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Extremal functions for capacities

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Abstract. The extremal function c_K for the variational 2-capacity $\mathrm{cap}(K)$ of a compact subset K of the Royden harmonic boundary δR of an open Riemann surface R relative to an end W of R, referred to as the capacitary function of K, is characterized as the Dirichlet finite harmonic function h on W vanishing continuously on the relative boundary ∂W of W satisfying the following three properties: the normal derivative measure $*dh$ of h exists on δR with $*dh \geqq 0$ on δR; $*dh = 0$ on $\delta R \setminus K$; $h = 1$ quasieverywhere on K. As a simple application of the above characterization, we will show the validity of the following inequality

$$\mathrm{hm}(K) \leqq \kappa \cdot \mathrm{cap}(K)^{1/2}$$

for every compact subset K of δR, where $\mathrm{hm}(K)$ is the harmonic measure of K calculated at a fixed point a in W and κ is a constant depending only upon the triple (R, W, a).

The Dirichlet space $L^{1,2}(R)$ on an open Riemann surface R is the real linear space of functions $f \in W^{1,2}_{loc}(R)$ with finite Dirichlet integral $D(f;R) := \int_R df \wedge *df$ of f taken over R (cf. e.g. [**3**]). Recall that every function f in $L^{1,2}(R) \cap C(R)$ is $[-\infty, +\infty]$-valued continuous on the Royden compactification R^* of R and the extended function will be denoted by the same notation f. The PWB (i.e. Perron-Wiener-Brelot) solution on R with continuous boundary values φ on the Royden boundary $\gamma R := R^* \setminus R$ is denoted by H^R_φ as usual. Then the Royden *harmonic boundary* δR is nothing but the set of regular points ζ in γR so that $\lim_{z\to\zeta} H^R_\varphi(z) = \varphi(\zeta)$ for every $\varphi \in C(\gamma R)$. Recall a Royden theorem (cf. [**7**]) that $\delta R \neq \emptyset$ if and only if R is *hyperbolic* (i.e. nonparabolic) characterized by the existence of the Green kernel $G(\cdot,\cdot;R)$ on R. Based upon this result, to avoid the trivial case of $\delta R = \emptyset$ from our standpoint, we always assume the hyperbolicity of R throughout this paper. Let W be an analytic *end* of R in the sense that W is a subregion (i.e. connected and open subset) of R such that $R \setminus \overline{W}$ ($\overline{W}$ being the closure of W taken in R^*) is an analytic subregion of R, i.e. a relatively compact subregion of R whose relative boundary consists of a finite number of mutually

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disjoint analytic Jordan (i.e. simple and closed) curves. Thus W is bounded by the relative boundary ∂W of W and the Royden boundary γR which is identical with the ideal boundary $\overline{W}\setminus(W\cup\partial W)$ of W. For $\varphi\in C(\gamma R)$ we extend it to $(\gamma R)\cup(\partial W)$ by setting $\varphi=0$ on ∂W and consider the PWB solution H_φ^W on W with boundary data φ so extended as described above. Then the set of regular points for these solutions H_φ^W is $(\partial W)\cup(\delta R)$ and hence H_φ^W vanishes continuously on ∂W. For fundamentals related to Royden compactifications R^* we refer to e.g. [**7**] (see also [**2**], [**5**], etc.).

In addition to the Dirichlet integrals $D(f;R)$ we also consider the mutual Dirichlet integrals $D(f,g;R):=\int_R df\wedge *dg$ of f and g in $L^{1,2}(R)$ taken over R. We occasionally write $D(f)$ and $D(f,g)$ for $D(f;R)$ and $D(f,g;R)$ omitting R if the integrating domain R is well understood. Let $HD(W;\partial W)$ be the class of harmonic functions u on W vanishing continuously on ∂W with finite Dirichlet integrals $D(u;W)$. Recall that the real linear space $HD(W;\partial W)$ with $D(\cdot,\cdot;W)$ as its inner product forms a Hilbert space. It has the reproducing kernel given by the Bergman kernel

$$B(z,w;W):=N(z,w;W)-G(z,w;W),$$

where $N(z,w;W)$ is the Neumann kernel on W with vanishing continuous boundary values on ∂W as the function of z and $G(z,w;W)$ is the Green kernel on W. Since $B(z,w;W)$ is separately harmonic on $(W\cup\partial W)\times(W\cup\partial W)$, it is jointly harmonic on $(W\cup\partial W)\times(W\cup\partial W)$ by the Hartogs theorem in the harmonic version (cf. [**4**]) and in particular $z\mapsto B(z,z;W)$ is continuous on $W\cup\partial W$ so that $k_E:=\sup_{z\in E}B(z,z;W)^{1/2}<+\infty$ for any compact subset $E\subset W\cup\partial W$. In view of the reproducing property of $B(z,w;W)$, we have $u(z)=D(u,B(\cdot,z;W);W)$ for every $u\in HD(W;\partial W)$ and $z\in W\cup\partial W$, the Schwarz inequality implies that $|u(z)|\leqq B(z,z;W)^{1/2}D(u;W)^{1/2}$ since $D(B(\cdot,z;W);W)=D(B(\cdot,z;W),B(\cdot,z;W);W)=B(z,z;W)$ so that

$$\sup_E|u(z)|\leqq k_E\cdot D(u;W)^{1/2} \tag{1}$$

for every $u\in HD(W;\partial W)$ and for any compact subset $E\subset W\cup\partial W$. As one of direct consequences of (1) we see that not only a strongly but also a weakly convergent sequence $(u_n)_{n\geqq1}$ to u in the Hilbert space $HD(W;\partial W)$ is locally uniformly convergent to u on $W\cup\partial W$. In fact, replacing u_n by u_n-u, suppose $(u_n)_{n\geqq1}$ converges to zero weakly in $HD(W;\partial W)$. Then, first, $(D(u_n;W))_{n\geqq1}$ is a bounded sequence as a result of weak convergence of $(u_n)_{n\geqq1}$ and a fortiori (1) assures that $(u_n)_{n\geqq1}$ is locally uniformly bounded on $W\cup\partial W$. Since $u_n(z)=$

$D(u_n, B(\cdot, z; W); W)$ yields the pointwise convergence of $(u_n)_{n \geqq 1}$ to zero on $W \cup \partial W$, the Montel theorem in the harmonic version implies the required conclusion.

