_0\Lambda'_K W_I((d))$. By (5), this has the same image as Λ_K under the projection $\Phi_{\beta(I)}$. Therefore, in either case, we have

$$\Phi_{\beta(I)}(\textstyle\sum_{\mathscr{B}\in\mathscr{P}_K} W_I(\mathscr{B})) = \Phi_{\beta(I)}(\Lambda_K). \tag{18}$$

Note that $W_I(B) = W'_I(B)$ for any block B in $\mathscr{B} \in \mathscr{P}_K$ with $\underline{B}$ disjoint from $\underline{B(1)} \cup \cdots \cup \underline{B(t)}$. Also, letting $\alpha_r = \alpha(B(r)), \alpha'_r = \alpha'(B(r))$ and $\omega_r = \omega(B(r))$, we have

$$\begin{aligned} X_r + Y_r &= g_{i_{k_r}}^{\alpha_r} a_{n-1}^{\omega_r} + \Gamma_{i_{k_r}}^{\alpha_r} (\Gamma_{i_{k_r+1}} a_2^{-1})^{\alpha'_r} a_{n-1}^{\omega_r} \\ &= \sum_{(j,j')\in S_{B(r)}(I)} g_j^{\alpha_r} g_{j'}^{\alpha'_r} a_{n-1}^{\omega_r} \\ &= W'_I(B(r)). \end{aligned}$$

Hence, we have the equality

$$\Lambda_K = \sum_{\mathscr{B}\in\mathscr{P}_K\cap\mathscr{P}'(d)} W'_I(\mathscr{B}).$$

This and (18) imply that $\sum_{\mathscr{B}\in\mathscr{P}_K} W_I(\mathscr{B})$ and $\sum_{\mathscr{B}\in\mathscr{P}_K\cap\mathscr{P}'(d)} W'_I(\mathscr{B})$ have the same image under $\Phi_{\beta(I)}$. Furthermore, since the disjoint unions $\cup_{K\in\mathscr{K}(d)}\mathscr{P}_K$ and $\cup_{K\in\mathscr{K}(d)}(\mathscr{P}_K\cap\mathscr{P}'(d))$ coincide with $\mathscr{P}(d)$ and $\mathscr{P}'(d)$ respectively, we have

$$\sum_{\mathscr{B}\in\mathscr{P}(d)} W_I(\mathscr{B}) = \sum_{K\in\mathscr{K}(d)} \sum_{\mathscr{B}\in\mathscr{P}_K} W_I(\mathscr{B}), \quad \sum_{K\in\mathscr{K}(d)} \sum_{\mathscr{B}\in\mathscr{P}_K\cap\mathscr{P}'(d)} W'_I(\mathscr{B}) = \sum_{\mathscr{B}\in\mathscr{P}'(d)} W'_I(\mathscr{B}).$$

Therefore, $\sum_{\mathscr{B}\in\mathscr{P}(d)} W_I(\mathscr{B})$ and $\sum_{\mathscr{B}\in\mathscr{P}'(d)} W'_I(\mathscr{B})$ have the same image under $\Phi_{\beta(I)}$, and (i) follows from Theorem 1.

In the case of $d < n-1$, it is trivial that $\mathscr{P}'(d) = \mathscr{P}(d)$ and $W'_I(\mathscr{B}) = W_I(\mathscr{B})$ for any partition $\mathscr{B}$. Hence, the formula (i) is identical with that in Theorem 1.

(ii) Let B be a block. Then, by the definition of $\lambda_B(J)$, we see that $W'_I(B)$ is written in the form $W'_I(B) = \sum_{J\in S_B(I)} \epsilon(J)\lambda_B(J)$, where $\epsilon(J)$ are integers. We have

