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Asymptotic dimension of invariant subspace in tensor product representation of compact Lie group

By Taro Suzuki and Tatsuru Takakura

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Abstract. We consider asymptotic behavior of the dimension of the invariant subspace in a tensor product of several irreducible representations of a compact Lie group G. It is equivalent to studying the symplectic volume of the symplectic quotient for a direct product of several coadjoint orbits of G. We obtain two formulas for the asymptotic dimension. The first formula takes the form of a finite sum over tuples of elements in the Weyl group of G. Each term is given as a multiple integral of a certain polynomial function. The second formula is expressed as an infinite series over dominant weights of G. This could be regarded as an analogue of Witten's volume formula in 2-dimensional gauge theory. Each term includes data such as special values of the characters of the irreducible representations of G associated to the dominant weights.

1. Introduction.

Let G be a connected, simply-connected, compact simple Lie group. For a representation V of G, the symbol V^G denotes the subspace of V of all G-invariant elements. Let P_+ be the set of dominant weights of G and denote the complex irreducible representation of G with highest weight $\lambda \in P_+$ by V_λ. Fix a positive integer n. For $\lambda_1, \dots, \lambda_n \in P_+$, we set

$$\mathscr{Q} = \mathscr{Q}(\lambda_1, \dots, \lambda_n) := \dim_{\boldsymbol{C}}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^G,$$
$$\mathscr{V} = \mathscr{V}(\lambda_1, \dots, \lambda_n) := \limsup_{k \to \infty} \frac{1}{k^d} \dim_{\boldsymbol{C}}(V_{k\lambda_1} \otimes \cdots \otimes V_{k\lambda_n})^G,$$

where the number d is the expected degree of the leading term in $\dim_{\boldsymbol{C}}(V_{k\lambda_1} \otimes \cdots \otimes V_{k\lambda_n})^G$ as a function of positive integers k (see Sections 2 and 3 for the details). The purpose of this paper is to study evaluations of $\mathscr{Q}$ and $\mathscr{V}$, in particular, to find some explicit formulas for $\mathscr{V}$. This should be a fundamental

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problem in representation theory. Although an exact evaluation of $\mathscr{Q}$, which is almost equivalent to determining all the multiplicities in a multiple tensor product representation, may be involved with somewhat complicated combinatorics, that of $\mathscr{V}$ has a chance to become more accessible.

As we will see in Section 3, these quantities have geometric counterparts. Namely, $\mathscr{Q}$ and $\mathscr{V}$ correspond to the Riemann-Roch number and the symplectic volume, respectively, of the symplectic quotient

$$\mathscr{M} = \mathscr{M}(\lambda_1, \ldots, \lambda_n) := \{(x_1, \ldots, x_n) \in \mathscr{O}_{\lambda_1} \times \cdots \times \mathscr{O}_{\lambda_n} \mid x_1 + \cdots + x_n = 0\}/G,$$

where $\mathscr{O}_{\lambda_i}$ is the coadjoint orbit of G associated to λ_i. This fact is indeed a motivation for us to study the problem above. It is expected that an explicit formula for $\mathscr{V}$ contains much information about the cohomology intersection pairings of $\mathscr{M}$.

In the case of $G = SU(2)$, explicit formulas for $\mathscr{Q}$ and $\mathscr{V}$ have been investigated from various points of view. Especially, when $\lambda_1 = \cdots = \lambda_n$, they are closely related to the classical invariant theory for binary forms (see, e.g., [**9**]). We refer to [**21**], [**22**], [**23**] as prototypes of this paper, where an application to the cohomology intersection pairings of $\mathscr{M}$ was also given. We refer to [**14**], [**17**], [**15**] for more geometric approaches.

The case of $G = SU(3)$ was studied by the authors in [**20**]. Explicit evaluations for $\mathscr{Q}$ and $\mathscr{V}$ were done there. Related results were also obtained in [**17**] and [**7**].

On the other hand, as far as the authors know, there have been few explicit results on $\mathscr{V}$ for other Lie groups. Our main results in this paper are two kinds of formulas of $\mathscr{V}$ for general G. It might be interesting that these two are quite different from each other, whereas they give the same answer. In order to obtain them, we restrict ourselves to the case that all the weights $\lambda_1, \ldots, \lambda_n$ are in the root lattice and that they are regular in the sense that they belong to the interior of a Weyl chamber. (See Sections 4 and 7 for the details about our other assumptions.)

The first formula is given in Theorem 4.11, which takes the form of a finite sum as follows:

$$\mathscr{V}(\lambda_1, \ldots, \lambda_n) = \frac{(-1)^{|\Delta_+|}}{|W|} \sum_{(w_1, \ldots, w_n) \in W^n} \varepsilon(w_1) \cdots \varepsilon(w_n)\, I(\lambda_1, \ldots, \lambda_n; w_1, \ldots, w_n), \quad (1.1)$$

where Δ_+ is the set of all positive roots of G, W is the Weyl group, and $\varepsilon(w_i)$ is the signature of $w_i \in W$. The summand $I(\lambda_1, \ldots, \lambda_n; w_1, \ldots, w_n)$ is the value of a multiple integral of a polynomial over a convex polytope. See Section 4 for the details. We only mention here that the polynomial and the convex polytope are determined by the root system Δ of G and the element $w_1(\lambda_1)+\cdots+w_n(\lambda_n)$. This formula is a generalization of the ones obtained in [**22**] for $G=SU(2)$, and in [**20**] for $G=SU(3)$. However, a concrete evaluation of the integral $\mathscr{I}(\lambda_1,\dots,\lambda_n;w_1,\dots,w_n)$ for general G is still another problem, while we were able to do it explicitly for $G=SU(2),SU(3)$. The case of $G=\mathit{Spin}(5)$ is treated in [**24**] from the viewpoint of Gel'fand-Kapranov-Zelevinsky hypergeometric functions.

The second formula is given in Theorem 7.3, which takes the form of an infinite series as follows:

$$\begin{aligned}&\mathscr{V}(\lambda_1,\dots,\lambda_n)\\&\quad=\frac{|P^\vee/Q^\vee|}{|P/Q^\vee|}L^d\left(\prod_{i=1}^n\prod_{\alpha\in\Delta_+}2\sin\frac{\pi(\lambda_i|\alpha)}{L}\right)\sum_{\mu\in P_+}\frac{\prod_{i=1}^n\chi_\mu\left(e^{\frac{-2\pi\sqrt{-1}\lambda_i}{L}}\right)}{\left(\prod_{\alpha\in\Delta_+}2\pi(\mu+\rho|\alpha)\right)^{n-2}}.\end{aligned}\tag{1.2}$$

Here P, $P^\vee$, and $Q^\vee$ is the weight lattice, coweight lattice, and coroot lattice, respectively. Besides, θ is the highest root, ρ is the half of the sum of all positive roots, χ_μ is the character of the irreducible representation V_μ, $(\,|\,)$ is the normalized symmetric bilinear form on P, and $L=(\lambda_1+\cdots+\lambda_n|\theta)$. This formula could be regarded as an analogue of the Witten's volume formula in [**26**] for the 2-dimensional gauge theory (see also [**19**], [**16**]). The fusion coefficients and the Verlinde formula for the affine Lie algebra associated to G play an essential role in the proof. The main idea is to generalize the argument in [**26**, Section 3] for $G=SU(2)$ to a general compact Lie group G. The point there is to explain why the factor $|P^\vee/Q^\vee|$ arises. We mention that our proof shows that L can be replaced by any number that is greater than L.

Let us make a comment for the case where some of $\lambda_1,\dots,\lambda_n$ are not regular. Also in such a case, we can proceed similarly to some extent. For example, in the case $G=SU(3)$ the first formula of $\mathscr{V}$ like as (1.1) for non-regular weights was obtained in [**20**]. It suggests that more careful consideration than the one in this paper will be needed to obtain the corresponding formula in non-regular cases for general G. On the other hand, taking into account of the Witten's volume formula, which is established also for non-regular case, one might expect that the second formula like as (1.2) would hold in a slightly modified form (see Remarks 7.4 and 7.5). However, our argument to prove (1.2) indeed requires the regularity assumption. We thus do not pursue this issue in this paper.

This paper is organized as follows. Section 2 is devoted to clarifying our notation on root systems, compact Lie groups, complex simple Lie algebras, and their representations. We explain the geometric background of our problem in Section 3. The discussion there is essentially the same with the one given in [**20**]. In Section 4, we first give a combinatorial expression for $\mathscr{Q}$ by the Weyl character formula and the Weyl integration formula. After introducing the assumptions on weights $\lambda_1, \dots, \lambda_n$, we derive the first formula (1.1) for $\mathscr{V}$. We also discuss the integrals appearing in this formula for some Lie groups with low rank.

In Section 5, we review the fusion coefficients and the Verlinde formula for a complex simple Lie algebra (or, more precisely, for an affine Lie algebra of split type), and give another expression for $\mathscr{Q}$. In Section 6, we study some details on a root system, especially on symmetry of a certain simplex, called an alcove, associated to the root system. We also prepare some technical estimates. In Section 7, we establish the second formula (1.2) for $\mathscr{V}$ using the results in Sections 5 and 6. Finally, we write out the formula more concretely in the special case that all of $\lambda_1, \dots, \lambda_n$ are proportional to ρ. As we will illustrate for $G = SU(2)$ or $SU(3)$, we then have another kind of formula that expresses $\mathscr{V}$ as an integral over an unbounded domain.

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2. Preliminaries.

In this section, we review some standard facts about root systems and representations of compact Lie groups or complex simple Lie algebras, in order to fix our notation. The symbols introduced here will be used throughout this paper without extra notice. We refer to [**2**], [**5**], [**25**], and [**6**] for the details on the generalities stated below.

Let G be a connected and simply-connected compact simple Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}_{\boldsymbol{C}}$ be the complexification of $\mathfrak{g}$. Let T be a maximal torus of G, with Lie algebra $\mathfrak{t}$. We denote by l the dimension of T. The complexification $\mathfrak{h} = \mathfrak{t}_{\boldsymbol{C}}$ of $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}_{\boldsymbol{C}}$.

Let $\Delta \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}_{\boldsymbol{C}}$ with respect to $\mathfrak{h}$. Let Δ_+ (resp. Δ_-) be a set of positive (resp. negative) roots and let $\{\alpha_1, \dots, \alpha_l\} \subset \Delta_+$ be a set of simple roots. We introduce the normalized standard inner product $(\,|\,)$ on $\mathfrak{h}$ and $\mathfrak{h}^*$, which is a nondegenerate symmetric bilinear form, defined as the non zero scalar multiple of the Killing form $B(\,,\,)$ normalized as $(\theta|\theta) = 2$, where θ is the highest root. Note that $(\,|\,)$ is negative definite on $\mathfrak{t}$ and $\mathfrak{t}^*$. By means of the inner product $(\,|\,)$, we often identify $\mathfrak{h}$ and $\mathfrak{t}$ with $\mathfrak{h}^*$ and $\mathfrak{t}^*$ respectively. For instance, for $\alpha \in \mathfrak{h}^*$, define $H_\alpha \in \mathfrak{h}$ by $(H_\alpha|\cdot) = \langle \alpha, \cdot \rangle$. Then we often confuse α and H_α. Let $\{\alpha_1^\vee, \dots, \alpha_l^\vee\} \subset \mathfrak{h}$ the associated set of simple coroots, namely $\alpha_i^\vee = 2H_{\alpha_i}/(\alpha_i|\alpha_i)$. We set $\mathfrak{h}_{\boldsymbol{R}}^* := \sum_{i=1}^l \boldsymbol{R}\alpha_i$ and $\mathfrak{h}_{\boldsymbol{R}} := \sum_{i=1}^l \boldsymbol{R}\alpha_i^\vee$.

Let W be the Weyl group of Δ. The Coxeter number h of Δ is defined as the order of the element $s_1 \cdots s_l$ in W, where s_i $(i = 1, \dots, l)$ is the reflection on $\mathfrak{h}^*$ defined by $s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i$ for $x \in \mathfrak{h}^*$. We can show that if we write the highest root θ as $\theta = n_1\alpha_1 + \cdots + n_l\alpha_l$ with $n_1, \dots, n_l \in \boldsymbol{Z}_{>0}$, then $h = n_1 + \cdots + n_l + 1$. (See [**2**, Chapter VI, Section 1, no. 11].) Under the identification $\mathfrak{h}^* = \mathfrak{h}$, we can also write the highest root θ as $\theta = n_1^\vee \alpha_1^\vee + \cdots + n_l^\vee \alpha_l^\vee$, with $n_1^\vee, \dots, n_l^\vee \in \boldsymbol{Z}_{>0}$. Then the number $g = n_1^\vee + \cdots + n_l^\vee + 1$ is called the dual Coxeter number of Δ. The Coxeter number h and the dual Coxeter number g for each root system is given by the table below. Note that g is not necessarily equal to the Coxeter number of the dual root system $\Delta^\vee$.

Table 1. Coxeter number h and Dual Coxeter number g.

Δ	A_l	B_l	C_l	D_l	E_6	E_7	E_8	F_4	G_2
h	$l+1$	$2l$	$2l$	$2l-2$	12	18	30	12	6
g	$l+1$	$2l-1$	$l+1$	$2l-2$	12	18	30	9	4

The fundamental weights $\Lambda_1, \dots, \Lambda_l \in \mathfrak{h}_{\boldsymbol{R}}^*$ are defined by $\langle \Lambda_j, \alpha_k^\vee \rangle = \delta_{jk}$. Similarly, the fundamental coweights $\Lambda_1^\vee, \dots, \Lambda_l^\vee \in \mathfrak{h}_{\boldsymbol{R}}$ are defined by $\langle \alpha_j, \Lambda_k^\vee \rangle = \delta_{jk}$, or equivalently by $\Lambda_k^\vee = 2H_{\Lambda_k}/(\alpha_k|\alpha_k)$. In view of $\mathfrak{h}_{\boldsymbol{R}}^* = \sqrt{-1}\mathfrak{t}^*$ and $\mathfrak{h}_{\boldsymbol{R}} = (1/\sqrt{-1})\mathfrak{t}$, let us set

$$\omega_i := \frac{1}{2\pi\sqrt{-1}}\Lambda_i \in \mathfrak{t}^*, \quad a_i^\vee := 2\pi\sqrt{-1}\alpha_i^\vee \in \mathfrak{t} \tag{2.1}$$

for $i = 1, \dots, l$. Since G is simply-connected, $a_1^\vee, \dots, a_l^\vee$ form a basis of the integral lattice $\mathrm{Ker}(\exp : \mathfrak{t} \to T)$ (see [**5**, Chapter V, Section 7]).

Let $\rho := (1/2)\sum_{\alpha \in \Delta_+} \alpha$. It is easy to see that

$$\rho - w\rho = \sum_{\alpha \in \Delta_+ \cap w\Delta_-} \alpha \tag{2.2}$$

for each $w \in W$. It is also well known that $\rho = \Lambda_1 + \cdots + \Lambda_l$, which implies that $g = (\rho|\theta) + 1$.

Remark 2.1. Here are some technical remarks which will be used later. Let $\alpha \in \Delta_+$. According to our normalization of the inner product,

$$(\alpha|\alpha) = \frac{2}{3}, 1, 2. \tag{2.3}$$

It follows that $(\rho|\alpha) \geq 1/3$, since $(\rho|\alpha) \geq (\Lambda_i|\alpha_i) = (\alpha_i|\alpha_i)/2 \geq 1/3$, where we picked an $i \in \{1, \ldots, l\}$ such that $(\Lambda_i|\alpha) > 0$. We thus have

$$(\mu + \rho|\alpha) \geq \frac{1}{3} \tag{2.4}$$

for each $\mu \in P_+$, since $(\mu + \rho|\alpha) \geq (\rho|\alpha)$.

Let us consider lattices

$$Q := \sum_{i=1}^{l} \boldsymbol{Z}\alpha_i, \quad Q^\vee := \sum_{i=1}^{l} \boldsymbol{Z}\alpha_i^\vee, \quad P := \sum_{i=1}^{l} \boldsymbol{Z}\Lambda_i, \quad P^\vee := \sum_{i=1}^{l} \boldsymbol{Z}\Lambda_i^\vee.$$

It is easy to see that $Q^\vee \subset Q \subset P$ and $Q^\vee \subset P^\vee \subset P$ under the identification $\mathfrak{h}_{\boldsymbol{R}}^* = \mathfrak{h}_{\boldsymbol{R}}$. The finite abelian groups P/Q and $P^\vee/Q^\vee$ are in duality each other and the order $|P/Q| = |P^\vee/Q^\vee|$ is called the connection index of the root system Δ (see [**2**, Chapter VI, Section 1, no. 9]). Since we assumed that G is simply connected, the group $P^\vee/Q^\vee$ is canonically isomorphic to the center $Z(G)$ of G (see [**3**, Chapter IX, Section 4, no. 9]).

Table 2. Connection index $|P/Q|$.