From the view point of the classification theory of Riemann surfaces, our primary concern is not about the space $HD(W; \partial W)$ but the space $HD(R)$ of harmonic functions u on the whole surface R with finite Dirichlet integrals $D(u; R) < +\infty$ taken over R. However, since $D(c; R) = 0$ for every constant function c on R, the space $HD(R)$ with $D(\cdot, \cdot; R)$ as its inner product is only a pre-Hilbert space and not Hilbert space. To compensate this drawback, one traditional way is to consider $dHD(R) = \{du : u \in HD(R)\}$ with $(du, dv) := D(u, v; R)$ as its inner product. Then it certainly forms a Hilbert space but, on the other hand, we are loosing so much because the important subspace $\boldsymbol{R}$ of real numbers disappears. Therefore, instead, we consider $HD(W; \partial W)$ in place of $HD(R)$. Our justification of doing this lies in the following two points: first, there is a linear bijection $T : HD(W; \partial W) \to HD(R)$ characterized by $Tu|\delta R = u|\delta R$ so that $HD(R)$ and $HD(W; \partial W)$ are identical at least from the view point of linear structures; second, $HD(W_1; \partial W_1)$ and $HD(W_2; \partial W_2)$ are bicontinuously linear isomorphic as Hilbert spaces by the mapping $T_2^{-1} \circ T_1$ where $T_j : HD(W_j; \partial W_j) \to HD(R)$ $(j = 1, 2)$ are T considered above so that the choice of the end W is immaterial.

Let K be any compact subset of δR. We now consider the *capacity* of K, or more precisely the variational 2-capacity of K, denoted by $\mathrm{cap}(K)$ relative to an end W given by

$$\mathrm{cap}(K) := \inf_{f \in \mathscr{V}(K)} D(f; R), \tag{2}$$

where $\mathscr{V}(K)$ is the class of functions $f \in L^{1,2}(R) \cap C(R)$ such that $f \geqq 1$ on K and $f \leqq 0$ on $R \setminus W$. Starting from the capacities $\mathrm{cap}(K)$ for compact subsets $K \subset \delta R$, we define as usual the outer (inner, resp.) capacity $\mathrm{cap}^*(X)$ ($\mathrm{cap}_*(X)$, resp.) for general subset $X \subset \delta R$ and then define the capacity $\mathrm{cap}(X)$ for general subset $X \subset \delta R$, if X is capacitable in the sense that $\mathrm{cap}^*(X) = \mathrm{cap}_*(X)$, by the common value $\mathrm{cap}^*(X) = \mathrm{cap}_*(X)$. Here recall the Choquet theorem that analytic subsets (and, in particular, Borel subsets) of δR are capacitable so that $\mathrm{cap}(K)$ in the general sense coincides with the original capacity $\mathrm{cap}(K)$ given by (2) for compact subsets $K \subset \delta R$ since $\mathrm{cap}(K)$ is seen to be a Choquet capacity (cf. e.g. [**3**]). A subset $X \subset \delta R$ is said to have capacity zero if $\mathrm{cap}^*(X) = 0$. In this case X is capacitable and $\mathrm{cap}(X) = 0$ so that $\mathrm{cap}(K) = 0$ for every compact subset $K \subset X$ and vice versa. We also say that a general set $X \subset \delta R$ is *polar* if X is of capacity zero. A subset $X \subset \delta R$ which is not polar is said to be *nonpolar*. A property on δR

is said to hold *quasieverywhere* (abbreviated as q.e.) on δR if it holds on δR except for a polar subset of δR. We can also consider $\mathrm{cap}(K)$ for compact subsets $K \subset \gamma R$ in exactly the same fashion as above but as is easily seen we have $\mathrm{cap}(K) = 0$ for every $K \subset \gamma R \setminus \delta R$ so that after all $\mathrm{cap}(\gamma R \setminus \delta R) = 0$. This is the reason why we confine ourselves to stay on δR in considering the capacity $\mathrm{cap}(X)$ only for subsets X of δR.

Let $\mathscr{W}(K)$ be the set of $u \in HD(W; \partial W)$ such that $0 \leqq u \leqq 1$ on W and $u = 1$ on the compact subset K of δR so that we can view $\mathscr{W}(K) \subset \mathscr{V}(K)$ by setting $u = 0$ on $R_0 := R \setminus \overline{W}$ for every $u \in HD(W; \partial W)$. Observe that $g := \max\{\min\{f, 1\}, 0\} \in \mathscr{V}(K)$ along with any $f \in \mathscr{V}(K)$ and $D(g; R) \leqq D(f; R)$. By the Royden decomposition (cf. e.g. [**7**]) of g on W, there is a unique $u \in HD(W; \partial W)$ with $u = g = 1$ on K and $D(u; R) \leqq D(g; R)$. Therefore for any $f \in \mathscr{V}(K)$ there is a $u \in \mathscr{W}(K) \subset \mathscr{V}(K)$ such that $D(u; W) = D(u; R) \leqq D(f; R)$. This implies the following relation