$$\epsilon(J) = (-1)^{|B|+e(\lambda_B(J))}. \tag{19}$$

In fact, if $|B| < n-1$, we have $\epsilon(J) = (-1)^{J+1}$ and this is equal to $(-1)^{|B|+e(\lambda_B(J))}$ since $e(\lambda_B(J)) = J + |B| + 1$. Also, if $|B| = n-1$, $\epsilon(J) = (-1)^{j+j'}$ and this is equal to $(-1)^{|B|+e(\lambda_B(J))}$ since $e(\lambda_B(J)) = j + j' + n - 1 = j + j' + |B|$. Let $\mathscr{B} = \{B_1, \dots, B_s\} \in \mathscr{P}'(d)$ and $\mathscr{J} = \{J_1, \dots, J_s\} \in S_{\mathscr{B}}(I)$. Then, since $|B_1| + \cdots + |B_s| = d$ and $e(\lambda_{B_1}(J_1)) + \cdots + e(\lambda_{B_s}(J_s)) = e(\lambda_{\mathscr{B}}(\mathscr{J}))$, the coefficient of $\lambda_{\mathscr{B}}(\mathscr{J})$ in $W'_I(\mathscr{B})$ is equal to $\epsilon(J_1)\cdots\epsilon(J_s)$, which is equal to $(-1)^{d+e(\lambda_{\mathscr{B}}(\mathscr{J}))}$ by (19). Thus, (ii) is proved.

(iii) Let $\Gamma(I)$ be the set of pairs $(\mathscr{B}, \mathscr{J})$ with $\mathscr{B} \in \mathscr{P}'(d)$, $\mathscr{J} \in S_{\mathscr{B}}(I)$. We say two elements $(\mathscr{B}, \mathscr{J})$, $(\mathscr{B}', \mathscr{J}') \in \Gamma(I)$ are equivalent if $\lambda_{\mathscr{B}}(\mathscr{J})$ is Reidemeister equivalent to $\lambda_{\mathscr{B}'}(\mathscr{J}')$ with respect to $\beta(I)$. This defines an equivalence relation on $\Gamma(I)$. Denote by $[(\mathscr{B}, \mathscr{J})]$ the equivalence class represented by $(\mathscr{B}, \mathscr{J})$. Let $n(\mathscr{B}, \mathscr{J})$ be the coefficient of $\Phi_{\beta(f)}(a_n^{\mu}\lambda_{\mathscr{B}}(\mathscr{J})^{\gamma})$ in $-\mathscr{L}(f)$. Then, by (i), (ii) of this lemma and Lemma 2(i), $n(\mathscr{B}, \mathscr{J})$ is equal to the sum of the coefficient of $\lambda_{\mathscr{B}'}(\mathscr{J}')$ in $W_I'(\mathscr{B}')$ taken over the elements $(\mathscr{B}', \mathscr{J}')$ of $[(\mathscr{B}, \mathscr{J})]$. Since this coefficient is equal to $(-1)^{d+e(\lambda_{\mathscr{B}'}(\mathscr{J}'))}$ by (ii), we have

$$n(\mathscr{B}, \mathscr{J}) = \sum_{(\mathscr{B}', \mathscr{J}') \in [(\mathscr{B}, \mathscr{J})]} (-1)^{d+e(\lambda_{\mathscr{B}'}(\mathscr{J}'))}. \tag{20}$$

For any $(\mathscr{B}', \mathscr{J}') \in [(\mathscr{B}, \mathscr{J})]$, we have $e(\lambda_{\mathscr{B}'}(\mathscr{J}')) = e(\lambda_{\mathscr{B}}(\mathscr{J}))$, since $\lambda_{\mathscr{B}'}(\mathscr{J}')$ is Reidemeister equivalent to $\lambda_{\mathscr{B}}(\mathscr{J})$ and the exponent sum of an element of F_n is preserved under the action of B_n on F_n. Therefore, (20) implies that $n(\mathscr{B}, \mathscr{J})$ is equal to $(-1)^{d+e(\lambda_{\mathscr{B}}(\mathscr{J}))}\sharp[(\mathscr{B}, \mathscr{J})]$, which is clearly non-zero. $\square$

Consider next the case where $n = 3$ and $i_1, \ldots, i_d \geq 3$. Let $\boldsymbol{Z}^d(I)$ denote the set of $J = (j_1, \ldots, j_d) \in \boldsymbol{Z}^d$ which satisfy $2 \leq j_l \leq i_l$ for any $1 \leq l \leq d$, and let $S(I)$ be the set of $J = (j_1, \ldots, j_d) \in \boldsymbol{Z}^d(I)$ with $(j_l, j_{l+1}) \neq (i_l, 2)$ for any $1 \leq l \leq d$, where $j_{d+1} = j_1$. For $J = (j_1, \ldots, j_d) \in \boldsymbol{Z}^d$, let $|J| = j_1 + \cdots + j_d$, $c(J) = c_{j_1}^{\beta_1} \cdots c_{j_d}^{\beta_d}$ and $\gamma(J) = g_{j_1}^{\beta_1} \cdots g_{j_d}^{\beta_d}$, where $\beta_l = \beta_l(I)$.

