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A characterization of the homogeneous minimal ruled real hypersurface in a complex hyperbolic space

Dedicated to Professor Masao Hashiguchi on the occasion of his 77th birthday

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Abstract. It is well-known that there exist no homogeneous *ruled real hypersurfaces* in a complex projective space. On the contrary there exists the unique homogeneous ruled real hypersurface in a complex hyperbolic space. Moreover, it is minimal. We characterize geometrically this minimal homogeneous real hypersurface by properties of extrinsic shapes of some curves.

1. Introduction.

For a non-zero constant c we denote by $M_n(c)$ a complex n-dimensional complete and simply connected Kähler manifold of constant holomorphic sectional curvature c. It is hence holomorphically isometric to a complex projective space $\boldsymbol{C}P^n(c)$ when $c > 0$, and is holomorphically isometric to a complex hyperbolic space $\boldsymbol{C}H^n(c)$ when $c < 0$. In the study of real hypersurfaces of $\widetilde{M} = M_n(c)$ there can be the following two problems.

(1) Classify homogeneous real hypersurfaces in $M_n(c)$ ($c \neq 0$) and characterize such examples in the class of all real hypersurfaces.
(2) Construct non-homogeneous *nice* real hypersurfaces in $M_n(c)$ ($c \neq 0$) and characterize such examples in the class of all real hypersurfaces.

A real hypersurface M is called *homogeneous* in $\widetilde{M} = M_n(c)$ if it is given as an orbit under some subgroup of the full isometry group $\mathrm{I}\,(M_n(c))$ of the ambient

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space $\widetilde{M} = M_n(c)$. In his paper, Takagi ([**14**]) classified all homogeneous real hypersurfaces in $\boldsymbol{C}P^n(c)$. After about 30 years Berndt and Tamaru ([**4**]) classified such hypersurfaces in $\boldsymbol{C}H^n(c)$. Those hypersurfaces are treated as typical examples in the theory of real hypersurfaces in $M_n(c)$ $(c \neq 0)$. There are many characterizations of such real hypersurfaces (for instance, see [**1**], [**3**], [**6**], [**7**], [**10**], [**12**], [**13**]).

In this paper, we treat ruled real hypersurfaces of $\widetilde{M} = M_n(c)$ $(c \neq 0)$ (for the definition of ruled real hypersurfaces and other fundamental knowledge on such hypersurfaces, see Section 2). It is well-known that every ruled real hypersurface in $\boldsymbol{C}P^n(c)$ is not homogeneous (see [**14**]). This fact implies that we study ruled real hypersurfaces in $\boldsymbol{C}P^n(c)$ only from the viewpoint of Problem (2). On the other hand, there exists the unique homogeneous (minimal) ruled real hypersurface in $\boldsymbol{C}H^n(c)$ (see [**4**]). Needless to say, there exist also many non-homogeneous ruled real hypersurfaces in $\boldsymbol{C}H^n(c)$. Hence we can investigate ruled real hypersurfaces in $\boldsymbol{C}H^n(c)$ from both of the viewpoints of Problems (1) and (2), which enriches the study of real hypersurfaces in $\boldsymbol{C}H^n(c)$. The purpose of this paper is to give a geometric characterization of the homogeneous minimal ruled real hypersurface of $\boldsymbol{C}H^n(c)$, which is based on the viewpoint of Problem (1).

2. Ruled real hypersurfaces in a complex space form.

For a real hypersurface M^{2n-1} with unit normal local vector field $\mathscr{N}$ in a Kähler manifold $(\widetilde{M}, J, \langle\ ,\ \rangle)$, we can naturally define an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ as $\eta(v) = \langle \xi, v\rangle$ and $\phi(v) = Jv - \eta(v)\mathscr{N}$ with the characteristic vector field $\xi = -J\mathscr{N}$, where $\langle\ ,\ \rangle$ is the Riemannian metric on M induced from the metric $\langle\ ,\ \rangle$ of the ambient Kähler manifold $\widetilde{M}$. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}$ and ∇ of M are related by the following formulas of Gauss and Weingarten

$$\begin{cases} \widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle \mathscr{N}, \\ \widetilde{\nabla}_X \mathscr{N} = -AX \end{cases} \tag{2.1}$$

for vector fields X and Y on M, where A is the shape operator of M in $\widetilde{M}$. Thus we have

