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Irreducible plane sextics with large fundamental groups

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Abstract. We compute the fundamental group of the complement of each irreducible sextic of weight eight or nine (in a sense, the largest groups for irreducible sextics), as well as of 169 of their derivatives (both of and not of torus type). We also give a detailed geometric description of sextics of weight eight and nine and of their moduli spaces and compute their Alexander modules; the latter are shown to be free over an appropriate ring.

1. Introduction.

1.1. Sextics of torus type.

Recall that a plain sextic B is said to be of *torus type* (more precisely, $(2,3)$-*torus type*) if its equation can be represented in the form $p^3+q^2=0$, where p and q are some homogeneous polynomials of degree 2 and 3, respectively. A singular point P of B is called *inner* (with respect to a given torus structure (p,q)) if it belongs to the conic $\{p=0\}$; then P also belongs to the cubic $\{q=0\}$. Otherwise, P is called *outer*. A sextic of torus type is called *tame* if all its singular points are inner.

In spite of the algebraic definition, the property of being of torus type is of a purely topological nature; in particular, it is invariant under equisingular deformations. For example, an irreducible sextic is of torus type if and only if its Alexander polynomial (see A. Libgober [**16**] and Definition 4.6.1 below) is nontrivial, see [**7**] and [**8**]; the latter condition can be restated as $|\pi'/\pi''|=\infty$, where $\pi=\pi_1(\boldsymbol{P}^2\smallsetminus B)$ and $'$ stands for the derived group. Alternatively, an irreducible sextic is of torus type if and only if its fundamental group factors to the dihedral group $\boldsymbol{D}_6$, see [**7**] and [**8**] or H. Tokunaga [**26**] (where reducible sextics with simple singularities are considered as well).

Define the *weight* $w(P)$ of a simple singular point P as follows: $w(\mathbf{A}_{3k-1})=k$, $k\in\boldsymbol{Z}$, $w(\mathbf{E}_6)=2$, and $w(P)=0$ for all other types. The *weight* $w(B)$ of a sextic B with simple singularities is the total weight of its singular points. According to [**7**], any sextic of torus type has weight $6\le w\le 9$, and any irreducible sextic of weight

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$w \geq 7$ is of torus type. All sets of singularities of maximal weight 8 or 9 are classified in [**7**]: each sextic of weight nine has nine cusps (and is dual to a nonsingular cubic), and sextics of weight eight have one of the following sets of singularities:

$$\begin{gathered}\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2, \quad \mathbf{E}_6 \oplus 6\mathbf{A}_2, \quad 2\mathbf{A}_5 \oplus 4\mathbf{A}_2, \\ \mathbf{A}_5 \oplus 6\mathbf{A}_2 \oplus \mathbf{A}_1, \quad \mathbf{A}_5 \oplus 6\mathbf{A}_2, \quad 8\mathbf{A}_2 \oplus \mathbf{A}_1, \quad 8\mathbf{A}_2.\end{gathered} \tag{1.1.1}$$

In addition, it seems reasonable to assign weight eight to the following two non-simple sets of singularities (see the discussion on the Alexander polynomial below):

$$\mathbf{J}_{2,3} \oplus 3\mathbf{A}_2, \quad \mathbf{J}_{2,0} \oplus 4\mathbf{A}_2. \tag{1.1.2}$$

(We use Arnol'd's notation [**1**] for non-simple singularity types: $\mathbf{J}_{2,p}$ is a point of simplest tangency of a smooth branch and a singularity of type $\mathbf{A}_{p+3}$, $p \geq 0$.) The lists (1.1.1) and (1.1.2) appeared first in M. Oka [**20**]: they are the sets of singularities realizing the Alexander polynomial $(t^2 - t + 1)^2$.

From the results of Oka [**20**], it follows that the Alexander polynomial $\Delta_B(t)$ of an irreducible sextic B of torus type is $(t^2 - t + 1)^s$, where $s = 1$ if $w(B) = 6$ and $s = w(B) - 6$ if $w(B) \geq 7$. (A simple proof of the fact that $s = 1$ for $w(B) \leq 7$ is given in Section 5.1, see Corollary 5.1.2.) Among irreducible sextics with a non-simple singular point, only the two listed in (1.1.2) have Alexander polynomial $(t^2 - t + 1)^2$ (see [**20**] or [**7**] and [**8**]); that is why they are assigned weight eight.

1.2. Principal results.

Since $\deg \Delta_B = \dim_{\boldsymbol{C}}(\pi'/\pi'') \otimes \boldsymbol{C}$, where $\pi = \pi_1(\boldsymbol{P}^2 \setminus C)$, one can conclude that, in a sense, the groups of sextics of weight eight and nine are largest possible for irreducible sextics. The principal result of the present paper is the computation of these groups.

THEOREM 1.2.1. *The fundamental group* $\pi_1(\boldsymbol{P}^2 \setminus B)$ *of a plane sextic* B *with a set of singularities* Σ *as in* (1.1.1) *or* (1.1.2) *is as follows*:

- *the group* G_2 *given by* (4.4.11), *if* $\Sigma = \mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$;
- *the group* G_1 *given by* (4.4.15), *if* $\Sigma = \mathbf{A}_5 \oplus 6\mathbf{A}_2 \oplus \mathbf{A}_1$;
- *the group* G_0 *given by* (4.4.7) *otherwise.*

The curves with the set of singularities $\mathbf{J}_{2,3} \oplus 3\mathbf{A}_2$ are tame; their fundamental group G_0 is found in M. Oka, D. T. Pho [**23**].

REMARK. There are two obvious perturbation epimorphisms $G_2 \twoheadrightarrow G_1 \twoheadrightarrow G_0$. The latter is proper, see Corollary 5.4.5. At present, I do not know whether the former epimorphism $G_2 \twoheadrightarrow G_1$ is proper or not.

THEOREM 1.2.2. *The fundamental group $\pi_1(\boldsymbol{P}^2 \smallsetminus B)$ of a plane sextic B with the set of singularities $9\mathbf{A}_2$ is the group G_3 given by* (4.5.3).

Theorems 1.2.1 and 1.2.2 are proved in Sections 4.4 and 4.5, respectively. The fundamental group of the nine-cuspidal sextic (Theorem 1.2.2) was first computed by Zariski [**28**]; yet another computation and further generalizations can be found in J. I. Cogolludo [**2**].

As a necessary preliminary step, we describe an explicit geometric construction of the curves in question and their torus structures, see Section 3.5, and study their moduli space. (The key ingredient here is Theorem 2.1.1, stating that each sextic of weight eight or nine is symmetric.) In particular, we prove the uniqueness of an equisingular deformation family realizing each of the sets of singularities listed in (1.1.1). (For (1.1.2) and for $9\mathbf{A}_2$ the uniqueness is known.)

THEOREM 1.2.3. *Each set of singularities listed in* (1.1.1) *is realized by a single equisingular deformation family of plane sextics.*

Theorem 1.2.3 admits a refinement, taking into account the torus structure.

THEOREM 1.2.4. *With one exception, a pair* $(B, \{\text{torus structure}\})$*, where B is an irreducible sextic of weight eight, is determined up to equisingular deformation of torus structures by the combinatorial type of the pair* $(\Sigma, \Sigma_{\text{in}})$*, where Σ is the set of singularities of B and Σ_{in} is the set of its inner singularities. The exception is the pair* $(\Sigma, \Sigma_{\text{in}}) = (\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2, \mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2)$*, which is realized by two complex conjugate equisingular deformation families.*

It should not be very difficult to deduce Theorems 1.2.3 and 1.2.4 arithmetically, using [**6**]. However, we give a geometric proof, see Section 3.6, based on a detailed description of the curves and their moduli space.

In Section 3.7, we discuss geometric properties of the twelve torus structures of a sextic of weight nine and prove an analogue of Theorem 1.2.4.

As a first application of Theorems 1.2.1 and 1.2.2, we compute the Alexander modules (in the sense of Libgober [**17**], see Definition 4.6.1 below) of all sextics of weight eight and nine. The following statement is proved in Section 4.6.

THEOREM 1.2.5. *The Alexander module of a sextic of weight eight* (*nine*) *is a free module on two* (*respectively, three*) *generators over the ring* $\Lambda = \boldsymbol{Z}[t]/(t^2 - t + 1)$.

As another application, we compute the fundamental groups of the perturbations of sextics of weight eight. Altogether, we consider 47 sets of singularities of torus type, see Theorem 5.6.1, and 122 sets of singularities that are not of torus type and not covered by M. V. Nori's theorem [**19**], see Theorem 5.6.3. For about half of these sets of singularities, the uniqueness of an equisingular deformation family is known. Among them are 17 so called *classical Zariski pairs*, see Corollary 5.6.4; we show that, as in the original example by O. Zariski [**27**], the fundamental groups of the two curves constituting each pair are $\boldsymbol{Z}_2 * \boldsymbol{Z}_3$ and $\boldsymbol{Z}_6$.

It is worth mentioning that irreducible sextics of torus type, their moduli, and fundamental groups have been a subject of intensive study, so that some of the results of this paper may overlap with results obtained by other authors. We will cite Oka, Pho [**22**] (classification of singularities of torus type and moduli of maximal sextics of torus type), Oka, Pho [**23**] (fundamental groups of tame sextics), Oka [**21**] (Zariski pairs involving sextics of torus types), and, for further references, recent paper Eyral, Oka [**12**].

1.3. Digression: geometry of a nine cuspidal sextic.

As a by-product, we apply the results obtained in the paper to study the geometry of a nine cuspidal sextic (sextic of weight nine). Any such sextic B is an elliptic curve. We show that, if B is generic (no complex multiplication), then the group $\operatorname{Aut}(\boldsymbol{P}^2, B)$ of projective automorphisms of B is a semi-direct product $(\boldsymbol{Z}_3 \times \boldsymbol{Z}_3) \rtimes \boldsymbol{Z}_2$, see Section 2.6. Then, in Section 3.7, we characterize the twelve torus structures of B, see [**7**] or [**25**], in terms of their stabilizers, which are dihedral subgroups $\boldsymbol{D}_6 \subset \operatorname{Aut}(\boldsymbol{P}^2, B)$, as well as in terms of the inflection points of the dual cubic curve. Finally, we prove an analogue of Theorem 1.2.4: pairs consisting of a sextic B of weight nine and a torus structure on B form a connected deformation family.

Some of these results may be known.

1.4. Contents of the paper.

The starting point of our computation is Section 2: we apply the characterization found in [**7**], V. V. Nikulin's theory of discriminant forms, and the theory of periods of $K3$-surfaces to show that each sextic B of weight eight or nine with simple singularities is symmetric (Theorem 2.1.1), thus folding B to a very special trigonal curve $\bar{B}$ in Hirzebruch surface Σ_2 (quadratic cone). It appears that the symmetry constructed is the only non-trivial projective automorphism of a generic curve of weight eight. In the case of weight nine, the situation is different: we show that the automorphism group of a generic nine cuspidal sextic has order 18.

In Section 3, we represent the trigonal curve $\bar{B}$ obtained above by explicit equations and use them to analyze the automorphisms of $\bar{B}$, its real structures, torus structures, and special (tangent, double tangent, inflection tangent, etc.) sections. The results are applied to prove Theorems 1.2.3 and 1.2.4, as well as to the study of generic nine cuspidal sextics.

In Section 4, we use the equations developed in Section 3 to visualize the braid monodromy of $\bar{B}$ and to write down presentations for the fundamental groups. Theorems 1.2.1, 1.2.2, and 1.2.5 are proved here.

Finally, in Section 5, we use the presentations obtained above to compute the groups of a number of other sextics. The key ingredient here is Proposition 5.1.1, which states that any 'thinkable' perturbation of singularities of a plane sextic is indeed realizable. We discuss briefly the uniqueness of equisingular deformation families realizing the sets of singularities for which the groups are found.

2. The construction.

In this section, we show that any irreducible sextic of weight eight or nine can be obtained as a double covering of a certain trigonal curve $\bar{B}$ in a geometrically ruled rational surface Σ_2.

2.1. Statements.

Recall that any involutive automorphism c: $\boldsymbol{P}^2 \to \boldsymbol{P}^2$ has a fixed line L_c and an isolated fixed point O_c, and the quotient $\boldsymbol{P}^2(O_c)/c$ is the rational geometrically ruled surface Σ_2. The images in Σ_2 of O_c and L_c are, respectively, the exceptional section E and a generic section $\bar{L}$, so that $\boldsymbol{P}^2(O_c)$ is the double covering of Σ_2 ramified at $\bar{L}+E$. Alternatively, $\boldsymbol{P}^2$ is the double covering of the quadratic cone Σ_2/E ramified at $\bar{L}$ and E/E. (Here and below, we use the notation $\boldsymbol{P}^2/c$ for the orbit space of c, and $\cdot/E$ stands for the contraction of E, so that E/E is the vertex of the cone Σ_2/E.)

The principal result of this section is the following theorem.

THEOREM 2.1.1. *Let $B \subset \boldsymbol{P}^2$ be a sextic of weight eight or nine and with simple singularities only. Then $\boldsymbol{P}^2$ admits an involution c preserving B, so that $O_c \notin B$, and the image of B in $\Sigma_2 = \boldsymbol{P}^2(O_c)/c$ is a trigonal curve $\bar{B}$, disjoint from E, with four cusps $\mathbf{A}_2$. Conversely, given a trigonal curve $\bar{B} \subset \Sigma_2$ as above and a section $\bar{L}$ of Σ_2 disjoint from E, the double covering of $\bar{B}$ ramified at $\bar{L}$ and E/E is a plane sextic of weight eight or nine and with simple singularities only.*

Theorem 2.1.1 is proved at the end of this section, in 2.5 below.

REMARK. According to [**8**], the two deformation families of weight 8 with non-simple singular points, see (1.1.2), can also be obtained from a four cuspidal trigonal curve, but by a birational transformation rather than double covering.

2.2. Discriminant forms.

An (*integral*) *lattice* is a finitely generated free abelian group L supplied with a symmetric bilinear form $b\colon L \otimes L \to \boldsymbol{Z}$. We abbreviate $b(x, y) = x \cdot y$ and $b(x, x) = x^2$. A lattice L is called *even* if $x^2 = 0 \bmod 2$ for all $x \in L$. As the transition matrix between two integral bases has determinant ± 1, the determinant $\det L \in \boldsymbol{Z}$ (i.e., the determinant of the Gram matrix of b in any basis of L) is well defined. A lattice L is called *nondegenerate* if $\det L \neq 0$; it is called *unimodular* if $\det L = \pm 1$.

Given a lattice L, the bilinear form extends to a form $(L \otimes \boldsymbol{Q}) \otimes (L \otimes \boldsymbol{Q}) \to \boldsymbol{Q}$. If L is nondegenerate, the dual group $L^* = \operatorname{Hom}(L, \boldsymbol{Z})$ can be identified with the subgroup

$$\{x \in L \otimes \boldsymbol{Q} \mid x \cdot y \in \boldsymbol{Z} \text{ for all } x \in L\}.$$

In particular, $L \subset L^*$ is a finite index subgroup. The quotient L^*/L is called the *discriminant group* of L and is denoted by $\operatorname{discr} L$ or $\mathscr{L}$. The discriminant group inherits from $L \otimes \boldsymbol{Q}$ a symmetric bilinear form $b_{\mathscr{L}}\colon \mathscr{L} \otimes \mathscr{L} \to \boldsymbol{Q}/\boldsymbol{Z}$, called the *discriminant form*, and, if L is even, its quadratic extension $q_{\mathscr{L}}\colon \mathscr{L} \to \boldsymbol{Q}/2\boldsymbol{Z}$. One has $\#\mathscr{L} = |\det L|$; in particular, $\mathscr{L} = 0$ if and only if L is unimodular.