Δ	A_l	B_l	C_l	D_l	E_6	E_7	E_8	F_4	G_2
$\lvert P/Q\rvert$	$l+1$	2	2	4	3	2	1	1	1

Let us set

$$C_+ := \sum_{i=1}^{l} \boldsymbol{R}_{\geq 0}\Lambda_i, \quad \mathfrak{t}_+^* := \sum_{i=1}^{l} \boldsymbol{R}_{\geq 0}\omega_i, \quad P_+ := \sum_{i=1}^{l} \boldsymbol{Z}_{\geq 0}\Lambda_i, \quad P_{++} := \sum_{i=1}^{l} \boldsymbol{Z}_{>0}\Lambda_i.$$

Elements in P_+ are called dominant weights of Δ. For $\lambda \in P_+$, let V_λ be the finite dimensional irreducible representation of G or $\mathfrak{g}^{\boldsymbol{C}}$ with highest weight λ, and $\chi_\lambda : G \to \boldsymbol{C}$ the character of V_λ. By the Weyl character formula, χ_λ is given by

$$\chi_\lambda(t) = \frac{\sum_{w \in W} \varepsilon(w) e^{\langle w(\lambda+\rho), X\rangle}}{e^{\langle \rho, X\rangle} \prod_{\alpha \in \Delta_+}(1 - e^{-\langle \alpha, X\rangle})} \tag{2.5}$$

for $t = \exp X \in T$ with $X \in \mathfrak{t}$, where $\varepsilon(w) = \pm 1$ is the signature of $w \in W$. If we set

$$A_\mu(X) := \sum_{w \in W} \varepsilon(w) e^{\langle w(\mu), X\rangle} \tag{2.6}$$

for $\mu \in P$ and $X \in \mathfrak{t}$ (or, more generally, $X \in \mathfrak{h}$), then the Weyl denominator formula tells us that

$$A_\rho(X) = e^{\langle \rho, X\rangle} \prod_{\alpha \in \Delta_+} (1 - e^{-\langle \alpha, X\rangle}) = (\sqrt{-1})^{|\Delta_+|} \prod_{\alpha \in \Delta_+} 2 \sin \frac{(X|\alpha)}{2\sqrt{-1}} \tag{2.7}$$

and (2.5) is also written as

$$\chi_\lambda(t) = \frac{A_{\lambda+\rho}(X)}{A_\rho(X)}. \tag{2.8}$$

Note that, in our convention, weights of a representation of G are regarded as elements in $P \subset \mathfrak{h}^*_{\boldsymbol{R}}$, not in $\mathrm{Hom}(T, \boldsymbol{C}^*)$ or $\mathfrak{t}^*$.

Given a representation V of G or $\mathfrak{g}_{\boldsymbol{C}}$ and $\lambda \in P_+$, let $\mathrm{Mult}(V, V_\lambda)$ be the multiplicity of the irreducible representation V_λ in V. It is obvious that $\dim_{\boldsymbol{C}} V^G = \mathrm{Mult}(V, V_0)$, where $V_0 = \boldsymbol{C}$ is the trivial 1-dimensional representation. Now, we define $\mathscr{Q}(\lambda_1, \ldots, \lambda_n)$ and $\mathscr{V}(\lambda_1, \ldots, \lambda_n)$ more precisely, although they have already been introduced in Section 1. For $\lambda \in P_+$, denote

$$\Delta_+^\lambda := \{\alpha \in \Delta_+ \mid (\alpha|\lambda) = 0\}. \tag{2.9}$$

DEFINITION 2.2. Fix a positive integer n. For $\lambda_1, \ldots, \lambda_n \in P_+$, we define

$$\begin{aligned} \mathscr{Q}(\lambda_1, \ldots, \lambda_n) : &= \dim_{\boldsymbol{C}} (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^G \\ &= \mathrm{Mult}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}, V_0). \end{aligned}$$

Supposing that k runs over positive integers, we set

$$\mathscr{V}(\lambda_1, \ldots, \lambda_n) := \limsup_{k \to \infty} \frac{1}{k^d} \dim_{\boldsymbol{C}} \mathscr{Q}(k\lambda_1, \ldots, k\lambda_n),$$

where the integer $d = d(\lambda_1, \ldots, \lambda_n)$ is defined by

$$d := \sum_{i=1}^{n}(|\Delta_+| - |\Delta_+^{\lambda_i}|) - \dim_{\boldsymbol{R}} G = (n-2)|\Delta_+| - \sum_{i=1}^{n}|\Delta_+^{\lambda_i}| - l \tag{2.10}$$

and we suppose that n is large enough that $d \geq 0$.

A geometric meaning of the number d will be explained in Section 3. Our purpose in this paper is to seek formulas which express $\mathscr{Q}(\lambda_1, \dots, \lambda_n)$ and $\mathscr{V}(\lambda_1, \dots, \lambda_n)$ as explicitly as possible. Later in Sections 4 and 7, we restrict ourselves to the case that $\Delta_+^{\lambda_i} = 0$ for all $i = 1, \dots, n$. More details on the assumptions on $\lambda_1, \dots, \lambda_n$ will be given in Sections 4 and 7.

Remark 2.3. More generally, let H be a closed subgroup of G. Then, we can also consider

$$\mathscr{Q}^H(\lambda_1, \dots, \lambda_n) := \dim_{\boldsymbol{C}}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^H,$$
$$\mathscr{V}^H(\lambda_1, \dots, \lambda_n) := \limsup_{k \to \infty} \frac{1}{k^{d'}} \mathscr{Q}^H(k\lambda_1, \dots, k\lambda_n)$$

where in this case we set $d' = \sum_{i=1}^{n}(|\Delta_+| - |\Delta_+^{\lambda_i}|) - \dim_{\boldsymbol{R}} H$.

3. Geometric background.

In this section, we will explain geometric counterparts for $\mathscr{Q}(\lambda_1, \dots, \lambda_n)$ and $\mathscr{V}(\lambda_1, \dots, \lambda_n)$. See [**10**] and [**20**, Section 2] for the details of the subjects explained below.

The left coadjoint action of G on $\mathfrak{g}^*$ is defined by $g \cdot f := \mathrm{Ad}^*(g^{-1})f$ for $g \in G$ and $f \in \mathfrak{g}^*$, where $\langle \mathrm{Ad}^*(g^{-1})f, X \rangle = \langle f, \mathrm{Ad}(g^{-1})X \rangle$ for $X \in \mathfrak{g}$. If we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by the inner product $(\,|\,)$, coadjoint orbits correspond to adjoint orbits. By the identification

$$\mathfrak{t}^* = \{f \in \mathfrak{g}^* \mid t \cdot f = f \ (\forall t \in T)\},$$

we regard $\mathfrak{t}^*$ as a subset of $\mathfrak{g}^*$, so that $\mathfrak{t}_+^*$ and Λ_+ become subsets of $\mathfrak{g}^*$. For $\lambda \in C_+$, let $\mathscr{O}_\lambda$ denote the coadjoint orbit through $(1/2\pi\sqrt{-1})\lambda \in \mathfrak{t}_+^*$. Then $\mathscr{O}_\lambda \cap \mathfrak{t}^*$ is the W-orbit of $(1/2\pi\sqrt{-1})\lambda$ and the set $\mathscr{O}_\lambda \cap \mathfrak{t}_+^*$ consists of the single point $(1/2\pi\sqrt{-1})\lambda$.

Lemma 3.1. *For* $\lambda \in C_+$, *we have* $\dim_{\boldsymbol{R}} \mathscr{O}_\lambda = 2(|\Delta_+| - |\Delta_+^\lambda|)$, *where* Δ_+^λ *is as in* (2.9).

Proof. Let G_λ be the isotropy subgroup at $(1/2\pi\sqrt{-1})\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$ with respect to the coadjoint action, and let $\mathfrak{g}_\lambda$ be its Lie algebra. Under the identification $\mathfrak{t}^* \cong \mathfrak{t}$, suppose that $(1/2\pi\sqrt{-1})\lambda \in \mathfrak{t}^*$ corresponds to $X \in \mathfrak{t}$. Then we have $\mathfrak{g}_\lambda = \mathrm{Ker}(\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g})$. The root space decomposition $\mathfrak{g}_{\boldsymbol{C}} = \mathfrak{h} \oplus \bigoplus_{\pm\alpha\in\Delta_+}(\mathfrak{g}_{\boldsymbol{C}})_\alpha$ of $\mathfrak{g}_{\boldsymbol{C}}$ shows that

$$\mathfrak{g}_\lambda \otimes \boldsymbol{C} = \mathrm{Ker}(\mathrm{ad}(X) : \mathfrak{g}_{\boldsymbol{C}} \to \mathfrak{g}_{\boldsymbol{C}}) = \mathfrak{h} \oplus \bigoplus_{\pm\alpha\in\Delta_+^X} (\mathfrak{g}_{\boldsymbol{C}})_\alpha,$$

where $\Delta_+^X := \{\alpha \in \Delta_+ \mid \langle \alpha, X \rangle = 0\}$, which is equal to Δ_+^λ. Thus we have $\dim_{\boldsymbol{C}}(\mathfrak{g}_\lambda \otimes \boldsymbol{C}) = l + 2|\Delta_+^\lambda|$, and hence $\dim_{\boldsymbol{R}} G_\lambda = \dim_{\boldsymbol{R}} \mathfrak{g}_\lambda = l + 2|\Delta_+^\lambda|$. Now it follows that

$$\dim_{\boldsymbol{R}} \mathscr{O}_\lambda = \dim_{\boldsymbol{R}} G - \dim_{\boldsymbol{R}} G_\lambda = 2(|\Delta_+| - |\Delta_+^\lambda|)$$

as required. □

On the coadjoint orbit $\mathscr{O}_\lambda$, there is a G-invariant symplectic structure ω_λ, called the Kirillov-Kostant-Souriau symplectic structure, defined by $(\omega_\lambda)_x(\tilde{X}, \tilde{Y}) := \langle x, [X, Y] \rangle$ for $x \in \mathscr{O}_\lambda$ and $X, Y \in \mathfrak{g}$, where $\tilde{X}$ denotes the vector field over $\mathscr{O}_\lambda$ given by $\tilde{X}_x = \left.\dfrac{d}{dt}\right|_{t=0} (\exp tX) \cdot x$. And then, the action of G on $\mathscr{O}_\lambda$ becomes Hamiltonian and the moment map is given by the inclusion $\iota : \mathscr{O}_\lambda \hookrightarrow \mathfrak{g}^*$. Namely, we have $d\langle \iota, X \rangle(\cdot) = \omega_\lambda(\tilde{X}, \cdot)$. Besides, there is a G-invariant complex structure J_λ on $\mathscr{O}_\lambda$, which is compatible with the symplectic structure ω_λ (i.e. $\omega_\lambda(\cdot, J_\lambda \cdot)$ is a Riemannian metric), so that $\mathscr{O}_\lambda$ becomes a Kähler manifold.

Moreover, when $\lambda \in P_+$, there is a G-equivariant holomorphic line bundle L_λ over $\mathscr{O}_\lambda$ such that $c_1(L_\lambda) = [\omega_\lambda]$. The Borel-Weil theorem shows that

$$H^0(\mathscr{O}_\lambda, L_\lambda) = V_\lambda, \quad H^i(\mathscr{O}_\lambda, L_\lambda) = 0 \quad (i > 0)$$

as representations of G, where $H^i(\mathscr{O}_\lambda, L_\lambda)$ denotes the i-th cohomology group of $\mathscr{O}_\lambda$ with coefficients in the sheaf of germs of holomorphic sections of L_λ.

For $\lambda_1, \ldots, \lambda_n \in C_+$, consider the diagonal action of G on the direct product $\mathscr{O}_{\lambda_1} \times \cdots \times \mathscr{O}_{\lambda_n}$ of the coadjoint orbits also becomes Hamiltonian and its moment map $\Phi : \mathscr{O}_{\lambda_1} \times \cdots \times \mathscr{O}_{\lambda_n} \to \mathfrak{g}^*$ is given by $\Phi(x_1, \ldots, x_n) = x_1 + \cdots + x_n$. Now consider the symplectic quotient

$$\begin{aligned}\mathscr{M} = \mathscr{M}(\lambda_1, \dots, \lambda_n) : &= \Phi^{-1}(0)/G \\ &= \{(x_1, \dots, x_n) \in \mathscr{O}_{\lambda_1} \times \cdots \times \mathscr{O}_{\lambda_n} \mid x_1 + \cdots + x_n = 0\}/G.\end{aligned}$$

We assume that 0 is a regular value of the moment map Φ and $\mathscr{M}$ is a non-empty smooth manifold. Then $\mathscr{M}$ has a natural symplectic structure $\omega = \omega(\lambda_1, \dots, \lambda_n)$ and a compatible complex structure induced from those on $\mathscr{O}_{\lambda_1} \times \cdots \times \mathscr{O}_{\lambda_n}$, which make $\mathscr{M}$ a Kähler manifold. By Lemma 3.1, the number $d = d(\lambda_1, \dots, \lambda_n)$ in (2.10) is the complex dimension of $\mathscr{M}$.

Now suppose $\lambda_1, \dots, \lambda_n \in P_+$. Let $\mathrm{pr}_i : \mathscr{O}_{\lambda_1} \times \cdots \times \mathscr{O}_{\lambda_n} \to \mathscr{O}_{\lambda_i}$ be the projection to the i-th factor and let

$$\mathscr{L} = \mathscr{L}(\lambda_1, \dots, \lambda_n) := \left(\mathrm{pr}_1^* L_{\lambda_1} \otimes \cdots \otimes \mathrm{pr}_n^* L_{\lambda_n}|_{\Phi^{-1}(0)}\right)\Big/ G.$$

Although $\mathscr{L}$ is in general an orbifold holomorphic line bundle over $\mathscr{M}$, we assume here that $\mathscr{L}$ is a genuine holomorphic line bundle. Then we have $c_1(\mathscr{L}) = [\omega]$.

The following proposition gives geometric interpretations of $\mathscr{Q} = \mathscr{Q}(\lambda_1, \dots, \lambda_n)$ and $\mathscr{V} = \mathscr{V}(\lambda_1, \dots, \lambda_n)$.

PROPOSITION 3.2. *Suppose that* $(\lambda_1, \dots, \lambda_n) \in (P_+)^n$ *satisfies the assumptions as above. Then we have*

$$\begin{aligned}\mathscr{Q} &= \int_{\mathscr{M}} \mathrm{ch}(\mathscr{L})\,\mathrm{td}(\mathscr{M}) = \int_{\mathscr{M}} \exp(\omega)\,\mathrm{td}(\mathscr{M}), \\ \mathscr{V} &= \lim_{k\to\infty} \frac{1}{k^d} \int_{\mathscr{M}} \exp(k\omega)\,\mathrm{td}(\mathscr{M}) = \int_{\mathscr{M}} \frac{\omega^d}{d!},\end{aligned}$$

where $\mathrm{ch}(\mathscr{L})$ *denotes the Chern character of* $\mathscr{L}$ *and* $\mathrm{td}(\mathscr{M})$ *denotes the Todd class of* $\mathscr{M}$.

In other words, $\mathscr{Q}$ is the Riemann-Roch number of $(\mathscr{M}, \mathscr{L})$, whereas $\mathscr{V}$ is the symplectic volume of $(\mathscr{M}, \omega)$. The proof of this proposition is the same as that of [**20**, Proposition 2.5]. It is essential that

$$(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^G \cong \sum_{i=0}^{d} (-1)^i H^i(\mathscr{M}, \mathscr{L})$$

as virtual vector spaces, by the theorem of Guillemin-Sternberg [**8**] and its generalization (see, e.g., [**18**]). We mention that $H^i(\mathscr{M}, \mathscr{L}) = 0$ for $i > 0$ (see [**4**]).

Remark 3.3. Let H be a closed subgroup of G. Then $\mathscr{Q}^H(\lambda_1,\dots,\lambda_n)$ and $\mathscr{V}^H(\lambda_1,\dots,\lambda_n)$ given in Remark 2.3 correspond to the characteristic numbers of the symplectic quotient of the H-action on $\mathscr{O}_{\lambda_1}\times\cdots\times\mathscr{O}_{\lambda_n}$.

4. The first formula.

In this section, we describe $\mathscr{Q}(\lambda_1,\dots,\lambda_n)=\dim_{\boldsymbol{C}}(V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n})^G$ in a combinatorial form and use it to obtain the first formula for $\mathscr{V}(\lambda_1,\dots,\lambda_n)$. The content of this section is a generalization of the one given in [**20**], where the case $G=SU(3)$ was considered. We also refer to [**24**], which is closely related to the discussion below.

4.1. Combinatorial expression.

Let $\lambda,\lambda_1,\dots,\lambda_n\in P_+$. As an element in $\boldsymbol{C}[e^{\Lambda_1},\dots,e^{\Lambda_l}][[e^{-\Lambda_1},\dots,e^{-\Lambda_l}]]$, let us set

$$\chi_\lambda=\frac{\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\rho)}}{e^\rho\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})},\quad D=e^\rho\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})$$

and define

$$\begin{aligned}F_{\lambda_1,\dots,\lambda_n}:&=\frac{(-1)^{|\Delta_+|}}{|W|}\cdot\chi_{\lambda_1}\cdots\chi_{\lambda_n}\cdot D^2\\&=\frac{(-1)^{|\Delta_+|}}{|W|}\frac{\left(\sum_{w_1\in W}\varepsilon(w_1)e^{w_1(\lambda_1+\rho)}\right)\cdots\left(\sum_{w_n\in W}\varepsilon(w_n)e^{w_n(\lambda_n+\rho)}\right)}{\left(e^\rho\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})\right)^{(n-2)}}.\end{aligned}\tag{4.1}$$

As in the Weyl character formula (2.5), we also regard them as functions on T.

Proposition 4.1. *For $\lambda_1,\dots,\lambda_n\in P_+$, $\mathscr{Q}(\lambda_1,\dots,\lambda_n)$ is equal to the coefficient of e^0 (i.e., the constant term) in $F_{\lambda_1,\dots,\lambda_n}$.*

Proof. Let $d\mu_G$ and $d\mu_T$ be the normalized invariant measure on G and T, respectively. Then, by the Weyl integration formula, we have

$$\begin{aligned}\dim_{\boldsymbol{C}}(V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n})^G&=\int_G\chi_{\lambda_1}(g)\cdots\chi_{\lambda_n}(g)d\mu_G\\&=\frac{1}{|W|}\int_T\chi_{\lambda_1}(t)\cdots\chi_{\lambda_n}(t)|D(t)|^2d\mu_T\\&=\frac{(-1)^{|\Delta_+|}}{|W|}\int_T\chi_{\lambda_1}(t)\cdots\chi_{\lambda_n}(t)\cdot D(t)^2d\mu_T.\end{aligned}\tag{4.2}$$

Here note that for the denominator $D(t)=e^{\langle\rho,X\rangle}\prod_{\alpha\in\Delta_+}(1-e^{-\langle\alpha,X\rangle})$ in the Weyl character formula (2.5), we have $\overline{D(t)}=(-1)^{|\Delta_+|}D(t)$, and hence $|D(t)|^2=(-1)^{|\Delta_+|}D(t)^2$.