$$(\nabla_X \phi)Y = \eta(Y)AX - \langle AX, Y\rangle \xi, \tag{2.2}$$

$$\nabla_X \xi = \phi AX. \tag{2.3}$$

In this paper we study ruled real hypersurfaces in $\widetilde{M} = M_n(c)$. A real hypersurface M in the ambient space $\widetilde{M} = M_n(c)$ $(n \geqq 2,\ c \neq 0)$ is said to be a *ruled real hypersurface* if the holomorphic distribution T^0M, which is a subbundle of TM defined by $T^0M = \bigcup_{x\in M}\{v \in T_xM \mid v \perp \xi_x\}$, is integrable and each of its maximal integral manifolds is locally congruent to a totally geodesic complex hypersurface $M_{n-1}(c)$ in the ambient space $\widetilde{M} = M_n(c)$. It is known that every ruled real hypersurface is constructed in the following manner. We take an arbitrary regular (real) curve $\gamma : I \to M_n(c)$ defined on some open interval I. At each point $\gamma(s)$ $(s \in I)$ we attach a totally geodesic complex hypersurface M_s which is locally congruent to $M_{n-1}(c)$ of $\widetilde{M} = M_n(c)$ and is orthogonal to the plane spanned by $\{\dot\gamma(s),\ J\dot\gamma(s)\}$ at the point $\gamma(s)$. We then get a ruled real hypersurface $M = \bigcup_{s\in I} M_s$ in $\widetilde{M} = M_n(c)$.

Ruled real hypersurfaces in $\widetilde{M} = M_n(c)$ are characterized by properties of their shape operators in the following manner (see [**13**] and Proposition 2 in [**11**]).

Lemma 1. *For a real hypersurface M in $\widetilde{M} = M_n(c)$ $(n \geqq 2,\ c \neq 0)$, the following conditions are mutually equivalent.*

(1) *M is a ruled real hypersurface.*
(2) *The shape operator A of M satisfies $\langle Av, w\rangle = 0$ for arbitrary tangent vectors $v, w \in T_xM$ orthogonal to ξ_x at any point $x \in M$.*
(3) *If we define differentiable functions μ, ν on M by $\mu = \langle A\xi, \xi\rangle$ and $\nu = \|A\xi - \mu\xi\|$, then they satisfy the following two conditions.*
   i) *The set $M_1 = \{x \in M \mid \nu(x) \neq 0\}$ is an open dense subset of M.*
   ii) *With a unit vector field U on M_1 orthogonal to ξ the shape operator A of M satisfies*

$$A\xi = \mu\xi + \nu U, \qquad AU = \nu\xi, \quad Av = 0 \tag{2.4}$$

   on M_1 for an arbitrary tangent vector $v \in T_xM$ orthogonal to ξ_x and U_x.

Remark 1.

(1) We treat a ruled real hypersurface *locally*, because generally this hypersurface has self-intersections and singularities. Moreover, we usually omit points where ξ is principal. That is, when we study a ruled real hypersurface M, we suppose that M_1 $(:= \{x \in M \mid \nu(x) \neq 0\})$ coincides with M.
(2) Every leaf M_s of a ruled real hypersurface $M = \bigcup_{s\in I} M_s$ in $\widetilde{M} = M_n(c)$ is a totally geodesic submanifold of M.

We say a real hypersurface M to be a Hopf hypersurface if ξ is a principal curvature vector of M at its each point in the ambient space $\widetilde{M} = M_n(c)$, namely for the shape operator A of M in $\widetilde{M} = M_n(c)$, it satisfies $A\xi = \alpha\xi$ with a function α on M. It is well-known that this function α is automatically locally constant on each Hopf hypersurface M and that tubes of sufficiently small constant radius around Kähler submanifolds in $\widetilde{M} = M_n(c)$ are Hopf hypersurfaces. The following proposition characterizes geometrically all Hopf hypersurfaces of $\widetilde{M} = M_n(c)$.