$$\mathrm{cap}(K) = \inf_{u \in \mathscr{W}(K)} D(u; W). \tag{3}$$

As remarked above hereafter we simply denote by $D(u)$ and $D(u, v)$ for $D(u; W)$ and $D(u, v; W)$ omitting W as far as it is clear that the integration is taken over W. Observe that $\mathscr{W}(K)$ is a convex subset of the Hilbert space $HD(W; \partial W)$ whose norm $D(\cdot)^{1/2}$ satisfies the parallelogram law: $D(u + v) + D(u - v) = 2(D(u) + D(v))$. This assures the following important property. Any sequence $(u_i)_{i \geqq 1}$ in $\mathscr{W}(K)$ is said to be a *minimal* sequence if $D(u_i) \to \mathrm{cap}(K)\ \ (i \to \infty)$. We denote by $\overline{\mathscr{W}(K)}$ the closure of $\mathscr{W}(K)$ in $HD(W; \partial W)$. Clearly $\overline{\mathscr{W}(K)}$ is also a convex subset of $HD(W; \partial W)$ along with $\mathscr{W}(K)$.

Proposition 4. *Any minimal sequence $(u_i)_{i \geqq 1}$ in $\mathscr{W}(K)$ is a Cauchy sequence in $HD(W; \partial W)$ and the limit function $u := \lim_{i \to \infty} u_i \in \overline{\mathscr{W}(K)}$ does not depend on the choice of minimal sequences in $\mathscr{W}(K)$ so that u is the unique function in $\overline{\mathscr{W}(K)}$ with $D(u) = \mathrm{cap}(K)$.*

Proof. Let $(u_j)_{j \geqq 1}$ be a minimal sequence in $\mathscr{W}(K)$. By the parallelogram law we see that

$$D(u_i - u_j) = 2\big(D(u_i) + D(u_j)\big) - 4D\left(\frac{u_i + u_j}{2}\right)$$

for every i and j. Since the convex combination $(u_i + u_j)/2 \in \mathscr{W}(K)$, we have $D((u_i + u_j)/2) \geqq \mathrm{cap}(K)$ and both of $D(u_i)$ and $D(u_j)$ tend to $\mathrm{cap}(K)$ as i and j

tend to ∞. Hence

$$D(u_i - u_j) \leqq 2\big(D(u_i) + D(u_j)\big) - 4\mathrm{cap}(K) \to 0 \quad (i, j \to \infty).$$

Thus any minimal sequence $(u_i)_{i \geqq 1}$ is a Cauchy sequence in $HD(W; \partial W)$. We set $u := \lim_{i\to\infty} u_i \in \overline{\mathscr{W}(K)}$. Choose another minimal sequence $(v_i)_{i\geqq 1}$ and put $v := \lim_{i\to\infty} v_i$. For any real number t, $u_i + t(u_i - v_i) \in \mathscr{V}(K)$ and $u_i + t(u_i - v_i)$ converges to $u + t(u - v)$ as $i \to \infty$ so that $D(u_i + t(u_i - v_i)) \geqq D(u)$ and then $D(u + t(u - v)) \geqq D(u)$. Hence

$$2D(u, u - v)t + D(u - v)t^2 \geqq 0$$

for every t, which implies that $D(u, u - v) = 0$. Changing the roles of u and v, we also have $D(v, v - u) = 0$ and thus we can conclude that $D(u - v) = 0$. This shows that $u = v$ on W, as desired. □

We have thus seen that the extremum problem

$$\mathrm{cap}(K) = \inf_{u \in \mathscr{W}(K)} D(u) = \min_{u \in \overline{\mathscr{W}(K)}} D(u) \tag{5}$$

has a unique solution $u \in \overline{\mathscr{W}(K)}$ with $D(u) = \mathrm{cap}(K)$. We will denote this extremal function by c_K and call it as the *capacitary function* for $K \subset \delta R$. Hence $c_K \in \overline{\mathscr{W}(K)} \subset HD(W; \partial W)$ and

$$\mathrm{cap}(K) = D(c_K) = \min_{u \in \overline{\mathscr{W}(K)}} D(u) = \inf_{u \in \mathscr{W}(K)} D(u). \tag{6}$$

The purpose of this paper is to characterize c_K as the solution of a certain mixed boundary value problem and the following assertion will play a decisive role for the aim. As usual for a class $\mathscr{F}$ of functions on a space X we set $\mathscr{F}^+ := \{f \in \mathscr{F} : f \geqq 0 \text{ on } X\}$.

Proposition 7 (Fundamental Lemma). *For an arbitrarily given compact subset $K \subset \delta R$ and an arbitrarily given positive number $\varepsilon > 0$ there exists an $h_\varepsilon \in \mathscr{W}(K)$ such that*