Let $a_1^\vee,\dots,a_l^\vee\in\mathfrak{t}$ be the basis of the integral lattice in $\mathfrak{t}$ as in (2.1), and write an element t in T as $t=\exp(x_1a_1^\vee+\cdots+x_la_l^\vee)$ with $x_i\in[0,1]$. Let us set $u_i=e^{2\pi\sqrt{-1}x_i}$ and define an isomorphism $T\cong U(1)^l$ by $t\mapsto(u_1,\dots,u_l)$. Then we have

$$d\mu_T=dx_1\cdots dx_l=\frac{du_1}{2\pi\sqrt{-1}u_1}\cdots\frac{du_l}{2\pi\sqrt{-1}u_l}.$$

Hence (4.2) is equal to the coefficient of $u_1^0\cdots u_l^0$ in $F_{\lambda_1,\dots,\lambda_n}(t)$, which is regarded as a Laurent series of $(u_1,\dots,u_l)$. If we write $F_{\lambda_1,\dots,\lambda_n}$ as

$$F_{\lambda_1,\dots,\lambda_n}=\sum C_{m_1,\dots,m_l}e^{m_1\Lambda_1+\cdots+m_l\Lambda_l},$$

then we have

$$F_{\lambda_1,\dots,\lambda_n}(u_1,\dots,u_l)=\sum C_{m_1,\dots,m_l}u_1^{m_1}\cdots u_l^{m_l}.$$

Therefore, (4.2) is equal to the coefficient of e^0 in $F_{\lambda_1,\dots,\lambda_n}$. □

PROPOSITION 4.2. *For $(\lambda_1,\dots,\lambda_n)\in(P_+)^n$, we have*

$$\mathscr{Q}(\lambda_1,\dots,\lambda_n)=\frac{(-1)^{|\Delta_+|}}{|W|}\sum_{(w_1,\dots,w_n)\in W^n}\varepsilon(w_1)\cdots\varepsilon(w_n)C(\lambda_1,\dots,\lambda_n;w_1,\dots,w_n),$$

where for $(w_1,\dots,w_n)\in W^n$, we define

$$C(\lambda_1,\dots,\lambda_n;w_1,\dots,w_n):=\sum_{(j_1,\dots,j_{|\Delta_+|})}\binom{j_1+n-3}{n-3}\cdots\binom{j_{|\Delta_+|}+n-3}{n-3},\tag{4.3}$$

the sum over all $(j_1,\dots,j_{|\Delta_+|})\in(\boldsymbol{Z}_{\geq0})^{|\Delta_+|}$ that satisfy the condition

$$\begin{aligned}w_1(\lambda_1)+\cdots+w_n(\lambda_n)+w_1(\rho)+\cdots+w_n(\rho)&\\-(n-2)\rho-j_1\alpha_1-\cdots-j_{|\Delta_+|}\alpha_{|\Delta_+|}&=0.\end{aligned}\tag{4.4}$$

PROOF. Applying the generalized binomial theorem to $(e^\rho\prod_{\alpha\in\Delta_+}(1-e^{-\alpha}))^{2-n}$ in (4.1), we have a power series expansion

$$\begin{aligned}F_{\lambda_1,\dots,\lambda_n} &= \frac{(-1)^{|\Delta_+|}}{|W|}\sum_{(w_1,\dots,w_n)}\sum_{(j_1,\dots,j_{|\Delta_+|})}\varepsilon(w_1)\cdots\varepsilon(w_n)(-1)^{j_1+\cdots+j_{|\Delta_+|}}\binom{2-n}{j_1}\cdots\binom{2-n}{j_{|\Delta_+|}}\\ &\qquad\times e^{w_1(\lambda_1+\rho)+\cdots+w_n(\lambda_n+\rho)-j_1\alpha_1-\cdots-j_{|\Delta_+|}\alpha_{|\Delta_+|}-(n-2)\rho}\\ &= \frac{(-1)^{|\Delta_+|}}{|W|}\sum_{(w_1,\dots,w_n)}\sum_{(j_1,\dots,j_{|\Delta_+|})}\varepsilon(w_1)\cdots\varepsilon(w_n)\binom{j_1+n-3}{n-3}\cdots\binom{j_{|\Delta_+|}+n-3}{n-3}\\ &\qquad\times e^{w_1(\lambda_1+\rho)+\cdots+w_n(\lambda_n+\rho)-j_1\alpha_1-\cdots-j_{|\Delta_+|}\alpha_{|\Delta_+|}-(n-2)\rho}.\end{aligned}$$

Now our claim follows form Proposition 4.1. □

REMARK 4.3. Since $w(\rho)-\rho\in Q$ for any $w\in W$ by (2.2) and $2\rho\in Q$, we see $w_1(\rho)+\cdots+w_n(\rho)-(n-2)\rho\in Q$. Therefore, if $w_1(\lambda_1)+\cdots+w_n(\lambda_n)\notin Q$, then there are no $j_1,\dots,j_{|\Delta_+|}\in \boldsymbol{Z}_{\geq 0}$ which satisfy (4.4), and hence $C(\lambda_1,\dots,\lambda_n;w_1,\dots,w_n)=0$.

REMARK 4.4. Similarly, $\mathscr{Q}^T(\lambda_1,\dots,\lambda_n)=\dim_{\boldsymbol{C}}(V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n})^T$ is equal to the coefficient of e^0 in

$$F^T_{\lambda_1,\dots,\lambda_n}:=\chi_{\lambda_1}\cdots\chi_{\lambda_n}=\frac{\left(\sum_{w_1\in W}\varepsilon(w_1)e^{w_1(\lambda_1+\rho)}\right)\cdots\left(\sum_{w_n\in W}\varepsilon(w_n)e^{w_n(\lambda_n+\rho)}\right)}{\left(e^{\rho}\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})\right)^n}$$

and we have

$$\mathscr{Q}^T(\lambda_1,\dots,\lambda_n)=\sum_{(w_1,\dots,w_n)\in W^n}\varepsilon(w_1)\cdots\varepsilon(w_n)C^T(\lambda_1,\dots,\lambda_n;w_1,\dots,w_n),$$

where for $(w_1,\dots,w_n)\in W^n$, we define

$$C^T(\lambda_1,\dots,\lambda_n;w_1,\dots,w_n):=\sum_{(j_1,\dots,j_{|\Delta_+|})}\binom{j_1+n-1}{n-1}\cdots\binom{j_{|\Delta_+|}+n-1}{n-1},$$

the sum over all $(j_1,\dots,j_{|\Delta_+|})\in(\boldsymbol{Z}_{\geq 0})^{|\Delta_+|}$ that satisfy the condition

$$w_1(\lambda_1)+\cdots+w_n(\lambda_n)+w_1(\rho)+\cdots+w_n(\rho)-n\rho-j_1\alpha_1-\cdots-j_{|\Delta_+|}\alpha_{|\Delta_+|}=0.$$

4.2. The formula.

In the sequel of this section, let n be an integer with $n\geq 3$, and assume that

$\lambda_1, \ldots, \lambda_n \in P_+$ satisfy the following three conditions:

(A1) $\langle w_1(\lambda_1) + \cdots + w_n(\lambda_n), \Lambda_i^\vee \rangle \neq 0$, for each $w_1, \ldots, w_n \in W$ and each $i = 1, \ldots, l$,

(A2) $\lambda_1, \ldots, \lambda_n \in Q$,

(A3) $\lambda_1, \ldots, \lambda_n \in P_{++}$.

It follows from (A3) that $\Delta_+^{\lambda_i} = \emptyset$ for each i, and hence we have $d = (n-2)|\Delta_+| - l$ in (2.10).

Remark 4.5. The assumption (A1) might be a technical assumption, whereas it simplifies the arguments below. Note that even if $\lambda_1, \ldots, \lambda_n$ do not satisfy (A2), after simultaneously multiplied by $|P/Q|$, they do satisfy (A2). On the other hand, we have to say that the assumption (A3) is essential in our argument below.

In the following, let us set $N = |\Delta_+|$ for brevity and enumerate all the elements of Δ_+ as $\alpha_1, \ldots, \alpha_l, \alpha_{l+1}, \ldots, \alpha_N$, where $\alpha_1, \ldots, \alpha_l$ are the fixed simple roots.

Definition 4.6. Let $\lambda_1, \ldots, \lambda_n \in P_+$ and $w_1, \ldots, w_n \in W$.

- Define $l \times (N-l)$-matrix R by

  $$(\alpha_{l+1}, \ldots, \alpha_N) = (\alpha_1, \ldots, \alpha_l)R.$$

  For $i = 1, \ldots, l$, the i-th row of the matrix R is denoted by R_i.

- For $i = 1, \ldots, l$, we define integers

  $$p_i = p_i(\lambda_1, \ldots, \lambda_n; w_1, \ldots, w_n), \quad q_i = q_i(w_1, \ldots, w_n)$$

  by

  $$w_1(\lambda_1) + \cdots + w_n(\lambda_n) = p_1\alpha_1 + \cdots + p_l\alpha_l, \tag{4.5}$$

  $$w_1(\rho) + \cdots + w_n(\rho) - (n-2)\rho = q_1\alpha_1 + \cdots + q_l\alpha_l. \tag{4.6}$$

- Define the subset $\mathscr{W} = \mathscr{W}(\lambda_1, \ldots, \lambda_n)$ of W^n by

  $$\mathscr{W} := \{(w_1, \ldots, w_n) \in W^n \mid p_1 > 0, \ldots, p_l > 0\}. \tag{4.7}$$

Remark 4.7. By means of the fundamental coweights, we can write as

$$\begin{aligned} R_i &= (\langle \alpha_{l+1}, \Lambda_i^\vee \rangle, \dots, \langle \alpha_{|\Delta_+|+1}, \Lambda_i^\vee \rangle), \\ p_i &= \langle w_1(\lambda_1) + \cdots + w_n(\lambda_n), \Lambda_i^\vee \rangle, \\ q_i &= \langle w_1(\rho) + \cdots + w_n(\rho) - (n-2)\rho, \Lambda_i^\vee \rangle. \end{aligned}$$

All the entries of R are nonnegative integers. The assumption (A1) shows $p_1, \dots, p_l \neq 0$, whereas (A2) shows that $p_1, \dots, p_l$ are integers.

Now that the condition (4.4) is written as

$$j_1 = -R_1{}^t(j_{l+1}, \dots, j_N) + p_1 + q_1, \dots, j_l = -R_l{}^t(j_{l+1}, \dots, j_N) + p_l + q_l,$$

we have the following from Proposition 4.2.

Proposition 4.8. *Let $C(\lambda_1, \dots, \lambda_n; w_1, \dots, w_n)$ be as in* (4.3) *and let p_i, q_i be as in* (4.5), (4.6). *Then we have*

$$C(\lambda_1, \dots, \lambda_n; w_1, \dots, w_n) = \sum_{(j_{l+1}, \dots, j_N)} \prod_{i=1}^{l} \binom{-R_i{}^t(j_{l+1}, \dots, j_N) + p_i + q_i + n - 3}{n-3} \prod_{i=l+1}^{N} \binom{j_i + n - 3}{n-3},$$

the sum over all $(j_{l+1}, \dots, j_N) \in (\boldsymbol{Z}_{\geq 0})^{N-l}$ which satisfies

$$R_1{}^t(j_{l+1}, \dots, j_N) \leq p_1 + q_1, \dots, \; R_l{}^t(j_{l+1}, \dots, j_N) \leq p_l + q_l.$$

Now, in order to seek a formula for $\mathscr{V}(\lambda_1, \dots, \lambda_n) = \limsup_{k \to \infty} (1/k^d) \mathscr{Q}(k\lambda_1, \dots, k\lambda_n)$, we consider the asymptotic behavior of $C(k\lambda_1, \dots, k\lambda_n; w_1, \dots, w_n)$ as a function of a positive integer k. We will see below that the assumptions (A1), (A2), and (A3) imply that the limit $\lim_{k \to \infty} (1/k^d) \mathscr{Q}(k\lambda_1, \dots, k\lambda_n)$ indeed exists.

Definition 4.9. For $\xi_1, \dots, \xi_l \in \boldsymbol{R}_{>0}$, we define the convex polytope $S = S(\xi_1, \dots, \xi_l)$ in $\boldsymbol{R}^{N-l}$ as the set consisting of all $(t_{l+1}, \dots, t_N) \in \boldsymbol{R}^{N-l}$ which satisfy the condition

$$t_{l+1} \geq 0, \dots, t_N \geq 0, \quad R_1{}^t(t_{l+1}, \dots, t_N) \leq \xi_1, \dots, R_l{}^t(t_{l+1}, \dots, t_N) \leq \xi_l.$$

For $r \in \boldsymbol{Z}_{\geq 0}$, we define

$$I_r(\xi_1, \ldots, \xi_l) \\ := \frac{1}{(r!)^N} \int_{S(\xi_1, \ldots, \xi_l)} \prod_{i=1}^{l} (\xi_i - R_i\, {}^t(t_{l+1}, \ldots, t_N))^r \prod_{i=l+1}^{N} (t_i)^r \, dt_{l+1} \cdots dt_N. \quad (4.8)$$

If one of $\xi_1, \ldots, \xi_l$ is nonpositive, then $S(\xi_1, \ldots, \xi_l)$ degenerates, and hence in such a case we define

$$I_r(\xi_1, \ldots, \xi_l) = 0.$$

Proposition 4.10. *Let* $\lambda_1, \ldots, \lambda_n \in P_+$ *satisfy* (A1), (A2), (A3) *and let* $w_1, \ldots, w_n \in W$. *Then we have*

$$\lim_{k \to \infty} \frac{1}{k^d} C(k\lambda_1, \ldots, k\lambda_n; w_1, \ldots, w_n) = I_{n-3}(p_1, \ldots, p_l), \quad (4.9)$$

where $p_i = p_i(\lambda_1, \ldots, \lambda_n; w_1, \ldots, w_n)$ *is as in* (4.5).

Proof. It follows from Proposition 4.2 that

$$C(k\lambda_1, \ldots, k\lambda_n; w_1, \ldots, w_n) \\ = \sum_{(j_{l+1}, \ldots, j_N)} \prod_{i=1}^{l} \binom{-R_i\, {}^t(j_{l+1}, \ldots, j_N) + kp_i + q_i + n - 3}{n-3} \prod_{i=l+1}^{N} \binom{j_i + n - 3}{n-3},$$

the sum over $(j_{l+1}, \ldots, j_N) \in (\boldsymbol{Z}_{\geq 0})^{N-l}$ satisfying the condition

$$R_1\, {}^t(j_{l+1}, \ldots, j_N) \leq kp_1 + q_1, \ldots, R_l\, {}^t(j_{l+1}, \ldots, j_N) \leq kp_l + q_l. \quad (4.10)$$

Case 1: If $(w_1, \ldots, w_n) \notin \mathscr{W}$, then one of $p_1, \ldots, p_l$ is negative (see (4.7) and Remark 4.7). Since entries of $R_1, \ldots, R_l$ are nonnegative integers, for each sufficiently large k, any $(j_{l+1}, \ldots, j_N) \in (\boldsymbol{Z}_{\geq 0})^{N-l}$ can not satisfy (4.10). Hence we have

$$C(k\lambda_1, \ldots, k\lambda_n; w_1, \ldots, w_n) = 0.$$

On the other hand, we have $I_{n-3}(p_1, \ldots, p_l) = 0$ by definition. Thus both sides of (4.9) are equal to 0.

Case 2: If $(w_1,\dots,w_n)\in\mathscr{W}$, then we have

$$\begin{aligned}
&C(k\lambda_1,\dots,k\lambda_n;w_1,\dots,w_n)\\
&\sim\sum_{(j_{l+1},\dots,j_N)}\prod_{i=1}^{l}\frac{(kp_i-R_i{}^t(j_{l+1},\dots,j_N))^{n-3}}{(n-3)!}\prod_{i=l+1}^{N}\frac{(j_i)^{n-3}}{(n-3)!}\\
&=\frac{k^{(n-2)N-l}}{((n-3)!)^N}\sum_{(j_{l+1},\dots,j_N)}\prod_{i=1}^{l}\left(p_i-R_i{}^t\left(\frac{j_{l+1}}{k},\dots,\frac{j_N}{k}\right)\right)^{n-3}\prod_{i=l+1}^{N}\left(\frac{j_i}{k}\right)^{n-3}\left(\frac{1}{k}\right)^{N-l},
\end{aligned}$$

which implies (4.9). □

Thus, from Propositions 4.8 and 4.10, we obtain the first formula for

$$\mathscr{V}(\lambda_1,\dots,\lambda_n)=\lim_{k\to\infty}\frac{1}{k^d}\mathscr{Q}(k\lambda_1,\dots,k\lambda_n)$$

as follows.

Theorem 4.11. *For $\lambda_1,\dots,\lambda_n\in P_+$ satisfying* (A1), (A2), *and* (A3), *we have*

$$\mathscr{V}(\lambda_1,\dots,\lambda_n)=\frac{(-1)^{|\Delta_+|}}{|W|}\sum_{(w_1,\dots,w_n)\in W^n}\varepsilon(w_1)\cdots\varepsilon(w_n)\,I_{n-3}(p_1,\dots,p_l),\tag{4.11}$$

where $p_1,\dots,p_l$ and $I_{n-3}(p_1,\dots,p_l)$ are as in (4.5) *and* (4.8).

