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# Some remarks on CM-triviality

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**Abstract.** We show that any rosy CM-trivial theory has weak canonical bases, and CM-triviality in the real sort is equivalent to CM-triviality with geometric elimination of imaginaries. We also show that CM-triviality is equivalent to the modularity in O-minimal theories with elimination of imaginaries.

## 1. Introduction.

CM-triviality is a geometric notion of the forking independence relation. It is introduced by Hrushovski [**H**] where he disproves Zilber's conjecture on strongly minimal sets. CM-triviality forbids a point-line-plane incident system. The usual definition for CM-triviality needs canonical bases of types. Since canonical bases do not necessarily exist in rosy theories as in Lemma 2.8 of [**P1**], from [**H**] we choose another definition for CM-triviality in rosy theories, which does not need canonical bases. In the next section we show that any CM-trivial rosy theory has weak canonical bases. In third section we investigate the geometric elimination of imaginaries by the strict independence relation in rosy theories. Many generic structures have CM-triviality and weak elimination of imaginaries as in [**H**],[**B**], [**Y**],[**VY**] and [**E**]. In fourth section we define *CM-triviality in the real sort*, and we show that CM-triviality in the real sort is equivalent to CM-triviality with geometric elimination of imaginaries in rosy theories. This gives a direct way to show CM-triviality of generic relational structures. We also show that one-basedness implies CM-triviality in rosy theories having weak canonical bases, and we refer to a one-based but non-CM-trivial O-minimal theory. It is known that infinite type-definable stable [**P**] or supersimple [**N**] fields give a witness for non-CM-triviality. In fifth section we check that the Nubling's proof works for superrosy fields of monomial $U^{\mathfrak{p}}$-rank. In Zariski geometries (which are strongly minimal structures having a generalized Zariski topology), CM-triviality is

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equivalent to one-basedness(=local modularity). In O-minimal theories, local modularity is a strictly strong notion to one-basedness(=CF-property) as in [**LP**]. In the last section we show that CM-triviality is equivalent to the modularity in O-minimal theories with elimination of imaginaries, by using Peterzil-Starchenko's trichotomy theorem and Pillay's consideration to weak canonical bases in O-minimal theories. Nubling [**N**] shows that CM-triviality is preserved under reducts in finite U-rank theories. We show that CM-triviality is not preserved under reducts in O-minimal theories. As O-minimal theories are finite $\mathrm{U}^{\mathfrak{p}}$-rank theories, CM-triviality is not preserved under reducts in finite $\mathrm{U}^{\mathfrak{p}}$-rank theories.

Our notation is standard. Let $T$ be a complete $L$-theory, and let $\mathscr{M}$ be the big model of $T$. We work in $\mathscr{M}^{\mathrm{eq}}$, consisting of imaginary elements, which are classes of equivalence relations definable over the empty set. $\bar{a}, \bar{b}, \ldots \subset_\omega \mathscr{M}$ denote finite sequences in $\mathscr{M}^{\mathrm{eq}}$. $A, B, \ldots$ denote small subsets of $\mathscr{M}^{\mathrm{eq}}$ and $AB$ denotes $A \cup B$. For $a \in \mathscr{M}^{\mathrm{eq}}$ and $A \subset \mathscr{M}^{\mathrm{eq}}$, we write $a \in \mathrm{dcl}^{\mathrm{eq}}(A)$ if $a$ is fixed by any automorphism fixing $A$ pointwise. And we write $a \in \mathrm{acl}^{\mathrm{eq}}(A)$ if the orbit of $a$ by automorphisms fixing $A$ pointwise is finite. We write $B \equiv_A C$ for $\mathrm{tp}(B/A) = \mathrm{tp}(C/A)$ in $T^{\mathrm{eq}}$. For definitions and basic properties of rosy theories, we refer the reader to [**A**] and [**O**]. The author would like to thank the referee for his/her kind comments.

## 2. The existence of weak canonical bases in rosy CM-trivial theories.

Following [**A**], recall that a ternary relation $* \downarrow_* *$ between small subsets of $\mathscr{M}^{\mathrm{eq}}$ is a *strict independence relation* if the following nine conditions hold.

(1) invariance: If $A \downarrow_B C$ and $ABC \equiv A'B'C'$, then $A' \downarrow_{B'} C'$.
(2) monotonicity: If $A \downarrow_B C$, $A' \subseteq A$ and $C' \subseteq C$, then $A' \downarrow_B C'$.
(3) (right) base monotonicity: If $A \downarrow_B D$ and $B \subseteq C \subseteq D$, then $A \downarrow_C D$.
(4) (left) transitivity: If $B \subseteq C \subseteq D$, $D \downarrow_C A$ and $C \downarrow_B A$, then $D \downarrow_B A$.
(5) (left) normality: $A \downarrow_B C$ implies $AB \downarrow_B C$.
(6) extension: If $A \downarrow_B C$ and $C \subseteq D$, then there exists $A'(\equiv_{BC} A)$ such that $A' \downarrow_B D$.
(7) (left) finite character: If $\bar{a} \downarrow_B C$ for each $\bar{a} \subset_\omega A$, then $A \downarrow_B C$.
(8) local character: For any $A$ there is a cardinal $\kappa(A)$ such that, for any $B$ there exists $B_0 \subseteq B$ with $|B_0| < \kappa(A)$ and $A \downarrow_{B_0} B$.
(9) anti-reflexivity: $A \downarrow_B A$ implies $A \subseteq \mathrm{acl}^{\mathrm{eq}}(B)$.

Note that (1)-(8) imply symmetry : $A \downarrow_B C \Leftrightarrow B \downarrow_A C$.
(Theorem 1.14 in [**A**])

REMARK 2.1. Let $A, B, C, A', B', C' \subset \mathscr{M}^{\mathrm{eq}}$ be such that $\mathrm{acl}^{\mathrm{eq}}(A') = \mathrm{acl}^{\mathrm{eq}}(A), \mathrm{acl}^{\mathrm{eq}}(B') = \mathrm{acl}^{\mathrm{eq}}(B), \mathrm{acl}^{\mathrm{eq}}(C') = \mathrm{acl}^{\mathrm{eq}}(C)$. Then $A \mathop{\smile\!\!\!\!\!\!\mid}_B C \Leftrightarrow A' \mathop{\smile\!\!\!\!\!\!\mid}_{B'} C'$.