PROPOSITION 1. *For a real hypersurface M in $\widetilde{M} = M_n(c)$ $(n \geqq 2,\ c \neq 0)$, the following two conditions are mutually equivalent.*

(1) *M is a Hopf hypersurface in $\widetilde{M}$.*
(2) *At each point $x \in M$, if we take a totally geodesic holomorphic line $M_1(c)$ in $\widetilde{M}$ through x whose tangent space $T_xM_1(c)$ is the complex one dimensional linear subspace of $T_x\widetilde{M}$ spanned by ξ_x, then the normal section $N_x = M \cap M_1(c)$ given by $M_1(c)$ is the integral curve through the point x for the characteristic vector field ξ of M.*

PROOF. It follows from the first equality in (2.1) and (2.3) that $\widetilde{\nabla}_\xi \xi = \phi A\xi + \langle A\xi, \xi\rangle \mathscr{N}$. This equation implies that the condition (1) in our proposition is equivalent to saying that

$$\widetilde{\nabla}_\xi \xi = \langle A\xi, \xi\rangle \mathscr{N} = \langle A\xi, \xi\rangle J\xi,$$

which is nothing but the condition (2). □

Ruled real hypersurfaces are typical examples of non-Hopf hypersurfaces in $M_n(c)$ $(c \neq 0)$ (see Lemma 1). We compute the function ν for a ruled real hypersurface.

LEMMA 2 (c.f. [**8**]). *For a ruled real hypersurface M in $\widetilde{M} = M_n(c)$, the function ν is of the following form on each geodesic ρ with $\dot\rho(0) = \phi U_{\rho(0)}$, which is the integral curve of ϕU through the point $\rho(0)$. When $c > 0$,*

$$\nu\big(\rho(s)\big) = \left(\frac{\sqrt{c}}{2}\right) \tan\left(\frac{\sqrt{c}(s+a)}{2}\right)$$

with some constant a and when $c < 0$,

$$\nu\big(\rho(s)\big) = -\left(\frac{\sqrt{|c|}}{2}\right) \tanh\left(\frac{\sqrt{|c|}(s+a)}{2}\right) \quad \text{or} \quad \nu\big(\rho(s)\big) = \frac{\sqrt{|c|}}{2}$$

with some constant a. In particular, every ruled real hypersurface in a complex *projective space is not complete.*

PROOF. By use of the Codazzi equation for a hypersurface in $\widetilde{M} = M_n(c)$ which is written as

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2\langle \phi X, Y\rangle \xi\}, \tag{2.5}$$

we have $(\nabla_\xi A)\phi U - (\nabla_{\phi U} A)\xi = -(c/4)U$. On the other hand, from (2.2), (2.3) and (2.4) we find that

$$\begin{aligned}(\nabla_\xi A)\phi U - (\nabla_{\phi U} A)\xi &= \nabla_\xi (A\phi U) - A\nabla_\xi(\phi U) - \nabla_{\phi U}(A\xi) + A\nabla_{\phi U}\xi \\ &= -A\big((\nabla_\xi \phi)U + \phi\nabla_\xi U\big) - \nabla_{\phi U}(\mu\xi + \nu U) \\ &= -A\big(\eta(U)A\xi - \langle A\xi, U\rangle\xi + \phi\nabla_\xi U\big) - (\phi U\mu)\xi \\ &\qquad - \mu\nabla_{\phi U}\xi - (\phi U\nu)U - \nu\nabla_{\phi U}U \\ &= \nu(\mu\xi + \nu U) - A\phi\nabla_\xi U - (\phi U\mu)\xi - (\phi U\nu)U - \nu\nabla_{\phi U}U,\end{aligned}$$

so that

$$\nu\mu\xi + \left(\nu^2 + \frac{c}{4}\right)U - A\phi\nabla_\xi U - (\phi U\mu)\xi - (\phi U\nu)U - \nu\nabla_{\phi U}U = 0. \tag{2.6}$$

Taking the inner product of each side of (2.6) and U, we see by (2.4) that $\phi U\nu = \nu^2 + (c/4)$. Solving this differential equation, we find the form of the function ν on each integral curve ρ of ϕU.