Remark 4.12. Let $\lambda_1,\dots,\lambda_n\in P_+$ be as in Theorem 4.11. In the same way, we see from Remark 4.4 that

$$\begin{aligned}
\mathscr{V}^T(\lambda_1,\dots,\lambda_n)&=\lim_{k\to\infty}\frac{1}{k^{d'}}\mathscr{Q}^T(k\lambda_1,\dots,k\lambda_n)\\
&=\sum_{(w_1,\dots,w_n)\in W^n}\varepsilon(w_1)\cdots\varepsilon(w_n)\,I_{n-1}(p_1,\dots,p_l),
\end{aligned}\tag{4.12}$$

where $d'=n|\Delta_+|-l$.

Remark 4.13.

(1) The sum in the right-hand side of (4.11) or (4.12) is equal to the sum over all $(w_1,\dots,w_n)\in\mathscr{W}$.

(2) More generally than (4.8), it might be significant to consider the integral of the form

$$\frac{1}{(r_1)!\cdots(r_N)!}\int_{S(\xi_1,\dots,\xi_l)}\prod_{i=1}^{l}(\xi_i - R_i{}^t(t_{l+1},\dots,t_N))^{r_i} \times \prod_{i=l+1}^{N}(t_i)^{r_i}\,dt_{l+1}\cdots dt_N \tag{4.13}$$

for $r_1,\dots,r_N \in \boldsymbol{Z}_{\geq 0}$. In fact, if the assumption (A3) is not satisfied, we encounter such an integral to express $\mathscr{V}(\lambda_1,\dots,\lambda_n)$ (see [**20**] for the case $G = SU(3)$). The integral (4.8) or (4.13) is a kind of hypergeometric integral. We refer to [**24**] for more details, where these integrals are studied from the point of view of Gel'fand-Kapranov-Zelevinsky hypergeometric functions.

(3) Although our method here is quite combinatorial, the data arising in (4.11) and (4.12) have geometric meanings, under the interpretation explained in Section 3. For example, $(w_1\lambda_1,\dots,w_n\lambda_n)$ with $w_1,\dots,w_n \in W$ corresponds to a fixed point of the diagonal action of T on $\mathscr{O}_{\lambda_1}\times\cdots\times\mathscr{O}_{\lambda_n}$. It might be interesting to compare the formulas (4.11) and (4.12) with the residue formula due to Jeffrey-Kirwan [**12**] and with the result of Martin [**17**].

4.3. Examples.

Let us compute concretely the integral $I_{n-3}(\xi_1,\dots,\xi_l)$ as in (4.8) for some G. As we will see below, it will be quite complicated, even if the rank of G is not so large. Consequently, it is still difficult to make the formula (4.11) more explicit for general G.

Example 4.14. When Δ is of type A_1, i.e., $G = SU(2)$, we see that $l = |\Delta_+| = 1$, and hence the matrix R does not appear. In this case $I_{n-3}(\xi_1)$ is not an integral but just a number;

$$I_{n-3}(\xi_1) = \frac{1}{(n-3)!}\xi_1^{n-3}$$

for $\xi_1 > 0$. (Recall that we have defined $I_{n-3}(\xi_1) = 0$ for $\xi_1 \leq 0$.) Next, let us consider the formula (4.11) for $\mathscr{V}(\lambda_1,\dots,\lambda_n)$. In view of the assumptions (A2) and (A3), let us set

$$\lambda_i = m_i\Lambda_1 = \frac{m_i}{2}\alpha_1 \quad (m_i \in 2\boldsymbol{Z}_{>0})$$

for $i = 1, \dots, n$. Since $W \cong \{\pm 1\}$, we have

$$w_1(\lambda_1) + \cdots + w_n(\lambda_n) = \left(\varepsilon_1 \frac{m_1}{2} + \cdots + \varepsilon_n \frac{m_n}{2}\right)\alpha_1,$$

where $\varepsilon_i = \pm 1$, and hence

$$p_1 = p_1(\lambda_1, \dots, \lambda_n; w_1, \dots, w_n) = \varepsilon_1 \frac{m_1}{2} + \cdots + \varepsilon_n \frac{m_n}{2}.$$

The assumption (A1) means that $\varepsilon_1 m_1/2 + \cdots + \varepsilon_n m_n/2 \neq 0$ for any $(\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^n$. The set $\mathscr{W}$ in (4.7) consists of all $(\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^n$ such that $\varepsilon_1 m_1/2 + \cdots + \varepsilon_n m_n/2 > 0$. Thus (4.11) becomes

$$\mathscr{V}(\lambda_1, \dots, \lambda_n) = -\frac{1}{2(n-3)!} \sum_{(\varepsilon_1, \dots, \varepsilon_n) \in \mathscr{W}} \varepsilon_1 \cdots \varepsilon_n \left(\varepsilon_1 \frac{m_1}{2} + \cdots + \varepsilon_n \frac{m_n}{2}\right)^{n-3}. \tag{4.14}$$

This is nothing but the formula for the symplectic volume of $\mathscr{M}(\lambda_1, \dots, \lambda_n)$ in [**22**]. (See also [**15**].)

Example 4.15. When Δ is of type A_2, i.e., $G = SU(3)$, we see that $l = 2$, $|\Delta_+| = 3$, and $R = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence

$$\begin{aligned} I_{n-3}(\xi_1, \xi_2) &= \frac{1}{((n-3)!)^3} \int_0^{\min(\xi_1, \xi_2)} \{(\xi_1 - t_3)(\xi_2 - t_3)t_3\}^{n-3} dt_3 \\ &= \begin{cases} \dfrac{1}{(3n-8)!} \displaystyle\sum_{c=0}^{n-3} \binom{3n-8}{c} \binom{2n-6-c}{n-3} \xi_1^{3n-8-c} (\xi_2 - \xi_1)^c & (\text{if } 0 < \xi_1 \leq \xi_2), \\ \dfrac{1}{(3n-8)!} \displaystyle\sum_{c=0}^{n-3} \binom{3n-8}{c} \binom{2n-6-c}{n-3} \xi_2^{3n-8-c} (\xi_1 - \xi_2)^c & (\text{if } 0 < \xi_2 < \xi_1). \end{cases} \end{aligned}$$

The formula (4.11) for $\mathscr{V}(\lambda_1, \dots, \lambda_n)$ is explicitly given in [**20**], where the cases that $\lambda_1, \dots, \lambda_n$ do not satisfy the assumption (A3) are also studied.

Example 4.16. When Δ is of type B_2, i.e., $G = Spin(5)$, we see $l = 2$,

$|\Delta_+| = 4$, and $R = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Hence

$$I_{n-3}(\xi_1, \xi_2) = \frac{1}{((n-3)!)^4} \int_{S(\xi_1,\xi_2)} \{(\xi_1 - t_3 - t_4)(\xi_2 - t_3 - 2t_4)t_3 t_4\}^{n-3} dt_3 dt_4.$$

Suppose $0 < \xi_1 < \xi_2 < 2\xi_1$ for simplicity. Then the evaluation of this integral was done in [**24**], which shows that

$$\begin{aligned} (4n-10)! \cdot I_{n-3}(\xi_1, \xi_2) = & \sum_{c=-3n+7}^{-1} \sum_{i=0}^{-c-1} \binom{4n-10}{n-c-3} \binom{2n+c-5}{n+c+i-2} \binom{-c-1}{i} \\ & \qquad \times (-1)^{n+c-2} 2^{n-i-3} \xi_1^{3n+c-7} \xi_2^{n-c-3} \\ & + \sum_{c=0}^{n-3} \sum_{i=0}^{n-3} \binom{4n-10}{n-c-3} \binom{2n+c-5}{n-i-3} \binom{c+i}{c} \\ & \qquad \times (-1)^{n+c+i-2} 2^{n-i-3} \xi_1^{3n+c-7} \xi_2^{n-c-3}. \end{aligned}$$

EXAMPLE 4.17. When Δ is of type G_2, then $l = 2$, $|\Delta_+| = 6$, and $R = \begin{pmatrix} 1 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 \end{pmatrix}$. Hence

$$\begin{aligned} I_{n-3}(\xi_1, \xi_2) = \frac{1}{((n-3)!)^6} \int_{S(\xi_1,\xi_2)} & \{(\xi_1 - t_3 - 2t_4 - 3t_5 - 3t_6) \\ & \times (\xi_2 - t_3 - t_4 - t_5 - 2t_6) t_3 t_4 t_5 t_6\}^{n-3} dt_3 dt_4 dt_5 dt_6. \end{aligned}$$

The evaluation might become more complicated.

5. A consequence of the Verlinde formula.

In this section, by means of the Verlinde formula for the fusion coefficients of the complex simple Lie algebra $\mathfrak{g}_C$ (or, more precisely, of the corresponding affine Lie algebra of sprit type), we will obtain another formula for $\mathscr{Q}(\lambda_1, \ldots, \lambda_n)$ in Proposition 5.9, which is quite different from the one given in Proposition 4.2.

5.1. Fusion coefficients and Verlinde formula.

We review some generalities about fusion coefficients for the complex simple Lie algebra $\mathfrak{g}_C$ (or the corresponding affine Lie algebra of split type). We refer to [**13**], [**25**], and [**6**] for more details. Most of the description below is based on Chapters 4 and 5 in [**25**].

Let $t \in \boldsymbol{R}_{>0}$ and

$$C_+^t := \{x \in C_+ \mid (x|\theta) \leq t\}, \quad P_+^t := C_+^t \cap P = \{x \in P_+ \mid (x|\theta) \leq t\},$$

where θ is the highest root of the root system Δ and C_+, P, and P_+ are as in Section 2. The set C_+^t is referred to as an alcove of Δ. We are mainly interested in the case where t is a positive integer. In such a case, we will use the letter m instead of t.

Recall from Section 2 that under the identification $\mathfrak{h}_{\boldsymbol{R}} = \mathfrak{h}_{\boldsymbol{R}}^*$ via the standard inner product $(\,|\,)$, we have $Q^\vee \subset Q \subset P \subset \mathfrak{h}_{\boldsymbol{R}}^*$.

Lemma 5.1. *Let us fix $m \in \boldsymbol{Z}_{>0}$. Consider the action of the group $W \ltimes mQ^\vee$ on $\mathfrak{h}_{\boldsymbol{R}}^*$, given by $x \mapsto w(x) + m\alpha^\vee$ for $x \in \mathfrak{h}_{\boldsymbol{R}}^*$, $w \in W$, and $\alpha^\vee \in Q^\vee$. Then the set C_+^m is a fundamental domain for this action.*

See [**2**, Chapter VI, Section 2] for the proof. Although the case $m = 1$ is considered there, the proof works also for general m. The group $W \ltimes mQ^\vee$ is referred to as the affine Weyl group at level m.

Definition 5.2. Let $m \in \boldsymbol{Z}_{>0}$. For $\lambda, \mu, \nu \in P_+^m$, we define

$$N_{\lambda,\mu}^\nu := \sum_\gamma \varepsilon(w) \mathrm{Mult}(V_\lambda \otimes V_\mu, V_\gamma), \tag{5.1}$$

the sum over all $\gamma \in P_+$ such that $\gamma + \rho$ is in the $(W \ltimes (m+g)Q^\vee)$-orbit through $\nu + \rho$, namely $\gamma + \rho \equiv w(\nu + \rho) \bmod (m+g)Q^\vee$ for some $w \in W$, where g is the dual Coxeter number of Δ. The number $N_{\lambda,\mu}^\nu$ is called the fusion coefficient. In addition, we define

$$\begin{aligned} a(\lambda, \mu) :&= (\sqrt{-1})^{|\Delta_+|} |P/(m+g)Q^\vee|^{-\frac{1}{2}} \sum_{w \in W} \varepsilon(w) \exp\left(-\frac{2\pi\sqrt{-1}}{m+g} (\lambda + \rho | w(\mu + \rho)) \right), \\ a(\lambda) :&= a(\lambda, 0) \\ &= (\sqrt{-1})^{|\Delta_+|} |P/(m+g)Q^\vee|^{-\frac{1}{2}} \sum_{w \in W} \varepsilon(w) \exp\left(-\frac{2\pi\sqrt{-1}}{m+g} (\lambda + \rho | w(\rho)) \right). \end{aligned}$$

Due to (2.6), (2.7), and (2.8), we can write them as

$$\begin{aligned} a(\lambda) &= (\sqrt{-1})^{|\Delta_+|}|P/(m+g)Q^\vee|^{-\frac{1}{2}}A_\rho\Big(-2\pi\sqrt{-1}\,\frac{\lambda+\rho}{m+g}\Big) \\ &= |P/(m+g)Q^\vee|^{-\frac{1}{2}}\prod_{\alpha\in\Delta_+}2\sin\frac{\pi(\lambda+\rho|\alpha)}{m+g}, \qquad (5.2) \end{aligned}$$

$$\begin{aligned} a(\lambda,\mu) &= (\sqrt{-1})^{|\Delta_+|}|P/(m+g)Q^\vee|^{-\frac{1}{2}}A_{\mu+\rho}\Big(-2\pi\sqrt{-1}\,\frac{\lambda+\rho}{m+g}\Big) \qquad (5.3) \\ &= |P/(m+g)Q^\vee|^{-\frac{1}{2}}\Bigg(\prod_{\alpha\in\Delta_+}2\sin\frac{\pi(\lambda+\rho|\alpha)}{m+g}\ \chi_\mu\Bigg) \\ &\quad\times\Bigg(\exp\Big(-2\pi\sqrt{-1}\,\frac{\lambda+\rho}{m+g}\Big)\Bigg), \qquad (5.4) \end{aligned}$$

where χ_μ is the character of the irreducible representation V_μ and $\exp(-2\pi\sqrt{-1}(\lambda+\rho)/(m+g))$ is regarded as an element in the maximal torus T of G. See [**25**, Section 4.3] for the proof of the following lemma.

LEMMA 5.3. *Let $m\in\boldsymbol{Z}_{>0}$ and $\lambda,\mu\in P_+^m$. Then we have*

$$a(\lambda,\mu)=a(\mu,\lambda),\quad a({}^t\lambda,\mu)=\overline{a(\lambda,\mu)},\quad \sum_{\nu\in P_+^m}a(\lambda,\nu)a({}^t\nu,\mu)=\delta_{\lambda,\mu},$$

where ${}^t\lambda$ is the transpose of λ.

REMARK 5.4. Let us denote by w_0 the unique element in W that sends Δ_+ to Δ_-. Then we have ${}^t\lambda=-w_0\lambda$ and it is the highest weight of the contragredient representation V_λ^* of V_λ. It is easy to see that $\lambda\in P_+^m$ implies ${}^t\lambda\in P_+^m$.

Now we quote the following theorem from [**25**, Chapter 5].

THEOREM 5.5 (Verlinde formula). *For $\lambda,\mu,\nu\in P_+^m$, we have*

$$N_{\lambda,\mu}^\nu=\sum_{x\in P_+^m}\frac{a(\lambda,x)a(\mu,x)a({}^t\nu,x)}{a(x)}.$$

As a consequence, we obtain the following.

PROPOSITION 5.6. *For $\lambda_1,\dots,\lambda_n,\nu\in P_+^m$, we have*

$$\sum_{\nu_3,\dots,\nu_n\in P_+^m} N_{\lambda_1,\lambda_2}^{\nu_3} N_{\nu_3,\lambda_3}^{\nu_4}\cdots N_{\nu_{n-1},\lambda_{n-1}}^{\nu_n} N_{\nu_n,\lambda_n}^{\nu}$$
$$= \sum_{\mu\in P_+^m} \frac{a(\lambda_1,\mu)a(\lambda_2,\mu)\cdots a(\lambda_n,\mu)a({}^t\nu,\mu)}{a(\mu)^{n-1}}. \tag{5.5}$$

Proof. It follows from Theorem 5.5 that the left-hand side of (5.5) is equal to

$$\sum_{\nu_3,\dots,\nu_n}\left(\sum_{x_2}\frac{a(\lambda_1,x_2)a(\lambda_2,x_2)a({}^t\nu_3,x_2)}{a(x_2)}\sum_{x_3}\frac{a(\nu_3,x_3)a(\lambda_3,x_3)a({}^t\nu_4,x_3)}{a(x_3)}\cdots\right.$$
$$\left.\cdots\sum_{x_n}\frac{a(\nu_n,x_n)a(\lambda_n,x_n)a({}^t\nu,x_n)}{a(x_n)}\right)$$
$$=\sum_{x_2,\dots,x_n}\left(\frac{a(\lambda_1,x_2)a(\lambda_2,x_2)a(\lambda_3,x_3)\cdots a(\lambda_n,x_n)a({}^t\nu,x_n)}{a(x_2)a(x_3)\cdots a(x_n)}\right.$$
$$\left.\times\sum_{\nu_3,\dots,\nu_n} a({}^t\nu_3,x_2)a(\nu_3,x_3)\cdots a({}^t\nu_n,x_{n-1})a(\nu_n,x_n)\right),$$

where $\nu_3,\dots,\nu_n$ and $x_2,\dots,x_n$ are supposed to run over P_+^m. Since $\sum_{\nu\in P_+^m} a({}^t\nu,x)a(\nu,y)=\delta_{x,y}$ by Lemma 5.3, the term

$$\sum_{\nu_3,\dots,\nu_n} a({}^t\nu_3,x_2)a(\nu_3,x_3)\cdots a({}^t\nu_n,x_{n-1})a(\nu_n,x_n)$$

becomes nonzero only if $x_2=\cdots=x_n$. Denoting it by μ, we obtain (5.5). □

5.2. Relation to the Littlewood-Richardson coefficients.

For $\lambda,\mu,\nu\in P_+$, let us set

$$n_{\lambda,\mu}^{\nu} := \mathrm{Mult}(V_\lambda\otimes V_\mu, V_\nu), \tag{5.6}$$

so that $V_\lambda\otimes V_\mu=\sum_{\nu\in P_+} n_{\lambda,\mu}^{\nu}V_\nu$. The integer $n_{\lambda,\mu}^{\nu}$ is called the Littlewood-Richardson coefficient.