PROOF. Suppose $A \mathop{\smile\!\!\!\!\!\!\mid}_B C$. By symmetry and normality, we may assume $B \subseteq C, B' \subseteq C'$. By local character and base monotonicity, for any $A, D$, we have $A \mathop{\smile\!\!\!\!\!\!\mid}_D D$. By extension and invariance, we have $A \mathop{\smile\!\!\!\!\!\!\mid}_D \mathrm{acl}^{\mathrm{eq}}(D)$. So, by symmetry and transitivity, we have $A \mathop{\smile\!\!\!\!\!\!\mid}_{B'} \mathrm{acl}^{\mathrm{eq}}(C')$. By monotonicity again, we see $A \mathop{\smile\!\!\!\!\!\!\mid}_{B'} C'$. By symmetry, we also see $A' \mathop{\smile\!\!\!\!\!\!\mid}_{B'} C'$. □

We say that $T$ is *rosy* if there exists a strict independence relation on $\mathscr{M}^{\mathrm{eq}}$. And we say that an algebraically closed set $C$ is the $\mathop{\smile\!\!\!\!\!\!\mid}$-*weak canonical base* of $\mathrm{tp}(\bar{a}/B)$ if $C$ is the smallest algebraically closed subset of $\mathrm{acl}^{\mathrm{eq}}(B)$ with $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_C B$. As in [**A**], $\mathrm{wcb}_{\mathop{\smile\!\!\!\!\!\!\mid}}(\bar{a}/B)$ denotes the $\mathop{\smile\!\!\!\!\!\!\mid}$-weak canonical base of $\mathrm{tp}(\bar{a}/B)$ if it exists. We also say that a rosy theory $T$ has the $\mathop{\smile\!\!\!\!\!\!\mid}$-weak canonical bases if there exists the $\mathop{\smile\!\!\!\!\!\!\mid}$-weak canonical base for each type.

FACT 2.2. *Let $\mathop{\smile\!\!\!\!\!\!\mid}$ be a strict independence relation on $\mathscr{M}^{\mathrm{eq}}$.*

(1) *Any type has the $\mathop{\smile\!\!\!\!\!\!\mid}$-weak canonical base if and only if $\mathop{\smile\!\!\!\!\!\!\mid}$ has the* eq-*intersection property* : *$\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_A B$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_B A$ imply $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_{A \cap B} AB$ for any $\bar{a}, A, B \subset \mathscr{M}^{\mathrm{eq}}$ such that $A = \mathrm{acl}^{\mathrm{eq}}(A)$ and $B = \mathrm{acl}^{\mathrm{eq}}(B)$.* (*Theorem* 3.20 *in* [**A**])

(2) *If $\mathop{\smile\!\!\!\!\!\!\mid}$ has the* eq-*intersection property, then $\mathop{\smile\!\!\!\!\!\!\mid}$ coincides with the thorn independence relation.* (*Theorem* 3.3 *in* [**A**])

Suppose that $\mathop{\smile\!\!\!\!\!\!\mid}$ is a strict independence relation on eq-structures. For now, we do not assume the existence of $\mathop{\smile\!\!\!\!\!\!\mid}$-weak canonical bases, we choose the definition for CM-triviality as follows.

DEFINITION 2.3. We say that a rosy theory $T$ is *CM-trivial* with respect to $\mathop{\smile\!\!\!\!\!\!\mid}$ if $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_A B$ implies $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_{A \cap \mathrm{acl}^{\mathrm{eq}}(\bar{a}, B)} B$ for any $\bar{a}, A, B \subset \mathscr{M}^{\mathrm{eq}}$ such that $A = \mathrm{acl}^{\mathrm{eq}}(A)$ and $B = \mathrm{acl}^{\mathrm{eq}}(B)$.

THEOREM 2.4. *If $T$ is CM-trivial with respect to $\mathop{\smile\!\!\!\!\!\!\mid}$, then $T$ has the $\mathop{\smile\!\!\!\!\!\!\mid}$-weak canonical bases, and $\mathop{\smile\!\!\!\!\!\!\mid}$ coincides with the thorn independence relation.*

PROOF. To apply Fact 2.2, we show that $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_A B$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_B A$ with $A = \mathrm{acl}^{\mathrm{eq}}(A)$ and $B = \mathrm{acl}^{\mathrm{eq}}(B)$ imply $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_{A \cap B} AB$. By CM-triviality, we have $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{acl}^{\mathrm{eq}}(\bar{a}, B) \cap A} B$. By $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_B A$ and anti-reflexivity, we see $\mathrm{acl}^{\mathrm{eq}}(\bar{a}, B) \cap AB = B$. As $A \cap B \subseteq A \cap \mathrm{acl}^{\mathrm{eq}}(\bar{a}, B) \subseteq AB \cap \mathrm{acl}^{\mathrm{eq}}(\bar{a}, B) = B$, we see

$$\mathrm{acl}^{\mathrm{eq}}(\bar{a}, B) \cap A = A \cap B.$$

By $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}^{\mathrm{eq}}(\bar{a},B)\cap A} B$ and $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_B A$, we see $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{A\cap B} AB$. ☐

REMARK 2.5. Let $T$ be a rosy theory with a strict independence relation $\mathop{\smile\!\!\!\!\!\mid}$. The following are equivalent.

(1) $T$ is CM-trivial with respect to $\mathop{\smile\!\!\!\!\!\mid}$.

(2) $T$ has the $\mathop{\smile\!\!\!\!\!\mid}$-weak canonical bases and $\mathrm{wcb}_{\mathop{\smile\!\!\!\!\!\mid}}(\bar{a}/A) \subseteq \mathrm{wcb}_{\mathop{\smile\!\!\!\!\!\mid}}(\bar{a}/B)$ holds for any $\bar{a}, A, B \subset \mathscr{M}^{\mathrm{eq}}$ such that $\mathrm{acl}^{\mathrm{eq}}(\bar{a}, A) \cap B = A$ with $A = \mathrm{acl}^{\mathrm{eq}}(A)$ and $B = \mathrm{acl}^{\mathrm{eq}}(B)$.

PROOF.

(1)⇒(2): Suppose that $\mathrm{acl}^{\mathrm{eq}}(\bar{a}, A) \cap B = A$ with $A = \mathrm{acl}^{\mathrm{eq}}(A)$ and $B = \mathrm{acl}^{\mathrm{eq}}(B)$. By Theorem 2.4, $T$ has weak canonical bases, so let $D := \mathrm{wcb}_{\mathop{\smile\!\!\!\!\!\mid}}(\bar{a}/B)$. Then $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_D A$ follows from $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_D A$ and $A \subseteq B$. By CM-triviality, we see $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}^{\mathrm{eq}}(\bar{a},A)\cap D} A$. As $D \subseteq B$ and $\mathrm{acl}^{\mathrm{eq}}(\bar{a}, A) \cap B = A$, we have $\mathrm{acl}^{\mathrm{eq}}(\bar{a}, A) \cap D = A \cap D$. So, we have $\mathrm{wcb}_{\mathop{\smile\!\!\!\!\!\mid}}(\bar{a}/A) \subseteq A \cap D \subseteq D = \mathrm{wcb}_{\mathop{\smile\!\!\!\!\!\mid}}(\bar{a}/B)$.