For each vector X which is orthogonal to both ξ and U, taking the inner product of each side of the equation (2.6) and X, we see by (2.4) that $\langle \nabla_{\phi U}U, X\rangle = 0$. This, together with the fact that $\langle \nabla_{\phi U}U, U\rangle = \langle \nabla_{\phi U}U, \xi\rangle = 0$, implies $\nabla_{\phi U}U = 0$. Thus we find by (2.2) that $\nabla_{\phi U}(\phi U) = 0$, so that every integral curve ρ of ϕU is a geodesic on our real hypersurface M. □

Since a complex hyperbolic space $\boldsymbol{C}H^n$ is a Hadamard manifold, being different from the case of a complex projective space $\boldsymbol{C}P^n$, we have complete ruled real hypersurfaces in $\boldsymbol{C}H^n$. In this paper, we pay particular attention to the homogeneous minimal ruled real hypersurface in $\boldsymbol{C}H^n(c)$. In order to explain our result, we here recall such a hypersurface (cf. [**9**]). We take a circle γ in $\boldsymbol{C}H^n(c)$ (with Riemannian connection $\widetilde{\nabla}$), which lies on some totally real totally geodesic real hyperbolic plane $\boldsymbol{R}H^2(c/4)$ and whose curvature is $\sqrt{|c|}/2$. That is, the curve γ is a smooth curve parameterized by its arclength which satisfies $\widetilde{\nabla}_{\dot\gamma}\dot\gamma = (\sqrt{|c|}/2)Y$,

$\widetilde{\nabla}_{\dot\gamma}Y = -(\sqrt{|c|}/2)\dot\gamma$ with unit principal normal vector field Y along γ orthogonal to complex lines spanned by $\dot\gamma$. We should note this circle is an unbounded curve and lies on a horosphere (see for example [**2**], [**5**]). It is known that the ruled real hypersurface obtained by this circle is the unique minimal homogeneous ruled real hypersurface in $\boldsymbol{C}H^n(c)$. This hypersurface is characterized as follows.

Lemma 3 ([**4**], [**9**]). *A ruled real hypersurface M in $\boldsymbol{C}H^n(c)$ ($n \geqq 2$) is the homogeneous minimal hypersurface in $\boldsymbol{C}H^n(c)$ if and only if the shape operator A of M in $\boldsymbol{C}H^n(c)$ satisfies*

$$A\xi = \left(\frac{\sqrt{|c|}}{2}\right)U, \quad AU = \left(\frac{\sqrt{|c|}}{2}\right)\xi, \quad AX = 0. \tag{2.7}$$

The following figure is a section of the homogeneous minimal ruled real hypersurface in the ball model of $\boldsymbol{C}H^n(c)$. The figure shows its image cutted by totally real totally geodesic $\boldsymbol{R}H^2(c/4)$. In this figure, the solid line denotes a circle of positive curvature $\sqrt{|c|}/2$ on $\boldsymbol{R}H^2(c/4)$ and every dotted line denotes a geodesic on $\boldsymbol{R}H^2(c/4)$. We note that when a solid line crosses a dotted line at some point these curves cross orthogonally at this point.

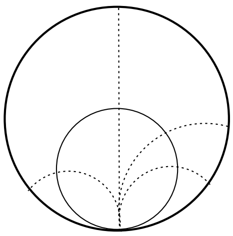

Figure 1. The image of the homogeneous minimal ruled real hypersurface in $\boldsymbol{C}H^n$.

The construction of the homogeneous minimal ruled real hypersurface M in $\boldsymbol{C}H^n(c)$ tells us that the isomery group $\mathrm{I}(M)$ of M is a direct product of the isometry group $\mathrm{I}(\boldsymbol{C}H^{n-1}(c))$ of totally geodesic $\boldsymbol{C}H^{n-1}(c)$ and a one-parameter subgroup $\{\varphi_s\}$ (of the isometry group $\mathrm{I}(\boldsymbol{C}H^2(c))$ of totally geodesic $\boldsymbol{C}H^2(c)$) whose orbit is a circle of curvature $\sqrt{|c|}/2$ on totally geodesic $\boldsymbol{R}H^2(c/4)$. This means that the totally geodesic embedding of every leaf M_s into our homogeneous minimal ruled real hypersurface M is an equivariant mapping.

3. Extrinsic shapes of some curves on the homogeneous minimal ruled real hypersurface.

In this section we give a characterization of the homogeneous minimal ruled real hypersurface in $\boldsymbol{C}H^n$ from the viewpoint of extrinsic shapes of some geodesics and all integral curves of the characteristic vector field ξ on this real hypersurface.