Lemma 5.7. *Let $\lambda,\mu\in P_+$. If $m\geq(\lambda+\mu|\theta)$, then we have*

$$n_{\lambda,\mu}^{\nu}=\begin{cases} N_{\lambda,\mu}^{\nu} & (\textit{if } \nu\in P_+^m),\\ 0 & (\textit{if } \nu\notin P_+^m).\end{cases}$$

Proof. Notice that $n^{\nu}_{\lambda,\mu} \neq 0$ implies $\lambda + \mu - \nu \in \boldsymbol{Z}_{\geq 0}\alpha_1 + \cdots + \boldsymbol{Z}_{\geq 0}\alpha_l$ (see, e.g., [**5**, Chapter VI, Lemma 2.8]). Since the highest root θ is in P_+, we have $(\nu|\theta) \leq (\lambda + \mu|\theta) \leq m$, and hence $\nu \in P^m_+$. Similarly,

$$(\nu + \rho|\theta) \leq (\lambda + \mu|\theta) + g - 1 < m + g$$

shows that $\nu + \rho$ in an interior point in C^{m+g}_+. Now suppose that $\gamma \in P^m_+$ satisfies $\gamma + \rho \equiv w(\nu + \rho) \bmod (m + g)Q^\vee$ for some $w \in W$ as in the definition (5.1) of $N^{\nu}_{\lambda,\mu}$, and that $n^{\gamma}_{\lambda,\mu} \neq 0$. Then just as above, $n^{\gamma}_{\lambda,\mu} \neq 0$ tells us that $\gamma + \rho$ is an interior point of C^{m+g}_+. Since, by Lemma 5.1, C^{m+g}_+ is a fundamental domain of the action of the affine Weyl group $W \ltimes (m + g)Q^\vee$ on $\mathfrak{h}^*_{\boldsymbol{R}}$, the condition $\gamma + \rho \equiv w(\nu + \rho) \bmod (m + g)Q^\vee$ for some $w \in W$ implies that $\gamma + \rho = \nu + \rho$, and hence $\gamma = \nu$. Then we have also $w = e$. Thus, we have $n^{\nu}_{\lambda,\mu} = N^{\nu}_{\lambda,\mu}$ by the definitions (5.1) and (5.6). □

Corollary 5.8. *Let $\lambda_1, \lambda_2 \in P_+$ and $m \geq (\lambda_1 + \lambda_2|\theta)$. Then we have*

$$V_{\lambda_1} \otimes V_{\lambda_2} = \sum_{\nu \in P^m_+} N^{\nu}_{\lambda_1,\lambda_2} V_\nu.$$

Similarly, if $\lambda_1, \dots, \lambda_n \in P_+$ and $m \geq (\lambda_1 + \cdots + \lambda_n|\theta)$, then we have

$$V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} = \sum_{\nu_1,\dots,\nu_{n-1} \in P^m_+} N^{\nu_1}_{\lambda_1,\lambda_2} N^{\nu_2}_{\nu_1,\lambda_3} \cdots N^{\nu_{n-1}}_{\nu_{n-2},\lambda_n} V_{\nu_{n-1}}. \tag{5.7}$$

Consequently, we obtain the following expression for $\mathscr{Q}(\lambda_1, \dots, \lambda_n) = \mathrm{Mult}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}, V_0)$.

Proposition 5.9. *If $m \geq (\lambda_1 + \cdots + \lambda_n|\theta)$, then we have*

$$\mathscr{Q}(\lambda_1, \dots, \lambda_n) = \sum_{\mu \in P^m_+} \frac{a(\lambda_1, \mu) a(\lambda_2, \mu) \cdots a(\lambda_n, \mu)}{a(\mu)^{n-2}}.$$

Proof. By (5.7) we see $\mathscr{Q}(\lambda_1, \dots, \lambda_n) = \sum_{\nu_1,\dots,\nu_{n-2}} N^{\nu_1}_{\lambda_1,\lambda_2} N^{\nu_2}_{\nu_1,\lambda_3} \cdots \cdots N^{\nu_{n-2}}_{\nu_{n-3},\lambda_{n-1}} N^{0}_{\nu_{n-2},\lambda_n}$. It is equal to

$$\sum_{\mu \in P^m_+} \frac{a(\lambda_1, x\mu) a(\lambda_2, \mu) \cdots a(\lambda_n, \mu) \cdot a({}^t 0, \mu)}{a(\mu)^{n-1}} = \sum_{\mu \in P^m_+} \frac{a(\lambda_1, \mu) a(\lambda_2, \mu) \cdots a(\lambda_n, \mu)}{a(\mu)^{n-2}}$$

by Proposition 5.6. □

6. Some details on root systems and alcoves.

In this section, we discuss some details about root systems. In particular, we will study a certain group of transformations on an alcove. After that, we prove somewhat technical estimates which will be used to prove our second formula for $\mathscr{V}(\lambda_1, \dots, \lambda_n)$ in the next section.

6.1. Special indices associated to the highest root.

Let us write the highest root θ as

$$\theta = n_1\alpha_1 + \cdots + n_l\alpha_l, \tag{6.1}$$

with $n_1, \dots, n_l \in \boldsymbol{Z}_{>0}$ and let us set

$$J := \{i \in \{1, \dots, l\} \mid n_i = 1\}.$$

Definition 6.1. Let $j \in J$. Denote by Δ_j the root system generated by all α_i ($i \in \{1, \dots, l\} - \{j\}$) and denote by W_j the Weyl group of Δ_j, which is regarded as a subgroup of W. Let $\Delta_{j+} = \Delta_j \cap \Delta_+$ and $\Delta_{j-} = \Delta_j \cap \Delta_-$ be the set of positive and negative roots of Δ_j, respectively. We denote by w_j the unique element in W_j that sends Δ_{j+} to Δ_{j-}, whereas w_0 is the unique element in W that sends Δ_+ to Δ_- as in Remark 5.4.

Remark 6.2.

(1) The set Δ_j is also written as $\Delta^{\Lambda_j^\vee}$ in view of the notation (2.9).

(2) If $\alpha \in \Delta_+ - \Delta_{j+}$, then the coefficient of α_j in α is 1.

(3) One has $w_0^2 = w_j^2 = 1$, $\varepsilon(w_0) = (-1)^{|\Delta_+|}$, $\varepsilon(w_j) = (-1)^{|\Delta_{j+}|}$, $w_0(\rho) = -\rho$, and $w_0(\theta) = -\theta$.

Lemma 6.3. *Let $j \in J$. If $\alpha \in \Delta_+ - \Delta_{j+}$, then $w_j(\alpha) \in \Delta_+ - \Delta_{j+}$. In particular, $w_j(\theta) \in \Delta_+ - \Delta_{j+}$.*

Proof. $\alpha \in \Delta_+ - \Delta_{j+}$ is of the form

$$\alpha = \alpha_j + \sum_{i \neq j} p_i\alpha_i.$$

Since the Weyl group W_j of Δ_j is generated by the reflections $s_i(x) = x - \langle \alpha_i^\vee, x\rangle\alpha_i$ ($i \in \{1, \dots, l\} - \{j\}$) on $\mathfrak{h}_{\boldsymbol{R}}^*$, w_j is a composition of them. Hence $w_j(\alpha)$ is of the form

$$w_j(\alpha) = \alpha_j + \sum_{i \neq j} q_i \alpha_i,$$

namely the coefficient of α_j in $w_j(\alpha)$ is 1. On the other hand, we know that $w_j(\alpha) \in \Delta$ since $w_j \in W_j \subset W$. Therefore, $w_j(\alpha)$ must be in Δ_+, and hence in $\Delta_+ - \Delta_{j+}$. □

COROLLARY 6.4. *Let us fix $j \in J$. Then we have the following.*

(1) $\Delta_+ \cap w_j(\Delta_+) = \Delta_+ - \Delta_{j+}$, $\Delta_+ \cap w_j(\Delta_-) = \Delta_{j+}$.

(2) $\{w_j(\alpha) \,|\, \alpha \in \Delta_{j+}\} \cup \{-w_j(\alpha) \,|\, \alpha \in \Delta_+ - \Delta_{j+}\} = \Delta_-$.

PROOF. By Lemma 6.3, we see $\{\alpha \in \Delta_+ \,|\, w_j(\alpha) \in \Delta_+\} = \Delta_+ - \Delta_{j+}$ and $\{\alpha \in \Delta_+ \,|\, w_j(\alpha) \in \Delta_-\} = \Delta_{j+}$, which is equivalent to (1) since $w_j^2 = 1$. Part (2) follows immediately. □

LEMMA 6.5. *For $j \in J$, we have $w_j(\alpha_j) = \theta$, or equivalently, $w_j(\theta) = \alpha_j$.*

PROOF. Let $\beta = w_j(\alpha_j)$. Then $\beta \in \Delta_+ - \Delta_{j+}$ by Lemma 6.3, and hence we see

$$\theta - \beta \in \sum_{i \neq j} \boldsymbol{Z}_{\geq 0} \alpha_i.$$

Since $w_j(\alpha_i) \in \Delta_{j-}$ for $i \neq j$, we have

$$w_j(\theta - \beta) \in \sum_{i \neq j} \boldsymbol{Z}_{\leq 0} \alpha_i. \tag{6.2}$$

On the other hand, $w_j(\theta - \beta) = w_j(\theta) - w_j(\beta) = w_j(\theta) - \alpha_j$ shows that

$$w_j(\theta - \beta) \in \sum_{i \neq j} \boldsymbol{Z}_{\geq 0} \alpha_i, \tag{6.3}$$

since $w_j(\theta) \in \Delta_+ - \Delta_{j+}$ by Lemma 6.3. Now (6.2) and (6.3) implies that $w_j(\theta - \beta) = 0$, that is, $\beta = \theta$. □

REMARK 6.6. In particular, we have $(\alpha_j|\alpha_j) = (\theta|\theta) = 2$, namely α_j is a long root, for $j \in J$. (It can be also checked by the classification of root systems (see, e.g., the table in [**2**]).) Hence we have $\alpha_j^\vee = \alpha_j$ and $\Lambda_j^\vee = \Lambda_j$.

LEMMA 6.7. *For $j \in J$, we have $w_j \Lambda_j = \Lambda_j$.*

PROOF. Let us prove that $w_j \Lambda_j^\vee = \Lambda_j^\vee$. Let $i \in \{1, \dots, l\}$. If $i \neq j$, then $w_j(\alpha_i) \in \Delta_{j-}$ and hence $(\Lambda_j^\vee | w_j(\alpha_i)) = 0$. On the other hand, $(\Lambda_j^\vee | w_j(\alpha_j)) = (\Lambda_j^\vee | \theta) = 1$ by Lemma 6.5. Thus, we have $(w_j(\Lambda_j^\vee) | \alpha_i) = (\Lambda_j^\vee | w_j(\alpha_i)) = \delta_{ij}$, which means that $w_j(\Lambda_j^\vee) = \Lambda_j^\vee$. □

LEMMA 6.8. *For $j \in J$, we have $w_j(\rho) + \rho = g\Lambda_j$.*

PROOF. Since $\Delta_+ \cap w_j(\Delta_-) = \Delta_{j+}$ by Corollary 6.4, we see from (2.2) that

$$w_j \rho = \rho - \sum_{\alpha \in \Delta_{j+}} \alpha = \rho - 2\rho_j,$$

where $\rho_j = (1/2) \sum_{\alpha \in \Delta_{j+}} \alpha$, and hence $w_j \rho + \rho = 2(\rho - \rho_j)$. It follows that for $i \neq j$,

$$(w_j(\rho) + \rho | \alpha_i^\vee) = 2((\rho | \alpha_i^\vee) - (\rho_j | \alpha_i^\vee)) = 2(1 - 1) = 0.$$

On the other hand, we see from Lemma 6.5 that

$$(w_j(\rho) + \rho | \alpha_j^\vee) = (w_j(\rho) | \alpha_j) + (\rho | \alpha_j^\vee) = (\rho | w_j(\alpha_j)) + 1 = (\rho | \theta) + 1 = g.$$

Thus, we have $w_j(\rho) + \rho = g\Lambda_j$. □

6.2. Symmetry of the alcove.

DEFINITION 6.9. For $m \in \boldsymbol{Z}_{>0}$ and $j \in J$, let $\gamma_j^m : \mathfrak{h}_{\boldsymbol{R}}^* \to \mathfrak{h}_{\boldsymbol{R}}^*$ be the map defined by

$$\gamma_j^m(x) := w_j w_0(x) + m\Lambda_j \quad (= w_j w_0(x) + m\Lambda_j^\vee).$$

LEMMA 6.10. *For any $j \in J$, γ_j^m is a bijection from C_+^m (resp. P_+^m) to itself.*

PROOF. For the fact that γ_j^m is a bijection from C_+^m to itself, see [**2**, Chapter VI, Section 2, no. 3]. (The case that $m = 1$ is considered there, but the same proof works for general m.) On the other hand, we see that $\gamma_j^m(\mu) = w_j w_0(\mu) + m\Lambda_j$ is in P if and only if μ is in P. Since $P_+^m = C_+^m \cap P$, it follows that γ_j^m is a bijection from P_+^m to itself. □

LEMMA 6.11. *Let us fix $m \in \boldsymbol{Z}_{>0}$.*

(1) *The set* $\{1\} \cup \{\gamma_j^m \mid j \in J\}$ *forms a group, which is isomorphic to* $P^\vee/Q^\vee$ $(\cong P/Q)$. *In particular,* $|J| + 1 = |P^\vee/Q^\vee|$ $(= |P/Q|)$.

(2) *If* $j \in J$, *then* $(\gamma_j^m)^{-1} = \gamma_s^m$ *for some* $s \in J$.

Proof. See [**2**, Chapter VI, Section 2, no. 3] for (1). Part (2) immediately follows from (1). ☐

Remark 6.12. It follows from a simple calculation that $(\gamma_j^m)^{-1}(x) = w_0 w_j(x) - m w_0 w_j \Lambda_j$, and hence that

$$(\gamma_j^m)^{-1}(x) = w_0 w_j(x) - m w_0 \Lambda_j \tag{6.4}$$

by Lemma 6.7. Then the condition $(\gamma_j^m)^{-1} = \gamma_s^m$ in Lemma 6.11 (2) means that $w_0 w_j = w_s w_0$ and $-w_0 \Lambda_j = \Lambda_s$.

Now we observe the following.

Proposition 6.13. *Let* $m \in \boldsymbol{Z}_{>0}$ *and* $j \in J$. *For* $\lambda \in Q \cap P_+^m$ *and* $\mu \in P_+^m$, *we have*

$$a(\lambda, \gamma_j^m(\mu)) = a(\lambda, \mu), \quad a(\gamma_j^m(\mu)) = a(\mu).$$

In order to prove it, we will prepare several lemmas.

Lemma 6.14. *For* $j \in J$, *let* $\Gamma_j^m : \mathfrak{h}_{\boldsymbol{R}}^* \to \mathfrak{h}_{\boldsymbol{R}}^*$ *be the map defined by*

$$\Gamma_j^m(\mu) := \gamma_j^{m+g}(\mu + \rho) - \rho.$$

Then we have $\gamma_j^m = \Gamma_j^m$.

Proof. From $w_0 \rho = -\rho$ and Lemma 6.8, we obtain

$$\Gamma_j^m(x) = w_j w_0(x + \rho) + (m + g)\Lambda_j - \rho = \gamma_j^m(x) + g\Lambda_j - w_j \rho - \rho = \gamma_j^m(x)$$

as claimed. ☐

Lemma 6.15. *Let* $j \in J$. *For any* $\lambda \in Q$ *and* $w \in W$, *we have*

$$\exp\Big(-2\pi\sqrt{-1}(\lambda + \rho | w\Lambda_j)\Big) = (-1)^{|\Delta_+ - \Delta_{j+}|} \; (= \varepsilon(w_0)\varepsilon(w_j)).$$

Proof. It follows that

$$\begin{aligned}\exp\Bigl(-2\pi\sqrt{-1}(\lambda+\rho|w\Lambda_j)\Bigr) &= \exp\Bigl(-2\pi\sqrt{-1}(w^{-1}\lambda|\Lambda_j)\Bigr)\exp\Bigl(-2\pi\sqrt{-1}(w^{-1}\rho|\Lambda_j)\Bigr)\\ &= \exp\Bigl(-2\pi\sqrt{-1}(w^{-1}(\rho)|\Lambda_j)\Bigr), \qquad (6.5)\end{aligned}$$

since the assumption $\lambda \in Q$ (and hence $w^{-1}\lambda \in Q$) implies $(w^{-1}\lambda|\Lambda_j) = (w^{-1}\lambda|\Lambda_j^\vee) \in \boldsymbol{Z}$. In addition, by (2.2), we know $w^{-1}\rho - \rho \in Q$ for $w \in W$. Hence we have $(w^{-1}\rho|\Lambda_j) - (\rho|\Lambda_j) = (w^{-1}\rho - \rho|\Lambda_j) = (w^{-1}\rho - \rho|\Lambda_j^\vee) \in \boldsymbol{Z}$. Thus (6.5) is equal to

$$\exp\Bigl(-2\pi\sqrt{-1}(\rho|\Lambda_j)\Bigr) = \exp\left(-\pi\sqrt{-1}\sum_{\alpha\in\Delta_+}(\alpha|\Lambda_j)\right). \qquad (6.6)$$

Since we see for $j \in J$

$$(\alpha|\Lambda_j) = (\alpha|\Lambda_j^\vee) = \begin{cases} 0 & (\text{if } \alpha \in \Delta_{j+}), \\ 1 & (\text{if } \alpha \in \Delta_+ - \Delta_{j+}), \end{cases}$$

(6.6) is equal to $(-1)^{|\Delta_+ - \Delta_{j+}|} = \varepsilon(w_0)\varepsilon(w_j)$. □

PROOF OF PROPOSITION 6.13. It is enough to show that $a(\lambda, \Gamma_j^m(\mu)) = a(\lambda, \mu)$ and $a(\Gamma_j^m(\mu)) = a(\mu)$ in view of Lemma 6.14. Let $\tilde{a}(\lambda,\mu) = (\sqrt{-1})^{-|\Delta_+|}|P/(m+g)Q^\vee|^{\frac{1}{2}}a(\lambda,\mu)$. Then Lemma 6.15 shows that

$$\begin{aligned}&\tilde{a}(\lambda, \Gamma_j^m(\mu))\\ &\quad= \sum_{w\in W}\varepsilon(w)\exp\frac{-2\pi\sqrt{-1}}{m+g}\bigl(\lambda+\rho|w(w_jw_0(\mu+\rho)+(m+g)\Lambda_j)\bigr)\\ &\quad= \sum_{w\in W}\varepsilon(w)\exp\Bigl(-2\pi\sqrt{-1}\bigl(\lambda+\rho|w(\Lambda_j)\bigr)\Bigr)\exp\left(\frac{-2\pi\sqrt{-1}}{m+g}\bigl(\lambda+\rho|ww_jw_0(\mu+\rho)\bigr)\right)\\ &\quad= \sum_{w\in W}\varepsilon(w)\varepsilon(w_j)\varepsilon(w_0)\left(\exp\frac{-2\pi\sqrt{-1}}{m+g}\bigl(\lambda+\rho|ww_jw_0(\mu+\rho)\bigr)\right)\\ &\quad= \sum_{w'\in W}\varepsilon(w')\left(\exp\frac{-2\pi\sqrt{-1}}{m+g}\bigl(\lambda+\rho|w'(\mu+\rho)\bigr)\right) = \tilde{a}(\lambda,\mu).\end{aligned}$$

Thus we conclude $a(\lambda, \Gamma_j^m(\mu)) = a(\lambda, \mu)$. By substituting $\lambda = 0$, we obtain $a(\Gamma_j^m(\mu)) = a(\mu)$. □

6.3. Technical estimates.

Below, we prove some rather technical inequalities, which will be used in the next section. First, we observe the following.