Lemma 7. *Let $n = 3$. Assume $\beta(f) = \gamma^{-1}\theta^{\mu}\beta(I)\gamma$, where $\mu \in \boldsymbol{Z}, \gamma \in B_3$, $I = (i_1, \ldots, i_d)$ with $i_1, \ldots, i_d \geq 3$. Then, we have*

(i) $\mathscr{L}(f) = (-1)^{d+1} \sum_{J \in S(I)} (-1)^{|J|} \Phi_{\beta(f)}(a_n^{\mu} c(J)^{\gamma})$.

(ii) *For any $J \in S(I)$, the coefficient of $\Phi_{\beta(f)}(a_n^{\mu} c(J)^{\gamma})$ in $\mathscr{L}(f)$ is non-zero.*

Proof. (i) For a partition $\mathscr{B}$, let $K(\mathscr{B})$ be the set of $l \in \boldsymbol{Z}_d$ with $(l, l+1) \in \mathscr{B}$. For $J \in \boldsymbol{Z}^d(I)$, let $L(J)$ be the set of $l \in \boldsymbol{Z}_d$ with $(j_l, j_{l+1}) = (i_l, 2)$. Also, let $\mathscr{P}_J$ be the set of $\mathscr{B} \in \mathscr{P}(d)$ with $K(\mathscr{B}) \subset L(J)$. For $l \in \boldsymbol{Z}_d$, we have $W_I((l)) = \sum_{j=2}^{i_l} g_j^{\beta_l}$ and $W_I((l, l+1)) = g_{i_l}^{\beta_l} a_2^{\beta_{l+1}} = -g_{i_l}^{\beta_l} g_2^{\beta_{l+1}}$, where we put $\beta_{l+1} = e$ if $l = d$. Therefore, for any partition $\mathscr{B}$, we have $W_I(\mathscr{B}) = (-1)^{\sharp K(\mathscr{B})} \sum_{J : \mathscr{B} \in \mathscr{P}_J} \gamma(J)$, and hence

$$\sum_{\mathscr{B} \in \mathscr{P}(d)} W_I(\mathscr{B}) = \sum_{\mathscr{B} \in \mathscr{P}(d)} \sum_{J : \mathscr{B} \in \mathscr{P}_J} (-1)^{\sharp K(\mathscr{B})} \gamma(J) = \sum_{J \in \boldsymbol{Z}^d(I)} \epsilon(J) \gamma(J),$$

where $\epsilon(J) = \sum_{\mathscr{B} \in \mathscr{P}_J} (-1)^{\sharp K(\mathscr{B})}$. If $J \in \boldsymbol{Z}^d(I) - S(I)$, then $\epsilon(J)$ is equal to some mutiple of the sum of $(-1)^{\sharp A}$ over the subsets A of $L(J)$. Since $L(J)$ is not empty, this sum is equal to zero. If $J \in S(I)$, $\mathscr{P}_J$ consists of a single partition $\{(1), \ldots, (d)\}$, and hence $\epsilon(J) = 1$. Therefore, Theorem 1 and the equality

$\gamma(J) = (-1)^{d+|J|}c(J)$ imply the equality (i).

(ii) can be proved similarly as Lemma 6(iii). □

7. Proof of Proposition 2.

We first show that it is enough for the proof to consider the case of $\mu = 0$, namely the case where $\beta(\varphi)$ is conjugate to $\beta(I)$. The reason is given as follows: Note that the period $m(\varphi)$ does not depend on the choice of an isotopy $\{\varphi_t\}$, but the braid $\beta(\varphi)$ and the integer $\nu(\varphi)$ depend on it. To clarify the dependence on an isotopy, denote them by $\beta(\varphi, \{\varphi_t\})$ and $\nu(\varphi; \{\varphi_t\})$ respectively. Let $R_t : D \to D$ be the rotation of the disk with angle $2\pi t$. Then, if we denote by $\{\varphi'_t\}$ the composition of the isotopies $\{\varphi_t\}$ and $\{R_{-\mu t}\}$, then $\beta(\varphi, \{\varphi'_t\})$ is equal to $\theta^{-\mu}\beta(\varphi, \{\varphi_t\})$, and hence it is conjugate to $\beta(I)$. Therefore, if the proposition is proved in the case of $\mu = 0$, then $\nu(\varphi, \{\varphi'_t\}) = \mathrm{LCM}\{d, n-2\}/(n-2)$, and hence $\nu(\varphi, \{\varphi_t\}) = m(\varphi)\mu + \nu(\varphi, \{\varphi'_t\}) = m(\varphi)\mu + \mathrm{LCM}\{d, n-2\}/(n-2)$. Thus, we can assume $\beta(\varphi) = \gamma^{-1}\beta(I)\gamma$ for some $\gamma \in B_n$.