$$D(w, h_\varepsilon; W) \geqq 0 \tag{8}$$

for every $w \in HD(W; \partial W)^+$ and

$$D(h_\varepsilon; W) < \operatorname{cap}(K) + \varepsilon. \tag{9}$$

Proof. We can find a $p \in \mathscr{W}(K)$ such that $D(p) < \operatorname{cap}(K) + \varepsilon/2$. By the Sard theorem that the set of critical values of p is of Lebesgue measure zero, or rather at the present two dimensional analytic case by the fact that the set of critical points of p is discrete, we can find a real number $\tau > 1$ enough close to 1 such that $D(\tau p) < \operatorname{cap}(K) + \varepsilon/2$ and the open subset $G := \{z \in W : \tau p(z) > 1\}$ has the relative boundary ∂G consisting of a countable number of mutually disjoint open analytic arcs without end points in W not accumulating in W. Let F be the closure of G taken in R^* so that $F = \{\zeta \in W \cup (\gamma R) : \tau p(\zeta) \geqq 1\}$ and $F \supset K$. We consider one more function $q := \min\{\tau p, 1\}$. Fix an analytic regular exhaustion $(R_i)_{i \geqq 0}$ of R with $R \setminus \overline{W} = R_0$, i.e. each R_i $(i \geqq 0)$ is an analytic subregion of R with the complement $R \setminus \overline{R_i}$ each component of which is relatively noncompact, $\overline{R}_i \subset R_{i+1}$ $(i \geqq 0)$ and $R = \cup_{i \geqq 0} R_i$. Then set $W_i := W \cap R_i$ $(i \geqq 1)$. Note that $\overline{R}_0 \cap F = \emptyset$. We put $\alpha_i := (\partial G) \cap \overline{W}_i$ and $\beta_i := (\partial W_i) \setminus (F \cup \partial W)$ so that $\partial(W_i \setminus F) = \alpha_i \cup \beta_i \cup \partial W$ for every $i \geqq 1$. We define a continuous function u_i on $\overline{W}_i$ for each $i \geqq 1$ as follows. First $u_i(z) = 1$ for $z \in \overline{W}_i \cap F$. On $\overline{W}_i \setminus F$ we require that u_i be given as the harmonic function on $W_i \setminus F$ having the following mixed boundary condition: $u_i | \partial W = 0$, $u_i | \alpha_i = 1$, and $*du_i | \beta_i = 0$.

One of simple ways to construct such a harmonic function u_i on $W_i \setminus F$ is as follows. Recall that the *double* $\hat{R}_i$ of the analytic subregion R_i along its relative boundary ∂R_i is the closed Riemann surface characterized by the following 4 conditions: i) $R_i \cup \partial R_i$ is embedded into $\hat{R}_i$; ii) the embedding map in i) is conformal on R_i; iii) each component of the image of ∂R_i under the embedding map in i) is an analytic curve in $\hat{R}_i$; iv) there exists an anticonformal selfmapping $\sigma : \hat{R}_i \to \hat{R}_i$ such that $\sigma \circ \sigma$ is the identity mapping on $\hat{R}_i$ and $\sigma | \partial R_i$ is the identity mapping on ∂R_i. Hence

$$\hat{R}_i = R_i \cup (\partial R_i) \cup \sigma(R_i).$$

Roughly speaking the double $\hat{R}_i$ of R_i along ∂R_i is a symmetric extension of R_i across ∂R_i and easily constructed based upon the reflection principle (cf. e.g. [**1**]). We can now consider the double $\hat{W}_i$ of W_i formed only along ∂R_i which is nothing but the subregion $\hat{W}_i := \hat{R}_i \setminus (\overline{R}_0 \cup \sigma(\overline{R}_0))$ of $\hat{R}_i$ so that

$$\hat{W}_i = W_i \cup (\partial R_i) \cup \sigma(W_i)$$

considered in $\hat{R}_i$. Let $\hat{F} := (F \cap \overline{W}_i) \cup \sigma(F \cap \overline{W}_i)$, $\hat{\alpha} := \alpha_i \cup \sigma(\alpha_i)$, $\beta = \partial R_i \setminus F = \beta_i$ in $\hat{W}_i$, $\hat{\omega} = (\partial W) \cup \sigma(\partial W)$, and $S := \hat{W}_i \setminus \hat{F}$. Then S is a subregion of R_i with

$\partial S = \hat{\omega} \cup \hat{\alpha}$, which is regular in the sense that every point in ∂S is regular with respect to the Dirichlet problem on S with respect to ∂S. Consider the boundary data $\varphi \in C(\partial S)$ such that $\varphi|\hat{\omega} = 0$ and $\varphi|\hat{\alpha} = 1$. The function $\hat{u}_i := H^S_\varphi$ is then harmonic on S with the boundary values 1 on $\hat{\alpha} = \alpha_i \cup \sigma(\alpha_i)$ and 0 on $\hat{\omega} = (\partial W) \cup \sigma(\partial W)$. Observe that $\hat{u}_i \circ \sigma$ is also harmonic on S and having the boundary values 1 on $\hat{\alpha}$ and 0 on $\hat{\omega}$ and thus $\hat{u}_i \circ \sigma = H^S_\varphi$. Therefore $\hat{u}_i \circ \sigma = \hat{u}_i$ on S so that $\hat{u}_i$ is symmetric about β and then the outer normal derivative $\partial \hat{u}_i/\partial n = 0$ on β with respect to the region $W_i \setminus F$ (and also with respect to $\sigma(W_i \setminus F)$). Since $*d\hat{u}_i = (\partial \hat{u}_i/\partial n)ds = 0$ on β with the line element ds on β. Then $u_i := \hat{u}_i|(W_i \setminus F)$ is the required one.

For the time being we view $R' := R \setminus F$ as the whole basic surface, $W'_i := W_i \setminus F$ as an end of R'. But $R'_i := R_i \setminus F$ is not relatively compact in R' and hence $(R'_i)_{i \geqq 0}$ is not an exhaustion of R' but still exhausts R'. The same is true of $W'_i := W_i \setminus F$. We denote by $D'(\cdot)$ and $D'(\cdot,\cdot)$ for $D(\cdot; W \setminus F)$ and $D(\cdot,\cdot; W \setminus F)$; $D'_i(\cdot)$ and $D'_i(\cdot,\cdot)$ for $D(\cdot; W_i \setminus F)$ and $D(\cdot,\cdot; W_i \setminus F)$. Then we can consider $HD(W'; \partial W')$ viewing R' as the whole basic surface, and similarly, viewing R'_i as the whole surface so that W'_i as an end of R'_i, we can also consider $HD(W'_i; \partial W'_i)$. Recall that $q = \min\{\tau p, 1\}$. Then

$$D'_i(u_i - q, u_i) = \int_{\alpha_i \cup \beta_i \cup \partial W} (u_i - q) * du_i = 0$$