LEMMA 6.16. *Let $\mu \in P_+^m$ and $\alpha \in \Delta_+$. Then we have*

$$\frac{2}{3(m+g)} < \sin \frac{\pi(\mu+\rho|\alpha)}{m+g} < \frac{\pi(\mu+\rho|\alpha)}{m+g}.$$

PROOF. Since $x > \sin x$ for $x > 0$, it immediately follows that $\pi(\mu+\rho|\alpha)/(m+g) > \sin(\pi(\mu+\rho|\alpha)/(m+g))$. Recall from (2.4) that $(\mu+\rho|\alpha) \geq 1/3$. Combining it with

$$(\mu+\rho|\alpha) \leq (\mu+\rho|\theta) \leq m+g-1 < m+g-\frac{1}{3},$$

we see that $\sin(\pi(\mu+\rho|\alpha)/(m+g)) \geq \sin(\pi/(3(m+g)))$. In addition, due to the fact that $\sin x > (2/\pi)x$ for $0 < x < \pi/2$, we have

$$\sin \frac{\pi}{3(m+g)} > \frac{2}{\pi} \cdot \frac{\pi}{3(m+g)} = \frac{2}{3(m+g)}.$$

Thus we obtain $\sin(\pi(\mu+\rho|\alpha)/(m+g)) > 2/(3(m+g))$. □

Next, let $\delta > 0$ be sufficiently small and let us set $t = \delta m$. We consider the small alcove P_+^t.

PROPOSITION 6.17. *Let $0 < \delta < 1/h$, where h is the Coxeter number of the root system Δ, and let $t = \delta m$. For $\mu \in P_+^m$, the following two conditions are equivalent.*

(i) $\mu \in P_+^t \cup \bigcup_{j \in J} \gamma_j^m(P_+^t)$.

(ii) *For any $\alpha \in \Delta_+$, either $(\mu|\alpha) \leq t$ or $(\mu|\alpha) \geq m-t$ holds.*

PROOF OF (i) $\Longrightarrow$ (ii). If $\mu \in P_+^t$, then $(\mu|\theta) \leq t$ implies that $(\mu|\alpha) \leq (\mu|\theta) \leq t$ for any $\alpha \in \Delta_+$. Hence (ii) certainly holds. Suppose next that $\mu \in \gamma_j^m(P_+^t)$ for some $j \in J$. Since $(\gamma_j^m)^{-1} = \gamma_s^m$ for some $s \in J$ by Lemma 6.11 (2), we have $\gamma_s^m(\mu) \in P_+^t$, and hence $(\gamma_s^m(\mu)|\theta) \leq t$. Therefore, $(\gamma_s^m(\mu)|\alpha) \leq t$, namely

$$(w_s w_0(\mu) + m\Lambda_s|\alpha) \leq t \tag{6.7}$$

for any $\alpha \in \Delta_+$. Note that $s \in J$ implies that $(\Lambda_s|\alpha) = (\Lambda_s^\vee|\alpha)$ is either 0 or 1.

(1) If $(\Lambda_s|\alpha) = 0$, namely $\alpha \in \Delta_{s+}$, then (6.7) becomes $(w_s w_0(\mu)|\alpha) \leq t$, and hence we have $(\mu|w_0 w_s(\alpha)) \leq t$.

(2) If $(\Lambda_s|\alpha) = 1$, namely $\alpha \in \Delta_+ - \Delta_{s+}$, then (6.7) becomes $(w_s w_0(\mu)|\alpha) + m \leq t$, and hence we have $(\mu| - w_0 w_s(\alpha)) \geq m - t$.

Since we know from Corollary 6.4 (2) that

$$\{w_0 w_s(\alpha) \,|\, \alpha \in \Delta_{s+}\} \cup \{-w_0 w_s(\alpha) \,|\, \alpha \in \Delta - \Delta_{s+}\} = \Delta_+,$$

(1) and (2) above show that for each $\beta \in \Delta_+$, either $(\mu|\beta) \leq t$ or $(\mu|\beta) \geq m - t$ holds. Thus we obtain (ii). □

Before beginning a proof of the converse, we mention the following.

LEMMA 6.18. *Let* $0 < \delta < 1/2$. *Suppose* $\mu \in P_+^m$ *satisfies the condition* (ii) *in Proposition* 6.17.

(1) *If* $i \in \{1, \dots, l\} - J$, *then* $(\mu|\alpha_i) \leq t$.

(2) *If* $(\mu|\alpha_j) \geq m - t$ *for some* $j \in J$, *then* $(\mu|\alpha_i) \leq t$ *for any* $i \in J - \{j\}$.

PROOF. By the assumption in (1), we have $n_i \geq 2$ in (6.1). If $(\mu|\alpha_i) \geq m - t$, then

$$(\mu|\theta) \geq n_i(\mu|\alpha_i) \geq 2(m - t) > m,$$

since $\delta < 1/2$ implies $2(m - t) = 2(1 - \delta)m > m$. However, it contradicts to the fact that $\mu \in P_+^m$. Hence we obtain (1).

If $(\mu|\alpha_i) \geq m - t$ and $(\mu|\alpha_j) \geq m - t$ for some distinct $i, j \in J$, then we have

$$(\mu|\theta) \geq (\mu|\alpha_i) + (\mu|\alpha_j) \geq 2(m - t) > m.$$

Again, it contradicts to the fact that $\mu \in P_+^m$. Thus we obtain (2). □

PROOF OF (ii) $\Longrightarrow$ (i) IN PROPOSITION 6.17. Let us assume that $\mu \in P_+^m$ satisfies (ii). We consider two cases.

Case 1: Suppose $(\mu|\alpha_j) \leq t$ for all $j \in \{1, \dots, l\}$. Then in view of (6.1), we are led to

$$(\mu|\theta) = n_1(\mu|\alpha_1) + \cdots + n_l(\mu|\alpha_l) \leq (n_1 + \cdots + n_l)t = (h - 1)t.$$

By the assumption $\delta < 1/h$, we have $(\mu|\theta) < m - t$. It follows form the condition (ii) that $(\mu|\theta) \leq t$, and hence $\mu \in P_+^t$.

Case 2: Suppose $(\mu|\alpha_j) \geq m - t$ for some $j \in \{1, \ldots, l\}$. Then by Lemma 6.18 we have $j \in J$ and $(\mu|\alpha_i) \leq t$ for $i \in \{1, \ldots, l\} - \{j\}$. Now we claim that $\mu \in \gamma_j^m(P_+^t)$, namely $((\gamma_j^m)^{-1}(\mu)|\theta) \leq t$. From (6.4) and Lemma 6.5, we obtain

$$\begin{aligned}((\gamma_j^m)^{-1}(\mu)|\theta) &= (w_0 w_j(\mu) - m w_0 \Lambda_j|\theta) = (\mu|w_j w_0(\theta)) - m(\Lambda_j|w_0(\theta)) \\ &= -(\mu|w_j(\theta)) + m(\Lambda_j|\theta) = -(\mu|\alpha_j) + m \\ &\leq -(m-t) + m = t\end{aligned}$$

as claimed. ☐

Corollary 6.19. *Suppose that* $m > g$ *and let* δ *and* t *be as in Proposition 6.17. Then for any* $\mu \in P_+^m - \left(P_+^t \cup \bigcup_{j \in J} \gamma_j^m(P_+^t)\right)$, *there exists at least one* $\alpha \in \Delta_+$ *such that* $\pi\delta/2 < \pi(\mu + \rho|\alpha)/(m+g) < \pi - \pi\delta/2$, *and hence*

$$\sin \frac{\pi(\mu+\rho|\alpha)}{m+g} > \sin \frac{\pi\delta}{2}.$$

Proof. By Proposition 6.17, there exists at least one $\alpha \in \Delta_+$ such that $t < (\mu|\alpha) < m - t$. Then we have

$$\frac{(\mu+\rho|\alpha)}{m+g} > \frac{t + (\rho|\alpha)}{m+g} > \frac{t}{m+g} = \frac{\delta}{1 + g/m} > \frac{\delta}{2}$$

and

$$\begin{aligned}\frac{(\mu+\rho|\alpha)}{m+g} &< \frac{m - t + (\rho|\alpha)}{m+g} \leq \frac{m - t + (\rho|\theta)}{m+g} = \frac{m + g - t - 1}{m+g} \\ &< 1 - \frac{t}{m+g} = 1 - \frac{\delta}{1 + g/m} < 1 - \frac{\delta}{2}.\end{aligned}$$

This completes the proof. ☐

We conclude this section by the following observation.

Lemma 6.20. *Suppose* $0 < \delta < 1/2$ *and* $m > (g-2)/(1-2\delta)$, *and let* $\mu \in P_+^t$. *Then for any* $\alpha \in \Delta_+$, *we have the following.*

(1) $0 < \dfrac{\pi(\mu+\rho|\alpha)}{m+g} < \dfrac{\pi}{2}$,

(2) $\sin \dfrac{\pi(\mu+\rho|\alpha)}{m+g} > \dfrac{2}{\pi} \cdot \dfrac{\pi(\mu+\rho|\alpha)}{m+g}$, *and*

(3) $\dfrac{1}{\sin \dfrac{\pi(\mu+\rho|\alpha)}{m+g}} - \dfrac{m+g}{\pi(\mu+\rho|\alpha)} < 1 - \dfrac{2}{\pi}$.

Proof. In fact, we have

$$0 < (\mu+\rho|\alpha) \leq (\mu+\rho|\theta) \leq t+g-1 = \delta m+g-1,$$

and hence

$$0 < \frac{\pi(\mu+\rho|\alpha)}{m+g} \leq \pi \frac{\delta m+g-1}{m+g}.$$

If $m > (g-2)/(1-2\delta)$, the right-hand side is less than $\pi/2$. This completes the proof of (1). Since $(2/\pi)x < \sin x$ and $1/\sin x - 1/x < 1 - 2/\pi$ for $0 < x < \pi/2$, we obtain (2) and (3). $\square$

7. The second formula.

In what follows, unless otherwise stated, we suppose that weights $\lambda_1, \ldots, \lambda_n \in P_+$ satisfy the assumptions

(A2) $\lambda_1, \ldots, \lambda_n \in Q$,

(A3) $\lambda_1, \ldots, \lambda_n \in P_{++}$,

as in Section 4.2. In particular, one has $d = (n-2)|\Delta_+| - l$. Further, we will introduce a new condition

(A4) $n \geq \max\{\, l+3,\, 5\,\}$.

In this section, we will establish our second formula for $\mathscr{V}(\lambda_1, \ldots, \lambda_n)$. The formula itself is given in Section 6.1 together with some related remarks. Section 6.2 is devoted to the proof of the formula. Although our proof becomes somewhat long and technical, the main idea is to generalize the argument in [**26**, Section 3] for $G = SU(2)$ to a general compact Lie group G.

In Section 7.3, we consider special cases where all of $\lambda_1, \dots, \lambda_n$ are proportional to ρ, and write out our formula more explicitly. Moreover, as we will illustrate for the root systems of type A_1 and A_2, we have another kind of formula that expresses $\mathscr{V}(\lambda_1, \dots, \lambda_n)$ as an integral over an unbounded domain.

7.1. The formula and related remarks.

In view of Proposition 5.9, let us consider the asymptotic behavior of

$$\mathscr{Q}(k\lambda_1, \dots, k\lambda_n) = \sum_{\mu \in P_+^m} \frac{a(k\lambda_1, \mu) a(k\lambda_2, \mu) \cdots a(k\lambda_n, \mu)}{a(\mu)^{n-2}},$$

as $k \to \infty$, where k runs over positive integers. We need the condition $m \geq k(\lambda_1 + \cdots + \lambda_n | \theta)$ according to the assumption in Proposition 5.9. So from now on, let us set

$$m = k(\lambda_1 + \cdots + \lambda_n | \theta) \tag{7.1}$$

and let us denote

$$L := (\lambda_1 + \cdots + \lambda_n | \theta)$$

for simplicity, so that $m = kL$. Note that L is a positive integer.

First of all, we observe the following.

Lemma 7.1. *Fix $\mu \in P_+$. As $k \to \infty$, and hence $m \to \infty$, we have*

$$\frac{1}{a(\mu)} \sim k^{\frac{l}{2} + |\Delta_+|} \cdot L^{\frac{l}{2} + |\Delta_+|} \, |P/Q^\vee|^{\frac{1}{2}} \, \frac{1}{\prod_{\alpha \in \Delta_+} 2\pi(\mu + \rho | \alpha)}, \tag{7.2}$$

$$a(k\lambda_i, \mu) \sim k^{-\frac{l}{2}} \cdot (\sqrt{-1})^{|\Delta_+|} L^{-\frac{l}{2}} \, |P/Q^\vee|^{-\frac{1}{2}} \, A_{\mu+\rho}\left(\frac{-2\pi\sqrt{-1}\lambda_i}{L} \right) \tag{7.3}$$

$$= k^{-\frac{l}{2}} \cdot L^{-\frac{l}{2}} \, |P/Q^\vee|^{-\frac{1}{2}} \left(\prod_{\alpha \in \Delta_+} 2 \sin \frac{\pi(\lambda_i | \alpha)}{L} \right) \chi_\mu \left(\exp \frac{-2\pi\sqrt{-1}\lambda_i}{L} \right). \tag{7.4}$$

Proof. Since $|P/(m+g)Q^\vee| = (m+g)^l |P/Q^\vee| = (kL+g)^l |P/Q^\vee|$, it follows from (5.2) that

$$\frac{1}{a(\mu)} = (m+g)^{\frac{l}{2}} |P/Q^\vee|^{\frac{1}{2}} \prod_{\alpha \in \Delta_+} \frac{1}{2 \sin \frac{\pi(\mu+\rho|\alpha)}{m+g}} \sim (m+g)^{\frac{l}{2}} |P/Q^\vee|^{\frac{1}{2}} \prod_{\alpha \in \Delta_+} \frac{m+g}{2\pi(\mu+\rho|\alpha)}$$

$$\sim k^{\frac{l}{2} + |\Delta_+|} \cdot L^{\frac{l}{2} + |\Delta_+|} \, |P/Q^\vee|^{\frac{1}{2}} \, \frac{1}{\prod_{\alpha \in \Delta_+} 2\pi(\mu + \rho | \alpha)},$$

while (7.3) and (7.4) immediately follow from (5.3) and (5.4). $\square$

REMARK 7.2. If $\lambda_1, \dots, \lambda_n$ does not satisfy the assumption (A3), namely if $\lambda_i \in P_+ - P_{++}$ for some i, then $A_{\mu+\rho}(-2\pi\sqrt{-1}\lambda_i/L) = A_\rho(-2\pi\sqrt{-1}\lambda_i/L) = 0$. Hence we need to be more careful. In this case, instead of (7.4) we have

$$\begin{aligned} a(k\lambda_i, \mu) \sim k^{-\frac{l}{2}-|\Delta_+^{\lambda_i}|} \cdot |P/Q^\vee|^{-\frac{1}{2}} L^{-\frac{l}{2}} & \left(\prod_{\alpha \in \Delta_+^{\lambda_i}} \frac{2\pi(\rho|\alpha)}{L} \prod_{\alpha \in \Delta_+ - \Delta_+^{\lambda_i}} 2\sin\frac{\pi(\lambda_i|\alpha)}{L} \right) \\ & \times \chi_\mu\left(\exp\frac{-2\pi\sqrt{-1}\lambda_i}{L} \right), \end{aligned}$$

where $\Delta_+^{\lambda_i}$ is as in (2.9). This follows from (5.4) since one has

$$2\sin\frac{\pi(k\lambda_i + \rho|\alpha)}{kL+g} \sim \begin{cases} \dfrac{1}{k} \cdot \dfrac{2\pi(\rho|\alpha)}{L} & (\text{if } (\lambda_i|\alpha) = 0) \\ 2\sin\dfrac{\pi(\lambda_i|\alpha)}{L} & (\text{if } (\lambda_i|\alpha) \neq 0). \end{cases}$$

Now let us define

$$\begin{aligned} T(\mu) :&= \frac{(\sqrt{-1})^{n|\Delta_+|} L^d}{|P/Q^\vee|} \cdot \frac{\prod_{i=1}^n A_{\mu+\rho}\left(\exp\frac{-2\pi\sqrt{-1}\lambda_i}{L}\right)}{\left(\prod_{\alpha\in\Delta_+} 2\pi(\mu+\rho|\alpha)\right)^{n-2}} \\ &= \frac{L^d}{|P/Q^\vee|} \left(\prod_{i=1}^n \prod_{\alpha\in\Delta_+} 2\sin\frac{\pi(\lambda_i|\alpha)}{L} \right) \frac{\prod_{i=1}^n \chi_\mu\left(\exp\frac{-2\pi\sqrt{-1}\lambda_i}{L}\right)}{\left(\prod_{\alpha\in\Delta_+} 2\pi(\mu+\rho|\alpha)\right)^{n-2}}. \end{aligned} \tag{7.5}$$

By Lemma 7.1, for a fixed $\mu \in P_+$ and for a sufficiently large k, we see

$$\frac{a(k\lambda_1, \mu) \cdots a(k\lambda_n, \mu)}{a(\mu)^{n-2}} \sim k^d \cdot T(\mu).$$

Now our second formula for $\mathscr{V}(\lambda_1, \dots, \lambda_n)$ is the following. It might be interesting to compare it with the first formula (4.11) in Theorem 4.11.