THEOREM. *A real hypersurface M in $\boldsymbol{C}H^n(c)$ $(n \geqq 2)$ is the minimal homogeneous ruled real hypersurface if and only if it satisfies the following three conditions.*

i) *At an arbitrary point $x \in M$, there exist such orthonormal vectors $v_1, \dots, v_{2n-2}$ $(\in T_xM)$ orthogonal to the characteristic vector ξ_x that every geodesic $\gamma_{ij,x}$ on M through the point $\gamma_{ij,x}(0) = x$ in the direction of $v_i + v_j$ $(1 \leqq i \leqq j \leqq 2n-2)$ is an extrinsic geodesic, namely $\gamma_{ij,x}$ is also a geodesic in the ambient space $\boldsymbol{C}H^n(c)$.*
ii) *At an arbitrary point $x \in M$, the integral curve γ_x of the characteristic vector field ξ on M through $\gamma_x(0) = x$ lies locally on a totally real totally geodesic 2-dimensional real hyperbolic space $\boldsymbol{R}H^2(c/4)$ of constant sectional curvature $c/4$ in $\boldsymbol{C}H^n(c)$.*
iii) *The curvature function $\kappa_x = \|\widetilde{\nabla}_{\dot{\gamma}_x}\dot{\gamma}_x\|$ of the curve γ_x in* ii) *does not depend on the choice of γ_x, where $\widetilde{\nabla}$ is the Riemannian connection of the ambient space $\boldsymbol{C}H^n(c)$. This means that for any curves γ_x, γ_y in* ii) *their curvature functions $\kappa_x(s)$ and $\kappa_y(s)$ satisfy the following equality with some constant s_0 : $\kappa_x(s) = \kappa_y(s + s_0)$ for $-\infty < \forall s < \infty$.*

In order to prove our Theorem we here recall some fundamental properties of ruled real hypersurfaces in a nonflat complex space form.

From the viewpoint of extrinsic shapes of geodesics we have the following results.

PROPOSITION 2. *On a ruled real hypersurface M in $\widetilde{M} = M_n(c)$ $(c \neq 0)$, every geodesic ρ whose initial vector $\dot{\rho}(0)$ is orthogonal to the characteristic vector $\xi_{\rho(0)}$ is an extrinsic geodesic.*

PROOF. Let M_0 be the leaf through the point $\rho(0)$ for the holomorphic distribution T^0M. We here take a geodesic ρ_1 on M_0 with the same initial condition that $\rho_1(0) = \rho(0)$ and $\dot{\rho}_1(0) = \dot{\rho}(0)$. Since M_0 is locally congruent to a totally geodesic complex hypersurface $M_{n-1}(c)$ of $\widetilde{M} = M_n(c)$, we see that the curve ρ_1 is also a geodesic in the ambient space $\widetilde{M} = M_n(c)$, which implies that the curve ρ_1 is a geodesic on our ruled real hypersurface M. Hence the uniqueness theorem on geodesics tells us that these two curves ρ and ρ_1 are coincidental. Thus we get the desirable conclusion. □

We should note that the tangent vector $\dot\rho(s)$ of a geodesic ρ in this proposition is orthogonal to $\xi_{\rho(s)}$ at each point $\rho(s)$. Ruled real hypersurfaces are characterized by such a property in Proposition 2.

Lemma 4. *A real hypersurface M of $\widetilde{M} = M_n(c)$ $(c \neq 0)$ is a ruled real hypersurface if and only if the following condition holds: At an arbitrary point $x \in M$ there exist such orthonormal vectors $v_1, \dots, v_{2n-2} \in T_xM$ orthogonal to the characteristic vector ξ_x that every geodesic $\gamma_{ij,x}$ on M through the point $\gamma_{ij,x}(0) = x$ in the direction of $v_i + v_j$ $(1 \leqq i \leqq j \leqq 2n-2)$ is an extrinsic geodesic.*

Proof. It suffices to show the "if" part. We take a geodesic $\gamma_{ii,x} = \gamma_{ii,x}(s)$ $(1 \leqq i \leqq 2n-2)$ on M with initial condition that $\gamma_{ii,x}(0) = x$ and $\dot\gamma_{ii,x}(0) = v_i$. Then it follows from the first equality in (2.1) that $\langle A\dot\gamma_{ii,x}(s), \dot\gamma_{ii,x}(s)\rangle = 0$ for every s. Hence, in particular at the point $x(= \gamma_{ii,x}(0)) \in M$ we have

$$\langle Av_i, v_i\rangle = 0 \quad \text{for } 1 \leqq i \leqq 2n-2. \tag{3.1}$$