since $u_i - q = 0$ on $\alpha_i \cup \partial W$ and $*du_i = 0$ on β_i and thus $D'_i(u_i) = D'_i(q, u_i)$. By the Schwarz inequality, $D'_i(u_i) = D'_i(u_i, q) \leqq D'_i(u_i)^{1/2} D'_i(q)^{1/2}$ and hence

$$D'_i(u_i) \leqq D'_i(q) \leqq D(\tau p; W) < \operatorname{cap}(K) + \frac{\varepsilon}{2} \tag{10}$$

for every $i \in \boldsymbol{N}$, the set of positive integers. Let $i < j$. Similarly as above

$$D'_i(u_j - u_i; u_i) = \int_{\alpha_i \cup \beta_i \cup \partial W} (u_j - u_i) * du_i = 0$$

and thus $D'_i(u_j, u_i) = D'_i(u_i)$. This implies that

$$D'_i(u_j) - D'_i(u_i) = D'_i(u_j - u_i) \geqq 0.$$

Hence $D'_j(u_j) \geqq D'_i(u_j) \geqq D'_i(u_i)$ and the sequence $(D'_i(u_i))_{i \geqq 1}$ is increasing and bounded by (10) so that it is convergent. Therefore

$$\lim_{i<j, i\to\infty} D_i'(u_i - u_j) = 0. \tag{11}$$

This in particular shows that $(u_i)_{i\geqq i_0}$ is a Cauchy sequence in the Hilbert space $HD(W'_{i_0}; \partial W'_{i_0})$ for every fixed $i_0 \in \boldsymbol{N}$ and we can find a $u \in HD(W'; \partial W')$ such that

$$\lim_{i\to\infty} D_i'(u_i - u) = 0$$

and in particular

$$u = \lim_{i\to\infty} u_i \tag{12}$$

locally uniformly on $W' = W \setminus F$ by virtue of (1). However, since $0 \leqq u_i \leqq 1$ on W_i and the boundary values of positive harmonic functions u_i on ∂W are zero and also the boundary values of positive harmonic functions $1 - u_i$ on $(\partial F) \cap W_i = (\partial G) \cap W_i$ are zero, the boundary Harnack inequality assures that the convergence in (12) is in fact locally uniform on $(\overline{W \setminus F}) \cap R = (W \setminus F) \cup \partial W \cup \partial F$. Hence $u \in HD(W \setminus F) \cap C(\overline{W \setminus F})$ with $u|\partial W = 0$ and $u|\partial F = 1$. We extend u to R by $u = 0$ on R_0 and $u = 1$ on G. Then $u \in C(R)$. By (10) and (11) we can conclude that

$$D(u; W) \leqq \mathrm{cap}(K) + \frac{\epsilon}{2}. \tag{13}$$

Take an arbitrary $w \in HD(W; \partial W)^+$ and we compute $D(w, u; W)$. Observe that $w|\partial W = 0$ and $*du_i|\beta_i = 0$ so that

$$D(w, u_i; W_i \setminus F) = \int_{\alpha_i \cup \beta_i \cup \partial W} w * du_i = \int_{\alpha_i} w * du_i.$$

Since $0 < u_i < 1$ on $W_i \setminus F$ and $u_i|\alpha_i = 1$, we see that the outer normal derivative $\partial u_i/\partial n \geqq 0$ with respect to $W_i \setminus F$ and thus $*du_i|\alpha_i \geqq 0$. Hence $w * du_i \geqq 0$ on α_i and $D(w, u_i; W_i \setminus F) \geqq 0$. Then, in view of (11) and $u|F = 1$, we have

$$\begin{aligned} D(w, u; W) &= \lim_{i\to\infty} D(w, u; W_i) \\ &= \lim_{i\to\infty} D(w, u; W_i \setminus F) = \lim_{i\to\infty} D(w, u_i; W_i \setminus F) \geqq 0, \end{aligned}$$

that is, we have seen that

$$D(w, u; W) \geqq 0 \tag{14}$$

for every $w \in HD(W; \partial W)^+$.

We are now in the final stage to construct the required $h_\varepsilon \in \mathscr{W}(K)$ with (8) and (9). We apply the Royden decomposition theorem (cf. [**7**]) to the function u, which belongs to $L^{1,2}(R) \cap C(R)$. Let h_ε be the harmonic part and g be the potential part of u considered on W:

$$u = h_\varepsilon + g, \tag{15}$$

where $h_\varepsilon \in HD(W; \partial W) \cap [L^{1,2}(R) \cap C(R)]$ with $h_\varepsilon | \overline{R}_0 = 0$ and $g \in L^{1,2}(R) \cap C(R)$ with $g | \overline{R}_0 \cup \delta R = 0$ and satisfies

$$D(v, g; W) = 0 \tag{16}$$

for every $v \in HD(W; \partial W)$ so that

$$D(u; W) = D(h_\varepsilon; W) + D(g; W). \tag{17}$$

By the above (17) and (13) we deduce

$$D(h_\varepsilon; W) \leqq D(u; W) \leq \operatorname{cap}(K) + \frac{\varepsilon}{2} < \operatorname{cap}(K) + \varepsilon,$$

which shows that h_ε satisfies (9). Next take any $w \in HD(W; \partial W)^+$ and observe by (15) and (16) that

$$D(u, w; W) - D(h_\varepsilon, w; W) = D(g, w; W) = 0.$$

Then we deduce by (14) that

$$D(w, h_\varepsilon; W) = D(h_\varepsilon, w; W) = D(u, w; W) \geqq 0,$$

which shows the validity of (8). □

A function $u \in HD(W; \partial W)$ is said to have the *normal derivative measure* $*du$ on γR supported by δR if $*du$ is a Radon measure on γR (in general signed) whose support is contained in δR such that