From now on, *all lattices considered are even.*

An *extension* of a lattice L is another lattice M containing L, so that the form on L is the restriction of that on M. An *isomorphism* between two extensions $M_1 \supset L$ and $M_2 \supset L$ is an isometry $M_1 \to M_2$ whose restriction to L is the identity. In what follows, we are only interested in the case when $[M : L] < \infty$. Next two theorems are found in Nikulin [**18**].

THEOREM 2.2.1. *Given a nondegenerate lattice L, there is a canonical one-to-one correspondence between the set of isomorphism classes of finite index extensions $M \supset L$ and the set of isotropic subgroups $\mathscr{K} \subset \mathscr{L}$. Under this correspondence, $M = \{x \in L^* \mid x \bmod L \in \mathscr{K}\}$ and* $\operatorname{discr} M = \mathscr{K}^\perp/\mathscr{K}$.

The isotropic subgroup $\mathscr{K} \subset \mathscr{L}$ as in Theorem 2.2.1 is called the *kernel* of the extension $M \supset L$. It can be defined as the image of M/L under the homomorphism induced by the natural inclusion $M \hookrightarrow L^*$.

THEOREM 2.2.2. *Let $M \supset L$ be a finite index extension of a nondegenerate lattice L, and let $\mathscr{K} \subset \mathscr{L}$ be its kernel. Then, an auto-isometry $L \to L$ extends to M if and only if the induced automorphism of $\mathscr{L}$ preserves $\mathscr{K}$.*

We will use Theorem 2.2.2 in the following form.

COROLLARY 2.2.3. *Let $L \subset M$ be a nondegenerate sublattice of a unimodular lattice M, and let $\mathscr{K} \subset \mathscr{L}$ be the kernel of the extension $\tilde{L} \supset L$, where $\tilde{L}$ is the primitive hull of L in M. Consider an auto-isometry c: $L \to L$. Then, $c \oplus \mathrm{id}_{L^\perp}$ extends to M if and only if c preserves $\mathscr{K}$ and the auto-isometry of $\mathscr{K}^\perp/\mathscr{K}$ induced by c is the identity.*

PROOF. We apply Theorem 2.2.2 twice: first, to the extension $\tilde{L} \supset L$, then to the extension $M \supset \tilde{L} \oplus L^\perp$. The condition that c should preserve $\mathscr{K}$ is necessary and sufficient for c to extend to an isometry $\tilde{c}$ of $\tilde{L}$. If it does extend, the auto-isometry of the discriminant

$$\operatorname{discr}(\tilde{L} \oplus L^\perp) = (\mathscr{K}^\perp/\mathscr{K}) \oplus \operatorname{discr} L^\perp$$

induced by $\tilde{c} \oplus \mathrm{id}_{L^\perp}$ is $\tilde{c}_{\mathscr{K}} \oplus \mathrm{id}_{\operatorname{discr} L^\perp}$, where $\tilde{c}_{\mathscr{K}}$ is the automorphism induced by $\tilde{c}$ (or c) on $\mathscr{K}^\perp/\mathscr{K}$. Since the kernel of the extension $M \supset \tilde{L} \oplus L^\perp$ is the graph of a certain anti-isometry $\mathscr{K}^\perp/\mathscr{K} \to \operatorname{discr} L^\perp$, see Nikulin [**18**], it is preserved by the automorphism $\tilde{c}_{\mathscr{K}} \oplus \mathrm{id}_{\operatorname{discr} L^\perp}$ above (the condition necessary and sufficient for the auto-isometry $\tilde{c} \oplus \mathrm{id}_{L^\perp}$ to extend further to M) if and only if $\tilde{c}_{\mathscr{K}} = \mathrm{id}_{\mathscr{K}^\perp/\mathscr{K}}$. $\square$

2.3. Sextics of weight eight.

Let B be a sextic with one of the sets of singularities listed in (1.1.1). Denote by $\tilde{X}$ the resolution of singularities of the double covering $X \to \boldsymbol{P}^2$ ramified at B. It is a $K3$-surface. For each singular point P of B, let Γ_P be the incidence graph of the exceptional divisors in $\tilde{X}$ over P (it is a Dynkin diagram of the same name as the type of P), and let $\Sigma_P \subset H_2(\tilde{X})$ be the sublattice spanned by the vertices of Γ_P. Let $\Gamma = \bigcup_P \Gamma_P$ and $\Sigma = \bigoplus_P \Sigma_P$, and denote by $\tilde{\Sigma}$ the primitive hull of Σ in $H_2(\tilde{X})$.

For future references, introduce also the class $h \in H_2(\tilde{X})$ realized by the pull-back of a generic line in $\boldsymbol{P}^2$. Observe that, regarded as an element of the Picard group, h is the linear system defining the projection $\tilde{X} \to \boldsymbol{P}^2$.

Split all singular points of B of positive weight into four groups of total weight two each, and let Γ_i and Σ_i, $i = 1, \ldots, 4$, be the corresponding subgraphs of Γ and sublattices of Σ, respectively. Each Σ_i is either $2\mathbf{A}_2$ or $\mathbf{A}_5$ or $\mathbf{E}_6$. Let c_i: $\Sigma_i \to \Sigma_i$ be the automorphism induced by a non-trivial symmetry of Γ_i: in the cases $\mathbf{A}_5$ and $\mathbf{E}_6$, such a symmetry is unique; in the case $2\mathbf{A}_2$, there are two symmetries transposing the two components, and we pick one of them. There is a unique pair of c_i-skew-invariant nonzero elements $\pm x_i \in \operatorname{discr} \Sigma_i$; one has $x_i^2 = 2/3 \bmod 2\boldsymbol{Z}$. (In the cases $\mathbf{A}_5$ and $\mathbf{E}_6$, $\pm x_i$ are the generators of

$\operatorname{discr}\Sigma_i \otimes \boldsymbol{F}_3 = \boldsymbol{F}_3$; in the case $2\mathbf{A}_2$, there are two pairs of opposite elements of square $2/3 \bmod 2\boldsymbol{Z}$, and the choice of one of them determines c_i.) Choose one of them and denote it by x_i.

Now, the description of sextics of weight 8 given in [**7**] can be restated as follows: the sublattices Σ_i, involutions c_i, and elements x_i, $i = 1, \ldots, 4$, as above can be chosen so that the kernel $\mathscr{K} \subset \operatorname{discr}\Sigma$ of the extension $\tilde\Sigma \supset \Sigma$ is spanned by $x_1 + x_2 + x_3$ and $x_1 - x_2 + x_4$. Then, extending $\bigoplus_i c_i$ identically to the rest of Σ, we obtain an involution $c_\Sigma\colon \Sigma \to \Sigma$ with the following properties: c_Σ acts identically on the p-primary part of $\operatorname{discr}\Sigma$ for any prime $p \neq 3$, and $\mathscr{K}$ is a maximal (half dimension) isotropic subspace of the (-1)-eigenspace of c_Σ in the 3-primary part $\operatorname{discr}\Sigma \otimes \boldsymbol{F}_3$. Hence, $\mathscr{K}^\perp/\mathscr{K}$ can be identified with the c_Σ-invariant part of $\operatorname{discr}\Sigma$, and, due to Corollary 2.2.3, $c_\Sigma \oplus \operatorname{id}_{\Sigma^\perp}$ extends to an involution $\tilde c_*$ on $H_2(\tilde X)$.

2.4. Sextics of weight nine.

Let B be a sextic with the set of singularities $9\mathbf{A}_2$. Pick one of the cusps and treat it as an ordinary point (of weight zero). Applying the construction of Section 2.3, we obtain a splitting of the remaining eight cusps into four groups and an involution $c_\Sigma\colon \Sigma \to \Sigma$. According to [**7**], the kernel $\mathscr{K}$ is spanned by the elements $x_1 + x_2 + x_3$ and $x_1 - x_2 + x_4$ introduced above and by $y_1 + y_2 + y_3 + y_4 + z_9$, where y_i is an appropriately chosen generator of $x_i^\perp$ in $\operatorname{discr}\Sigma_i$, $i = 1, \ldots, 4$, and z_9 is a generator of the discriminant of the ninth cusp. Thus, $\mathscr{K}^\perp/\mathscr{K}$ can be identified with a certain subquotient of the c_Σ-invariant part of $\operatorname{discr}\Sigma$, and still $c_\Sigma \oplus \operatorname{id}_{\Sigma^\perp}$ extends to an involution $\tilde c_*$ on $H_2(\tilde X)$.

2.5. Proof of Theorem 2.1.1.

Consider the $K3$-surface $\tilde X$ introduced in 2.3 and the involution $\tilde c_*$ on $H_2(\tilde X)$ constructed in 2.3 and 2.4. From the construction, it follows that $\tilde c_*$ preserves Γ (as a set), the class $h \in H_2(\tilde X)$ of the pull-back of a line, see 2.3, and the class $\omega \in H_2(\tilde X; \boldsymbol{C})$ of a holomorphic 2-form on $\tilde X$ (as both h and ω are orthogonal to Σ). Then, $\tilde c_*$ also preserves the positive cone of $\tilde X$ (as it can be expressed in terms of Γ, h, and ω, cf. [**10**]) and hence $\tilde c_*$ is induced by a unique involution $\tilde c\colon \tilde X \to \tilde X$; the latter is symplectic (i.e., preserving holomorphic 2-forms) and commutes with the deck translation of the covering $\tilde X \to \boldsymbol{P}^2$ (as $\tilde c_*$ preserves h). The descent of $\tilde c$ to $\boldsymbol{P}^2$ is the desired involution $c\colon \boldsymbol{P}^2 \to \boldsymbol{P}^2$.

The image $\bar B$ of B in $\Sigma_2 = \boldsymbol{P}^2(O_c)/c$ can easily be studied similar to [**10**]. (In particular, all necessary facts relating the singularities of B and those of $\bar B + \bar L$ are found in [**10**]; the relevant part is represented in Table 1 in Section 3.) For the converse statement, one observes that each cusp $\bar P$ of $\bar B$ gives rise to either two cusps of B (if $\bar P \notin \bar L$) or one singular point of B of type $\mathbf{A}_5$ or $\mathbf{E}_6$ (if $\bar P \in \bar L$ and, for

$\mathbf{E}_6$, $\bar{L}$ is tangent to $\bar{B}$ at $\bar{P}$). The intersection points $\bar{L} \cap \bar{B}$ nonsingular for $\bar{B}$ produce type $\mathbf{A}$ singularities of B. Hence, all singularities of B are simple and the total weight of B is at least eight. $\square$

2.6. Automorphisms of nine cuspidal sextics.

Let $B \subset \boldsymbol{P}^2$ be a nine cuspidal sextic. As follows from 2.4 and 2.5, for each cusp P of B, there is an involution of the pair $(\boldsymbol{P}^2, P)$ preserving P and transposing the other eight cusps. This classical fact has a transparent geometric explanation. Consider the nonsingular cubic $C \subset \boldsymbol{P}^2$ dual to B. Then, the automorphisms of $(\boldsymbol{P}^2, B)$ are those of $(\boldsymbol{P}^2, C)$, which in turn are the automorphisms of the abstract elliptic curve C preserving its nine inflection points. (Note that C is the normalization of B; as abstract curves they can be identified.) Hence, the following statement holds.

LEMMA 2.6.1. *If $B \cong C$ is generic (no complex multiplication), then the group $\operatorname{Aut}(\boldsymbol{P}^2, B)$ of projective automorphisms of B is a semi-direct product of $\boldsymbol{Z}_3 \times \boldsymbol{Z}_3$ (affine shifts by the vectors $(m/3, n/3) \in B \cong \boldsymbol{R}/\boldsymbol{Z} \times \boldsymbol{R}/\boldsymbol{Z}$) and $\boldsymbol{Z}_2$ (multiplication by (-1) in B), $\boldsymbol{Z}_2$ acting on $\boldsymbol{Z}_3 \times \boldsymbol{Z}_3$ via $z \mapsto -z$.*

The nine involutions mentioned above are the nine elements of the non-trivial coset modulo $\boldsymbol{Z}_3 \times \boldsymbol{Z}_3$. These are all order 2 elements in $\operatorname{Aut}(\boldsymbol{P}^2, B)$.

The automorphisms of C can be seen from its Hesse's normal form

$$z_0^3 + z_1^3 + z_2^3 = \lambda z_0 z_1 z_2.$$

The group $\operatorname{Aut}(\boldsymbol{P}^2, C)$ is generated by the permutations of the coordinates and the automorphisms $(z_0 : z_1 : z_2) \mapsto (z_0 : \epsilon z_1 : \epsilon^2 z_2)$, $\epsilon^3 = 1$.

If C has a complex multiplication, then $\operatorname{Aut}(\boldsymbol{P}^2, C)$ is a semi-direct product of $\boldsymbol{Z}_3 \times \boldsymbol{Z}_3$ and $\boldsymbol{Z}_4$ or $\boldsymbol{Z}_6$; in particular, the set of involutions in $\operatorname{Aut}(\boldsymbol{P}^2, C)$ is the same. As a consequence, we have the following statement.

LEMMA 2.6.2. *For a nine cuspidal sextic $B \subset \boldsymbol{P}^2$, each involutive automorphism $c \in \operatorname{Aut}(\boldsymbol{P}^2, B)$ has a single fixed cusp P_c, and c is determined by P_c uniquely.*

The action of $\operatorname{Aut}(\boldsymbol{P}^2, B)$ on the torus structures of B is discussed in 3.7 below.

3. The trigonal curve $\bar{B}$.

In this section, we study the geometry of the four cuspidal trigonal curve $\bar{B} \subset \Sigma_2$. Most calculations involving equations were performed using Maple.

3.1. The geometric description.

A trigonal curve $\bar{B} \subset \Sigma_2$ with the set of singularities $4\mathbf{A}_2$ is a maximal trigonal curve in the sense of [**9**]; in particular, it is unique up to automorphism of Σ_2. The skeleton $\operatorname{Sk} \subset \boldsymbol{P}^1$ of $\bar{B}$ (see [**9**]) is the 1-skeleton of a regular tetrahedron Δ (assuming that $\boldsymbol{P}^1$ is regarded as the surface of Δ). Hence, the group of Klein automorphisms of $\bar{B}$ (i.e., holomorphic or anti-holomorphic automorphisms $\Sigma_2 \to \Sigma_2$ preserving $\bar{B}$) is the full symmetric group $\boldsymbol{S}_4$ (the group of symmetries of Δ), and the group of holomorphic automorphisms of $\bar{B}$ is the subgroup $\boldsymbol{A}_4 \subset \boldsymbol{S}_4$ (the group of rotations of Δ). Both groups act faithfully on the set of cusps of $\bar{B}$ (barycenters of the faces of Δ). Explicit generators for the group $\boldsymbol{A}_4$ are given in (3.2.4) and (3.2.5) below.

As a consequence, $\bar{B}$ is real with respect to six different real structures on Σ_2 (transpositions in $\boldsymbol{S}_4$). Each real structure preserves exactly two of the four cusps of $\bar{B}$ and is uniquely determined by this pair of cusps. Below, we use two distinct real structures, see (3.3.1), to visualize the braid monodromy of $\bar{B}$.

Alternatively, $\bar{B}$ can be constructed as a birational transform of a three cuspidal plane quartic C. As is known, C has a unique double tangent L; one should pick one of the tangency points P, blow it up twice, and blow down the proper transform of L and one of the exceptional divisors over P. The inverse transformation is the stereographic projection of the quadratic cone Σ_2/E from one of the cusps of $\bar{B}$.