Let $\bar{d} = \mathrm{LCM}\{d, n-2\}$, $m = \bar{d}/d$, and $\nu = \bar{d}/(n-2)$. We shall prove that the coefficient of $\Phi_{\beta(\varphi^m)}(a_n^\nu)$ in $\mathscr{L}(\varphi^m)$ is non-zero. Then, since m and ν are relatively prime, the assertion of the proposition follows from Lemma 1. Let p be an integer with $1 \le p \le d-n+3$ and let $q = p+n-3$. Then, we have

$$a_1^{\beta_p} a_{n-1}^{\beta_q} = a_1^{\beta_p}(a_{n-1}^{\beta(i_q)})^{\beta_{q+1}} = a_1^{\beta_p}(a_n a_1^{-1})^{\beta_{q+1}} = a_1^{\beta_p} a_n (a_1^{\beta_{q+1}})^{-1}, \tag{21}$$

where we put $\beta_{d+1} = \beta_1$.

Assume that $n \ge 4$ and $i_1, \dots, i_d \ge 2$. Define integers $\bar{i}_1, \dots, \bar{i}_{\bar{d}}$ by $\bar{i}_l = i_{[l]}$, where $[l]$ is the integer with $1 \le [l] \le d$ and $[l] \equiv l$ modulo d. Let $\bar{I} = (\bar{i}_1, \dots, \bar{i}_{\bar{d}})$. By Lemma 6(iii), it is enough for the proof to show that $\Phi_{\beta(\varphi^m)}(a_n^\nu) = \Phi_{\beta(\varphi^m)}(\lambda_{\mathscr{B}}(\mathscr{J})^\gamma)$ for some $\mathscr{B} \in \mathscr{P}(\bar{d})$ and some $\mathscr{J} \in S_{\mathscr{B}}(\bar{I})$. We see by Lemma 2 that this equality is equivalent to

$$\Phi_{\beta(\bar{I})}(a_n^\nu) = \Phi_{\beta(\bar{I})}(\lambda_{\mathscr{B}}(\mathscr{J})). \tag{22}$$

For $1 \le r \le \nu$, let $p_r = (r-1)(n-2)+1$, $q_r = r(n-2)$ and $B_r = [p_r, q_r]$. Note that all of these blocks have length $n-2$, and $\{B_1, \dots, B_\nu\}$ is a partition of $\boldsymbol{Z}_{\bar{d}}$. Let

$$\mathscr{B}_r = \begin{cases} \{B_r\} & \text{if } \bar{i}_{p_r} \ge 3, \\ \{(p_r), [p_{r+1}, q_r]\} & \text{if } \bar{i}_{p_r} = 2. \end{cases}$$

Then $\mathscr{B}_r$ is a partition of $\{p_r, \dots, q_r\}$, and if we put $\mathscr{B} = \mathscr{B}_1 \cup \dots \cup \mathscr{B}_\nu$, we have

$\mathscr{B} \in \mathscr{P}'(\bar{d})$. Let $S_{\mathscr{B}_r}(\bar{I})$ be $S_{B_r}(\bar{I})$ if $\bar{i}_{p_r} \geq 3$, and be $S_{(p_r)}(\bar{I}) \times S_{[p_r+1,q_r]}(\bar{I})$ if $\bar{i}_{p_r} = 2$.