(2)⇒(1): Suppose that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A B$ with $A = \mathrm{acl}^{\mathrm{eq}}(A)$ and $B = \mathrm{acl}^{\mathrm{eq}}(B)$. Put $C := \mathrm{acl}^{\mathrm{eq}}(AB) \cap \mathrm{acl}^{\mathrm{eq}}(\bar{a}, B)$. Then we have $B \subseteq C$ and $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A C$. As $\mathrm{acl}^{\mathrm{eq}}(\bar{a}, C) \subseteq \mathrm{acl}^{\mathrm{eq}}(\bar{a}, AB) \cap \mathrm{acl}^{\mathrm{eq}}(\bar{a}, B)$ and $\mathrm{acl}^{\mathrm{eq}}(CA) \subseteq \mathrm{acl}^{\mathrm{eq}}(AB) \cap \mathrm{acl}^{\mathrm{eq}}(\bar{a}, AB)$, we see $C = \mathrm{acl}^{\mathrm{eq}}(\bar{a}, C) \cap \mathrm{acl}^{\mathrm{eq}}(CA)$. By our assumption, we have $\mathrm{wcb}_{\mathop{\smile\!\!\!\!\!\mid}}(\bar{a}/C) \subseteq \mathrm{wcb}_{\mathop{\smile\!\!\!\!\!\mid}}(\bar{a}/CA) = \mathrm{wcb}_{\mathop{\smile\!\!\!\!\!\mid}}(\bar{a}/A)$. As $\mathrm{wcb}_{\mathop{\smile\!\!\!\!\!\mid}}(\bar{a}/C) \subseteq C \cap A = \mathrm{acl}^{\mathrm{eq}}(\bar{a}, B) \cap A$, we see $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}^{\mathrm{eq}}(\bar{a},B)\cap A} C$. As $B \subseteq C$, we have $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}^{\mathrm{eq}}(\bar{a},B)\cap A} B$. ☐

## 3. Geometric elimination of imaginaries in rosy theories.

We say that $T$ has geometric elimination of imaginaries ($T$ has GEI) if for any $e \in \mathscr{M}^{\mathrm{eq}}$, there exists $\bar{b} \subset_\omega \mathscr{M}$ such that $e \in \mathrm{acl}^{\mathrm{eq}}(\bar{b})$ and $\bar{b} \in \mathrm{acl}^{\mathrm{eq}}(e)$.

Let $\mathop{\smile\!\!\!\!\!\mid}$ be a strict independence relation on $\mathscr{M}^{\mathrm{eq}}$. We say that $\mathop{\smile\!\!\!\!\!\mid}$ has the intersection property if $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A B$ and $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_B A$ imply $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{A\cap B} AB$ for any $\bar{a}, A, B \subset \mathscr{M}$ with $A = \mathrm{acl}(A)$ and $B = \mathrm{acl}(B)$.

LEMMA 3.1. *If $T$ has a strict independence relation having the intersection property, then $T$ has GEI.*

PROOF. Fix $e = \bar{a}_E \in \mathscr{M}^{\mathrm{eq}}$. Take $\bar{b}, \bar{c} \models \mathrm{tp}(\bar{a}/e)$ such that $\bar{b}, \bar{c}, \bar{a}$ are $\mathop{\smile\!\!\!\!\!\mid}$-independent over $e$. As $e = \bar{b}_E = \bar{c}_E$ and $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_e \bar{b}\bar{c}$, we have $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{\bar{b}} \bar{b}\bar{c}$ and $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{\bar{c}} \bar{b}\bar{c}$. Let $A = \mathrm{acl}(\bar{b}) \cap \mathrm{acl}(\bar{c})$. Then $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A \bar{b}\bar{c}$ by the intersection property of $\mathop{\smile\!\!\!\!\!\mid}$. By $e \in \mathrm{dcl}^{\mathrm{eq}}(\bar{a}) \cap \mathrm{dcl}^{\mathrm{eq}}(\bar{b}\bar{c})$ and anti-reflexivity, $e \in \mathrm{acl}^{\mathrm{eq}}(A)$. On the other hand, $A \subset \mathrm{acl}^{\mathrm{eq}}(e)$ follows from $\bar{b} \mathop{\smile\!\!\!\!\!\mid}_e \bar{c}$ and anti-reflexivity. ☐

LEMMA 3.2. *If $T$ has GEI, then we have*

$$\mathrm{acl}^{\mathrm{eq}}(A) \cap \mathrm{acl}^{\mathrm{eq}}(B) = \mathrm{acl}^{\mathrm{eq}}(A \cap B)$$

*for any $A, B \subset \mathscr{M}$ such that $A = \mathrm{acl}(A)$ and $B = \mathrm{acl}(B)$.*

PROOF. Let $e \in \mathrm{acl}^{\mathrm{eq}}(A) \cap \mathrm{acl}^{\mathrm{eq}}(B)$. By GEI, there exists $\bar{a} \subset_\omega \mathscr{M}$ such that $e \in \mathrm{acl}^{\mathrm{eq}}(\bar{a})$ and $\bar{a} \in \mathrm{acl}^{\mathrm{eq}}(e)$. As $\bar{a} \in \mathrm{acl}^{\mathrm{eq}}(A)$ and $\bar{a} \in \mathrm{acl}^{\mathrm{eq}}(B)$, we see $\bar{a} \subseteq A \cap B$. Thus, $e \in \mathrm{acl}^{\mathrm{eq}}(A \cap B)$. □

LEMMA 3.3. *If $\mathop{\smile\!\!\!\!\!\mid}$ has the intersection property, then it has the* eq*-intersection property.*

PROOF. Suppose that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A B$ and $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_B A$ with $A = \mathrm{acl}^{\mathrm{eq}}(A)$ and $B = \mathrm{acl}^{\mathrm{eq}}(B)$. By 3.1, there exist $\bar{a}', A' = \mathrm{acl}(A'), B' = \mathrm{acl}(B') \subseteq \mathscr{M}$ such that $\mathrm{acl}^{\mathrm{eq}}(\bar{a}') = \mathrm{acl}^{\mathrm{eq}}(\bar{a}), \mathrm{acl}^{\mathrm{eq}}(A') = \mathrm{acl}^{\mathrm{eq}}(A), \mathrm{acl}^{\mathrm{eq}}(B') = \mathrm{acl}^{\mathrm{eq}}(B)$. By remark 2.1, we have $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}_{A'} B'$ and $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}_{B'} A'$. So we see $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}_{A' \cap B'} A'B'$ by the intersection property. Since $A \cap B = \mathrm{acl}^{\mathrm{eq}}(A' \cap B')$ holds by Lemma 3.2, we see $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{A \cap B} AB$ by remark 2.1. □

PROPOSITION 3.4. *The following are equivalent.*

(1) *$T$ has GEI and a strict independence relation having the* eq*-intersection property.*
(2) *$T$ has a strict independence relation having the intersection property.*
(3) *$T$ has a strict independence relation having weak canonical bases in the real sort : weak canonical bases are interalgebraic with real elements.*

PROOF. (1)⇒(2) follows from remark 2.1 and Lemma 3.2. (2)⇒(1) follows from Lemma 3.1 and 3.3. (1)⇒(3) and (3)⇒(2) are clear. □

REMARK 3.5.

(1) Let $T$ be a simple theory with elimination of hyperimaginaries. As the forking independence relation in $T$ has the eq-intersection property, by Fact 2.2, we see that $T$ has GEI iff the forking independence relation in $T$ has the intersection property.