We next take a geodesic $\gamma_{ij,x} = \gamma_{ij,x}(s)$ $(1 \leqq i < j \leqq 2n-1)$ on M with initial condition that $\gamma_{ij,x}(0) = x$ and $\dot\gamma_{ij,x}(0) = (v_i + v_j)/\sqrt{2}$. Then, applying the above discussion to the curve $\gamma_{ij,x}$, we get

$$\left\langle \frac{A(v_i+v_j)}{\sqrt{2}}, \frac{v_i+v_j}{\sqrt{2}} \right\rangle = 0 \quad \text{for } 1 \leqq i < j \leqq 2n-2. \tag{3.2}$$

Thus, from (2) in Lemma 1, (3.1) and (3.2) we can see that M is a ruled real hypersurface. □

We are now in a position to prove our Theorem. We are enough to show that a real hypersurface satisfying these three conditions is ruled, minimal and homogeneous in the ambient space $\boldsymbol{C}H^n(c)$. We suppose that a real hypersurface M satisfies the three conditions. By Lemma 4 the first condition guarantees that M is a ruled real hypersurface in $\boldsymbol{C}H^n(c)$. By use of the equalities in (2.1), (2.3) and (2.4) we have

$$\widetilde\nabla_\xi \xi = \nabla_\xi \xi + \langle A\xi, \xi\rangle \mathscr{N} = \phi A\xi + \mu\mathscr{N} = \nu\phi U + \mu\mathscr{N}.$$

On the other hand, by the second condition we see that $\langle \widetilde\nabla_\xi\xi, \mathscr{N}\rangle = \langle \widetilde\nabla_\xi\xi, J\xi\rangle = 0$. Hence we find μ vanishes identically on M, so that M is minimal and $\widetilde\nabla_\xi\xi = \nu(\phi U)$.

We now compute the function ν. By the Gauss formula and the equalities (2.2), (2.4) we have

$$\begin{aligned}\widetilde{\nabla}_\xi(\phi U) &= \nabla_\xi(\phi U) + \langle A\xi, \phi U\rangle \mathscr{N} = (\nabla_\xi \phi)U + \phi(\nabla_\xi U) \\ &= \eta(U)A\xi - \langle A\xi, U\rangle\xi + \phi(\nabla_\xi U) = -\nu\xi + \phi(\nabla_\xi U).\end{aligned} \tag{3.3}$$

We here check $\nabla_\xi U = 0$. It is clear that $\langle \nabla_\xi U, \xi\rangle = 0 = \langle \nabla_\xi U, U\rangle$ from (2.3), (2.4) and the facts $\langle \xi, U\rangle = 0$, $\langle U, U\rangle = 1$. So we only need to verify that $\langle \nabla_\xi U, X\rangle = 0$ for each X perpendicular to ξ and U. We take such a vector X. For any vector Y orthogonal to ξ, the Codazzi equation (2.5) gives

$$(\nabla_\xi A)Y - (\nabla_Y A)\xi = (c/4)\phi Y. \tag{3.4}$$

On the other hand, from (2.3) and (2.4) we have

$$\begin{aligned}(\nabla_\xi A)X - (\nabla_X A)\xi &= \nabla_\xi(AX) - A\nabla_\xi X - \nabla_X(A\xi) + A\nabla_X \xi \\ &= -A\nabla_\xi X - \nabla_X(\nu U) + A\phi AX \\ &= -A\nabla_\xi X - (X\nu)U - \nu\nabla_X U.\end{aligned}$$

This, together with (3.4), yields

$$A\nabla_\xi X + (X\nu)U + \nu\nabla_X U + (c/4)\phi X = 0. \tag{3.5}$$

Taking the inner product of each side of this equality and ξ, we see from (2.4) and the fact that $\nu \neq 0$ that $\langle \nabla_\xi X, U\rangle + \langle \nabla_X U, \xi\rangle = 0$. On the other hand, from (2.3) and (2.4) we get

$$\langle \nabla_X U, \xi\rangle = -\langle U, \nabla_X \xi\rangle = -\langle U, \phi AX\rangle = 0.$$

Hence, from these equalities we have $\langle \nabla_\xi X, U\rangle = 0$, so that $\langle \nabla_\xi U, X\rangle = 0$. We hence obtain $\nabla_\xi U = 0$. Thus we find by (3.3) that $\widetilde{\nabla}_\xi(\phi U) = -\nu\xi$.