For $r = 1, \ldots, \nu$, let $\zeta_r = a_1^{\beta_{p_r}} a_{n-1}^{\beta_{q_r}}$. We shall show that there exists an element J_r of $S_{\mathscr{B}_r}(\bar{I})$ with $\lambda_{\mathscr{B}_r}(J_r) = \zeta_r$. In the case of $\bar{i}_{p_r} \geq 3$, let $J_r = 1$. Then, $\lambda_{\mathscr{B}_r}(J_r) = \lambda_{\mathscr{B}_r}(1) = c_1^{\beta_{p_r}} a_{n-1}^{\beta_{q_r}} = \zeta_r$. In the case of $\bar{i}_{p_r} = 2$, let $J_r = (0, 0)$. Then, $\lambda_{\mathscr{B}_r}(J_r) = \lambda_{(p)}(0)\lambda_{[p_r+1,q_r]}(0) = a_2^{\beta_{p_r}} a_{n-2}^{\beta_{q_r}}$. Since

$$(a_1^{-1}a_2)^{\beta(2)} = \left[(a_3a_1^{-1})(a_2a_1^{-1})^{-1}(a_3a_1^{-1})^{-1}\right]a_3a_1^{-1} = a_3a_2^{-1},$$

and hence

$$(a_1^{-1}a_2)^{\beta_{p_r}} = ((a_1^{-1}a_2)^{\beta(2)})^{\beta_{p_r+1}} = (a_3a_2^{-1})^{\beta_{p_r+1}} = (a_{n-1}a_{n-2}^{-1})^{\beta_{q_r}},$$

we have

$$a_2^{\beta_{p_r}} a_{n-2}^{\beta_{q_r}} = (a_1a_1^{-1}a_2)^{\beta_{p_r}} a_{n-2}^{\beta_{q_r}} = a_1^{\beta_{p_r}} (a_1^{-1}a_2)^{\beta_{p_r}} a_{n-2}^{\beta_{q_r}} = \zeta_r.$$

Therefore, $\lambda_{\mathscr{B}_r}(J_r) = \zeta_r$.

Let $\mathscr{J} = (J_1, \ldots, J_\nu)$. Then, $\mathscr{J} \in S_{\mathscr{B}}(\bar{I})$ and, since $\lambda_{\mathscr{B}_r}(J_r) = \zeta_r$, we have $\lambda_{\mathscr{B}}(\mathscr{J}) = \zeta_1 \cdots \zeta_\nu$. Applying (21) to each pair p_r, q_r, we have

$$\lambda_{\mathscr{B}}(\mathscr{J}) = \prod_{r=1}^{\nu} a_1^{\beta_{p_r}} a_n (a_1^{\beta_{q_r+1}})^{-1}.$$

Since $p_1 = 1, q_r + 1 = p_{r+1}, q_\nu + 1 = \bar{d} + 1$, this is equal to $a_1^{\beta_1} a_n^\nu a_1^{-1}$. Therefore, by (5), $\lambda_{\mathscr{B}}(\mathscr{J})$ is Reidemeister equivalent to $(a_1^{-1})^{\beta_1}(a_1^{\beta_1} a_n^\nu) = a_n^\nu$ with respect to $\beta(\bar{I})$. Therefore, (22) is proved.

Assume that $n = 3$ and $i_1, \ldots, i_d \geq 3$. In this case, $\bar{d} = d = \nu$ and $m = 1$. Using Lemma 7(ii) and Lemma 2, we see that it is enough for the proof to show that $\Phi_{\beta(I)}(a_n^\nu) = \Phi_{\beta(I)}(c(J))$ for some $J \in S(I)$. Let $J = (3, \ldots, 3) \in S(I)$. Then, by (21),

$$c(J) = c_3^{\beta_1} \cdots c_3^{\beta_d} = \prod_{l=1}^{d} a_1^{\beta_l} a_3 (a_1^{\beta_{l+1}})^{-1} = a_1^{\beta_1} a_3^d a_1^{-1}.$$

Therefore, $\Phi_{\beta(I)}(c(J)) = \Phi_{\beta(I)}(a_1^{\beta_1} a_3^d a_1^{-1}) = \Phi_{\beta(I)}(a_3^d)$. Since $d = \nu$, the proof is completed.

8. Bounds for the Nielsen number.

As a byproduct of Lemma 6 and Lemma 7, we can obtain the following upper and lower bounds for the Nielsen number $N(f)$.