$$D(v,u;W)=\int_{\delta R} v*du \tag{18}$$

holds for every $v\in HD(W;\partial R)$ (cf. e.g. [**5**]). A necessary and sufficient condition for a $u\in HD(W;\partial W)$ to have $*du$ on δR is

$$\sup_{v\in HD(W;\partial W)\setminus\{0\}} |D(v,u;W)|/\sup_W |v|<+\infty$$

but a simple sufficient condition easy to use is that

$$D(v,u;W)\geqq 0$$

for every $v\in HD(W;\partial W)^+$. If R is a compact bordered surface with analytic border ∂R and $u\in HD(W;\partial W)\cap C^1(W\cup\partial R)$, then the conjugate differential $*du$ of du on ∂R takes the form $*du=(\partial u/\partial n)ds$, where $\partial u/\partial n$ is the directional derivative of u on the border ∂R, to the direction of the outer normal with respect to W and ds the line element on ∂R, and the Stokes formula yields that

$$D(v,u;W)=\int_{\partial R} v*du$$

for every $v\in HD(W,\partial W)\cap C^1(W\cup\partial R)$. Appealing to this analogy we employ the impressive notation $*du$ though it superfacially has nothing to do with the exterior differential calculus and the term normal derivative measure in the above definition (18). We now show that the capacitary function satisfies a certain mixed boundary condition described below which will turn out to be the characterizing property for the capacitary function later.

Theorem 19. *The capacitary function c_K on an end W of an open Riemann surface R for the capacity* cap(K) *of any compact subset K of the Royden harmonic boundary δR of R relative to the end W of R has the following four properties: first, $c_K\in HD(W;\partial W)$; second, c_K has the nonnegative normal derivative measure $*dc_K\geqq 0$ on δR; third, $*dc_K=0$ on $\delta R\setminus K$; fourth and lastly, $c_K=1$ quasieverywhere on K.*

Proof. For each $i\in\boldsymbol{N}$ we take the $v_i:=h_{1/(i+k)}$ in Proposition 7 for sufficiently large fixed $k\in\boldsymbol{N}$. Then by (9) we have

$$\mathrm{cap}(K)\leqq D(v_i;W)\leqq \mathrm{cap}(K)+1/i$$

for every $i \in \boldsymbol{N}$ and thus $(v_i)_{i \geqq 1}$ is a minimal sequence in $\mathscr{W}(K)$. Thus by Proposition 4, $D(v_i - c_K; W) \to 0$ $(i \to \infty)$ and, by virtue of (1), $v_i \to c_K$ $(i \to \infty)$ locally uniformly on $W \cup \partial W$. Thus $c_K \in \overline{\mathscr{W}(K)}$ and in particular $c_K \in HD(W; \partial W)$ so that the first property for c_K is shown.

For any $w \in HD(W; \partial W)^+$, (8) implies $D(w, v_i; W) \geqq 0$ for every $i \in \boldsymbol{N}$ and therefore

$$D(w, c_K; W) = \lim_{i \to \infty} (w, v_i; W) \geqq 0. \tag{20}$$

Set $A := HD(W; \partial W)|\delta R$ and let $\rho : HD(W; \partial W) \to A$ be the restriction mapping. By the definition of the harmonic boundary δR and the maximum principle related to δR (cf. [**7**]), $\rho : HD(W; \partial W) \to A$ and its inverse $\rho^{-1} : A \to HD(W; \partial W)$ are bijective, order-preserving, and linear isomorphisms. Then the functional

$$\varphi \mapsto D(\rho^{-1}\varphi, c_K; W)$$

on A is linear and, by (20), positive. Since A is dense in $C(\delta R)$, the Risz representation theorem assures the existence of a positive Borel measure μ on δR such that

$$D(\rho^{-1}\varphi, c_K; W) = \int_{\delta R} \varphi d\mu$$

for every $\varphi \in A = \rho(HD(W; \partial W))$, or equivalently,

$$D(w, c_K; W) = \int_{\delta R} w d\mu \tag{21}$$

for every $w \in HD(W; \partial W)$. In view of (18), we see, by compareing (18) and (21), the existence of $*dc_K = d\mu \geqq 0$ on δR. Thus we have shown that the second property of c_K stated in the theorem is valid.

Choose any $g \in HD(W; \partial W)$ vanishing on K. For any $t \in \boldsymbol{R}$, the real number field, $v_i + tg \in \mathscr{V}(K)$ for every $i \in \boldsymbol{N}$ so that $D(v_i + tg; W) \geqq \mathrm{cap}(K)$ and by making $i \to \infty$ we deduce $D(c_K + tg; W) \geqq \mathrm{cap}(K)$. Since $D(c_K; W) = \mathrm{cap}(K)$, we conclude that

$$2D(g, c_K; W)t + D(g; W)t^2 \geqq 0$$

for every $t \in \boldsymbol{R}$. Thus we see that $D(g, c_K; W) = 0$ and a fortiori

$$\int_{\delta R} g * dc_K = 0$$

for every $g \in HD(W; \partial W)$ with $g|K = 0$. This shows, since $*dc_K \geqq 0$ on δR, that the support of the measure $*dc_K$ is contained in K, i.e. $*dc_K = 0$ on $\delta R \setminus K$, proving the validity of the third property of c_K.