(2) In rosy theories, GEI does not necessarily imply the intersection property: Let $T = \mathrm{Th}(\boldsymbol{R}, +, <, \pi|_{(-1,1)}(*))$, where $\pi|_{(-1,1)}(x) := \pi x$ for $x \in (-1, 1)$. Then $T$ is an o-minimal theory with elimination of imaginaries. Take $a, b, c \in \mathscr{M}$ be such that $a, b, c > \boldsymbol{R}$, $|a - b| < 1, |a - c| < 1$ and $\dim(a, b, c) = 3$. Then $\dim(a, \pi a/b, \pi b, c, \pi c) = \dim(a, \pi a/b, \pi b) = \dim(a, \pi a/c, \pi c) = 1 < 2 = \dim(a, \pi a)$ and $\mathrm{acl}(b, \pi b) \cap \mathrm{acl}(c, \pi c) = \mathrm{acl}(\emptyset)$. As $\mathrm{U}^{\mathfrak{p}}(*) = \dim(*)$ in O-minimal theories by [**O**], the thorn independence relation in $T$ does not have the intersection property.

## 4. CM-triviality in the real sort.

Definition 4.1. We say that $T$ is *CM-trivial in the real sort* with respect to $\mathop{\smile\!\!\!\!\!\mid}$ if $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A B$ implies $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{A \cap \mathrm{acl}(\bar{a},B)} B$ for any $\bar{a}, A, B \subset \mathscr{M}$ such that $A = \mathrm{acl}(A)$ and $B = \mathrm{acl}(B)$.

Theorem 4.2. *The following are equivalent.*

(1) *$T$ is CM-trivial with respect to $\mathop{\smile\!\!\!\!\!\mid}$ and has GEI.*

(2) *$T$ is CM-trivial in the real sort with respect to $\mathop{\smile\!\!\!\!\!\mid}$.*

Proof. (1)⇒(2): Clear. (2)⇒(1): By working in $\mathscr{M}$ and replacing $\mathrm{acl}^{\mathrm{eq}}$ with acl in the proof of Theorem 2.4, we see that $\mathop{\smile\!\!\!\!\!\mid}$ has the intersection property. By Lemma 3.1, GEI follows. □

Remark 4.3.

(1) Let $T$ be the theory of a rosy relational structure with a closure operator $\mathrm{cl}(*)$ and a strict independence relation $\mathop{\smile\!\!\!\!\!\mid}$ such that

- $\mathrm{cl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$ and $\mathrm{cl}(\mathrm{cl}(A) \cap \mathrm{cl}(B)) = \mathrm{cl}(A) \cap \mathrm{cl}(B)$,
- $A \mathop{\smile\!\!\!\!\!\mid}_{A\cap B} B \Leftrightarrow$ "$AB = \mathrm{cl}(AB)$ and $R^{AB} = R^A \cup R^B$ for any predicate $R$" for any algebraically closed sets $A, B \subset \mathscr{M}$.

Then $T$ is CM-trivial: By Theorem 4.2, we have only to show CM-triviality in the real sort. Suppose that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A B$. Let $C = \mathrm{acl}(\bar{a}, A), D = \mathrm{acl}(AB)$. As $C \mathop{\smile\!\!\!\!\!\mid}_A B$ and $C \cap B = A$, $\mathrm{cl}(CB) = CB$ and $R^{CB} = R^C \cup R^B$ for any predicate $R$. Let $E = \mathrm{acl}(\bar{a}, B)$. Then $\mathrm{cl}(CB \cap E) = CB \cap E$ and $R^{CB\cap E} = R^{C\cap E} \cup R^{B\cap E}$ for any predicate $R$. So, we see $C \cap E \mathop{\smile\!\!\!\!\!\mid}_{A\cap E} B \cap E$. As $\bar{a} \subset C \cap E, B \subset B \cap E$, $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{A\cap \mathrm{acl}(\bar{a},B)} B$ follows.

(2) CM-triviality does not imply *CM-triviality in the real sort.*
In [**E**], Evans gave an $\omega$-categorical CM-trivial structure $\mathfrak{C}$, defined below, of SU-rank one without weak elimination of imaginaries.

Here, we show that $\mathfrak{C}$ does not have GEI: Let $M$ be the $\omega$-categorical SU-rank two generic structure $M$ (a countable binary graph with a predimension $\delta(A) = 2|A| - |R^A|$) constructed by Evans such that no triangles, no squres in $M$, and points and adjacent pairs of points are closed in $M$, and $\mathrm{cl}(*) = \mathrm{acl}(*)$ in $M$. Fix $a \in M$. Let $C, D$ be the sets of vertices at distance $1, 2$ from $a$. Let $\mathfrak{C}$ be the canonical structure on $C$ such that $\mathrm{Aut}(\mathfrak{C})$ is homeomorphic to $\mathrm{Aut}(M/a)$. As $\mathfrak{C}$ and $(M, a)$ are biinterpretable, $\mathfrak{C}$ is of SU-rank one and CM-trivial.

Let $c \in C, d \in D$ be such that $M \models R(a, c) \wedge R(c, d)$. As no triangles and squares in $M$, we have $\mathrm{acl}(a, d) \cap C = \mathrm{cl}(a, d) \cap C = \{c\}$. If $\mathfrak{C}$ had GEI, then, as $d \in \mathfrak{C}^{\mathrm{eq}}$, we could find $\bar{c} \subset_\omega C$ such that $d \in \mathrm{acl}(a, \bar{c})$ and $\bar{c} \in \mathrm{acl}(a, d)$ in the sense of $M$. As $\mathrm{acl}(a, d) \cap C = \{c\}$, $\bar{c}$ must be the singleton $c$. Since $\mathrm{cl}(a, c) = \mathrm{acl}(a, c) =$

$\{a, c\}$ in $M$, so $d \notin \operatorname{acl}(a, c)$ in $M$, a contradiction.

By Theorem 4.2, $\mathfrak{C}$ is CM-trivial but not *CM-trivial in the real sort.*

REMARK 4.4.

(1) In rosy theories having weak canonical bases, we define one-basedness as usual: $\operatorname{wcb}(a/A) \subseteq \operatorname{acl}^{\mathrm{eq}}(a)$ holds for any $a, A \subset \mathscr{M}$ with $A = \operatorname{acl}^{\mathrm{eq}}(A)$. By Remark 2.5, we see that one-basedness implies CM-triviality: As $\operatorname{wcb}(\bar{a}/B) \subseteq \operatorname{acl}^{\mathrm{eq}}(\bar{a}) \cap B \subseteq A \subseteq B$, we have $\operatorname{wcb}(\bar{a}/B) = \operatorname{wcb}(\bar{a}/A)$.

(2) There exists a one-based but non-CM-trivial rosy theory: Let $T = \operatorname{Th}(\boldsymbol{R}, +, <, \ \pi|_{(-1,1)}(*))$. $T$ is an O-minimal theory with CF-property and elimination of imaginaries. As in [**P1**], CF-property is equivalent to one-basedness in O-minimal theories. By Remark 3.5 (2) and Theorem 4.2, $T$ is not CM-trivial.