3.2. The equation of $\bar{B}$.

In appropriate affine coordinates (x, y) in Σ_2 a curve $\bar{B}$ as above can be given by the polynomial

$$f(x, y) = 4y^3 - (24x^3 + 3)y + (8x^6 + 20x^3 - 1). \tag{3.2.1}$$

The discriminant of this expression with respect to y is

$$-(2)^{10}(3)^3x^3(x^3 - 1)^3; \tag{3.2.2}$$

hence, $\bar{B}$ does have four cusps.

The curve is rational; it can be parametrized by

$$x(t) = \frac{3t}{t^3 + 2}, \quad y(t) = -\frac{t^6 - 20t^3 - 8}{2(t^3 + 2)^2}. \tag{3.2.3}$$

The cusps of $\bar{B}$ are certain points $\bar{P}_0$ over $x_0 = 0$, $\bar{P}_1$ over $x_1 = 1$, and $\bar{P}_\pm$ over $x_\pm = \epsilon_\pm = (-1 \pm i\sqrt{3})/2$; the corresponding values of the parameter t are $t_0 = \infty$,

$t_1 = 1$, and $t_\pm = \epsilon_\pm$. The other points in the same fibers as the cusps correspond to the values $t'_0 = 0$, $t'_1 = -2$, and $t'_\pm = -2\epsilon_\pm$. The ordinate of $\bar{P}_0$ is $-1/2$; the ordinates of the other three cusps are $3/2$.

The curve given by (3.2.1) is plotted, e.g., in Figure 3 in Section 4.

The curve intersects the x-axis at the points $x = \epsilon r_\pm$, where $r_\pm = (-1 \pm \sqrt{3})/2$ and $\epsilon^3 = 1$; the corresponding values of the parameter are $t = \epsilon(1 \pm \sqrt{3})$. Denote the two real intersection points by $\bar{R}_\pm(r_\pm, 0)$.

To describe the symmetries of $\bar{B}$, we regard them as changes of coordinates (x, y), indicating as well the corresponding change of the parameter t in (3.2.3). Denote by $\epsilon = (-1 + i\sqrt{3})/2$ a primitive cubic root of unity. Then the group $\boldsymbol{A}_4$ of the holomorphic automorphisms of $\bar{B}$ is generated by the order 3 transformation

$$(x, y) = (\epsilon x', y'), \qquad t = \epsilon t', \tag{3.2.4}$$

and the order 2 transformation

$$(x, y) = \left(-\frac{x' - \epsilon}{2\epsilon^2 x' + 1}, -\frac{3y'}{(2\epsilon^2 x' + 1)^2}\right), \quad t = \frac{t' + 2\epsilon}{\epsilon^2 t' - 1}. \tag{3.2.5}$$

For the full group $\boldsymbol{S}_4$ of Klein automorphisms of $\bar{B}$, one adds to the generating set the real structure conj: $(x, y) \mapsto (\bar{x}, \bar{y})$.

3.3. Real structures.

In the sequel, we consider two real structures on Σ_2 with respect to which $\bar{B}$ is real:

$$\text{conj}\colon (x, y) \mapsto (\bar{x}, \bar{y}), \quad \text{and} \quad \text{conj}'\colon (x', y') \mapsto (\bar{x}', \bar{y}'), \tag{3.3.1}$$

where (x', y') are the coordinates introduced in (3.2.5) and bar stands for the complex conjugation. The real part $\{\operatorname{Im} x' = 0\}$ is the circle $|x + (1/2)| = 3/4$, see Figure 2; it contains $x_\pm$. One has $\{\operatorname{Im} x' = 0\} \cap \{\operatorname{Im} x = 0\} = \{\bar{R}_+, \bar{R}_-\}$.

Analyzing the sign of the discriminant (3.2.2), one can see that $\bar{B}$ has three (one) conj-real points over the inside (respectively, outside) of the segment $[x_0, x_1]$, and it has three (one) conj$'$-real points over the open arc $(x_-\, r_-\, x_+)$ (respectively, the open arc $(x_-\, r_+\, x_+)$).

We are interested in real sections of Σ_2 and their real points. Note that sections of Σ_2 are the parabolas of the form

$$y = s(x) = ax^2 + bx + c. \tag{3.3.2}$$

LEMMA 3.3.3. *A section* (3.3.2) *is real with respect to both* conj *and* conj′ *if and only if it has the form* $s(x) = -c(2x^2 + 2x - 1)$ *for some* $c \in \boldsymbol{R}$, *or, alternatively, if* a, b, c *are real and* $s(r_+) = s(r_-) = 0$.

PROOF. Under (3.2.5), a section (3.3.2) transforms to $y' = a'x'^2 + b'x' + c'$, where

$$a' = \frac{a - 2\epsilon^2 b + 4\epsilon c}{3}, \quad b' = \frac{-2\epsilon a + b + 4\epsilon^2 c}{3}, \quad c' = \frac{\epsilon^2 a + \epsilon b + c}{3}. \tag{3.3.4}$$

Equating the imaginary parts to zero, it is easy to see that all six parameters a, b, c, a', b', c' are real if and only if $a = b = -2c$, $c \in \boldsymbol{R}$. On the other hand, $a = b = -2c$ if and only if the section passes through $\bar{R}_\pm$. □

LEMMA 3.3.5. *A* conj*-real section* (3.3.2) *has a* conj′*-real point if and only if* $s(r_+)s(r_-) \geq 0$.

PROOF. Assuming a, b, c in (3.3.4) real, substituting a real value for x', and equating the imaginary part of the result to zero, one arrives at the equation

$$(2b + 4c)x'^2 - (2a + 4c)x' + b - a = 0.$$

Its discriminant is $16\, s(r_+)s(r_-)$. □

3.4. Special sections.

Let $\bar{B}$ be as above, and let $\bar{P}_0$, $\bar{P}_1$, and $\bar{P}_\pm$ be its cusps, as explained in 3.2.

A section (3.3.2) passes through $\bar{P}_0$ if and only if $c = -1/2$; it is tangent to $\bar{B}$ at $\bar{P}_0$ if and only if

$$c = -\frac{1}{2} \quad \text{and} \quad b = 0. \tag{3.4.1}$$

Such a section passes through $\bar{P}_1$ if and only if

$$(a, b, c) = \left(2, 0, -\frac{1}{2}\right). \tag{3.4.2}$$

A section (3.3.2) passes through both $\bar{P}_0$ and $\bar{P}_1$ if and only if

$$c = -\frac{1}{2} \quad \text{and} \quad a + b = 2. \tag{3.4.3}$$

Observe that one of these sections, with $(a, b, c) = (1, 1, -1/2)$, is both conj- and conj$'$-real, see Lemma 3.3.3. The section passes through three cusps $\bar{P}_1$, $\bar{P}_\pm$ if and only if $(a,b,c)=(0,0,3/2)$.

Equating $s(x(t))=y(t)$ and $s'_t=y'_t$, one can see that a section (3.3.2) is tangent to $\bar{B}$ at a point $(x(t),y(t))\in\bar{B}$, $t^3\neq 1$, if and only if $t\neq 0$ and

$$a=\frac{2(t^3+2)^2c+t^6+16t^3-8}{18t^2},\quad b=-\frac{2(t^3+2)c+t^3-4}{3t} \tag{3.4.4}$$

or $t=0$ and $b=0$, $c=1$. Note that, if $\bar{S}\in\Sigma_2$ is a fixed point not over $\bar{P}_0$ and $t\to t_0=\infty$, then the sections as in (3.4.4) passing through $\bar{S}$ tend to the section tangent to $\bar{B}$ at $\bar{P}_0$ (and passing through $\bar{S}$).

A section as in (3.4.4) passes through $\bar{P}_1$ if and only if

$$a=\frac{2(t^3-3t-1)}{(t-1)(t+2)^2},\quad b=\frac{6t(t+1)}{(t-1)(t+2)^2},\quad c=-\frac{(t^3+3t^2+8)}{2(t-1)(t+2)^2}. \tag{3.4.5}$$

Equating, in addition, $s''_t=y''_t$, one concludes that a section (3.3.2) is inflection tangent to $\bar{B}$ at a point $(x(t),y(t))\in\bar{B}$, $t^3\neq 1$, if and only if

$$a=\frac{t(t^3-4)}{2(t^3-1)},\quad b=\frac{3t^2}{(t^3-1)},\quad c=-\frac{(t^3+2)}{2(t^3-1)}. \tag{3.4.6}$$

Such a section cannot pass through a singular point of $\bar{B}$.

If a section as in (3.4.6) is conj-real, then it has conj$'$-real points, as one has

$$s(r_+)s(r_-)=\frac{(t^2-2t-2)^4}{16(t^3-1)^2}\geq 0,$$

see Lemma 3.3.5. Two of the sections inflection tangent to $\bar{B}$ are both conj- and conj$'$-real; they are obtained at $t=1\pm\sqrt{3}$ (the tangency points being $\bar{R}_\pm$), and their equations are

$$y=\frac{\pm\sqrt{3}(2x^2+2x-1)}{3}. \tag{3.4.7}$$

These sections are plotted in Figure 6 in Section 4.

3.5. Moduli and torus structures.

The four cuspidal trigonal curve $\bar{B}$ has four distinct ‘torus structures’; one of them is given by the decomposition

$$2f(x,y) = 8\left(y - \frac{3}{2}\right)^3 + (6y - 4x^3 - 5)^2, \tag{3.5.1}$$

where f is as in (3.2.1), and the others are obtained from (3.5.1) by applying a sequence of transformations (3.2.4) and (3.2.5). Formally, a torus structure for $\bar{B}$ should be defined as a representation of its equation in the form $\bar{p}^3 + e\bar{q}^2$, where $\bar{p}$, $\bar{q}$, and e are sections of $\mathscr{O}_{\Sigma_2}(E+2F)$, $\mathscr{O}_{\Sigma_2}(E+3F)$, and $\mathscr{O}_{\Sigma_2}(E)$, respectively, cf. [**8**]. Up to coefficient, $\bar{p}$ has the form $y - s(x)$, where $s(x)$ is a section of Σ_2 passing through three of the four cusps of $\bar{B}$. This triple of cusps determines the torus structure. In particular, with respect to any real structure preserving $\bar{B}$, two of the torus structures are real and two are complex conjugate. The stabilizer of each torus structure of $\bar{B}$ is a subgroup $\boldsymbol{Z}_3 \subset \boldsymbol{A}_4$ (respectively, a dihedral subgroup $\boldsymbol{D}_6 \subset \boldsymbol{S}_4$); the stabilizer of (3.5.1) in $\boldsymbol{A}_4$ is (3.2.4).

REMARK. The four torus structures on the four-cuspidal trigonal curve $\bar{B}$ were also studied in Tokunaga [**25**].

Theorem 2.1.1 implies that, in some affine coordinates (x,y) in $\boldsymbol{P}^2$, each sextic B of weight eight or nine is given by a polynomial of the form

$$f(x, y^2 + s(x)), \tag{3.5.2}$$

where f is as in (3.2.1) and $s(x)$ is an appropriate section (3.3.2), and (some of) the torus structures of B are obtained by substituting $y \mapsto y - s(x)$ to (3.5.1) and its three conjugates by the automorphism group $\boldsymbol{A}_4$. It follows that each of the four torus structures of a sextic of weight eight, see [**7**], is invariant under the involution c given by Theorem 2.1.1 and is indeed obtained from (3.5.1) and its

Table 1. The set of singularities of B vs. the position of $\bar{L}$.

Singularities	$\bar{B} \cap \bar{L}$	Condition	Data
$\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$	$\mathbf{A}_2^*, \mathbf{A}_2, \times 1$	(3.4.2)	$(\bar{P}_0, \bar{P}_1)$
$\mathbf{E}_6 \oplus 6\mathbf{A}_2$	$\mathbf{A}_2^*, \times 1, \times 1, \times 1$	(3.4.1)	$\{\bar{P}_0\}$
$2\mathbf{A}_5 \oplus 4\mathbf{A}_2$	$\mathbf{A}_2, \mathbf{A}_2, \times 1, \times 1$	(3.4.3)	$\{\bar{P}_0, \bar{P}_1\}$
$\mathbf{A}_5 \oplus 6\mathbf{A}_2 \oplus \mathbf{A}_1$	$\mathbf{A}_2, \times 2, \times 1, \times 1$	(3.4.5)	$\{\bar{P}_1\}$
$\mathbf{A}_5 \oplus 6\mathbf{A}_2$	$\mathbf{A}_2, \times 1, \times 1, \times 1, \times 1$	$c = -\frac{1}{2}$	$\{\bar{P}_0\}$
$8\mathbf{A}_2 \oplus \mathbf{A}_1$	$\times 2, \times 1, \times 1, \times 1, \times 1$	(3.4.4)	
$8\mathbf{A}_2$	$\times 1, \times 1, \times 1, \times 1, \times 1, \times 1$		
$9\mathbf{A}_2$	$\times 3, \times 1, \times 1, \times 1$	(3.4.7)	

conjugates. (The twelve torus structures of a sextic of weight nine are discussed in 3.7 below.)

The set of singularities of the sextic B covering $\bar{B}$ determines and is determined by the singularities of the divisor $\bar{L}+\bar{B}$, see, e.g., [**10**]. The relevant statements are cited in Table 1, where the mutual position of $\bar{L}$ and $\bar{B}$ is described by listing all intersection points:

- $\times n$ stands for a point of n-fold intersection where $\bar{B}$ is nonsingular,
- $\mathbf{A}_2$ stands for transversal intersection of $\bar{L}$ and $\bar{B}$ at a cusp of $\bar{B}$, and
- $\mathbf{A}_2^*$ stands for a cusp of $\bar{B}$ where $\bar{L}$ is tangent to $\bar{B}$.

In 3.4, we describe the conditions on (a,b,c) necessary for the section (3.3.2) to be in a prescribed position with respect to $\bar{B}$ (see 'Condition' in the table). If $\bar{B}$ is fixed, in each case it follows that the triples (a,b,c) constitute a Zariski open subset in one, four, six, or twelve irreducible families, a single family being selected by a choice of a cusp of $\bar{B}$, a pair of cusps, or an ordered pair of cusps (see 'Data' in the table, where $\{\cdot\}$ stands for a set and $(\cdot)$, for an ordered set).

3.6. Proof of Theorems 1.2.3 and 1.2.4.

Let B be a plane sextic of weight eight and with simple singularities, let c be the involution given by Theorem 2.1.1, and fix an isomorphism φ: $(\boldsymbol{P}^2(O_c)/c, B/c) \to (\Sigma_2, \bar{B})$. As explained in the previous section, the equisingular moduli space of pairs (B,φ) consists of one to twelve connected components, a single component being selected by a choice of a cusp of $\bar{B}$ or an (ordered) pair of cusps, see Table 1. Since the group $\boldsymbol{A}_4$ of automorphisms of $\bar{B}$ acts transitively on its cusps, pairs of cusps, and ordered pairs of cusps, ignoring φ results in a single deformation family, and Theorem 1.2.3 follows.

For Theorem 1.2.4 (in the case of simple singularities), one should also take into consideration the torus structure of $\bar{B}$, which can be identified by the cusp $\bar{P}$ that is *not* in the section $\{\bar{p}=0\}$. Analyzing Table 1, one can see that in all but one cases the additional data still reduce to a cusp or an (ordered) pair of cusps, thus giving rise to a single deformation family. For example, for the set of singularities $2\mathbf{A}_5 \oplus 4\mathbf{A}_2$, the additional data are either $(\{\bar{P}_0,\bar{P}_1\},\bar{P}_0)$ or $(\{\bar{P}_0,\bar{P}_1\},\bar{P}_+)$; in the four element set of the cusps of $\bar{B}$ they are equivalent to the ordered pairs $(\bar{P}_0,\bar{P}_1)$ and $(\bar{P}_+,\bar{P}_-)$, respectively. The exception is the ordered triple $((\bar{P}_0,\bar{P}_1),\bar{P}_+)$ resulting from $(\Sigma,\Sigma_{\mathrm{in}}) = (\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2, \mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2)$; such configurations form two $\boldsymbol{A}_4$-orbits which are interchanged by the transposition $(\bar{P}_+,\bar{P}_-) \in \boldsymbol{S}_4$, i.e., by the complex conjugation.