We turn to the final property. By checking the construction of h_ϵ in Proposition 7, we see that $0 \leqq h_\varepsilon \leqq 1$. Hence $0 \leqq v_i \leqq 1$ for every $i \in \boldsymbol{N}$. As c_K is the local uniform limit of v_i on $W \cup \partial W$, we see that $0 \leqq c_K \leqq 1$ on W. Thus, in particular, $0 \leqq c_K \leqq 1$ on K. The set

$$F_n := \{\zeta \in K : c_K(\zeta) \leqq 1 - 1/n\} \quad (n \in \boldsymbol{N})$$

is compact and

$$F := \{\zeta \in K : c_K(\zeta) < 1\} = \cup_{n \in \boldsymbol{N}} F_n$$

is capacitable as an F_σ-set. It is seen that $\mathrm{cap}(F_n) \uparrow \mathrm{cap}(F) \quad (n \uparrow \infty)$ (cf. e.g. [**3**]) so that we only have to derive a certain contradiction from the erroneous assumption $\mathrm{cap}(F_n) > 0$ for some n in order to maintain that $\mathrm{cap}(F) = 0$, which is nothing but the statement $c_K = 1$ quasieverywhere on K. Since $D(c_{F_n}; W) = \mathrm{cap}(F_n) > 0$, we see that $0 < c_{F_n} < 1$ on W. Let $(p_i)_{i \geqq 1}$ ($(q_i)_{i \geqq 1}$, resp.) be a minimal sequence in $\mathscr{W}(K)$ ($\mathscr{W}(F_n)$, resp.). Then on letting $i \to \infty$ in

$$D(q_i, c_{F_n}; W) = \int_{\delta R} q_i * dc_{F_n} = \int_{\delta R} * dc_{F_n},$$

we conclude that

$$\int_{\delta R} * dc_{F_n} = D(c_{F_n}; W) = \mathrm{cap}(F_n) > 0. \tag{22}$$

From $D(p_i, c_{F_n}; W) = \int_{\delta R} * dc_{F_n}$ and $D(c_K, c_{F_n}; W) = \int_{\delta R} c_K * dc_{F_n}$ it follows by making $i \to \infty$ that

$$D(c_K, c_{F_n}; W) = \int_{\delta R} * dc_{F_n} = \int_{\delta R} c_K * dc_{F_n}$$

so that, by noting $1 - c_K \geqq 0$ and $*dc_{F_n} \geqq 0$ on δR, we see that

$$0 = \int_{\delta R} (1 - c_K) * dc_{F_n} \geqq \int_{F_n} (1 - c_K) * dc_{F_n}$$
$$\geqq \frac{1}{n} \int_{F_n} *dc_{F_n} = \frac{1}{n} \int_{\delta R} *dc_{F_n}$$

or, equivalently, $\int_{\delta R} *dc_{F_n} \leqq 0$, which contradicts (22), and we are done. □

The following assertions are contained in the above proof. Namely, we have the following result.

COROLLARY 23. *The capacitary function c_K for* cap(K) *of a nonpolar compact subset $K \subset \delta R$ relative to an end W of R satisfies*

$$0 < c_K < 1 \tag{24}$$

on W and

$$\text{cap}(K) = \int_{\delta R} *dc_K. \tag{25}$$

We now come to the final stage to state and prove our main result of this paper to characterize the capacitary function as a solution of a certain mixed boundary value problem for the Laplace equation with boundary data on the Royden harmonic boundary (see the announcement [**6**]).

THEOREM 26 (The main theorem). *A function h is the capacitary function c_K for a compact subset K of the Royden harmonic boundary δR of an open Riemann surface R relative to an end W of R, i.e. $h = c_K$, if and only if h satisfies the following four conditions: first of all, $h \in HD(W; \partial W)$; second, h has the normal derivative measure $*dh \geqq 0$ on δR; third, $*dh = 0$ on $\delta R \setminus K$; fourth and lastly, $h = 1$ quasieverywhere on K.*

PROOF. If $h = c_K$, then, by Theorem 19, we can see that h satisfies the four conditions stated in the theorem. Therefore we only have to show that a function h on W coincides with c_K if h satisfies the four conditions in the theorem. Hence, on setting $p = h - c_K$, we are to show that $p \equiv 0$ on R. We also denote by q either h or c_K so that q also satisfies the four conditions in the theorem. Consider the set F of points ζ in K such that $p(\zeta) \neq 0$, or equivalently $|p(\zeta)| \neq 0$. Since F is covered by the union of two sets F_q $(q = h, c_K)$, where $F_q := \{\zeta \in K : q(\zeta) \neq 1\}$. By the assumption that cap$(F_q) = 0$, we see that

$$\operatorname{cap}(F) \leqq \operatorname{cap}(F_h \cup F_{c_K}) \leqq \operatorname{cap}(F_h) + \operatorname{cap}(F_{c_K}) = 0,$$

i.e. F is of capacity zero. For each $n \in \boldsymbol{N}$ we set $F_n := \{\zeta \in K : |p(\zeta)| \geqq 1/n\}$, which is a compact subset of the compact set K since $|p| \in C(\delta R) \subset C(K)$. Clearly $F_n \subset F$, and, for any $\zeta \in F$, we can find an $n \in \boldsymbol{N}$ with $\zeta \in F_n$ so that

$$F = \bigcup_{n \in \boldsymbol{N}} F_n. \tag{27}$$

For each $n \in \boldsymbol{N}$, we see that $\operatorname{cap}(F_n) = 0$ along with $\operatorname{cap}(F) = 0$. Fix arbitrarily an $n \in \boldsymbol{N}$. Then we can find a minimal sequence $(v_i)_{i \in \boldsymbol{N}}$ in $\mathscr{W}(F_n)$. We write $D(\cdot)$ and $D(\cdot, \cdot)$ for $D(\cdot; W)$ and $D(\cdot, \cdot; W)$. We infer that

$$\begin{aligned} 0 \leqq \int_{F_n} {*dq} = \int_{F_n} v_i * dq \\ = D(v_i, q) \leqq D(v_i)^{1/2} D(q)^{1/2} \to \operatorname{cap}(F_n)^{1/2} D(q)^{1/2} = 0 \end{aligned}$$

as $i \to \infty$. Hence we see that $\int_{F_n} {*dq} = 0$ for each n and therefore by (27) we conclude that

$$\int_F {*dq} = 0 \quad (q = h, c_K).$$

Recall that ${*dq} = 0$ on $\delta R \setminus K$ and $p = 0$ on $K \setminus F$. A fortiori we deduce that

$$D(p, q) = \int_{\delta R} p * dq = \int_K p * dq = \int_F p * dq = 0$$

for $q = h$ and $q = c_K$. Then

$$D(p) = D(p, p) = D(p, h - c_K) = D(p, h) - D(p, c_K) = 0,$$

i.e. $D(p; W) = 0$ and with $p|\partial W = 0$ we can conclude that $p \equiv 0$ on W, which was to be proved. □