## 5. Non-CM-triviality of superrosy fields of monomial rank.

Let $\downarrow$ be the thorn independence relation. We show that CM-triviality is equivalent to non-2-ampleness in rosy theories. We also show that superrosy fields of monomial $\mathrm{U}^{\mathfrak{p}}$-rank are 2-ample. It is unknown whether any superrosy (non-supersimple) field of infinite $\mathrm{U}^{\mathfrak{p}}$-rank exists. Any supersimple field has monomial $\mathrm{SU}(= \mathrm{U}^{\mathfrak{p}})$-rank. It is also unknown whether any superrosy field has monomial $\mathrm{U}^{\mathfrak{p}}$-rank.

DEFINITION 5.1. A rosy theory $T$ is $n$-ample if after naming some parameters, there exist $A_0, A_1, \ldots, A_n \subset \mathscr{M}^{eq}$ such that

(1) $\operatorname{acl}^{\mathrm{eq}}(A_{<r}A_r) \cap \operatorname{acl}^{\mathrm{eq}}(A_{<r}A_{r+1}) = \operatorname{acl}^{\mathrm{eq}}(A_{<r})$ for any $r \leq n-1$.

(2) $A_{r+1} \downarrow_{A_r} A_{\leq r}$ for any $r \leq n-1$.

(3) $A_n \not\downarrow A_0$

where $A_{\leq r} = A_0 A_1 \ldots A_r$ and $A_{<r} = A_0 A_1 \ldots A_{r-1}$.

LEMMA 5.2. *Let $T$ be rosy. Then the following are equivalent.*

(1) *For any $A_0, A_1, A_2 \subset \mathscr{M}^{\mathrm{eq}}$, $A_2 \downarrow_{A_1} A_0$ implies $A_2 \downarrow_{\operatorname{acl}^{\mathrm{eq}}(A_1) \cap \operatorname{acl}^{\mathrm{eq}}(A_2 A_0)} A_0$.*

(2) *For any $A_0, A_1, A_2, B \subset \mathscr{M}^{\mathrm{eq}}$, $\operatorname{acl}^{\mathrm{eq}}(BA_0) \cap \operatorname{acl}^{\mathrm{eq}}(BA_1) = \operatorname{acl}^{\mathrm{eq}}(B)$, $\operatorname{acl}^{\mathrm{eq}}(BA_0A_1) \cap \operatorname{acl}^{\mathrm{eq}}(BA_0A_2) = \operatorname{acl}^{\mathrm{eq}}(BA_0)$ and $A_2 \downarrow_{\operatorname{acl}^{\mathrm{eq}}(BA_1)} A_0$ imply $A_2 \downarrow_B A_0$.*

*Thus CM-triviality is equivalent to non-2-ampleness without assuming the existence of weak canonical bases.*

PROOF.

(1)⇒(2): We have $A_2 \downarrow_{\operatorname{acl}^{\mathrm{eq}}(BA_1)} A_0 B$ by $A_2 \downarrow_{\operatorname{acl}^{\mathrm{eq}}(BA_1)} A_0$.

By (1), we see $A_2 \downarrow_{\operatorname{acl}^{\mathrm{eq}}(BA_1) \cap \operatorname{acl}^{\mathrm{eq}}(BA_0A_2)} A_0 B$. On the other hand, we have

$\mathrm{acl}^{\mathrm{eq}}(BA_1) \cap \mathrm{acl}^{\mathrm{eq}}(BA_2A_0) \subseteq \mathrm{acl}^{\mathrm{eq}}(BA_1) \cap \mathrm{acl}^{\mathrm{eq}}(BA_0) = \mathrm{acl}^{\mathrm{eq}}(B)$. Thus we see $A_2 \mathop{\smile\!\!\!\!\!\mid}_B A_0$.

(2)⇒(1): Put $B = \mathrm{acl}^{\mathrm{eq}}(A_1) \cap \mathrm{acl}^{\mathrm{eq}}(A_0A_2) \subseteq \mathrm{acl}^{\mathrm{eq}}(A_1)$.

CLAIM 1. *We have* $\mathrm{acl}^{\mathrm{eq}}(BA_0) \cap \mathrm{acl}^{\mathrm{eq}}(BA_1) = \mathrm{acl}^{\mathrm{eq}}(B)(= B)$ *and* $\mathrm{acl}^{\mathrm{eq}}(BA_0A_1) \cap \mathrm{acl}^{\mathrm{eq}}(BA_0A_2) = \mathrm{acl}^{\mathrm{eq}}(BA_0)$.

By the definition of $B$, we see $\mathrm{acl}^{\mathrm{eq}}(BA_0) \subseteq \mathrm{acl}^{\mathrm{eq}}(A_0A_1) \cap \mathrm{acl}^{\mathrm{eq}}(A_0A_2) \subseteq \mathrm{acl}^{\mathrm{eq}}(BA_0)$, so $\mathrm{acl}^{\mathrm{eq}}(BA_0A_1) \cap \mathrm{acl}^{\mathrm{eq}}(BA_0A_2) = \mathrm{acl}^{\mathrm{eq}}(BA_0)$ follows.

$$\begin{aligned}
\mathrm{acl}^{\mathrm{eq}}(B) &\subseteq \mathrm{acl}^{\mathrm{eq}}(BA_0) \cap \mathrm{acl}^{\mathrm{eq}}(BA_1) \\
&= \mathrm{acl}^{\mathrm{eq}}(BA_0) \cap \mathrm{acl}^{\mathrm{eq}}(A_1) \\
&\subseteq \mathrm{acl}^{\mathrm{eq}}(A_0A_1) \cap \mathrm{acl}^{\mathrm{eq}}(A_0A_2) \cap \mathrm{acl}^{\mathrm{eq}}(A_1) \\
&\subseteq \mathrm{acl}^{\mathrm{eq}}(A_0A_1) \cap \mathrm{acl}^{\mathrm{eq}}(B) \subseteq \mathrm{acl}^{\mathrm{eq}}(B)
\end{aligned}$$

By $A_2 \mathop{\smile\!\!\!\!\!\mid}_{BA_1} A_0$ and (2), $A_2 \mathop{\smile\!\!\!\!\!\mid}_B A_0$ follows. □

From now on, we check that any superrosy field of monomial $\mathrm{U}^{\mathfrak{p}}$-rank is not CM-trivial (=2-ample) by following the Nubling's proof for $n$-ampleness of supersimple field. As the Nubling's proof works for superrosy field of monomial $\mathrm{U}^{\mathfrak{p}}$-rank, any superrosy field of monomial $\mathrm{U}^{\mathfrak{p}}$-rank is $n$-ample for any $n < \omega$.

Let $F$ be an infinite superrosy field. We say that $a_0, a_1, \dots, a_i, \dots \in F$ are independent generics over $A$ if $\mathrm{U}^{\mathfrak{p}}(a_0/A) = \mathrm{U}^{\mathfrak{p}}(a_1/A) = \cdots = \mathrm{U}^{\mathfrak{p}}(a_i/A) = \cdots = \mathrm{U}^{\mathfrak{p}}(F)$ and $a_0, a_1, \dots, a_i, \dots$ are thorn independent over $A$.