THEOREM 7.3. *Let us suppose* $\lambda_1, \dots, \lambda_n \in Q \cap P_{++}$ *and* $n \geq \max\{5, l+3\}$, *and let* $L = (\lambda_1 + \cdots + \lambda_n|\theta)$. *Then we have* $\mathscr{V}(\lambda_1, \dots, \lambda_n) = |P^\vee/Q^\vee| \sum_{\mu\in P_+} T(\mu)$, *namely*

$$\begin{aligned}
&\mathscr{V}(\lambda_1,\dots,\lambda_n)\\
&= (\sqrt{-1})^{n|\Delta_+|}\frac{|P^\vee/Q^\vee|}{|P/Q^\vee|}L^d\sum_{\mu\in P_+}\frac{\prod_{i=1}^n A_{\mu+\rho}\Bigl(\frac{-2\pi\sqrt{-1}\lambda_i}{L}\Bigr)}{\Bigl(\prod_{\alpha\in\Delta_+}2\pi(\mu+\rho|\alpha)\Bigr)^{n-2}} \qquad (7.6)\\
&= \frac{|P^\vee/Q^\vee|}{|P/Q^\vee|}L^d\Biggl(\prod_{i=1}^n\prod_{\alpha\in\Delta_+}2\sin\frac{\pi(\lambda_i|\alpha)}{L}\Biggr)\sum_{\mu\in P_+}\frac{\prod_{i=1}^n \chi_\mu\Bigl(\exp\frac{-2\pi\sqrt{-1}\lambda_i}{L}\Bigr)}{\Bigl(\prod_{\alpha\in\Delta_+}2\pi(\mu+\rho|\alpha)\Bigr)^{n-2}}. \qquad (7.7)
\end{aligned}$$

The proof of this theorem is given in Section 7.2. The main point there is to explain why the factor $|P^\vee/Q^\vee|$ arises. Before proceeding further, we collect some remarks concerned with this formula.

REMARK 7.4.

(1) From the proof and our convention (7.1), we will see that we can replace L in (7.6) or (7.7) to any larger number.

(2) In view of Remark 7.2, even if $\lambda_1,\dots,\lambda_n$ does not satisfy the assumption (A3), one might expect that $\mathscr{V}(\lambda_1,\dots,\lambda_n)/L^d$ is given by

$$\begin{aligned}
&\frac{|P/Q|}{|P/Q^\vee|}\Biggl(\prod_{i=1}^n\prod_{\alpha\in\Delta_+^{\lambda_i}}2\pi(\rho|\alpha)\prod_{\alpha\in\Delta_+-\Delta_+^{\lambda_i}}2\sin\frac{\pi(\lambda_i|\alpha)}{L}\Biggr)\\
&\qquad\times\sum_{\mu\in P_+}\frac{\prod_{i=1}^n \chi_\mu\Bigl(\exp\frac{-2\pi\sqrt{-1}\lambda_i}{L}\Bigr)}{\Bigl(\prod_{\alpha\in\Delta_+}2\pi(\mu+\rho|\alpha)\Bigr)^{n-2}}, \qquad (7.8)
\end{aligned}$$

where $d=(n-2)|\Delta_+|-\sum_{i=1}^n|\Delta_+^{\lambda_i}|-l$. (See also Remark 7.5 below.) Unfortunately, in this case our arguments in Section 7.2 will not work; we need some extra efforts. However, we will not go into this issue in this paper.

(3) The assumption (A4), namely $n\geq\max\{5,l+3\}$, might also be technical, although our proof indeed needs it.

REMARK 7.5. Theorem 7.3 is closely related to Witten's volume formula in 2-dimensional gauge theory. For $X_i\in\mathfrak{t}$ $(i=1,\dots,n)$, let denote by $\mathscr{C}_{X_i}$ the conjugacy class in G containing $\exp(X_i)$. Here X_i is not necessarily regular. Let $\mathscr{P}(X_1,\dots,X_n)$ be the moduli space of flat G connections over a punctured sphere $S^2-\{z_1,\dots,z_n\}$ such that the holonomy around the point z_i is in $\mathscr{C}_{X_i}$ for each

$i=1,\dots,n$. It is well known that this moduli space has a natural symplectic structure. Witten's volume formula claims that the symplectic volume of it is given by

$$\frac{|Z(G)|}{\operatorname{vol}(G)^2}\left(\prod_{i=1}^{n}\operatorname{vol}(\mathscr{C}_{X_i})\prod_{\alpha\in\Delta_+-\Delta_+^{X_i}}2\sin\frac{\sqrt{-1}}{2}(X_i|\alpha)\right)\sum_{x\in P_+}\frac{\prod_{i=1}^{n}\chi_x(\exp X_i)}{(\dim_{\boldsymbol{C}}V_x)^{n-2}} \tag{7.9}$$

(see, e.g., [**26**], [**16**], [**19**]). Here $\operatorname{vol}(G)$ and $\operatorname{vol}(\mathscr{C}_{X_i})$ denote the Riemannian volume of G and $\mathscr{C}_{X_i}$, respectively (see [**1**] for their concrete expressions), and $\Delta_+^{X_i}=\{\alpha\in\Delta_+\mid\langle\alpha,X_i\rangle=0\}$.

On the other hand, it is shown in [**11**] that we can identify the moduli space $\mathscr{P}(X_1,\dots,X_n)$ with our symplectic quotient $\mathscr{M}(\lambda_1,\dots,\lambda_n)$ as in Section 2 by letting $X_i=-2\pi\sqrt{-1}\lambda_i/L\in\mathfrak{t}^*\cong\mathfrak{t}$. (To be precise, we have to assume that $X_1,\dots,X_n$ are sufficiently close to 0. It is achieved by multiplying them simultaneously by a small positive constant. Further, we have to be careful to compare the symplectic forms of these two spaces.) Hence $\mathscr{V}(\lambda_1,\dots,\lambda_n)$, the volume of $\mathscr{M}(\lambda_1,\dots,\lambda_n)$, should be essentially the same with the volume of $\mathscr{P}(X_1,\dots,X_n)$. In fact, we can check that (7.8) in Remark 7.4 and Witten's formula (7.9) indeed coincide up to a constant factor. See also [**23**] and [**15**] for the case of $G=SU(2)$.

However, our proof of Theorem 7.3 is independent of the geometric context described above; it is based on the rather combinatorial results in Sections 5 and 6.

7.2. Proof of the formula.

7.2.1. Outline.

Recall that we have set $m=Lk$. Let δ be a positive real number and let us set $t=\delta m=\delta Lk$ as in Section 5. In the sequel, we will prove the following two propositions, which are the essential parts of the proof of Theorem 7.3.

Proposition 7.6. *Suppose $n\geq l+3$ and $0<\delta<1/h$. Then there exists a positive constant C_1, which might depend on $\lambda_1,\dots,\lambda_n$ and δ, such that for any positive integer k with $m>g$, one has*

$$\left|\frac{1}{k^d}\sum_{\mu\in P_+^m}\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}}-\frac{|P^\vee/Q^\vee|}{k^d}\sum_{\mu\in P_+^t}\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}}\right|\leq C_1\cdot\frac{1}{k}.$$

Proposition 7.7. *Suppose $n\geq 5$ and $0<\delta<1/2$. Then there exists a positive constant C_2, which might depend on $\lambda_1,\dots,\lambda_n$ and δ, such that for any* *positive integer k with $m > (g-2)/(1-2\delta)$, one has*

$$\left|\frac{1}{k^d}\sum_{\mu\in P_+^t}\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}} - \sum_{\mu\in P_+^t} T(\mu)\right| \leq C_2\cdot\frac{1}{k}.$$

Proof of Theorem 7.3 assuming Propositions 7.6 and 7.7. These two propositions imply that there exists a positive constant C such that for any sufficiently large k, the following holds:

$$\left|\frac{1}{k^d}\sum_{\mu\in P_+^m}\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}} - |P^\vee/Q^\vee|\sum_{\mu\in P_+^t} T(\mu)\right| \leq C\cdot\frac{1}{k}.$$

Therefore, we obtain

$$\begin{aligned}\mathscr{V}(\lambda_1,\dots,\lambda_n) &= \lim_{k\to\infty}\frac{1}{k^d}\mathscr{Q}(k\lambda_1,\dots,k\lambda_n) = \lim_{k\to\infty}\frac{1}{k^d}\sum_{\mu\in P_+^m}\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}}\\ &= \lim_{k\to\infty}|P^\vee/Q^\vee|\sum_{\mu\in P_+^t} T(\mu) = |P^\vee/Q^\vee|\sum_{\mu\in P_+} T(\mu)\end{aligned}$$

as claimed. □

Remark 7.8. We will also verify the existence of the limit $\lim_{k\to\infty}\sum_{\mu\in P_+^t} T(\mu) = \sum_{\mu\in P_+} T(\mu)$ in due course. See Remark 7.18.

7.2.2. Proof of Proposition 7.6.

Throughout this Section 7.2.2, let us suppose $n \geq l+3$ and $0<\delta<1/h$, and let k be a positive integer such that $m > g$. Due to Lemma 6.11 (1) and Proposition 6.13, we have

$$|P^\vee/Q^\vee|\sum_{\mu\in P_+^t}\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}} = \sum_{\mu\in P_+^t\cup\bigcup_{j\in J}\gamma_j^m(P_+^t)}\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}},$$

and hence

$$\begin{aligned}\sum_{\mu\in P_+^m}\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}}-|P^\vee/Q^\vee|\sum_{\mu\in P_+^t}\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}}\\ =\sum_{\mu\in P_+^m-(P_+^t\cup\bigcup_{j\in J}\gamma_j^m(P_+^t))}\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}}.\end{aligned}$$

Therefore, it is enough to show the following in order to prove Proposition 7.6.

Claim 7.9. *There exists a positive constant C_1 such that for any k, one has*

$$\sum_{\mu\in P_+^m-(P_+^t\cup\bigcup_{j\in J}\gamma_j^m(P_+^t))}\left|\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}}\right|\le C_1\cdot k^{d-1}.$$

The proof of this claim will be given after several lemmas.

Lemma 7.10. *There exists a positive constant C such that for any $\mu\in P_+^m-(P_+^t\cup\bigcup_{j\in J}\gamma_j^m(P_+^t))$ and m with $m>g$, one has*

$$\prod_{\alpha\in\Delta_+}\frac{1}{2\sin\frac{\pi(\mu+\rho|\alpha)}{m+g}}\le C\cdot m^{|\Delta_+|-1}\,(=CL^{|\Delta_+|-1}\cdot k^{|\Delta_+|-1}).$$

Proof. By Lemma 6.16, we see that for any $\alpha\in\Delta_+$,

$$\sin\frac{\pi(\mu+\rho|\alpha)}{m+g}>\frac{2}{3(m+g)}.$$

In addition, by Corollary 6.19 there exists at least one $\alpha\in\Delta_+$ such that

$$\sin\frac{\pi(\mu+\rho|\alpha)}{m+g}>\sin\frac{\pi\delta}{2}.$$

It thus follows that

$$\prod_{\alpha\in\Delta_+}\frac{1}{2\sin\frac{\pi(\mu+\rho|\alpha)}{m+g}}<\left(\frac{3}{4}(m+g)\right)^{|\Delta_+|-1}\frac{1}{2\sin\frac{\pi\delta}{2}}<C\cdot m^{|\Delta_+|-1}$$

for any $\mu\in P_+^m-\left(P_+^t\cup\bigcup_{j\in J}\gamma_j^m(P_+^t)\right)$, where $C=(3/2)^{|\Delta_+|-1}/(2\sin(\pi\delta/2))$. □

Lemma 7.11. *There exists a positive constant C such that for any k and $\mu \in P_+^m - (P_+^t \cup \bigcup_{j\in J} \gamma_j^m(P_+^t))$, one has*

$$\left|\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}}\right| \leq C\cdot k^{d-(n-2)}. \tag{7.10}$$

Proof. Lemma 7.10 above shows that there exists a positive constant C' such that

$$\frac{1}{a(\mu)} = \left|P/(m+g)Q^\vee\right|^{\frac{1}{2}} \prod_{\alpha\in\Delta_+} \frac{1}{2\sin\frac{\pi(\mu+\rho|\alpha)}{m+g}} \leq C'\cdot k^{|\Delta_+|+\frac{l}{2}-1}. \tag{7.11}$$

On the other hand, there exist a positive constant C'' such that for any $\mu\in P_+^m$ and each $i=1,\dots,n$,

$$|a(k\lambda_i,\mu)| \leq C''\cdot k^{-\frac{l}{2}} \tag{7.12}$$

holds, since

$$\begin{aligned}|a(k\lambda_i,\mu)| &= |P/Q^\vee|^{-\frac{1}{2}}(m+g)^{-\frac{l}{2}}\left|A_{\mu+\rho}\left(-2\pi\sqrt{-1}\,\frac{k\lambda_i+\rho}{m+g}\right)\right| \\ &\leq |P/Q^\vee|^{-\frac{1}{2}}(m+g)^{-\frac{l}{2}}|W|.\end{aligned}$$

Here note that

$$|A_{\mu+\rho}(-2\pi\sqrt{-1}x)| = \left|\sum_{w\in W}\varepsilon(w)e^{-2\pi\sqrt{-1}(w(\mu+\rho)|x)}\right| \leq |W|$$

for $x\in\mathfrak{h}_{\boldsymbol{R}}^*$. It follows from (7.11) and (7.12) that

$$\left|\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}}\right| \leq C\cdot k^{(n-2)\left(|\Delta_+|+\frac{l}{2}-1\right)-\frac{l}{2}n} = C\cdot k^{d-(n-2)}$$

with $C=(C')^{n-2}(C'')^n$. ☐

Now we are in a position to prove Claim 7.9.

Proof of Claim 7.9. Due to the assumption $n \geq l+3$, (7.10) implies that

$$\left|\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}}\right| \leq C \cdot k^{d-l-1}.$$

Since there exists a positive constant C' such that

$$\left|P_+^m - \left(P_+^t \cup \bigcup_{j\in J} \gamma_j^m(P_+^t)\right)\right| \leq |P_+^m| \leq C' \cdot m^l = C'L^l \cdot k^l,$$

we obtain

$$\sum_{\mu \in P_+^m - (P_+^t \cup \bigcup_{j\in J} \gamma_j^m(P_+^t))} \left|\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}}\right| \leq C \cdot k^{d-l-1} \cdot C'L^l \cdot k^l = CC'L^l \cdot k^{d-1}$$

as claimed. □

7.2.3. Proof of Proposition 7.7.

Throughout this Section 7.2.3, let us suppose $n \geq 5$ and $0 < \delta < 1/2$, and let k be a positive integer such that $m > (g-2)/(1-2\delta)$. In order to prove Proposition 7.7, it is enough to show that there exists a positive constant C_2 such that for any k

$$k \cdot \sum_{\mu \in P_+^t} \left|\frac{1}{k^d}\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}} - T(\mu)\right| \leq C_2$$

holds. Equivalently, we will show the following.