Next, we shall show that $\xi\nu = 0$. It follows from (2.3), (2.4), (2.5) and $\nabla_\xi U = 0$ that

$$\begin{aligned}(c/4)\phi U &= (\nabla_\xi A)U - (\nabla_U A)\xi \\ &= \nabla_\xi(AU) - A\nabla_\xi U - \nabla_U(A\xi) + A\nabla_U \xi \\ &= \nabla_\xi(\nu\xi) - \nabla_U(\nu U) + A\phi AU \\ &= (\xi\nu)\xi + \nu(\phi A\xi) - (U\nu)U - \nu\nabla_U U.\end{aligned}$$

Taking the inner product of ξ and each side of this equality, we get $\xi\nu - \nu\langle\nabla_U U, \xi\rangle = 0$, so that

$$\xi\nu = \nu\langle\nabla_U U, \xi\rangle = -\nu\langle U, \nabla_U \xi\rangle = -\nu\langle U, \phi AU\rangle = -\nu^2\langle U, \phi\xi\rangle = 0.$$

As we see $\widetilde{\nabla}_\xi \xi = \nu(\phi U)$, $\widetilde{\nabla}_\xi(\phi U) = -\nu\xi$ and $\xi\nu = 0$, we find every integral curve of the characteristic vector field ξ is a circle of positive constant curvature ν in the ambient space $\boldsymbol{C}H^n(c)$. This, combined with the third condition in the hypothesis, implies that ν is a constant function on M. So, setting $X = \phi U$ in (3.5), we get $A\nabla_\xi(\phi U) + \nu\nabla_{\phi U} U - (c/4)U = 0$. Taking the inner product of each side of this equality and U, we have from (2.2), (2.4) and $\nabla_\xi U = 0$

$$\begin{aligned} c/4 &= \langle A\nabla_\xi(\phi U), U\rangle = \nu\langle\nabla_\xi(\phi U), \xi\rangle = \nu\langle(\nabla_\xi\phi)U + \phi\nabla_\xi U, \xi\rangle \\ &= \nu\langle -\langle A\xi, U\rangle\xi, \xi\rangle = -\nu^2, \end{aligned}$$

so that $\nu = \sqrt{|c|}/2$, since $\nu > 0$. Then the shape operator of our real hypersurface M satisfies (2.7). We can hence conclude that M is the homogeneous minimal ruled real hypersurface. □

At the end of this paper we show the following congruency on some geodesics of the homogeneous minimal ruled real hypersurface M in $\boldsymbol{C}H^n(c)$.

Proposition 3. *At each point x of the homogeneous minimal ruled real hypersurface M in $\boldsymbol{C}H^n(c)$, up to the action of isometries of M there exists just one geodesic $\gamma_x = \gamma_x(s)$ on M through the point $x = \gamma_x(0)$ whose initial vector $\dot\gamma_x(0)$ is orthogonal to ξ_x.*

Proof. We take the leaf M_s through the point x for the holomorphic distribution T^0M on our ruled real hypersurface M. This leaf M_s is congruent to totally geodesic $\boldsymbol{C}H^{n-1}(c)$. If two geodesic γ_x^1, γ_x^2 on M through the point $x = \gamma_x^i(0)$ have initial vectors $\dot\gamma_x^1(0), \dot\gamma_x^2(0)$ orthogonal to the characteristic vector ξ_x, then they lie on the same leaf M_s (see the proof of Proposition 2). As the leaf M_s is a Riemannian symmetric space of rank one, these two geodesics are congruent to each other with respect to the isometry group $\mathrm{I}(M_s)$ of M_s. On the other hand, the canonical totally geodesic embedding of our leaf M_s into our ruled real hypersurface M is an equivariant mapping (see the comment on the isometry group $\mathrm{I}(M)$ of M). Hence these geodesics γ_x^1, γ_x^2 are congruent to each other with respect to the isometry group $\mathrm{I}(M)$ of M. □

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