There are many applications of the main theorem 26 (or the theorem 19) expected, among which we state here the following simple and direct one concerning the harmonic measure and the capacity on δR. The *harmonic measure* hm (hm_0, resp.) on γR supported by δR relative to (R, a) ((W, a), resp.), a being a reference point in R (W, resp.), is a Borel measure on γR such that

$$H_f^R(a) = \int_{\delta R} f d\mathrm{hm} \quad \left(H_f^W(a) = \int_{\delta R} f d\mathrm{hm}_0, \quad \text{resp.} \right) \tag{28}$$

for every $f \in C(\delta R)$. This has been an important tool in the application of the Royden compactification theory to the classification theory of Riemann surfaces (cf. e.g. [**7**]). For a fixed reference point $a \in W \subset R$, there is a constant $c = c(W, a) \in [1, +\infty)$ such that

$$\frac{1}{c} \leqq \liminf_{z \to \zeta} \frac{G(z, a; W)}{G(z, a; R)} \leqq \limsup_{z \to \zeta} \frac{G(z, a; W)}{G(z, a; R)} \leqq 1$$

for every $\zeta \in \gamma R$. Based upon this inequality we see at once that

$$\frac{1}{c} \mathrm{hm}(X) \leqq \mathrm{hm}_0(X) \leqq \mathrm{hm}(X) \tag{29}$$

for every Borel subset $X \subset \delta R$. Observe that the Bergman kernel $B(\cdot, a; W) \in HD(W; \partial W)$ satisfies

$$D(v, B(\cdot, a; W); W) = v(a) \geqq 0$$

for every $v \in HD(W; \partial W)^+$ so that the normal derivative measure $*dB(\cdot, a; W)$ of $B(\cdot, a; W)$ exists on δR. Hence, by (28), we see that

$$\int_{\delta R} u d\mathrm{hm}_0 = u(a) = D(u, B(\cdot, a; W); W) = \int_{\delta R} u * dB(\cdot, a; W)$$

for every $u \in HD(W; \partial W)$. Since $HD(W; \partial W)|\delta$ is uniformly dense in $C(\delta R)$, we can conclude that

$$d\mathrm{hm}_0 = *dB(\cdot, a; W) \tag{30}$$

as measures on δR.

We now have two important measurements $\mathrm{cap}(K)$ and $\mathrm{hm}(K)$ for every compact subset K of δR. In many instances we are usually interested not in the quantities $\mathrm{cap}(K)$ and $\mathrm{hm}(K)$ themselves but rather in the properties whether they vanish or not. Concerning the relation between $\mathrm{cap}(K)$ and $\mathrm{hm}(K)$, we are therefore satisfied in many cases with the standard knowledge that $\mathrm{cap}(K) = 0$ implies $\mathrm{hm}(K) = 0$. In view of this, one step further, one naturally expect that if

$(\mathrm{cap}(K_n))_{n\in\boldsymbol{N}}$ is a zero sequence, then $(\mathrm{hm}(K_n))_{n\in\boldsymbol{N}}$ is also a zero sequence, $\boldsymbol{N}$ being the set of positive integers, and then one might wish to know the relation between speeds of convergence of these two zero sequences above. The following inequality contributes to this question.

Fact 31. *There is a positive constant κ depending only upon the triple (R, W, a) such that*

$$\mathrm{hm}(K) \leqq \kappa \cdot \mathrm{cap}(K)^{1/2} \tag{32}$$

for every compact subset K of δR.

Proof. Suppose first that $\mathrm{cap}(K) = 0$. Then there is a sequence $(u_n)_{n\in\boldsymbol{N}} \subset \mathscr{W}(K)$ such that $D(u_n; W) \to 0$ $(n \to \infty)$ so that $u_n(a) \to 0$ $(n \to \infty)$. Observe that

$$\mathrm{hm}_0(K) = \int_K d\mathrm{hm}_0 \leqq \int_K u_n d\mathrm{hm}_0 = u_n(a) \to 0 \ (n \to \infty),$$

i.e. $\mathrm{hm}_0(K) = 0$ so that we have by (29) the relation $\mathrm{hm}(K) = 0$. A fortiori (32) is trivially true. Therefore, we have also obtained that a Borel subset X of δR is of harmonic measure zero if X is polar. Next we consider the case $\mathrm{cap}(K) > 0$. Then $c_K = 1$ hm_0-a.e. on K and thus, by (30), we have

$$\begin{aligned}\mathrm{hm}_0(K) &= \int_K d\mathrm{hm}_0 = \int_K c_K d\mathrm{hm}_0 = \int_K c_K * dB(\cdot, a; W)\\ &= D(c_K, B(\cdot, a; W); W) \leqq D(c_K; W)^{1/2} \cdot D(B(\cdot, a; W); W)^{1/2}.\end{aligned}$$

From (29) and the above relation it follows that

$$\frac{1}{c}\mathrm{hm}(K) \leqq \mathrm{hm}_0(K) \leqq B(a, a; W)^{1/2} \cdot \mathrm{cap}(K)^{1/2},$$

which implies (32) with $\kappa := cB(a, a; W)^{1/2}$. □

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