FACT 5.3. *Let $F$ be an infinite superrosy field.*

(1) *Let $a, b, c \in F$ be independent generics over $A$. Then $bc, a, c$ are independent generics over $A$ and $a + bc, a, c$ are independent generics over $A$.*
(2) *Let $a_1, \dots, a_i, \dots, b, c_1, \dots, c_i, \dots \in F$ be independent generics over $A$. Then $a_1 + bc_1, \dots a_i + bc_i, \dots, c_1, \dots, c_i, \dots$ are independent generics over $A$.*

PROOF. We may assume $A = \emptyset$.
(1) Since $bc$ and $b$ are interdefinable oner $c$, we see $\mathrm{U}^{\mathfrak{p}}(F) \geq \mathrm{U}^{\mathfrak{p}}(bc) \geq \mathrm{U}^{\mathfrak{p}}(bc/c, a) = \mathrm{U}^{\mathfrak{p}}(b/c, a) = \mathrm{U}^{\mathfrak{p}}(F)$. As $a + bc$ and $bc$ are interdefinable over $a$, we also see that $\mathrm{U}^{\mathfrak{p}}(F) \geq \mathrm{U}^{\mathfrak{p}}(a + bc) \geq \mathrm{U}^{\mathfrak{p}}(a + bc/a, c) = \mathrm{U}^{\mathfrak{p}}(bc/a, c) = \mathrm{U}^{\mathfrak{p}}(F)$.

(2) By (1), we have only to show
$a_{i+1} + bc_{i+1}, c_{i+1} \mathop{\smile\!\!\!\!\!\mid} a_0 + bc_0, \dots, a_i + bc_i, c_0 \dots, c_i$.
As $a_{i+1}, c_{i+1} \mathop{\smile\!\!\!\!\!\mid}_b a_0, \dots, a_i, c_0, \dots, c_i$, we have
$a_{i+1} + bc_{i+1}, c_{i+1} \mathop{\smile\!\!\!\!\!\mid}_b a_0 + bc_0, \dots, a_i + bc_i, c_0, \dots, c_i$.

Since $U^{\mathfrak{p}}(a_{i+1}+bc_{i+1}/b, c_{i+1}) = U^{\mathfrak{p}}(a_{i+1}/b, c_{i+1}) = U^{\mathfrak{p}}(F)$, we have $a_{i+1}+bc_{i+1} \mathop{\smile\!\!\!\!\!\!|}\, b, c_{i+1}$. As $b \mathop{\smile\!\!\!\!\!\!|}\, c_{i+1}$, we see $a_{i+1}+bc_{i+1}, c_{i+1} \mathop{\smile\!\!\!\!\!\!|}\, b$.
So we see the conclusion. □

Let $F$ be a superrosy field. To get a witness for non-CM-triviality, we define a plane $\mathbf{P}$ in $F^3$, a line $\mathbf{l}$ on $\mathbf{P}$, and a point $\mathbf{p}$ on $\mathbf{l}$ as follows.

Let $a_0^{0,0}, a_1^{0,0}, a_2^{0,0}$ be independent generics. Put $\mathbf{P} = \{(x_1,x_2,x_3) \in F^3 : a_0^{0,0} + a_1^{0,0}x_1 + a_2^{0,0}x_2 = x_3\}$. We consider $A_0 := \{a_0^{0,0}, a_1^{0,0}, a_2^{0,0}\}$ as parameters for $\mathbf{P}$.

Let $a_0^{1,0}, a_1^{1,0}$ be independent generics over previous elements. Put $B_1^{1,0} = \{(x_1,x_2,x_3) \in F^3 : a_0^{1,0} + a_1^{1,0}x_1 = x_2\}$ and Put $\mathbf{l} = \mathbf{P} \cap B_1^{1,0}$. Then $(x_1,x_2,x_3) \in \mathbf{l}$ iff $(a_0^{0,0} + a_2^{0,0}a_0^{1,0}) + (a_1^{0,0} + a_2^{0,0}a_1^{1,0})x_1 = x_3$. Put $a_0^{1,1} := a_0^{0,0} + a_2^{0,0}a_0^{1,0}$ and $a_1^{1,1} := a_1^{0,0} + a_2^{0,0}a_1^{1,0}$. Let $B_1^{1,1} = \{(x_1,x_2,x_3) \in F^3 : a_0^{1,1} + a_1^{1,1}x_1 = x_3\}$. Then $\mathbf{l} = B_1^{1,0} \cap B_1^{1,1}$ and we consider $A_1 := \{a_0^{1,0}, a_1^{1,0}, a_0^{1,1}, a_1^{1,1}\}$ as parameters for $\mathbf{l}$.

Let $a_0^{2,0}$ be generic over previous elements. Put $B_2^{2,0} := \{(x_1,x_2,x_3) \in F^3 : a_0^{2,0} = x_1\}$ and $B_2^{2,1} := B_2^{2,0} \cap B_1^{1,0}$ and $B_2^{2,2} := B_2^{2,0} \cap B_1^{1,1}$. Then $(x_1,x_2,x_3) \in B_2^{2,1}$ iff $a_0^{2,1} := a_0^{1,0} + a_1^{1,0}a_0^{2,0} = x_2$, and $(x_1,x_2,x_3) \in B_2^{2,2}$ iff $a_0^{2,2} := a_0^{1,1} + a_1^{1,1}a_0^{2,0} = x_3$.

Let $\mathbf{p} := B_2^{2,0} \cap B_2^{2,1} \cap B_2^{2,2} = B_2^{2,0} \cap \mathbf{l}$ and we consider $A_2 = \{a_0^{2,0}, a_0^{2,1}, a_0^{2,2}\}$ as parameters for $\mathbf{p}$.

Now we have the following lemma. (Here, we need not to assume that $F$ is of monomial $U^{\mathfrak{p}}$-rank.)

Lemma 5.4.
(1) $\mathrm{dcl}^{\mathrm{eq}}(A_1, A_2) = \mathrm{dcl}^{\mathrm{eq}}(A_1, a_0^{2,0})$.
(2) $\mathrm{dcl}^{\mathrm{eq}}(A_0, A_1) = \mathrm{dcl}^{\mathrm{eq}}(A_0, a_0^{1,0}, a_1^{1,0})$.
(3) $A_2 \mathop{\smile\!\!\!\!\!\!|}_{A_1} A_0$
(4) $a_0^{2,2} \in \mathrm{dcl}^{\mathrm{eq}}(A_0, a_0^{2,0}, a_0^{2,1})$ and $a_0^{0,0} \in \mathrm{dcl}^{\mathrm{eq}}(a_1^{0,0}, a_2^{0,0}, A_2)$.
(5) $A_0 \not\mathop{\smile\!\!\!\!\!\!|}\, A_2$.