It remains to consider pairs $(B,\{\text{torus structure}\})$, where B is a sextic of weight eight with a non-simple singular point, see (1.1.2). According to [**8**], B can be obtained from $\bar{B}$ by a birational transformation (stereographic projection), which is determined by its blow-up center $\bar{O} \in \Sigma_2 \smallsetminus (\bar{B} \cup E)$. This point either is generic (the set of singularities $\mathbf{J}_{2,0} \oplus 4\mathbf{A}_2$) or shares a fiber with a cusp of $\bar{B}$ (the set of singularities $\mathbf{J}_{2,3} \oplus 3\mathbf{A}_2$). Hence, even after an extra cusp identifying a torus structure of $\bar{B}$ is added, the data selecting a connected equisingular deformation family still form a single $\boldsymbol{A}_4$-orbit. □

3.7. Torus structures of nine cuspidal sextics.

Pick a generic nine cuspidal sextic $B \subset \boldsymbol{P}^2$ and let $C \subset \boldsymbol{P}^2$ be the dual cubic. We identify the set $\Sigma = \Sigma_B$ of the cusps of B with the set of inflection points of C.

A torus structure (p, q) of B can be characterized by the six point set $\Sigma_{(p,q)} \subset \Sigma$ of its inner cusps or, equivalently, by the triple $\bar{\Sigma}_{(p,q)} \subset \Sigma$ of its outer cusps. For each cusp $P \subset \Sigma$, there are four torus structures (p, q) with $\bar{\Sigma}_{(p,q)} \ni P$, and from the discussion in 3.5 it follows that they are all invariant under the involution $c_P \in \operatorname{Aut}(\boldsymbol{P}^2, B)$ determined by P, see Lemma 2.6.2. Hence, each torus structure (p, q) is stabilized by the three involutions c_P, $P \in \bar{\Sigma}_{(p,q)}$. On the other hand, an involution c_P with $P \in \Sigma_{(p,q)}$ cannot stabilize (p, q) as any invariant subset of c_P containing P has odd cardinality. Now, using the description of $\operatorname{Aut}(\boldsymbol{P}^2, B)$ given by Lemma 2.6.1 (and well known properties of the inflection points on a plane cubic curve), one can easily prove the following statements.

PROPOSITION 3.7.1. *If B is generic, the map $(p, q) \mapsto \{\text{stabilizer}\}$ establishes a one-to-one correspondence between the set of twelve torus structures of B and the set of twelve dihedral subgroups $\boldsymbol{D}_6 \subset \operatorname{Aut}(\boldsymbol{P}^2, B)$. Each dihedral subgroup $G \cong \boldsymbol{D}_6 \subset \operatorname{Aut}(\boldsymbol{P}^2, B)$ has two orbits Σ_3 and Σ_6 of cardinality 3 and 6, respectively, and the inverse map sends G to the torus structure (p, q) with $\Sigma_{(p,q)} = \Sigma_6$.*

COROLLARY 3.7.2. *The action of $\operatorname{Aut}(\boldsymbol{P}^2, B)$ on the set of torus structures of a generic nine cuspidal sextic B has four orbits, each consisting of three elements. Two torus structures belong to the same orbit if and only if their sets of outer singularities are disjoint.*

COROLLARY 3.7.3. *For any nine cuspidal sextic B, a triple $\bar{\Sigma} \subset \Sigma_B$ is the set of outer singularities of a torus structure if and only if the three points of $\bar{\Sigma}$ regarded as inflection points of the dual cubic $C \subset \boldsymbol{P}^2$ are collinear.*

COROLLARY 3.7.4. *The pairs $(B, \{\text{torus structure}\})$, where B is a sextic of weight nine, form a connected deformation family.*

4. The fundamental group.

Throughout this section, $\bar{B} \subset \Sigma_2$ is the four cuspidal trigonal curve given by (3.2.1), $\bar{L}$ is a section of Σ_2, and $B \subset \boldsymbol{P}^2$ is the plane sextic obtained as the pull-back of $\bar{B}$ under the double covering $\boldsymbol{P}^2 \to \Sigma_2/E$ ramified at $\bar{L}$ and E/E.

4.1. The braid group (see [**14**]).

Recall that the *braid group on n strands*, $n \geq 2$, can be defined as the group $\boldsymbol{B}_n$ of automorphisms of the free group $G = \langle \zeta_1, \ldots, \zeta_n \rangle$ sending each generator to a conjugate of another generator and leaving the product $\zeta_1 \cdots \zeta_n$ fixed. We assume that the action of $\boldsymbol{B}_n$ on G is *from the left.* One has

$$\boldsymbol{B}_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ [\sigma_i, \sigma_j] = 1 \text{ for } |i - j| > 1 \rangle,$$

where

$$\sigma_i\colon (\ldots, \zeta_i, \zeta_{i+1}, \ldots) \mapsto (\ldots, \zeta_i \zeta_{i+1} \zeta_i^{-1}, \zeta_i, \ldots), \quad i = 1, \ldots, n-1,$$

are the so called *standard generators* of $\boldsymbol{B}_n$. The center of $\boldsymbol{B}_n$, $n \geq 3$, is the infinite cyclic group generated by Δ^2, where $\Delta = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1)$.

There is a canonical homomorphism $\boldsymbol{B}_n \to \boldsymbol{S}_n$, $\sigma_i \mapsto (i, i+1)$, where $\boldsymbol{S}_n$ is the symmetric group on an n element set. Its kernel is called the *pure braid group*, its elements being *pure braids.*

4.2. Preliminary remarks.

In this section, we briefly outline a few observations made in [**10**] and concerning the computation of the fundamental group of a plane sextic represented as a double of a trigonal curve. A more formal exposition of the application of van Kampen's method to trigonal curves can be found in [**11**].

Consider the groups

$$\pi^\delta = \pi_1(\Sigma_2 \smallsetminus (\bar{B} \cup E \cup \bar{L})), \quad \pi^1 = \pi_1(\Sigma_2 \smallsetminus (\bar{B} \cup E)), \quad \text{and} \quad \pi = \pi_1(\boldsymbol{P}^2 \smallsetminus B).$$

To find π^δ and π^1, we apply van Kampen's method [**13**] to the vertical pencil $x = \text{const}$ (the ruling of Σ_2). We choose the initial fiber F over the point $x = r_-$, and the basis α, δ, β, γ for $\pi_F = \pi_1(F \smallsetminus (\bar{B} \cup E \cup \bar{L}))$ as shown in Figure 1, left. (The fiber is real with respect to both conj and conj$'$, and the grey lines represent the two real parts; all loops are oriented in the counterclockwise direction.) Note that δ plays a special rôle: it is a loop about $\bar{L} \cap F$.

The following statement is an immediate consequence of the double covering construction (the passage from $(\bar{B}, \bar{L})$ to B).

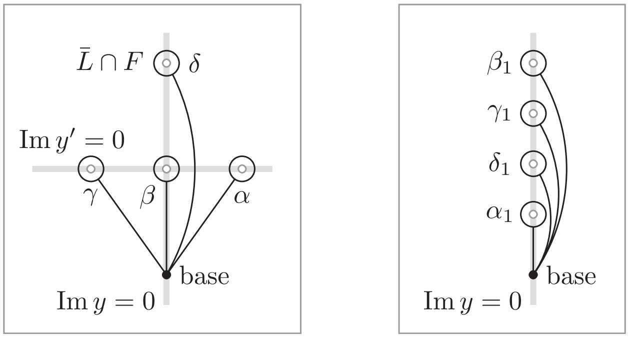

Figure 1. The bases in the fibers.

PROPOSITION 4.2.1. *One has* $\pi^1 = \pi^\delta/\delta$, *and* π *is the kernel of the homomorphism* $\pi^\delta/\delta^2 \to \boldsymbol{Z}_2$, $\alpha, \beta, \gamma \mapsto 0$, $\delta \mapsto 1$.

For the braid monodromy, we keep the base point in the fibers in a conj-real section $y = \text{const} \ll 0$. (Such a section is *proper* in the sense of [**11**].) The basis for $\pi_1(\boldsymbol{C}^1 \smallsetminus \{\text{singular fibers}\})$ is chosen as shown in Figure 2 (which represents the cases resulting in sextics of weight 8; the modifications for the case of weight 9 are explained in 4.5). In the figure, the bold grey line and circle represent the real parts of conj and conj′, respectively, solid (dotted) being the portions over which $\bar{B}$ has three (respectively, one) real points. The grey dots are the projections of the cusps of $\bar{B}$, and the white dot is the projection of a point of transversal intersection of $\bar{B}$ and $\bar{L}$, see Figures 3–5 below; this point gives a relation $[\beta, \delta] = 1$. For the other singular fibers (which all happen to be conj-real and located between x_0 and x_1, see 4.4 below for details), the loops for the monodromy are constructed similar to x_1 (see the global monodromy computation in 4.3 for more details).

Van Kampen's approach states that the group π^δ has a presentation

$$\pi^\delta = \langle \alpha, \beta, \gamma, \delta \mid m_i = \mathrm{id},\ i = 1, \ldots, n,\ (\alpha\delta\beta\gamma)^2 = 1 \rangle, \tag{4.2.2}$$

where n is the number of singular fibers, $m_i \colon \pi_F \to \pi_F$ is the braid monodromy about the singular fiber F_i, and each relation $m_i = \mathrm{id}$ should be understood as a quadruple of relations $m_i(\alpha) = \alpha$, $m_i(\beta) = \beta$, $m_i(\gamma) = \gamma$, $m_i(\delta) = \delta$. (For the *relation at infinity* $(\alpha\delta\beta\gamma)^2 = 1$, see [**10**].)

The following lemma reduces the number of singular fibers to be considered.

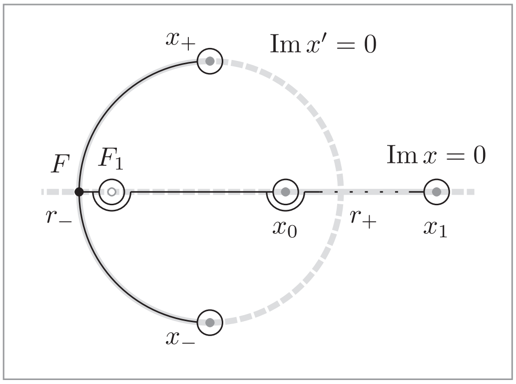

Figure 2. The basis for the monodromy.

LEMMA 4.2.3. *In the presentation* (4.2.2) *for* π^δ, (*any*) *one of the braid relations* $m_i = \mathrm{id}$ *can be ignored.*

In the sequel, we will usually ignore the monodromy about the point $x = x_1$, as the 'farthest' from F.

PROOF. The composition $m_1 \circ \cdots \circ m_n$ of all braid monodromies (in appropriate order) is the braid monodromy along a large circle encompassing all singular fibers (the so called monodromy at infinity). It is known to be $\Delta^4 \in \boldsymbol{B}_4$, i.e., the conjugation by $(\alpha\delta\beta\gamma)^2$, see, e.g., [**11**]. Since $(\alpha\delta\beta\gamma)^2 = 1$, the monodromy at infinity becomes the identity and any of m_i is a composition of the others. $\square$

In all cases considered below, the presentation (4.2.2) for π^δ contains a relation $[\beta, \delta] = 1$. Hence, the following corollary is a consequence of Proposition 4.2.1 and the standard algorithm for presenting a finite index subgroup, see, e.g., [**15**].

COROLLARY 4.2.4. *If* π^δ *is given by* $\langle \alpha, \beta, \gamma, \delta \mid R_j = 1,\ j = 1, \ldots, k \rangle$, *then*

$$\pi = \langle \alpha, \bar\alpha, \beta, \gamma, \bar\gamma \mid R'_j = \bar R'_j = 1,\ j = 1, \ldots, k \rangle,$$

where bar stands for the conjugation by δ, $\bar w = \delta w \delta$, *and* R'_j *is the relation obtained from* R_j, $j = 1, \ldots, k$, *by eliminating* δ (*using the extra relation* $\delta^2 = 1$).

REMARK. In the presentation (4.2.2) for π^δ, each relation contains an even number of copies of δ. Hence, after $\delta^2 = 1$ is added, each relation can be expressed in terms of α, $\bar{\alpha}$, β, γ, $\bar{\gamma}$.

REMARK. From Corollary 4.2.4, the fact that the curves used are real, and the construction of the basis, see Figure 1, left, it follows that each group G_0, G_1, G_2, G_3 introduced below has two involutive automorphisms:

$$\alpha \leftrightarrow \bar{\alpha}, \quad \beta \leftrightarrow \beta, \quad \gamma \leftrightarrow \bar{\gamma} \quad \text{(conjugation by } \delta\text{)},$$
$$\alpha \leftrightarrow \gamma^{-1}, \quad \bar{\alpha} \leftrightarrow \bar{\gamma}^{-1}, \quad \beta \leftrightarrow \beta^{-1} \quad \text{(induced by conj)}.$$

The conjugation by δ can as well be regarded as the automorphism induced by the deck translation of the ramified covering $\boldsymbol{P}^2 \to \Sigma_2/E$.

COROLLARY 4.2.5. *If δ is a central element of π^δ/δ^2, then the map $\alpha, \bar{\alpha} \mapsto \alpha$, $\beta \mapsto \beta$, $\gamma, \bar{\gamma} \mapsto \gamma$ establishes an isomorphism $\pi = \pi^1$.*

COROLLARY 4.2.6. *If $\bar{L}$ is transversal to $\bar{B}$, then the map $\alpha, \bar{\alpha} \mapsto \alpha$, $\beta \mapsto \beta$, $\gamma, \bar{\gamma} \mapsto \gamma$ establishes an isomorphism $\pi = \pi^1$.*

PROOF. With $\bar{B}$ fixed, all pairs $(\bar{B}, \bar{L})$ with $\bar{L}$ transversal to $\bar{B}$ are equisingular deformation equivalent. Hence, one can take for $\bar{L}$ a small perturbation of $E + 2F$. For such a section it is obvious that δ commutes with the other generators. Indeed, in a small tubular neighborhood T of F, the curve $\bar{B}$ is a union of three pairwise distinct almost constant sections, whereas $\bar{L}$ is a very 'sharp' quadratic parabola intersecting $\bar{B}$ transversally at six distinct points. Thus, the space $T \smallsetminus (\bar{B} \cup E \cup \bar{L})$ is diffeomorphic to the complement of $\boldsymbol{C}^2$ to the parabola $y = x^2$ and three distinct horizontal lines $y = a_i \neq 0$, $i = 1, 2, 3$. It is straightforward to see (e.g., using van Kampen's method) that the fundamental group of the latter space has the form

$$\langle \alpha_1, \beta_1, \gamma_1, \delta_1 \mid [\alpha_1, \delta_1] = [\beta_1, \delta_1] = [\gamma_1, \delta_1] = 1 \rangle,$$

where α_1, β_1, γ_1, δ_1 are appropriate generators in F similar to those shown in Figure 1. In particular, the element δ_1 realized by a small loop about $F \cap \bar{L}$ is in the center of the group. On the other hand, δ_1 is conjugate to δ, i.e., δ is also in the center. Then, Corollary 4.2.5 applies to produce the desired isomorphism. $\square$

REMARK. Corollaries 4.2.5 and 4.2.6 hold for any trigonal curve $\bar{B} \subset \Sigma_2$, not only the one given by (3.2.1).