THEOREM 2. *Assume that $\beta(f)$ is conjugate to $\theta^{\mu}\beta(I)$.*

(i) *If $n \geq 4$ and $i_1, \dots, i_d \geq 2$, then*

$$\sum_{\mathscr{B} \in \mathscr{P}'(d)} \sharp S_{\mathscr{B}}(I) - (2n-2) \leq N(f) \leq \sum_{\mathscr{B} \in \mathscr{P}'(d)} \sharp S_{\mathscr{B}}(I).$$

(ii) *If $n = 3$ and $i_1, \dots, i_d \geq 3$, then $\sharp S(I) - 4 \leq N(f) \leq \sharp S(I)$.*

PROOF. We prove (i). Let $\Psi : \Gamma(I) \to \mathscr{R}(\beta(f))$ be the map defined by $\Psi((\mathscr{B}, \mathscr{J})) = \Phi_{\beta(f)}(a_n^{\mu}\lambda_{\mathscr{B}}(\mathscr{J})^{\gamma})$. Let $\Gamma'(I)$ be the set of $(\mathscr{B}, \mathscr{J}) \in \Gamma(I)$ with $\sharp[(\mathscr{B}, \mathscr{J})] > 1$. Let $\mathscr{R}'(\beta(f))$ be the set of Reidemeister classes α with $\mathrm{Fix}_{\alpha}(f)$ having index less than -1. We shall show that $\mathscr{R}'(\beta(f))$ coincides with the image of $\Gamma'(I)$ under Ψ. As we have shown in the proof of Lemma 6 (iii), the coefficient $n(\mathscr{B}, \mathscr{J})$ of $\Psi((\mathscr{B}, \mathscr{J}))$ in $\mathscr{L}(f)$ is equal to $(-1)^{d+1+e(\lambda_{\mathscr{B}}(\mathscr{J}))}\sharp[(\mathscr{B}, \mathscr{J})]$. On the other hand, $n(\mathscr{B}, \mathscr{J})$ is equal to $\mathrm{ind}(\mathrm{Fix}_{\Psi((\mathscr{B},\mathscr{J}))}(f))$ by its definition. Hence, we have

$$\mathrm{ind}(\mathrm{Fix}_{\Psi((\mathscr{B},\mathscr{J}))}(f)) = (-1)^{d+1+e(\lambda_{\mathscr{B}}(\mathscr{J}))}\sharp[(\mathscr{B}, \mathscr{J})]. \tag{23}$$

This implies that $\Psi((\mathscr{B}, \mathscr{J})) \in \mathscr{R}'(\beta(f))$ if and only if $(-1)^{d+1+e(\lambda_{\mathscr{B}}(\mathscr{J}))} = -1$ and $\sharp[(\mathscr{B}, \mathscr{J})] > 1$. The former condition $(-1)^{d+1+e(\lambda_{\mathscr{B}}(\mathscr{J}))} = -1$ is redundant, since the index of any fixed point class of f is less than two (Jiang and Guo [**17**]). Thus we have proved the equality $\mathscr{R}'(\beta(f)) = \Psi(\Gamma'(I))$.

$\Gamma'(I)$ is a disjoint union of equivalence classes $[(\mathscr{B}_1, \mathscr{J}_1)], \dots, [(\mathscr{B}_m, \mathscr{J}_m)]$, where $m = \sharp\Psi(\Gamma'(I))$. We have the following inequality due to [**17**] (see the proof of Theorem 4.1 there):

$$\sum_{\alpha \in \mathscr{R}'(\beta(f))} (\mathrm{ind}(\mathrm{Fix}_{\alpha}(f)) + 1) \geq 2\chi(D_n) = 2 - 2n.$$

This inequality and (23) imply that $2 - 2n \leq \sum_{i=1}^{m}(-\sharp[(\mathscr{B}_i, \mathscr{J}_i)] + 1) = -\sharp\Gamma'(I) + m$, and hence we have $\sharp\Psi(\Gamma'(I)) = m \geq \sharp\Gamma'(I) + 2 - 2n$. Therefore, since Ψ is injective on $\Gamma(I) - \Gamma'(I)$, we have by Lemma 6(iii) that

$$N(f) = \sharp\Psi(\Gamma(I)) = \sharp(\Gamma(I) - \Gamma'(I)) + \sharp\Psi(\Gamma'(I)) \geq \sharp\Gamma(I) + 2 - 2n.$$

Also, it is obvious that $N(f) \leq \sharp\Gamma(I)$. Since $\sharp\Gamma(I) = \sum_{\mathscr{B}\in\mathscr{P}'(d)} \sharp S_{\mathscr{B}}(I)$, we have the bounds in (i).

(ii) can be proved similarly. $\square$

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