Proof. (1),(2) are clear. (3) follows from $a_0^{2,0} \mathop{\smile\!\!\!\!\!\!|}\, A_0, A_1$. (4) follows from

$$\begin{aligned} a_0^{2,2} &= a_0^{1,1} + a_1^{1,1}a_0^{2,0} \\ &= (a_0^{0,0} + a_2^{0,0}a_0^{1,0}) + (a_1^{0,0} + a_2^{0,0}a_1^{1,0})a_0^{2,0} \\ &= a_0^{0,0} + a_2^{0,0}(a_0^{1,0} + a_1^{1,0}a_0^{2,0}) + a_1^{0,0}a_0^{2,0} \\ &= a_0^{0,0} + a_2^{0,0}a_0^{2,1} + a_1^{0,0}a_0^{2,0} \end{aligned}$$

(5): If we had $A_0 \perp A_2$, then $a_0^{0,0} \perp_{a_1^{0,0}, a_2^{0,0}} A_2$, so $a_0^{0,0} \in \mathrm{acl}^{\mathrm{eq}}(a_1^{0,0}, a_2^{0,0})$ would hold. □

PROPOSITION 5.5. *If $F$ has a monomial $\mathrm{U}^{\mathfrak{p}}$-rank, then we have*

(1) $\mathrm{acl}^{\mathrm{eq}}(A_0) \cap \mathrm{acl}^{\mathrm{eq}}(A_1) = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$.
(2) $\mathrm{acl}^{\mathrm{eq}}(A_0 A_1) \cap \mathrm{acl}^{\mathrm{eq}}(A_0 A_2) = \mathrm{acl}^{\mathrm{eq}}(A_0)$.

PROOF. Let $\mathrm{U}^{\mathfrak{p}}(F) = \omega^\alpha k =: \beta$, where $\alpha$ is an ordinal and $k$ is a natural number.
(1): By Fact 5.3, $A_1$ consists of independent generics.

CLAIM 2. $\mathrm{U}^{\mathfrak{p}}(A_0/A_1) \geq \beta$.

$A_0, A_1$ and $A_0, a_0^{1,0}, a_1^{1,0}$ are interdefinable. So, we have $\beta 5 = \mathrm{U}^{\mathfrak{p}}(A_0 A_1) \leq \mathrm{U}^{\mathfrak{p}}(A_0/A_1) \oplus \mathrm{U}^{\mathfrak{p}}(A_1) = \mathrm{U}^{\mathfrak{p}}(A_0/A_1) \oplus \beta 4$. The claim follows.

CLAIM 3. *Take $A_0' \equiv_{\mathrm{acl}^{\mathrm{eq}}(A_1)} A_0$ with $A_0' \perp_{A_1} A_0$. Then $A_0' \perp A_0$.*

$$\begin{aligned} \mathrm{U}^{\mathfrak{p}}(A_0' A_0 A_1) &\geq \mathrm{U}^{\mathfrak{p}}(A_0' A_0/A_1) + \mathrm{U}^{\mathfrak{p}}(A_1) \\ &= (\mathrm{U}^{\mathfrak{p}}(A_0'/A_1) \oplus \mathrm{U}^{\mathfrak{p}}(A_0/A_1)) + \beta 4 \\ &\geq \beta 6 \end{aligned}$$

As $a_i^{1,1} = a_i^{0,0} + a_2^{0,0} a_i^{1,0} = a_i'^{0,0} + a_2'^{0,0} a_i^{1,0}$, we have

$$a_i^{1,0} = \frac{a_i^{0,0} - a_i'^{0,0}}{a_2'^{0,0} - a_2^{0,0}} \in \mathrm{dcl}^{\mathrm{eq}}(A_0' A_0),$$

so we have $A_1 \subseteq \mathrm{acl}^{\mathrm{eq}}(A_0' A_0)$.

$$\begin{aligned} \beta 6 &\leq \mathrm{U}^{\mathfrak{p}}(A_0' A_0 A_1) \\ &= \mathrm{U}^{\mathfrak{p}}(A_0' A_0) \\ &\leq \beta 6 \end{aligned}$$

As $\mathrm{U}^{\mathfrak{p}}(A_0) = \mathrm{U}^{\mathfrak{p}}(A_0') = \beta 3$, we see the claim.

As $\mathrm{acl}^{\mathrm{eq}}(A_0) \cap \mathrm{acl}^{\mathrm{eq}}(A_1) = \mathrm{acl}^{\mathrm{eq}}(A_0') \cap \mathrm{acl}^{\mathrm{eq}}(A_1) \subseteq \mathrm{acl}^{\mathrm{eq}}(A_0) \cap \mathrm{acl}^{\mathrm{eq}}(A_0')$ and $A_0' \perp A_0$, we see the conclusion.

(2): As $A_1$ and $a_0^{1,0}, a_1^{1,0}$ are interdefinable over $A_0$ by Lemma 5.4 (2), and $A_2$ and

$a_0^{2,0}, a_0^{2,1}$ are interdefinable over $A_0$ by Lemma 5.4 (4), working over $\mathrm{acl}^{\mathrm{eq}}(A_0)$, we need to prove $\mathrm{acl}^{\mathrm{eq}}(a_0^{1,0}, a_1^{1,0}) \cap \mathrm{acl}^{\mathrm{eq}}(a_0^{2,0}, a_0^{2,1}) = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$. Note that $\mathrm{U}^{\mathfrak{p}}(a_0^{2,0}, a_0^{2,1}) = \beta 2$ over $\mathrm{acl}^{\mathrm{eq}}(A_0)$ by Fact 5.3(2).

The rest is similar to (1) :
As $a_0^{2,1} \in \mathrm{dcl}^{\mathrm{eq}}(a_0^{1,0}, a_1^{1,0}, a_0^{2,0})$, $\mathrm{U}^{\mathfrak{p}}(a_0^{1,0}, a_1^{1,0}, a_0^{2,0}, a_0^{2,1}) = \beta 3$ follows.
As $\beta 3 = \mathrm{U}^{\mathfrak{p}}(a_0^{1,0}, a_1^{1,0}, a_0^{2,0}, a_0^{2,1}) \leq \mathrm{U}^{\mathfrak{p}}(a_0^{1,0}, a_1^{1,0}/a_0^{2,0}, a_0^{2,1}) \oplus \mathrm{U}^{\mathfrak{p}}(a_0^{2,0}, a_0^{2,1})$
$= \mathrm{U}^{\mathfrak{p}}(a_0^{1,0}, a_1^{1,0}/a_0^{2,0}, a_0^{2,1}) \oplus \beta 2$, we have $\mathrm{U}^{\mathfrak{p}}(a_0^{1,0}, a_1^{1,0}/a_0^{2,0}, a_0^{2,1}) \geq \beta$.
Take $a_0'^{1,0}, a_1'^{1,0} \equiv_{\mathrm{acl}^{\mathrm{eq}}(a_0^{2,0}, a_0^{2,1})} a_0^{1,0}, a_1^{1,0}$ with $a_0'^{1,0}, a_1'^{1,0} \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}^{\mathrm{eq}}(a_0^{2,0}, a_0^{2,1})} a_0^{1,0}, a_1^{1,0}$.