4.3. The braid monodromy.

If the base section is chosen as indicated in 4.2 (a proper section in the terminology of [**11**]), the braid monodromy does indeed act on π_F via braids.

As usual, the computation of the braid monodromy consists of two parts. First, for a singular fiber F_i, one computes the *local braid monodromy*, i.e., the braid monodromy along a small loop about F_i. For this purpose, one chooses appropriate 'standard' generators $\zeta_1, \zeta_2, \ldots$ in a nonsingular fiber F_i' near the singular point (similar to $\alpha, \beta, \gamma, \ldots$ in Figure 1, left or $\beta_1, \gamma_1, \ldots$ in Figure 1, right) and uses a local standard form of the singularity to trace the points of the curve when F_i' is dragged about F_i. (We assume that $\zeta_1, \zeta_2, \ldots$ are represented by small loops about the points of the curve that tend to the singular point when $F_i' \to F_i$; only these generators are involved in the braid relations resulting from F_i.)

In the present paper, we only need the singular fibers F_i passing through the singular points (of the curve $\bar{B} + \bar{L}$) of the following types:

(1) $\mathbf{A}_p$, $p \geq 1$: the local braid monodromy is σ_1^{p+1} (the two points make $(p+1)/2$ full turns about their center of gravity), resulting in the only nontrivial relation

$$\begin{cases} (\zeta_1\zeta_2)^k = (\zeta_2\zeta_1)^k, & \text{if } p = 2k-1 \text{ is odd,} \\ (\zeta_1\zeta_2)^k\zeta_1 = (\zeta_2\zeta_1)^k\zeta_2, & \text{if } p = 2k \text{ is even.} \end{cases} \tag{4.3.1}$$

(2) $\mathbf{D}_5$: the local braid monodromy is $\sigma_2^3\sigma_1\sigma_2^2\sigma_1$ (if it is ζ_1 that corresponds to the nonsingular branch) or $\sigma_1^3\sigma_2\sigma_1^2\sigma_2$ (if ζ_3 corresponds to the nonsingular branch): the two points in the singular branch make 1.5 full turns about their center of gravity, whereas the third point makes one turn about the same center. The resulting nontrivial relations are

$$[\zeta_1, \zeta_2\zeta_3] = 1, \quad \zeta_3\zeta_1\zeta_2\zeta_3 = \zeta_1\zeta_2\zeta_3\zeta_2, \quad \text{or} \tag{4.3.2}$$

$$[\zeta_3, \zeta_1\zeta_2] = 1, \quad \zeta_2\zeta_1\zeta_2\zeta_3 = \zeta_1\zeta_2\zeta_3\zeta_1, \tag{4.3.3}$$

respectively.

(3) $\mathbf{E}_7$: the local braid monodromy is $(\sigma_1\sigma_2\sigma_1)^3$ (the two points in the singular branch make 1.5 full turns about the third point, which is fixed), and the resulting nontrivial relations are

$$[\zeta_2, \zeta_1\zeta_2\zeta_3\zeta_1] = 1, \quad \zeta_3\zeta_1\zeta_2\zeta_3 = \zeta_1\zeta_2\zeta_3\zeta_1. \tag{4.3.4}$$

(Here, the generator corresponding to the nonsingular branch is ζ_2.)

The second step is computing the *global monodromy*, i.e., relating the generators $\zeta_1, \zeta_2, \ldots$ in F_i' to the original generators α, β, γ, δ in the fixed nonsingular fiber F and thus expressing the relations above in terms of the original basis. For this, one chooses a path ξ connecting F' and F and drags F along ξ, keeping the base point in the base section and tracing the point of intersection of F' and the curve. In each case considered in the paper, both the fibers F, F' and the curve are real (with respect to an appropriate real structure) and one can take for ξ a segment of the real line circumventing the interfering real singular fibers of the curve along small semicircles. Over each real point of ξ, all but at most two points of the curve are real, and they can easily be traced using plots (Figures 3–6 below); over the semicircles, the points can be traced using local normal forms of the singularities, similar to the computation of the local monodromy. We leave details to the reader.

4.4. Proof of Theorem 1.2.1: sextics of weight eight.

Let $\bar{L}$ be either the section $y = 2x^2 - 1/2$, see (3.4.2) and Figure 3, or its small conj-real perturbation. (In the plots, $\bar{B}$ and $\bar{L}$ are shown in black and grey, respectively.) Such a section intersects $\bar{B}$ transversally at a real point over $x = -4/5$ (or close); the monodromy about this point (the fiber F_1 in Figure 2) gives a relation

$$[\beta, \delta] = 1 \tag{4.4.1}$$

(relation (4.3.1) with $(\zeta_1, \zeta_2) = (\delta, \beta)$ and $p = 1$). Furthermore, due to Lemma 3.3.5, the section has no conj$'$-real points. Hence, over the loops about $x_\pm$ shown in Figure 2, the point $\bar{L} \cap \{\text{fiber}\}$ remains above the real part, whereas the points $\bar{B} \cap \{\text{fiber}\}$ remain real; hence, the two groups of points are not linked, and one can easily compute the monodromy. The resulting relations are

$$\alpha\beta\alpha = \beta\alpha\beta, \quad \gamma\beta\gamma = \beta\gamma\beta. \tag{4.4.2}$$

(The conj$'$-real part of $\bar{B}$ looks the same as its conj-real part, cf. Figure 3, but the x'-coordinate of F is r_+ rather than r_-; hence, the relations are (4.3.1) with $p = 2$ and $(\zeta_1, \zeta_2) = (\alpha, \beta)$ or (β, γ), respectively.) Finally, we have the relation at infinity

$$(\alpha\delta\beta\gamma)^2 = 1. \tag{4.4.3}$$

Relations (4.4.1)–(4.4.3) hold in any group π^δ obtained below. The corresponding relations for π, see Corollary 4.2.4, are

$$\alpha\beta\alpha = \beta\alpha\beta, \quad \bar{\alpha}\beta\bar{\alpha} = \beta\bar{\alpha}\beta, \tag{4.4.4}$$

$$\gamma\beta\gamma = \beta\gamma\beta, \quad \bar{\gamma}\beta\bar{\gamma} = \beta\bar{\gamma}\beta, \tag{4.4.5}$$

$$\beta\gamma\alpha\beta\bar{\gamma}\bar{\alpha} = 1. \tag{4.4.6}$$

Letting $\delta = 1$ and adding the relation $\alpha\gamma\alpha = \gamma\alpha\gamma$ resulting from the monodromy about the cusp $\bar{P}_0$ (over $x = x_0$), one obtains a presentation for the fundamental group $G_0 = \pi_1(\Sigma_2 \smallsetminus (\bar{B} \cup E))$:

$$G_0 = \langle \alpha, \beta, \gamma \mid \alpha\beta\alpha = \beta\alpha\beta,\ \beta\gamma\beta = \gamma\beta\gamma,\ \gamma\alpha\gamma = \alpha\gamma\alpha,\ (\alpha\beta\gamma)^2 = 1\rangle. \tag{4.4.7}$$

Note that, according to [**8**], G_0 is also the fundamental group of any sextic with the set of singularities $\mathbf{J}_{2,0} \oplus 4\mathbf{A}_2$ or $\mathbf{J}_{2,3} \oplus 3\mathbf{A}_2$.

Consider the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$, see Figure 3. The additional relations for π^δ resulting from the cusp over $x = 0$ are

$$[\delta, \alpha\delta\gamma\alpha] = [\delta, \gamma\alpha\delta\gamma] = 1, \tag{4.4.8}$$

$$\alpha\delta\gamma\alpha = \gamma\alpha\delta\gamma \tag{4.4.9}$$

(relations (4.3.4) with $(\zeta_1, \zeta_2, \zeta_3) = (\alpha, \delta, \gamma)$), their counterpart for π, see Corollary 4.2.4, being

$$\bar{\alpha}\gamma\alpha = \alpha\bar{\gamma}\bar{\alpha} = \gamma\alpha\bar{\gamma} = \bar{\gamma}\bar{\alpha}\gamma. \tag{4.4.10}$$

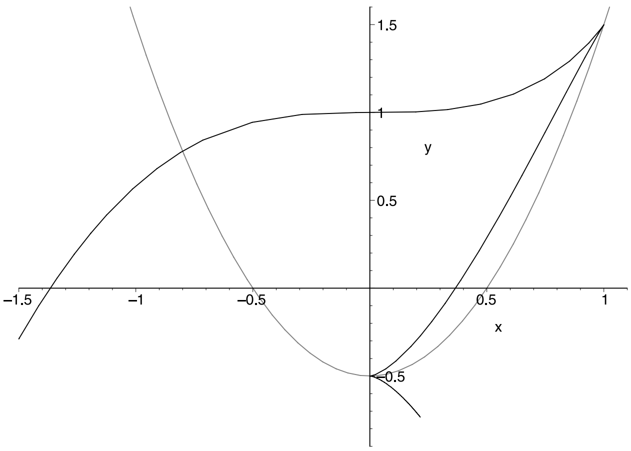

Figure 3. The set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$.

Hence, the fundamental group is

$$G_2 = \langle \alpha, \bar{\alpha}, \beta, \gamma, \bar{\gamma} \mid (4.4.4)\text{–}(4.4.6),\ (4.4.10) \rangle. \tag{4.4.11}$$

Note that any other curve B (or pair $\bar{B} + \bar{L}$) dealt with further in this section is a perturbation of the one just considered. Hence, the corresponding fundamental group π (respectively, π^δ) has relation (4.4.10) (respectively, (4.4.8) and (4.4.9)). Furthermore, similar to Lemma 4.2.3, in the presence of these relations one can ignore (any) one of the singular fibers resulting from the perturbation of the type $\mathbf{E}_7$ singular point of $\bar{B} + \bar{L}$ over $x = 0$.

To facilitate the further calculation, consider another real nonsingular fiber F' between $x = x_0$ and $x = x_1$ and the generators β_1, γ_1, δ_1, α_1 for the group $\pi_1(F' \smallsetminus (\bar{B} \cup E \cup \bar{L}))$ shown in Figure 1, right. Connecting F to F' by a real segment circumventing the point $x = 0$ along the semicircle $x = \epsilon e^{it}$, where ϵ is a sufficiently small positive constant and $t \in [-\pi, 0]$ (cf. the calculation of the global monodromy in 4.3), and taking into account (4.4.1), one obtains the following expressions

$$\beta_1 = \alpha\beta\alpha^{-1}, \quad \gamma_1 = \gamma, \quad \delta_1 = \gamma^{-1}\delta\gamma, \quad \alpha_1 = \gamma^{-1}\delta^{-1}\alpha\delta\gamma. \tag{4.4.12}$$

Clearly, (4.4.12) still holds for any small perturbation of $\bar{B} + \bar{L}$, as one can assume that the semicircle above also circumvents all singular fibers resulting from the perturbation of the type $\mathbf{E}_7$ point. We will express the remaining relations in terms of β_1, γ_1, δ_1, α_1 first, and then use (4.4.12) to convert them back to the original basis.

For the set of singularities $\mathbf{E}_6 \oplus 6\mathbf{A}_2$, perturb $\bar{L}$ to form two new points of transversal intersections of $\bar{L}$ and $\bar{B}$ next to and to the left from the right cusp of $\bar{B}$. The new relations $[\delta_1, \gamma_1] = [\delta_1, \beta_1] = 1$ (relations (4.3.1) with $p = 1$ and $(\zeta_1, \zeta_2) = (\gamma_1, \delta_1)$ or (β_1, δ_1), respectively) in terms of the old generators are

$$[\delta, \gamma] = 1 \text{ (hence, } \delta_1 = \delta), \quad [\delta, \alpha\beta\alpha^{-1}] = 1.$$

Since $\alpha\beta\alpha^{-1} = \beta^{-1}\alpha\beta$ and $[\delta, \beta] = 1$, see (4.4.2) and (4.4.1), respectively, the last relation implies $[\delta, \alpha] = 1$. Hence, δ is a central element and, due to Corollary 4.2.5, one has $\pi = G_0$. The same holds for the sets of singularities $\mathbf{A}_5 \oplus 6\mathbf{A}_2$, $8\mathbf{A}_2 \oplus \mathbf{A}_1$, and $8\mathbf{A}_2$, which are further perturbations of the curve just considered.

For the set of singularities $\mathbf{A}_5 \oplus 6\mathbf{A}_2 \oplus \mathbf{A}_1$, see Figure 4, we can start with the same fiber F' and move backwards toward the left real cusp of $\bar{B}$. The extra relations are

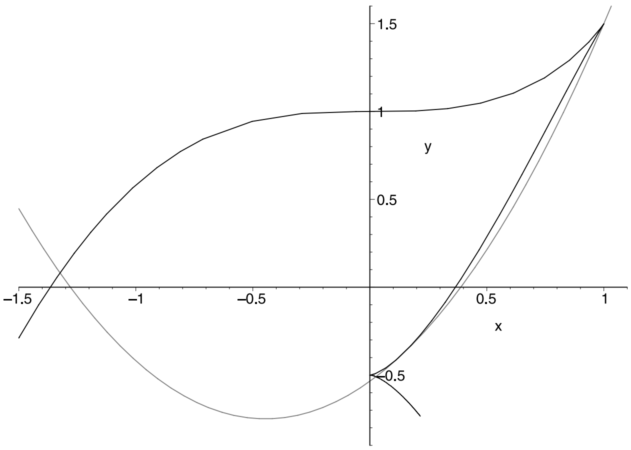

Figure 4. The set of singularities $\mathbf{A}_5 \oplus 6\mathbf{A}_2 \oplus \mathbf{A}_1$.

$$(\gamma_1\delta_1)^2 = (\delta_1\gamma_1)^2, \quad \text{i.e.,} \quad (\gamma\delta)^2 = (\delta\gamma)^2$$

(from the tangency point; relation (4.3.1) with $p = 3$ and $(\zeta_1, \zeta_2) = (\gamma_1, \delta_1)$) and

$$[(\gamma_1\delta_1)\delta_1(\gamma_1\delta_1)^{-1}, \alpha_1] = 1, \quad \text{i.e.,} \quad [\delta, (\delta\gamma)^{-1}\alpha(\delta\gamma)] = 1$$

(from the point of transversal intersection; this is relation (4.3.1) with $p = 1$ and generators $(\zeta_1, \zeta_2) = ((\gamma_1\delta_1)\delta_1(\gamma_1\delta_1)^{-1}, \alpha_1)$ obtained by circumventing the tangency point). As explained above, in the presence of (4.4.8) and (4.4.9), we can ignore the monodromy about the cusp of $\bar{B}$ over $x = 0$. Since $[\delta, (\delta\gamma)^2] = 1$, the last relation turns into

$$[\delta, (\delta\gamma)\alpha(\delta\gamma)^{-1}] = 1, \quad \text{or} \quad [\delta, \gamma\alpha\gamma^{-1}] = 1.$$

Then, (4.4.9) turns into the braid relation $\alpha\gamma\alpha = \gamma\alpha\gamma$ and, since

$$\gamma\alpha\delta\gamma = (\gamma\alpha\gamma^{-1})\delta^{-1}(\delta\gamma)^2,$$

the commutativity relation (4.4.8) becomes a tautology. Finally, the new (in addition to (4.4.1)–(4.4.3)) relations for π^δ are

$$(\gamma\delta)^2 = (\delta\gamma)^2, \quad \alpha\gamma\alpha = \gamma\alpha\gamma, \quad [\delta, \gamma\alpha\gamma^{-1}] = 1, \tag{4.4.13}$$

and their counterparts for π, see Corollary 4.2.4, are

$$\alpha\gamma\alpha = \gamma\alpha\gamma, \quad \bar\alpha\bar\gamma\bar\alpha = \bar\gamma\bar\alpha\bar\gamma, \quad \gamma\alpha\bar\gamma = \bar\gamma\bar\alpha\gamma, \quad [\gamma, \bar\gamma] = 1. \tag{4.4.14}$$