Claim 7.12. *There exists a positive constant C_2 such that for any k, one has*

$$(m+g) \cdot \sum_{\mu \in P_+^t} \left|\frac{1}{k^d}\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}} - T(\mu)\right| \leq C_2.$$

The proof will be completed at the end of Section 7.2.3. From (5.2) and (5.3), we see that

$$\frac{1}{k^d}\cdot\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}}=\frac{(\sqrt{-1})^{n|\Delta_+|}}{|P/Q^\vee|}\cdot\frac{1}{k^d(m+g)^l}\cdot\frac{\prod_{i=1}^n A_{\mu+\rho}\Big(-2\pi\sqrt{-1}\,\frac{k\lambda_i+\rho}{m+g}\Big)}{\Big(\prod_{\alpha\in\Delta_+}2\sin\frac{\pi(\mu+\rho|\alpha)}{m+g}\Big)^{n-2}}$$
$$=\frac{(\sqrt{-1})^{n|\Delta_+|}}{|P/Q^\vee|}\cdot\frac{(m+g)^d}{k^d}\cdot\frac{\prod_{i=1}^n A_{\mu+\rho}\Big(-2\pi\sqrt{-1}\,\frac{k\lambda_i+\rho}{m+g}\Big)}{\Big(\prod_{\alpha\in\Delta_+}2(m+g)\sin\frac{\pi(\mu+\rho|\alpha)}{m+g}\Big)^{n-2}}.$$

This together with the definition (7.5) of $T(\mu)$ shows that

$$\frac{|P/Q^\vee|}{(\sqrt{-1})^{n|\Delta_+|}}\left(\frac{1}{k^d}\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}}-T(\mu)\right)$$
$$=\frac{(m+g)^d}{k^d}\cdot\frac{\prod_{i=1}^n A_{\mu+\rho}\Big(-2\pi\sqrt{-1}\,\frac{k\lambda_i+\rho}{m+g}\Big)}{\Big(\prod_{\alpha\in\Delta_+}2(m+g)\sin\frac{\pi(\mu+\rho|\alpha)}{m+g}\Big)^{n-2}}-L^d\,\frac{\prod_{i=1}^n A_{\mu+\rho}\Big(\frac{-2\pi\sqrt{-1}\lambda_i}{L}\Big)}{\Big(\prod_{\alpha\in\Delta_+}2\pi(\mu+\rho|\alpha)\Big)^{n-2}}.$$

For simplicity, let us set $N=|\Delta_+|$ and denote

$$A=\frac{(m+g)^d}{k^d},\quad R_i=A_{\mu+\rho}\bigg(-2\pi\sqrt{-1}\,\frac{k\lambda_i+\rho}{m+g}\bigg),\quad X_j=\frac{1}{\Big(2(m+g)\sin\frac{\pi(\mu+\rho|\alpha_j)}{m+g}\Big)^{n-2}},$$
$$B=L^d,\quad S_i=A_{\mu+\rho}\left(\frac{-2\pi\sqrt{-1}\lambda_i}{L}\right),\quad Y_j=\frac{1}{\big(2\pi(\mu+\rho|\alpha_j)\big)^{n-2}},$$

where $i\in\{1,\dots,n\}$ and $j\in\{1,\dots,N\}$. Then

$$|P/Q^\vee|\cdot\left|\frac{1}{k^d}\cdot\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}}-T(\mu)\right|=\left|A\prod_{i=1}^n R_i\prod_{j=1}^N X_j-B\prod_{i=1}^n S_i\prod_{j=1}^N Y_j\right|$$
$$\le|A-B|\prod_{i=1}^n|R_i|\prod_{j=1}^N X_j+\sum_{p=1}^n B\left(\prod_{i=1}^{p-1}|S_i|\right)|R_p-S_p|\left(\prod_{i=p+1}^n|R_i|\right)\prod_{j=1}^N X_j$$
$$+\sum_{q=1}^N B\prod_{i=1}^n|S_i|\left(\prod_{j=1}^{q-1}Y_j\right)|X_q-Y_q|\prod_{j=q+1}^N X_j. \tag{7.13}$$

We have to estimate each term in (7.13) after multiplying it by $m+g$. First, observe the following.

Lemma 7.13. *There exists a positive constant C such that for any $\mu \in P_+^t$, i, j, and k, one has*

$$(0<)\, B \leq C, \quad |R_i| \leq C, \quad |S_i| \leq C, \quad (0<)\, X_j \leq C \cdot Y_j.$$

Proof. The first one is obvious. As in the proof of Lemma 7.11, we see $|R_i| \leq |W|$ and $|S_i| \leq |W|$. Finally, by Lemma 6.20 (2), we have $X_j \leq (\pi/2)^{n-2} Y_j$. □

Lemma 7.14. *There exists a positive constant C such that for any positive integer k, one has $(m+g)|A-B| \leq C$.*

Proof. Let us set $x = g/m$. One has $0 < x \leq g$ by recalling that m is a positive integer. We now see that

$$(m+g)|A-B| = gL^d(1+x)\,\frac{(1+x)^d - 1}{x} \leq gL^d(1+g)\,\frac{(1+g)^d - 1}{g},$$

since the function $(1+x)((1+x)^d - 1)/x$ is monotone increasing for $x > 0$. □

Lemma 7.15. *There exists a positive constant C such that for any $\mu \in P_+^t$, i, and k, one has*

$$(m+g)|R_i - S_i| \leq C(2\pi(\mu+\rho|\alpha_1) + \cdots + 2\pi(\mu+\rho|\alpha_l)).$$

Proof. We have

$$\begin{aligned}
&(m+g)|R_i - S_i| \\
&\quad \leq (m+g)\sum_{w\in W}\left|\varepsilon(w)e^{-2\pi\sqrt{-1}(w(\mu+\rho)|\lambda_i/L)}\left(e^{-2\pi\sqrt{-1}(w(\mu+\rho)|(k\lambda_i+\rho)/(m+g)-\lambda_i/L)} - 1\right)\right| \\
&\quad = \sum_{w\in W}\left|(m+g)\left(\exp\frac{-2\pi\sqrt{-1}}{m+g}\left(w(\mu+\rho)\middle|\rho - \frac{g}{L}\lambda_i\right) - 1\right)\right|. \qquad (7.14)
\end{aligned}$$

Since $|(e^{-2\pi\sqrt{-1}xt} - 1)/t| \leq 2\pi|x|$ for $x, t \in \boldsymbol{R}$ with $xt \neq 0$, the right-hand side of (7.14) is not greater than

$$\sum_{w\in W} 2\pi\left|\left(w(\mu+\rho)\;\middle|\;\rho - \frac{g}{L}\lambda_i\right)\right| = \sum_{w\in W} 2\pi\left|\left(\mu+\rho\;\middle|\;w^{-1}\left(\rho - \frac{g}{L}\lambda_i\right)\right)\right|.$$

Let us write $w^{-1}(\rho - (g/L)\lambda_i) = p_{i1}(w)\alpha_1 + \cdots + p_{il}(w)\alpha_l$ and let $C =$

$\max_{w,i,j} |p_{ij}(w)|/|W|$. Then we obtain

$$\sum_{w\in W} 2\pi \left| \left(\mu+\rho \middle| w^{-1}\left(\rho - \frac{g}{L}\lambda_i\right)\right)\right| \le 2\pi C((\mu+\rho|\alpha_1) + \cdots + (\mu+\rho|\alpha_l))$$

as claimed. □

Lemma 7.16. *There exists a positive constant C such that for any $\mu \in P_+^t$, j, and m with $m > (g-2)/(1-2\delta)$, one has*

$$(m+g)(X_j - Y_j) \le C \cdot \frac{1}{(2\pi(\mu+\rho|\alpha_j))^{n-3}}.$$

Proof. By Lemma 6.16, one has

$$\frac{3}{4} > \frac{1}{2(m+g)\sin\frac{\pi(\mu+\rho|\alpha)}{m+g}} > \frac{1}{2\pi(\mu+\rho|\alpha)} > 0$$

for any $\alpha \in \Delta_+$. Since

$$x^N - y^N = (x-y)(x^{N-1} + x^{N-2}y + \cdots + y^{N-1}) \le (x-y)Nx^{N-1}$$

for $1 > x \ge y > 0$, we have

$$\begin{aligned}(m+g)&\left(\frac{1}{\left(2(m+g)\sin\frac{\pi(\mu+\rho|\alpha)}{m+g}\right)^{n-2}} - \frac{1}{(2\pi(\mu+\rho|\alpha))^{n-2}}\right)\\ &\le (m+g)\left(\frac{1}{2(m+g)\sin\frac{\pi(\mu+\rho|\alpha)}{m+g}} - \frac{1}{2\pi(\mu+\rho|\alpha)}\right)\frac{n-2}{\left(2(m+g)\sin\frac{\pi(\mu+\rho|\alpha)}{m+g}\right)^{n-3}}.\end{aligned}$$

We see

$$(m+g)\left(\frac{1}{2(m+g)\sin\frac{\pi(\mu+\rho|\alpha)}{m+g}} - \frac{1}{2\pi(\mu+\rho|\alpha)}\right) < \frac{1}{2}\left(1 - \frac{2}{\pi}\right)$$

by Lemma 6.20 (3), while

$$\frac{1}{\left(2(m+g)\sin\frac{\pi(\mu+\rho|\alpha)}{m+g}\right)^{n-3}} < \left(\frac{\pi}{2}\right)^{n-3}\frac{1}{(2\pi(\mu+\rho|\alpha))^{n-3}}$$

by Lemma 6.20 (2). Therefore, by setting $C = (1-2/\pi)(n/2-1)(\pi/2)^{n-3}$ we obtain the conclusion. □

LEMMA 7.17. *If $s > 1$, the series $\displaystyle\sum_{\mu\in P_+}\frac{1}{\left(\prod_{\alpha\in\Delta_+}2\pi(\mu+\rho|\alpha)\right)^s}$ converges.*

PROOF. Recall from (2.4) that $0 < 1/(2\pi(\mu+\rho|\alpha)) \le 3/(2\pi) < 1$ for any $\mu \in P_+$ and $\alpha \in \Delta_+$. It implies

$$0 < \frac{1}{\left(\prod_{\alpha\in\Delta_+}2\pi(\mu+\rho|\alpha)\right)^s} \le \frac{1}{\left(\prod_{i=1}^{l}2\pi(\mu+\rho|\alpha_i)\right)^s}.$$

Therefore, it is enough to show that the series $\displaystyle\sum_{\mu\in P_+}\frac{1}{\left(\prod_{i=1}^{l}2\pi(\mu+\rho|\alpha_i)\right)^s}$ converges. Let us set $\mu+\rho = m_1\Lambda_1+\cdots+m_l\Lambda_l$. If μ runs over P_+, the tuple $(m_1,\ldots,m_l)$ runs over $(\boldsymbol{Z}_{>0})^l$. In view of (2.3) we have

$$2\pi(\mu+\rho|\alpha_i) = 2\pi m_i(\Lambda_i|\alpha_i) = \pi m_i(\alpha_i|\alpha_i) \ge \frac{2\pi}{3}m_i > m_i$$

for $i = 1,\ldots,l$, and hence

$$0 < \frac{1}{\left(\prod_{i=1}^{l}2\pi(\mu+\rho|\alpha_i)\right)^s} < \frac{1}{(m_1\cdots m_l)^s}.$$

Since $\displaystyle\sum_{(m_1,\ldots,m_l)\in(\boldsymbol{Z}_{>0})^l}\frac{1}{(m_1\cdots m_l)^s}$ indeed converges, so does $\displaystyle\sum_{\mu\in P_+}\frac{1}{\left(\prod_{i=1}^{l}2\pi(\mu+\rho|\alpha_i)\right)^s}$. □

REMARK 7.18. This lemma also shows that the convergence of the series $\sum_{\mu\in P_+}T(\mu)$, since

$$|T(\mu)| \leq C \cdot \frac{1}{\left(\prod_{\alpha\in\Delta_+} 2\pi(\mu+\rho|\alpha)\right)^{n-2}}$$

for some positive constant C.

Now we are ready to prove Claim 7.12.

PROOF OF CLAIM 7.12. By the inequality (7.13) and a series of the lemmas above, we see that there exist positive constants C and D such that

$$\begin{aligned}
&(m+g)\left|\frac{1}{k^d}\cdot\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}} - T(\mu)\right| \\
&\leq C\left(\prod_{j=1}^{N} Y_j\right)\left(1 + 2\pi(\mu+\rho|\alpha_1) + \cdots + 2\pi(\mu+\rho|\alpha_l) + \sum_{q=1}^{N}\frac{1}{Y_q(2\pi(\mu+\rho|\alpha_q))^{n-3}}\right) \\
&\leq C\cdot\frac{1}{\prod_{j=1}^{N}(2\pi(\mu+\rho|\alpha_j))^{n-2}} + D\cdot\frac{1}{\prod_{j=1}^{N}(2\pi(\mu+\rho|\alpha_j))^{n-3}}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&(m+g)\sum_{\mu\in P_+^t}\left|\frac{1}{k^d}\cdot\frac{a(k\lambda_1,\mu)\cdots a(k\lambda_n,\mu)}{a(\mu)^{n-2}} - T(\mu)\right| \\
&\leq C\sum_{\mu\in P_+^t}\frac{1}{\prod_{j=1}^{N}(2\pi(\mu+\rho|\alpha_j))^{n-2}} + D\sum_{\mu\in P_+^t}\frac{1}{\prod_{j=1}^{N}(2\pi(\mu+\rho|\alpha_j))^{n-3}} \\
&\leq C\sum_{\mu\in P_+}\frac{1}{\prod_{j=1}^{N}(2\pi(\mu+\rho|\alpha_j))^{n-2}} + D\sum_{\mu\in P_+}\frac{1}{\prod_{j=1}^{N}(2\pi(\mu+\rho|\alpha_j))^{n-3}}.
\end{aligned}$$

It follows from Lemma 7.17 that the right-hand side is finite if $n-3\geq 2$. □

7.3. Examples.

In order to make the formula in Theorem 7.3 more explicit, we need to write out the value of the character $\chi_\mu\big(\exp(-2\pi\sqrt{-1}\lambda_i/L)\big)$ or $A_{\mu+\rho}\big(-2\pi\sqrt{-1}\lambda_i/L\big)$, which itself seems to be quite complicated in general. Here, let us consider the very special case that $\lambda_i = m_i\rho$ with $m_i \in \boldsymbol{Z}_{>0}$ for all $i=1,\ldots,n$. In this case, we can apply the Weyl denominator theorem for $A_{\mu+\rho}\big(-2\pi\sqrt{-1}\lambda_i/L\big)$ in (7.6). Namely, it is equal to

$$A_\rho\left(\frac{-2\pi\sqrt{-1}m_i(\mu+\rho)}{L}\right)=(-\sqrt{-1})^{|\Delta_+|}\prod_{\alpha\in\Delta_+}2\sin\frac{\pi m_i(\mu+\rho|\alpha)}{L}.$$

Let $M=m_1+\cdots+m_n$. Since $L=M(\rho|\theta)=M(g-1)$, we obtain the following.

Corollary 7.19. *Suppose $\lambda_i=m_i\rho$ with $m_i\in\boldsymbol{Z}_{>0}$ and $\lambda_i\in Q$ for all $i=1,\dots,n$. Then we have*

$$\mathscr{V}(\lambda_1,\dots,\lambda_n)=\frac{|P^\vee/Q^\vee|}{|P/Q^\vee|}(M(g-1))^d\sum_{\mu\in P_+}\frac{\prod_{i=1}^n\prod_{\alpha\in\Delta_+}2\sin\frac{\pi m_i(\mu+\rho|\alpha)}{M(g-1)}}{\left(\prod_{\alpha\in\Delta_+}2\pi(\mu+\rho|\alpha)\right)^{n-2}},$$

where $d=(n-2)|\Delta_+|-l$ and $M=m_1+\cdots+m_n$.

Let us consider as typical examples the cases that the root system Δ is of type A_1 or A_2.

Example 7.20. When Δ is of type A_1, we see $\Delta_+=\{\alpha_1\}$, $\Lambda_1=\rho=\alpha_1/2$, $\theta=\alpha_1$, $g=2$, $(\alpha_1|\alpha_1)=2$, $P^\vee=P$, and $Q^\vee=Q$. Let us apply Corollary 7.19 to $\lambda_1=m_1\rho,\dots,\lambda_n=m_n\rho$, where $m_i\in 2\boldsymbol{Z}_{>0}$. For $\mu\in P_+=\boldsymbol{Z}_{\geq 0}\Lambda_1$, let us set $\mu+\rho=p\Lambda_1$ with $p\in\boldsymbol{Z}_{>0}$. Then we have

$$\mathscr{V}(\lambda_1,\dots,\lambda_n)=4M^{n-3}\sum_{p=1}^{\infty}\frac{\prod_{i=1}^n\sin\frac{\pi m_i p}{M}}{(\pi p)^{n-2}}, \tag{7.15}$$

which has already appeared in [**23**] and [**15**]. It is interesting to compare it with the result (4.14) in Example 4.14. Moreover in view of Remark 7.4 (1), by taking the limit as $L\to\infty$, we can also express the right-hand side of (7.15) as an integral as follows:

$$\frac{4}{\pi}\int_0^\infty\frac{\prod_{i=1}^n\sin m_i x}{x^{n-2}}\,dx.$$

Example 7.21. When Δ is of type A_2, $\Delta_+=\{\alpha_1,\alpha_2,\alpha_1+\alpha_2\}$, $\Lambda_1=(2\alpha_1+\alpha_2)/3$, $\Lambda_2=(\alpha_1+2\alpha_2)/3$, $\rho=\theta=\alpha_1+\alpha_2$, $g=3$, and

$$(\alpha_1|\alpha_1)=2,\quad(\alpha_2|\alpha_2)=2,\quad(\alpha_1|\alpha_2)=-1.$$

Note that $P^\vee=P$ and $Q^\vee=Q$. Let us apply Corollary 7.19 to $\lambda_1=m_1\rho,\dots,$

$\lambda_n = m_n\rho$, where $m_i \in \boldsymbol{Z}_{>0}$. For $\mu \in P_+$, let us set $\mu + \rho = p\Lambda_1 + q\Lambda_2$ with $p, q \in \boldsymbol{Z}_{>0}$. Then we have

$$\mathscr{V}(\lambda_1, \ldots, \lambda_n) = 2^6 \cdot (2M)^{3n-8} \sum_{p,q \in \boldsymbol{Z}_{>0}} \frac{\prod_{i=1}^{n} \sin \frac{\pi m_i p}{2M} \sin \frac{\pi m_i q}{2M} \sin \frac{\pi m_i (p+q)}{2M}}{(\pi p \cdot \pi q \cdot \pi(p+q))^{n-2}}.$$

As before, the right-hand side can be expressed in the form

$$\frac{2^6}{\pi^2} \int_{x,y \geq 0} \frac{\prod_{i=1}^{n} \sin m_i x \cdot \sin m_i y \cdot \sin m_i (x+y)}{(xy(x+y))^{n-2}} \, dxdy.$$

It would be interesting to compare them with the formula given in [**20**].

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Taro Suzuki
Graduate School of Science and Engineering
Chuo University
1-13-27 Kasuga, Bunkyo-ku
Tokyo 112-8551, Japan
E-mail: suzuki@gug.math.chuo-u.ac.jp

Tatsuru Takakura
Department of Mathematics
Chuo University
1-13-27 Kasuga, Bunkyo-ku
Tokyo 112-8551, Japan
E-mail: takakura@math.chuo-u.ac.jp