We have
$\mathrm{U}^{\mathfrak{p}}(a_0'^{1,0}, a_1'^{1,0}, a_0^{1,0}, a_1^{1,0}, a_0^{2,0}, a_0^{2,1}) \geq \mathrm{U}^{\mathfrak{p}}(a_0'^{1,0}, a_1'^{1,0}, a_0^{1,0}, a_1^{1,0}/a_0^{2,0}, a_0^{2,1}) + \mathrm{U}^{\mathfrak{p}}(a_0^{2,0}, a_0^{2,1})$
$= (\mathrm{U}^{\mathfrak{p}}(a_0^{1,0}, a_1^{1,0}/a_0^{2,0}, a_0^{2,1}) \oplus \mathrm{U}^{\mathfrak{p}}(a_0'^{1,0}, a_1'^{1,0}/a_0^{2,0}, a_0^{2,1})) + \beta 2 \geq \beta 4$.
As $a_0^{2,1} = a_0^{1,0} + a_1^{1,0} a_0^{2,0} = a_0'^{1,0} + a_1'^{1,0} a_0^{2,0}$ and

$$a_0^{2,0} = \frac{a_0^{1,0} - a_0'^{1,0}}{a_1'^{1,0} - a_1^{1,0}} \in \mathrm{dcl}^{\mathrm{eq}}(a_0'^{1,0}, a_1'^{1,0}, a_0^{1,0}, a_1^{1,0}),$$

we have $a_0^{2,0}, a_0^{2,1} \in \mathrm{dcl}(a_0'^{1,0}, a_1'^{1,0}, a_0^{1,0}, a_1^{1,0})$. So we see $\mathrm{U}^{\mathfrak{p}}(a_0'^{1,0}, a_1'^{1,0}, a_0^{1,0}, a_1^{1,0}) = \beta 4$ and $a_0'^{1,0}, a_1'^{1,0} \mathop{\smile\!\!\!\!\!\mid} a_0^{1,0}, a_1^{1,0}$.

Therefore we have $\mathrm{acl}^{\mathrm{eq}}(a_0^{1,0}, a_1^{1,0}) \cap \mathrm{acl}(a_0^{2,0}, a_0^{2,1}) = \mathrm{acl}^{\mathrm{eq}}(a_0'^{1,0}, a_1'^{1,0}) \cap \mathrm{acl}(a_0^{2,0}, a_0^{2,1}) \subseteq \mathrm{acl}^{\mathrm{eq}}(a_0^{1,0}, a_1^{1,0}) \cap \mathrm{acl}(a_0'^{1,0}, a_1'^{1,0}) = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$. □

THEOREM 5.6. *Let $T$ be a rosy theory. If $T$ interprets a superrosy field of monomial $\mathrm{U}^{\mathfrak{p}}$-rank, then $T$ is not CM-trivial.*

PROOF. If $T$ interprets a superrosy field of monomial $\mathrm{U}^{\mathfrak{p}}$-rank, then $T$ has a witness for non-CM-triviality by Lemma 5.4 and Proposition 5.5. □

## 6. CM-triviality in O-minimal theories.

We begin with the following facts on O-minimal theories.

FACT 6.1. *Let $T$ be O-minimal.*

(1) (*Peterzil-Starchenko,* [**PS**]) $T$ is not one-based iff $T$ has a definable real closed field of dimension 1 on some interval.
(2) (*Onshuus,* [**O**]) In O-minimal theories, the thorn independence relation coincides with the independence relation defined by dimension.

From now on, we work in O-minimal theories with elimination of imaginaries. (Any O-minimal theory having a group-operation eliminates imaginaries by definable choice.) Note that $\mathrm{dcl} = \mathrm{acl}^{\mathrm{eq}}$. In [**P1**], Pillay defines one-basedness in O-minimal theories by the germs of definable functions as follows. Let $f(\bar{x}, \bar{y})$ be an

$\emptyset$-definable function and let $\bar{a}$ be such that $\dim(\bar{a}) = |\bar{a}| = |\bar{x}|$. Let $E_{f,\bar{a}}$ be an $\bar{a}$-definable equivalence relation defined by $E_{f,\bar{a}}(\bar{b}_1, \bar{b}_2) \Leftrightarrow$ *either* there exists an open neighborhood $U$ of $\bar{a}$ such that $f(\bar{x}, \bar{b}_1), f(\bar{x}, \bar{b}_2)$ are defined on $U$ and $f(\bar{x}, \bar{b}_1)|U = f(\bar{x}, \bar{b}_2)|U$ *or* neither of $f(\bar{x}, \bar{b}_1), f(\bar{x}, \bar{b}_2)$ is defined on an open neiborhood of $\bar{a}$. An O-minimal theory is *one-based* (equivalent to CF-property, defined by Peterzil) if $\bar{b}_{E_{f,\bar{a}}} \in \mathrm{dcl}(\bar{a}, f(\bar{a}, \bar{b}))$ holds for any $\emptyset$-definable function $f(\bar{x}, \bar{y})$ and any $\bar{a}$ and $\bar{b}$ with $\dim(\bar{a}/\bar{b}) = |\bar{a}|$.

FACT 6.2. (*Pillay,* [**P1**]) If $T$ *has weak canonical bases, then one-basedness is equivalent to the modularity in* $T$.

THEOREM 6.3. *In O-minimal theories having elimination of imaginaries, CM-triviality is equivalent to the modularity.*

PROOF. Let $T$ be a CM-trivial O-minimal theory with elimination of imaginaries. By Fact 6.1 and Theorem 5.6, $T$ is one-based. By CM-triviality and Theorem 2.4, $T$ has weak canonical bases, so it must be modular by Fact 6.2. Conversely, let $T$ be a modular O-minimal theory with elimination of imaginaries. If $A_2 \perp_{A_1} A_0$, by modularity we have $A_2 \perp_{\mathrm{dcl}(A_2)\cap\mathrm{dcl}(A_1)} A_0 A_1$. As $\mathrm{dcl}(A_2) \cap \mathrm{dcl}(A_1) \subseteq \mathrm{dcl}(A_2 A_0) \cap \mathrm{dcl}(A_1) \subseteq \mathrm{dcl}(A_0 A_1)$, we have CM-triviality; $A_2 \perp_{\mathrm{dcl}(A_1)\cap\mathrm{dcl}(A_2 A_0)} A_0$. □

REMARK 6.4.

(1) CM-triviality is not equivalent to one-basedness in O-minimal theories in general: Let $T = \mathrm{Th}(\boldsymbol{R}, +, <, \pi(*)|(-1,1))$, where $\pi(x) = \pi x \in \mathrm{dcl}(x)$ for each $x \in (-1,1)$. Example 4.5 in [**LP**] and [**P1**] show that $T$ is one-based but non-locally modular and does not have weak canonical bases. So $T$ is a non-CM-trivial one-based theory.

(2) Neither local modularity nor CM-triviality are preserved under reducts in O-minimal theories: Let $T' = \mathrm{Th}(\boldsymbol{R}, +, <, \pi(*))$, where $\pi(x) = \pi x \in \mathrm{dcl}(x)$ for each $x$. Then $T'$ is locally modular and CM-trivial. But the reduct $T$ of $T'$ is non-locally modular and non-CM-trivial.

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