The fundamental group is

$$G_1 = \langle \alpha, \bar\alpha, \beta, \gamma, \bar\gamma \mid (4.4.4)\text{–}(4.4.6), (4.4.14)\rangle. \tag{4.4.15}$$

Finally, consider the set of singularities $2\mathbf{A}_5 \oplus 4\mathbf{A}_2$, see Figure 5. Starting from the same fiber F', one finds that $[\delta_1, \gamma_1] = 1$ (from the point of transversal intersection; relation (4.3.1) with $(\zeta_1, \zeta_2) = (\gamma_1, \delta_1)$ and $p = 1$) and $[\delta_1, \beta_1\gamma_1] = [\delta_1, \gamma_1\alpha_1] = 1$ (partial relations from the two cusps; these are, respectively, the commutativity relations in (4.3.3) and (4.3.2) with $(\zeta_1, \zeta_2, \zeta_3) = (\beta_1, \gamma_1, \delta_1)$ and $(\delta_1, \gamma_1, \alpha_1)$). Since β_1, γ_1, δ_1, α_1 also generate the group, δ_1 is a central element. Then $\delta = \delta_1$, see (4.4.12), and since this element is in the center of π^δ, from Corollary 4.2.5 it follows that $\pi = G_0$. □

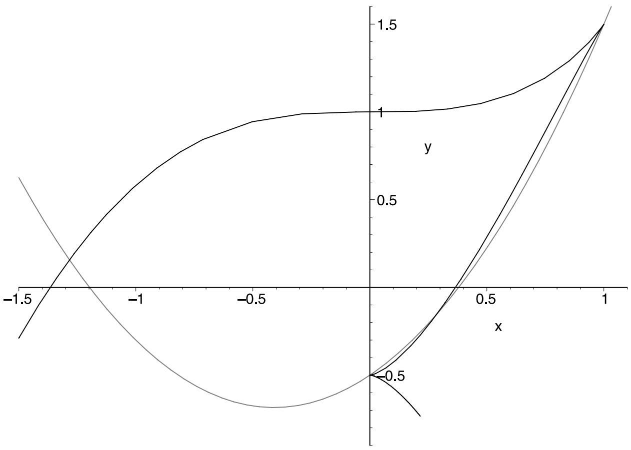

Figure 5. The set of singularities $2\mathbf{A}_5 \oplus 4\mathbf{A}_2$.

4.5. Proof of Theorem 1.2.1: sextics of weight nine.

Any curve of weight nine is a nine cuspidal sextic, see [**7**]; hence, $\bar L$ must be inflection tangent to $\bar B$. We take for $\bar L$ the section $y = \sqrt{3}(2x^2 + 2x - 1)/3$, see (3.4.7), which is real with respect to both conj and conj$'$. This section is plotted in Figure 6, where the solid grey line represents $\bar L$, and the dotted grey line represents the same section $\bar L$ in coordinates (x', y'). (Note that $\bar B$ itself looks the same in both coordinate systems, except that the points $\bar{R}_\pm$ trade rôles when the coordinates are changed.) Now, the fiber over $x = r_-$ is singular, and we choose for F a fiber over a conj-real point between r_- and x_0, cf. Figures 2 and 6. Then, the monodromy about r_- gives the commutativity relation (4.4.1), and hence the monodromy about $x_\pm$ still gives the braid relations (4.4.2): due to (4.4.1), the intertwining of β and δ that occurs when F moves towards one of the two singular fibers can be ignored.

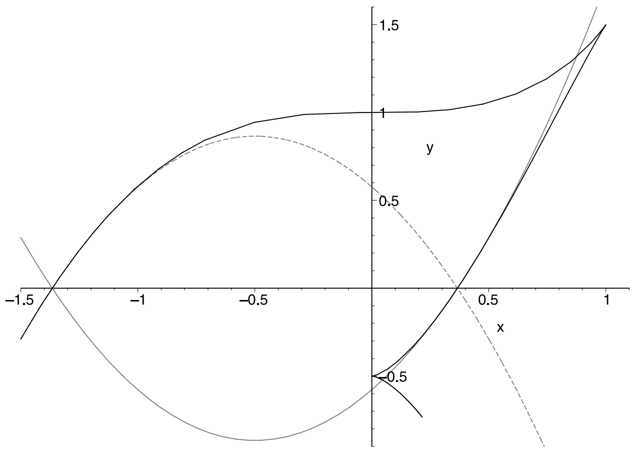

Figure 6. The set of singularities $9\mathbf{A}_2$.

Consider also the generators β_1, γ_1, α_1, δ_1 (cf. Figure 1, right) in a fiber F' over $x = \varepsilon$ for some small $\varepsilon > 0$. They can be expressed in terms of the original generators as explained in 4.3, by circumventing the cusp over $x = 0$. One has

$$\beta_1 = \alpha\beta\alpha^{-1}, \quad \gamma_1 = \delta\gamma\delta^{-1}, \quad \alpha_1 = (\delta\gamma\delta^{-1})^{-1}\alpha(\delta\gamma\delta^{-1}), \quad \delta_1 = \delta.$$

The additional relations are $\gamma_1\alpha_1\gamma_1 = \alpha_1\gamma_1\alpha_1$ (from the cusp; relation (4.3.1) with $p = 2$ and $(\zeta_1, \zeta_2) = (\gamma_1, \alpha_1)$), $(\delta_1\gamma_1)^3 = (\gamma_1\delta_1)^3$ (from the inflection tangency point; this is relation (4.3.1) with $p = 5$ and generators $(\zeta_1, \zeta_2) = (\gamma_1, \delta_1)$, which are obtained by taking into account the fact that δ_1 and α_1 commute, see below), and the commutativity relations $[\delta_1, \alpha_1] = [(\gamma_1\delta_1)^{-1}\delta_1(\gamma_1\delta_1), \beta_1] = 1$ (from the two points of transversal intersection; these are relations (4.3.1) with $p = 1$, where the generators $(\zeta_1, \zeta_2) = (\beta_1, (\gamma_1\delta_1)^{-1}\delta_1(\gamma_1\delta_1))$ at the second intersection point are obtained by circumventing the point of inflection tangency, cf. 4.3). Switching back to the original basis, we obtain

$$\begin{gathered} \delta\gamma\delta^{-1}\alpha\delta\gamma\delta^{-1} = \alpha\delta\gamma\delta^{-1}\alpha, \quad (\delta\gamma)^3 = (\gamma\delta)^3, \\ [\delta, (\delta\gamma)^{-1}\alpha(\delta\gamma)] = [\delta, (\gamma\alpha)\beta(\gamma\alpha)^{-1}] = 1. \end{gathered} \tag{4.5.1}$$

The corresponding relations for π, see Corollary 4.2.4, are

$$\begin{gathered} \bar{\gamma}\alpha\bar{\gamma} = \alpha\bar{\gamma}\alpha, \quad \gamma\bar{\alpha}\gamma = \bar{\alpha}\gamma\bar{\alpha}, \quad \gamma\bar{\gamma}\gamma = \bar{\gamma}\gamma\bar{\gamma}, \\ \bar{\gamma}^{-1}\alpha\bar{\gamma} = \gamma^{-1}\bar{\alpha}\gamma, \quad [\beta, \alpha^{-1}\gamma^{-1}\bar{\gamma}\bar{\alpha}] = 1, \end{gathered} \tag{4.5.2}$$

and the fundamental group is

$$G_3 = \langle \alpha, \bar{\alpha}, \beta, \gamma, \bar{\gamma} \mid (4.4.4)\text{–}(4.4.6),\ (4.5.2) \rangle. \tag{4.5.3}$$

□

4.6. Proof of Theorem 1.2.5: the Alexander module.

Let B be a plane sextic, and let $\pi = \pi_1(\boldsymbol{P}^2 \setminus B)$ be its fundamental group. If B is irreducible, its Alexander module A_B can be defined as follows (see Libgober [**17**] for details and generalizations).

DEFINITION 4.6.1. The *Alexander module* A_B of an irreducible plane curve B of degree m is the abelian group π'/π'' regarded as a $\boldsymbol{Z}[t]/(t^m - 1)$-module via the multiplication $t[x] = [a^{-1}xa]$, where $[x] \in A_B$ is the class of an element $x \in \pi'$ and $a \in \pi$ is any element generating $\pi/\pi' = \boldsymbol{Z}_m$. The *Alexander polynomial* $\Delta_B(t)$ is the order of the torsion $\boldsymbol{C}[t]/(t^m - 1)$-module $A_B \otimes \boldsymbol{C}$ or, alternatively, the characteristic polynomial of the operator $t\colon A_B \otimes \boldsymbol{C} \to A_B \otimes \boldsymbol{C}$. One has $\deg \Delta_B(t) = \operatorname{rk}_{\boldsymbol{Z}} A_B$.

Thus, the Alexander module is an invariant of the fundamental group π. Below, we compute it for the groups G_0, G_1, G_2, G_3 introduced in this section.

Consider the ring $\Lambda = \boldsymbol{Z}[t]/(t^2 - t + 1)$. Note that $t^3 = -1$ and $t^6 = 1$ in Λ; in particular, t is invertible. First, consider the 'universal' group

$$\tilde{G} = \langle \alpha, \bar{\alpha}, \beta, \gamma, \bar{\gamma} \mid (4.4.4)\text{–}(4.4.6) \rangle.$$

One has $\tilde{G}/\tilde{G}' = \boldsymbol{Z}_6$. A standard calculation (see, e.g., [**15**]) shows that the derived group $\tilde{G}'$ is generated by the elements

$$\begin{gathered} a_i = \beta^i\alpha\beta^{-i-1}, \quad c_i = \beta^i\gamma\beta^{-i-1}, \quad i \in \boldsymbol{Z}, \\ \bar{a}_i = \delta a_i\delta, \quad \bar{c}_i = \delta c_i\delta, \quad i \in \boldsymbol{Z}, \quad \text{and} \quad b = \beta^6, \end{gathered}$$

which are subject to the relations

$$bs_ib^{-1} = s_{i+6} \quad \text{for } s = a, c, \bar{a}, \bar{c} \text{ and } i \in \boldsymbol{Z}, \tag{4.6.2}$$

$$s_i = s_{i-1}s_{i+1} \quad \text{for } s = a, c, \bar{a}, \bar{c} \text{ and } i \in \boldsymbol{Z}, \tag{4.6.3}$$

$$a_i\bar{c}_{i+2}\bar{a}_{i+3}c_{i+5}b = \bar{a}_ic_{i+2}a_{i+3}\bar{c}_{i+5}b = 1, \quad i \in \boldsymbol{Z}. \tag{4.6.4}$$

Consider the abelianization $\tilde{A} = \tilde{G}'/\tilde{G}''$ (passing to the additive notation) and represent t as conjugation by β. One has $s_i = t^is_0$ for $s = a, c, \bar{a}, \bar{c}$, $i \in \boldsymbol{Z}$, and $tb = b$. Due to (4.6.4), $\tilde{A}$ is generated by a_0, $\bar{a}_0$, c_0, $\bar{c}_0$, and from the braid relations (4.6.3) it follows that $(t^2 - t + 1)$ annihilates $\tilde{A}$, i.e., $\tilde{A}$ is a Λ-module. Since $tb = b$, one has $b = 0$ (as $t - 1 = t^2$ is invertible in Λ). Then, in view of the fact that $t^3 = -1$ in Λ, the last relation (4.6.4) implies $\bar{a}_0 - a_0 = t^2(\bar{c}_0 - c_0)$. Thus, $\tilde{A}$ is a free Λ module with three generators, for example, a_0, c_0, and $\bar{c}_0$.

Consider the group G_2 corresponding to the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$, see (4.4.11). The additional relations for the derived group G_2' are

$$\bar{a}_{i-1}c_ia_{i+1} = a_{i-1}\bar{c}_i\bar{a}_{i+1} = c_{i-1}a_i\bar{c}_{i+1} = \bar{c}_{i-1}\bar{a}_ic_{i+1}, \quad i \in \boldsymbol{Z}.$$

Passing to the abelianization $A_2 = G_2'/G_2''$ and cancelling the term $a_0 + c_1 + a_2 = c_0 + a_1 + c_2$, one obtains

$$\bar{a}_0 - a_0 = t^2(\bar{a}_0 - a_0) + t(\bar{c}_0 - c_0) = t^2(\bar{c}_0 - c_0) = t(\bar{a}_0 - a_0) + (\bar{c}_0 - c_0).$$

Substituting $\bar{a}_0 - a_0 = t^2(\bar{c}_0 - c_0)$, one gets $t^2(\bar{c}_0 - c_0) = (t^3 + 1)(\bar{c}_0 - c_0) = 0$. Thus, $\bar{c}_0 = c_0$ and A_2 is the free Λ-module generated by a_0 and c_0.

Any other irreducible sextic B of weight 8 is a perturbation of the sextic with the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$. Hence, its Alexander module A_B is a quotient of A_2. On the other hand, $\operatorname{rk}_{\boldsymbol{Z}} A_B = \deg \Delta_B(t) = 4$; hence, there can be no other relations and $A_B = A_2$.

Consider the group G_3 corresponding to the set of singularities $9\mathbf{A}_2$, see (4.5.3). The extra relations for G_3' are

$$\bar{c}_ia_{i+1}\bar{c}_{i+2} = a_i\bar{c}_{i+1}a_{i+2}, \quad c_i\bar{a}_{i+1}c_{i+2} = \bar{a}_ic_{i+1}\bar{a}_{i+2}, \quad \bar{c}_ic_{i+1}\bar{c}_{i+2} = c_i\bar{c}_{i+1}c_{i+2},$$
$$\bar{c}_i^{-1}a_i\bar{c}_{i+1} = c_i^{-1}\bar{a}_ic_{i+1}, \quad \bar{c}_i\bar{a}_{i+1}\bar{a}_{i+2}^{-1}\bar{c}_{i+1}^{-1} = c_ia_{i+1}a_{i+2}^{-1}c_{i+1}^{-1},$$

$i \in \boldsymbol{Z}$. Passing to the Λ-module $A_3 = G_3'/G_3''$, the first three groups of relations result in a tautology $(t^2 - t + 1)(\ldots) = 0$, and the last two groups, considering that $t - 1 = t^2$ and $t(t-1) = -1$ in Λ, give the same relation $\bar{a}_0 - a_0 = t^2(\bar{c}_0 - c_0)$.

Hence, $A_3 = \tilde{A}$ is the free Λ-module generated by a_0, c_0, and $\bar{c}_0$. $\square$

5. Applications.

As above, we consider the four cuspidal trigonal curve $\bar{B} \subset \Sigma_2$ given by (3.2.1) and a section $\bar{L} \subset \Sigma_2$ and denote by $B \subset \boldsymbol{P}^2$ the plane sextic obtained as the pull-back of $\bar{B}$ under the double covering $\boldsymbol{P}^2 \to \Sigma_2/E$ ramified at $\bar{L}$ and E/E.

5.1. Perturbations.

Denote by Γ_B the combined Dynkin diagram of B, i.e., the union of the Dynkin diagrams of all singular points of B (cf. 2.3).

PROPOSITION 5.1.1. *Any induced subgraph $\Gamma' \subset \Gamma_B$ is the combined Dynkin diagram of an appropriate small perturbation B' of B. Conversely, the combined Dynkin diagram of any perturbation of B is an induced subgraph of Γ_B.*

REMARK. We do *not* assert that *any* sextic B' whose combined Dynkin diagram is Γ' is a perturbation of B. In general, this is not true, as the homological type of B' does not need to extend to that of B.

REMARK. Proposition 5.1.1 holds for any sextic with simple singularities; it does not need to be irreducible or obtained by the double covering construction.

PROOF. The statement follows from the description of the moduli space of plane sextics given in [**6**]. Let Σ_B and Σ' be the root systems spanned by the vertices of Γ_B and Γ', respectively, cf. 2.3. Then $\Sigma' \oplus \langle 2 \rangle \subset \Sigma_B \oplus \langle 2 \rangle$ is a primitive sublattice. Hence, it gives rise to a valid abstract homological type, and to construct a perturbation, one shifts the class ω of holomorphic 2-form of the covering $K3$-surface from the union of the hyperplanes orthogonal to the vertices in $\Gamma_B \smallsetminus \Gamma'$. Conversely, a perturbation of B means a perturbation of ω, and we merely remove from Σ_B the generators (former (-2)-curves) that are not orthogonal to ω. $\square$

COROLLARY 5.1.2. *The Alexander polynomial of an irreducible plane sextic B of torus type, with simple singularities, and of weight $w(B) \le 7$ is $t^2 - t + 1$.*

PROOF. The Alexander polynomial can be computed using linear systems of conics, see [**4**] or [**20**], and the problem reduces to the non-existence of a sextic of weight six (seven) with at least two (respectively, one) singular points of weight zero other than nodes $\mathbf{A}_1$. In view of Proposition 5.1.1, such a sextic would perturb to a sextic with the set of singularities $7\mathbf{A}_2 \oplus \mathbf{A}_3$. The latter does not exist due to [**6**] or J.-G. Yang [**29**]: in terms of [**6**], one would have $\ell(\operatorname{discr} \tilde{S}) \ge 5 > 4 = \operatorname{rk} S^{\perp}$.

(Here, $S = \Sigma \oplus \boldsymbol{Z}h \subset H_2(\tilde{X})$, cf. 2.3, $\tilde{S}$ is the primitive hull of S in $H_2(\tilde{X})$, and ℓ stands for the minimal number of generators of a group.) ☐

Pick a singular point P of B and perturb it to obtain another sextic B'. To describe the impact of this operation on the fundamental group, pick a Milnor ball D_P around P and consider a set of generators $\rho_1, \ldots, \rho_k$ for the fundamental group $\pi_1(\partial D_P \smallsetminus B) = \pi_1(D_P \smallsetminus B)$, the isomorphism being induced by the inclusion $\partial D_P \smallsetminus B \hookrightarrow D_P \smallsetminus B$. After the perturbation, the inclusion homomorphism becomes an epimorphism $\pi_1(\partial D_P \smallsetminus B) \twoheadrightarrow \pi_1(D_P \smallsetminus B)$, i.e., a number of relations $R_1, \ldots$ in the generators $\rho_1, \ldots, \rho_k$ is introduced. Clearly, one has

$$\pi_1(\boldsymbol{P}^2 \smallsetminus B') = \pi_1(\boldsymbol{P}^2 \smallsetminus B)/\langle R_1, \ldots \rangle, \tag{5.1.3}$$

where the generators ρ_i and relations $R_1, \ldots$ are regarded as elements of $\pi_1(\boldsymbol{P}^2 \smallsetminus B)$ via the inclusion $\partial D_P \smallsetminus B \hookrightarrow \boldsymbol{P}^2 \smallsetminus B$.

5.2. Perturbations of cusps.

Let P be a cusp of B over the cusp $\bar{P}_+$ of $\bar{B}$. It can be perturbed to either $\mathbf{A}_1$ or $\varnothing$, in each case the extra relation added to the group being $\alpha = \beta$. (We can choose P so that α and β generate $\pi_1(D_P \smallsetminus B)$, and the latter group becomes cyclic after the perturbation.)

If we perturb another cusp P' over the same point $\bar{P}_+$, the group gets one more relation $\bar{\alpha} = \beta$ (and, in view of Lemma 5.2.1 below, this extra perturbation does not affect the group). If we perturb another cusp P' over $\bar{P}_-$, the new relation is $\gamma = \beta$ or $\bar{\gamma} = \beta$, depending on the choice of P'; in view of Lemma 5.2.2 below, the resulting groups are isomorphic (and the resulting curve is not of torus type).

Remark. Note that, unlike Section 4, we are working with the group of a perturbed sextic B', which does not need to be symmetric. Hence, we cannot automatically produce new relations applying the conjugation by δ.

Lemma 5.2.1. *The map $\alpha, \bar{\alpha}, \beta \mapsto \sigma_1$, $\gamma, \bar{\gamma} \mapsto \sigma_2$ establishes isomorphisms*

$$G_m/\langle \alpha = \beta \rangle = G_m/\langle \alpha = \bar{\alpha} = \beta \rangle = \boldsymbol{B}_3/(\sigma_1\sigma_2\sigma_1)^2 \cong \boldsymbol{Z}_2 * \boldsymbol{Z}_3, \quad m = 0, 1, 2,$$

where G_m are the groups given by (4.4.7), (4.4.15), *and* (4.4.11).

Proof. For the 'smallest' group $G_0/\langle \alpha = \bar{\alpha} = \beta \rangle$ the statement is obvious. Hence, it suffices to consider the 'largest' group $G_2/\langle \alpha = \beta \rangle$.

From the relation $\gamma\alpha\bar{\gamma} = \bar{\gamma}\bar{\alpha}\gamma$, see (4.4.10), it follows that $\bar{\alpha} = \bar{\gamma}^{-1}\gamma\beta\bar{\gamma}\gamma^{-1}$. Substituting this expression and $\alpha = \beta$ to $\bar{\alpha}\gamma\alpha = \gamma\alpha\bar{\gamma}$, see (4.4.10), we obtain

$$\bar{\gamma}^{-1}\gamma\cdot\underline{\beta\bar{\gamma}\beta}=\gamma\beta\bar{\gamma},\quad\text{or}\quad\bar{\gamma}^{-1}\gamma\cdot\underline{\bar{\gamma}\beta\bar{\gamma}}=\gamma\beta\bar{\gamma},\quad\text{or}\quad\bar{\gamma}^{-1}\gamma\bar{\gamma}=\gamma.$$

Thus, $[\gamma,\bar{\gamma}]=1$ and $\bar{\alpha}=\bar{\gamma}^{-1}\gamma\beta\gamma^{-1}\bar{\gamma}$. Substitute this expression to the braid relation $\bar{\alpha}\beta\bar{\alpha}=\beta\bar{\alpha}\beta$, see (4.4.4):

$$\bar{\gamma}^{-1}\cdot\underline{\gamma\beta\gamma^{-1}}\cdot\bar{\gamma}\beta\bar{\gamma}^{-1}\cdot\underline{\gamma\beta\gamma^{-1}}\cdot\bar{\gamma}=\beta\bar{\gamma}^{-1}\cdot\underline{\gamma\beta\gamma^{-1}}\cdot\bar{\gamma}\beta.$$

Replacing the underlined expressions using braid relations (4.4.5), one obtains

$$\bar{\gamma}^{-1}\beta^{-1}\gamma\beta\cdot\underline{\bar{\gamma}\beta\bar{\gamma}^{-1}}\cdot\beta^{-1}\gamma\beta\bar{\gamma}=\underline{\beta\bar{\gamma}^{-1}\beta^{-1}}\cdot\gamma\cdot\underline{\beta\bar{\gamma}\beta}$$

and, once again,

$$\bar{\gamma}^{-1}\beta^{-1}\gamma\beta\beta^{-1}\bar{\gamma}\beta\beta^{-1}\gamma\beta\bar{\gamma}=\bar{\gamma}^{-1}\beta^{-1}\bar{\gamma}\gamma\bar{\gamma}\beta\bar{\gamma}.$$

Cancelling and using $[\gamma,\bar{\gamma}]=1$ gives $\bar{\gamma}=\gamma$. Hence also $\bar{\alpha}=\beta$, and the statement follows. □

LEMMA 5.2.2. *There are isomorphisms*

$$G_m/\langle\alpha=\beta=\gamma\rangle=G_m/\langle\alpha=\beta=\bar{\gamma}\rangle=\boldsymbol{Z}_6,\quad m=0,1,2,$$

where G_m are the groups given by (4.4.7), (4.4.15), *and* (4.4.11).

PROOF. Due to Lemma 5.2.1, all groups are quotients of $\boldsymbol{B}_3/\langle\sigma_1=\sigma_2\rangle=\boldsymbol{Z}$. □

5.3. Perturbations of $\mathbf{E}_6$.

Assume that $\bar{L}$ is tangent to $\bar{B}$ at $\bar{P}_0$ and let P be the type $\mathbf{E}_6$ singular point over $\bar{P}_0$. We consider one of the following perturbations:

$$\mathbf{D}_5,\ \mathbf{D}_4,\ \mathbf{A}_4\oplus\mathbf{A}_1,\ \mathbf{A}_4,\ \mathbf{A}_3\oplus\mathbf{A}_1,\ \mathbf{A}_3,\ \mathbf{A}_2\oplus k\mathbf{A}_1\ (k\le 2),\ k\mathbf{A}_1\ (k\le 3).\tag{5.3.1}$$

(Thus, we exclude the perturbations $\mathbf{A}_5$, $2\mathbf{A}_2\oplus\mathbf{A}_1$, and $2\mathbf{A}_2$, which can be realized by perturbing $\bar{L}$ in Σ_2 and lead to sextics of weight 8.) The group $\pi_1(D_P\smallsetminus B)$ is generated by α, $\bar{\alpha}$, γ, and $\bar{\gamma}$.

LEMMA 5.3.2. *The space $D_P\smallsetminus B$ is diffeomorphic to $\boldsymbol{P}^2\smallsetminus(C\cup N)$, where $C\subset\boldsymbol{P}^2$ is a plane quartic with a type $\mathbf{E}_6$ singular point, and N is a line with a single quadruple intersection point with C.*

PROOF. Since $D_P \smallsetminus B$ is a local object, one can replace B with C, assuming that P is the type $\mathbf{E}_6$ singular point of C and that D_P is a Milnor ball about P. The affine coordinates (x, y) in $\boldsymbol{C}^2 = \boldsymbol{P}^2 \smallsetminus N$ can be chosen so that C is the quartic $y^3 - x^4 = 0$, and then D_P can be replaced with the polydisk $\Delta_1 \cap \Delta_2$, where

$$\Delta_1 = \{(x, y) \in \boldsymbol{C}^2 \mid |x| < \epsilon_1\}, \quad \Delta_2 = \{(x, y) \in \boldsymbol{C}^2 \mid |y| < \epsilon_2\},$$

and $0 < \epsilon_1 \ll \epsilon_2$. Then $C \cap \Delta_1 \subset \Delta_1 \cap \Delta_2$, and contracting the y-axis to Δ_2 results in a diffeomorphism $(\Delta_1 \cap \Delta_2) \smallsetminus C \cong \Delta_1 \smallsetminus C$.

Now, consider the projection p: $\boldsymbol{C}^2 \to \boldsymbol{C}$, $(x, y) \mapsto x$, and its restriction p_C to C. Each pull-back $p_C^{-1}(x)$, $x \neq 0$, consists of exactly three distinct points (as p_C has no critical points other than P and C has no branches tending to infinity over a finite value of x); hence, the fibration p: $\boldsymbol{C}^2 \smallsetminus C \to \boldsymbol{C}$ is locally trivial outside $x = 0$, and contracting the x-axis $\boldsymbol{C}$ to Δ_1 lifts to a diffeomorphism $\boldsymbol{C}^2 \smallsetminus C \cong \Delta_1 \smallsetminus C$. □

In view of Lemma 5.3.2, the perturbations of B inside D_P can be regarded as perturbations of C keeping the point of quadruple intersection with N, see [**3**], and in all cases except those excluded above, for the perturbed quartic C' one has $\pi_1(\boldsymbol{P}^2 \smallsetminus (C' \cup N)) = \boldsymbol{Z}$, see [**5**]. Hence, the extra relations are $\alpha = \bar{\alpha} = \gamma = \bar{\gamma}$. From the lemma below, it follows that it suffices to add the relation $\alpha = \gamma$.

LEMMA 5.3.3. *The map $\alpha, \bar{\alpha}, \gamma, \bar{\gamma} \mapsto \sigma_1$, $\beta \mapsto \sigma_2$ establishes isomorphisms*

$$G_m / \langle \alpha = \gamma \rangle = \boldsymbol{B}_3 / (\sigma_1 \sigma_2 \sigma_1)^2 \cong \boldsymbol{Z}_2 * \boldsymbol{Z}_3, \quad m = 0, 1, 2,$$

where G_m are the groups given by (4.4.7)*,* (4.4.15)*, and* (4.4.11)*.*

PROOF. In the presence of $\alpha = \gamma$, relations (4.4.10) are rewritten in the form

$$\bar{\alpha}\alpha^2 = \alpha\bar{\gamma}\bar{\alpha} = \alpha^2\bar{\gamma} = \bar{\gamma}\bar{\alpha}\alpha.$$

Hence, $[\alpha, \bar{\gamma}\bar{\alpha}] = 1$, then $\bar{\gamma} = \alpha^{-1}\bar{\gamma}\bar{\alpha} = \bar{\gamma}\bar{\alpha}\alpha^{-1} = \bar{\alpha}$, and finally $\bar{\alpha} = \bar{\gamma} = \alpha = \gamma$. □

5.4. Perturbations of $\mathbf{A}_5$.

Assume that $\bar{L}$ passes through the cusp $\bar{P}_1$ of $\bar{B}$, and let P be the type $\mathbf{A}_5$ singular point of B over $\bar{P}_1$. We consider the following perturbations of P:

$$\mathbf{A}_3 \oplus \mathbf{A}_1, \ 3\mathbf{A}_1, \tag{5.4.1}$$

$$\mathbf{A}_4, \ \mathbf{A}_3, \ \mathbf{A}_2 \oplus \mathbf{A}_1, \ \mathbf{A}_2, \ k\mathbf{A}_1 \ (k \leq 2). \tag{5.4.2}$$

(Thus, we exclude the perturbation $2\mathbf{A}_2$, which can be realized by a shift of $\bar{L}$.) The group $\pi_1(D_P \smallsetminus B)$ is generated by the elements $\beta_1 = \alpha\beta\alpha^{-1}$ and $\gamma_1 = \gamma$ given by (4.4.12). For the series (5.4.1), the group $\pi_1(D_P \smallsetminus B')$ is abelian of rank two; hence, the extra relation is $[\beta_1, \gamma_1] = 1$. For the series (5.4.2), one has $\pi_1(D_P \smallsetminus B') = \boldsymbol{Z}$ and the extra relation is $\beta_1 = \gamma_1$.

LEMMA 5.4.3. *The map $\alpha \mapsto \sigma_1$, $\bar\alpha \mapsto \sigma_3$, $\beta \mapsto \sigma_2$, $\gamma \mapsto \sigma_2^{-1}\sigma_3\sigma_2$, $\bar\gamma \mapsto \sigma_2^{-1}\sigma_1\sigma_2$ establishes isomorphisms*

$$G_2/[\beta_1, \gamma_1] = G_1/[\beta_1, \gamma_1] = \boldsymbol{B}_4/\sigma_1^2\sigma_2\sigma_3^2\sigma_2,$$

where G_2 and G_1 are the groups given by (4.4.11) *and* (4.4.15)*, respectively, and $\beta_1 = \alpha\beta\alpha^{-1}$ and $\gamma_1 = \gamma$ are the generators given by* (4.4.12).

PROOF. As explained above, the relation $[\beta_1, \gamma_1] = 1$ means that B is perturbed so that the group $\pi_1(D_P \smallsetminus B')$ becomes abelian. On the other hand, the elements $\delta_1^{-1}\beta_1\delta_1$ and $\delta_1^{-1}\gamma_1\delta_1$ also belong to $\pi_1(D_P \smallsetminus B)$ and are conjugate to γ_1 and β_1, respectively. Hence, in addition, we get the relations

$$\delta_1^{-1}\gamma_1\delta_1 = \beta_1, \quad \delta_1^{-1}\beta_1\delta_1 = \gamma_1.$$

(In view of Lemma 4.2.3, the two additional relations should follow from $[\beta_1, \gamma_1] = 1$ and the relations for G_2, but we will not try to deduce them algebraically.) Passing back to the generators $\alpha, \ldots, \bar\gamma$, we obtain

$$[\alpha\beta\alpha^{-1}, \gamma] = 1, \quad \alpha\beta\alpha^{-1} = \gamma^{-1}\bar\gamma\gamma, \quad \bar\alpha\beta\bar\alpha^{-1} = \bar\gamma^{-1}\gamma\bar\gamma.$$

Comparing the first two relations, we conclude that $[\gamma, \bar\gamma] = 1$, and then the right hand sides of the last two relations are $\bar\gamma$ and γ, respectively. Due to (4.4.4), the last two relations can be rewritten in the form $\beta^{-1}\alpha\beta = \bar\gamma$ and $\beta^{-1}\bar\alpha\beta = \gamma$; hence, $[\alpha, \bar\alpha] = 1$ and the relation at infinity (4.4.6) becomes $\alpha^2\beta\bar\alpha^2\beta = 1$.

It remains to check that all other relations follow from those already listed and braid relations (4.4.4), so that the homomorphism in the statement is well defined. We will do this for the smaller group $G_1/[\beta_1, \gamma_1]$. Braid relations (4.4.5) are the conjugation of (4.4.4) by β, and in (4.4.14), after the substitution, the first two relations turn into (4.4.4) and the third one simplifies to $\alpha^2\beta\bar\alpha^2 = \bar\alpha^2\beta\alpha^2$, which follows from $\alpha^2\beta\bar\alpha^2\beta = 1$ (as both sides equal β^{-1}). □

LEMMA 5.4.4. *The map $\alpha, \bar\alpha \mapsto \sigma_1$, $\beta \mapsto \sigma_2$, $\gamma, \bar\gamma \mapsto \sigma_2^{-1}\sigma_1\sigma_2$ establishes isomorphisms*

$$G_0/[\beta_1,\gamma_1] = G_m/\langle\beta_1=\gamma_1\rangle = \boldsymbol{B}_3/(\sigma_1\sigma_2\sigma_1)^2 \cong \boldsymbol{Z}_2 * \boldsymbol{Z}_3, \quad m=0,1,2,$$

where G_m are the groups given by (4.4.7), (4.4.15), *and* (4.4.11), *and $\beta_1 = \alpha\beta\alpha^{-1}$ and $\gamma_1 = \gamma$ are the generators given by* (4.4.12).

PROOF. The statement follows from Lemma 5.4.3, as we get a relation $\sigma_3 = \sigma_1$ for the braid group $\boldsymbol{B}_4$, either from $\alpha = \bar\alpha$ in G_0, or from the relation $\beta_1 = \gamma_1$. □

COROLLARY 5.4.5. *The perturbation epimorphism $G_1 \twoheadrightarrow G_0$ is not one-to-one.*

PROOF. If the map $G_1 \twoheadrightarrow G_0 = G_1/\langle\alpha\bar\alpha^{-1},\gamma\bar\gamma^{-1}\rangle$ were one-to-one, so would be its quotient $\boldsymbol{B}_4/\sigma_1^2\sigma_2\sigma_3^2\sigma_2 \twoheadrightarrow \boldsymbol{B}_3/(\sigma_1\sigma_2\sigma_1)^2$, see Lemmas 5.4.3 and 5.4.4, which can be regarded as adding an extra relation $\sigma_3 = \sigma_1$. On the other hand, $\sigma_1^2\sigma_2\sigma_3^2\sigma_2$ is a pure braid; hence, its normal closure cannot contain $\sigma_3\sigma_1^{-1}$. □

5.5. Abelian perturbations.

Next corollary describes the result of perturbing two or more singular points of B. (In the case of two cusps only, they should be chosen over distinct cusps of $\bar B$.) The statement is an immediate consequence of Lemmas 5.2.1, 5.2.2, 5.3.3, and 5.4.3: one should take into account the description of the isomorphisms given by the lemmas and observe that adding a second relation results in the relation $[\sigma_1, \sigma_2] = 1$ in $\boldsymbol{B}_3$, hence, in a cyclic group.

COROLLARY 5.5.1. *Let G_m, $m=0,1,2$, be one of the groups given by* (4.4.7), (4.4.15), *or* (4.4.11), *and let R_1, R_2 be two distinct relations from the set $\alpha=\beta$, $\gamma=\beta$, $\alpha=\gamma$, $[\alpha\beta\alpha^{-1},\gamma]=1$. Then $G_m/\langle R_1,R_2\rangle = \boldsymbol{Z}_6$.*

5.6. Some sextics of weight ≤ 7.

We apply the results just obtained to produce a number of plane sextics with controlled fundamental group. Altogether, we obtain 47 sets of singularities of torus type and 122 sets of singularities not of torus type and not covered by Nori's theorem [**19**].

THEOREM 5.6.1. *Let Σ be a set of singularities obtained from one of those listed in Table* 2 *by removing several* (*possibly none*) *nodes $\mathbf{A}_1$. Then Σ is realized by an irreducible plane sextic of torus type whose fundamental group is $\boldsymbol{B}_4/\sigma_1^2\sigma_2\sigma_3^2\sigma_2$* (*for the four sets of singularities marked with a $*$ in the table*) *or $\boldsymbol{B}_3/(\sigma_1\sigma_2\sigma_1)^2 \cong \boldsymbol{Z}_2 * \boldsymbol{Z}_3$* (*for the other sets in the table and for all nontrivial perturbations*).

Table 2. Sextics of torus type.

$\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 3\mathbf{A}_2 \oplus \mathbf{A}_1$	$2\mathbf{A}_5 \oplus 3\mathbf{A}_2 \oplus \mathbf{A}_1$
$\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$	$2\mathbf{A}_5 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$
$\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus 4\mathbf{A}_2$	$\mathbf{A}_5 \oplus \mathbf{A}_4 \oplus 4\mathbf{A}_2 \oplus \mathbf{A}_1$
$* \ \mathbf{E}_6 \oplus \mathbf{A}_3 \oplus 4\mathbf{A}_2 \oplus \mathbf{A}_1$	$\mathbf{A}_5 \oplus \mathbf{A}_3 \oplus 4\mathbf{A}_2 \oplus \mathbf{A}_1$
$\mathbf{E}_6 \oplus 5\mathbf{A}_2 \oplus \mathbf{A}_1$	$\mathbf{A}_5 \oplus 5\mathbf{A}_2 \oplus 2\mathbf{A}_1$
$* \ \mathbf{E}_6 \oplus 4\mathbf{A}_2 \oplus 3\mathbf{A}_1$	$\mathbf{A}_5 \oplus 4\mathbf{A}_2 \oplus 3\mathbf{A}_1$
$\mathbf{D}_5 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$	$\mathbf{A}_4 \oplus 6\mathbf{A}_2 \oplus \mathbf{A}_1$
$\mathbf{D}_5 \oplus 6\mathbf{A}_2$	$* \ \mathbf{A}_3 \oplus 6\mathbf{A}_2 \oplus 2\mathbf{A}_1$
$\mathbf{D}_4 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$	$7\mathbf{A}_2 \oplus 2\mathbf{A}_1$
$\mathbf{D}_4 \oplus 6\mathbf{A}_2$	$* \ 6\mathbf{A}_2 \oplus 4\mathbf{A}_1$

PROOF. We start with one of the sets of singularities listed in (1.1.1), realize it by a sextic B of weight eight, and use Proposition 5.1.1 to perturb B to a new sextic B'. The fundamental group of B' is controlled using Lemmas 5.2.1, 5.3.3, 5.4.3, and 5.4.4. According to the lemmas, one can perturb up to two cusps of B (both over the same cusp of $\bar{B}$) or one type $\mathbf{E}_6$ or $\mathbf{A}_5$ singular point of B, so that the new group $\pi_1(\boldsymbol{P}^2 \smallsetminus B')$ would factor to $\boldsymbol{Z}_2 * \boldsymbol{Z}_3$. (In the case of a type $\mathbf{E}_6$ or $\mathbf{A}_5$ singular point, the allowed perturbations are those listed in (5.3.1) or (5.4.1), (5.4.2), respectively.) Since the latter group has nontrivial Alexander polynomial $t^2 - t + 1$, the new sextic B' is of torus type, see [**7**]. $\square$

REMARK. Theorem 5.6.1 covers five tame sextics:

$$\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2, \quad \mathbf{E}_6 \oplus 4\mathbf{A}_2, \quad 2\mathbf{A}_5 \oplus 2\mathbf{A}_2, \quad \mathbf{A}_5 \oplus 4\mathbf{A}_2, \quad 6\mathbf{A}_2.$$

The groups of these curves were first found in [**23**].

COROLLARY 5.6.2. *The Alexander module of each curve mentioned in Theorem* 5.6.1 *is a free module on one generator over the ring* $\Lambda = \boldsymbol{Z}[t]/(t^2 - t + 1)$.

PROOF. The Alexander module of each braid group $\boldsymbol{B}_n$, $n \geq 3$, is known to be a free Λ-module generated, e.g., by $\sigma_2\sigma_1^{-1}$. Since the Alexander polynomial of each curve is $t^2 - t + 1$, there can be no further relations. $\square$

THEOREM 5.6.3. *Let* Σ *be a set of singularities obtained from one of those listed in Table* 3 *by several* (*possibly none*) *perturbations* $\mathbf{A}_2 \to \mathbf{A}_1, \varnothing$ *or* $\mathbf{A}_1 \to \varnothing$.

Then Σ is realized by an irreducible plane sextic, not of torus type, whose fundamental group is $\boldsymbol{Z}_6$.

Table 3. Sextics with abelian fundamental group.

$\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$	$\mathbf{D}_4 \oplus 5\mathbf{A}_2 \oplus \mathbf{A}_1$
$\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus 3\mathbf{A}_2 \oplus \mathbf{A}_1$	$\mathbf{D}_4 \oplus 4\mathbf{A}_2 \oplus 3\mathbf{A}_1$
$\mathbf{E}_6 \oplus \mathbf{A}_3 \oplus 3\mathbf{A}_2 \oplus 2\mathbf{A}_1$	$2\mathbf{A}_5 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$
$\mathbf{E}_6 \oplus 4\mathbf{A}_2 \oplus 2\mathbf{A}_1$	$\mathbf{A}_5 \oplus \mathbf{A}_4 \oplus 3\mathbf{A}_2 \oplus 2\mathbf{A}_1$
$\mathbf{E}_6 \oplus 3\mathbf{A}_2 \oplus 4\mathbf{A}_1$	$\mathbf{A}_5 \oplus \mathbf{A}_3 \oplus 3\mathbf{A}_2 \oplus 2\mathbf{A}_1$
$\mathbf{D}_5 \oplus \mathbf{A}_5 \oplus 3\mathbf{A}_2 \oplus \mathbf{A}_1$	$\mathbf{A}_5 \oplus 4\mathbf{A}_2 \oplus 3\mathbf{A}_1$
$\mathbf{D}_5 \oplus \mathbf{A}_4 \oplus 4\mathbf{A}_2$	$2\mathbf{A}_4 \oplus 4\mathbf{A}_2 \oplus \mathbf{A}_1$
$\mathbf{D}_5 \oplus \mathbf{A}_3 \oplus 4\mathbf{A}_2 \oplus \mathbf{A}_1$	$\mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 4\mathbf{A}_2 \oplus 2\mathbf{A}_1$
$\mathbf{D}_5 \oplus 5\mathbf{A}_2 \oplus \mathbf{A}_1$	$\mathbf{A}_4 \oplus 5\mathbf{A}_2 \oplus 2\mathbf{A}_1$
$\mathbf{D}_5 \oplus 4\mathbf{A}_2 \oplus 3\mathbf{A}_1$	$\mathbf{A}_4 \oplus 4\mathbf{A}_2 \oplus 4\mathbf{A}_1$
$\mathbf{D}_4 \oplus \mathbf{A}_5 \oplus 3\mathbf{A}_2 \oplus \mathbf{A}_1$	$2\mathbf{A}_3 \oplus 4\mathbf{A}_2 \oplus 2\mathbf{A}_1$
$\mathbf{D}_4 \oplus \mathbf{A}_4 \oplus 4\mathbf{A}_2$	$\mathbf{A}_3 \oplus 5\mathbf{A}_2 \oplus 3\mathbf{A}_1$
$\mathbf{D}_4 \oplus \mathbf{A}_3 \oplus 4\mathbf{A}_2 \oplus \mathbf{A}_1$	$6\mathbf{A}_2 \oplus 3\mathbf{A}_1$
	$5\mathbf{A}_2 \oplus 5\mathbf{A}_1$

Proof. As in the previous proof, we start with a sextic B of weight eight (one of the sets of singularities listed in (1.1.1)) and use Proposition 5.1.1 to perturb two or more singular points of B. (If only two cusps are perturbed, they should be over distinct cusps of $\bar{B}$; the allowed perturbations of a type $\mathbf{E}_6$ or $\mathbf{A}_5$ singular point are those listed in (5.3.1) or (5.4.1), (5.4.2), respectively.) Then, due to Corollary 5.5.1, the new group is abelian. □

Remark. Since we only construct examples in the proofs, Theorems 5.6.1 and 5.6.3 are stated in the form of existence. Proving the uniqueness of the equisingular deformation family realizing each of the 169 sets of singularities covered by the theorem would require a lot of tedious calculations. (Certainly, when speaking about uniqueness, one should distinguish sextics of torus type and those not of torus type; in some cases of weight 6, see item (2) below, they may share the same sets of singularities.) We mention two series where the uniqueness is known.

(1) For 19 sets of singularities in Theorem 5.6.1 and for 60 sets of singularities in Theorem 5.6.3, the uniqueness of an equisingular deformation family follows directly from Theorem 5.2.1 in [**6**].
(2) According to A. Özgüner [**24**], the uniqueness takes place for any set of singularities of weight 6 with all points of weight zero of type $\mathbf{A}_1$. With few exceptions, these sets of singularities are realized by two deformation families: one of torus type and one not.

Comparing the two lists and using [**24**], we obtain the following corollary related to some of the so called *classical Zariski pairs*, i.e., pairs of sextics that share the same set of singularities but differ by the Alexander polynomial, see [**4**].

Corollary 5.6.4. *Each of the following* 17 *sets of singularities*

$$\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2 \oplus k\mathbf{A}_1, \quad \mathbf{E}_6 \oplus 4\mathbf{A}_2 \oplus k\mathbf{A}_1, \quad 2\mathbf{A}_5 \oplus 2\mathbf{A}_2 \oplus k\mathbf{A}_1 \quad (k \le 2),$$
$$\mathbf{A}_5 \oplus 4\mathbf{A}_2 \oplus k\mathbf{A}_1, \quad 6\mathbf{A}_2 \oplus k\mathbf{A}_1 \quad (k \le 3)$$

is realized by exactly two equisingular deformation families of irreducible plane sextics. In each case, the families differ by the fundamental group of the curves, which is either $\boldsymbol{Z}_2 * \boldsymbol{Z}_3$ (*torus type*) *or* $\boldsymbol{Z}_6$.

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