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On q-analoques of divergent and exponential series

By Tapani Matala-aho

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Abstract. We shall consider linear independence measures for the values of the functions $D_a(z)$ and $E_a(z)$ given below, which can be regarded as q-analogues of Euler's divergent series and the usual exponential series. For the q-exponential function $E_q(z)$, our main result (Theorem 1) asserts the linear independence (over any number field) of the values 1 and $E_q(\alpha_j)$ $(j = 1, \dots, m)$ together with its measure having the exponent $\mu = O(m)$, which sharpens the known exponent $\mu = O(m^2)$ obtained by a certain refined version of Siegel's lemma (cf. [**1**]). Let p be a prime number. Then Theorem 1 further implies the linear independence of the p-adic numbers $\prod_{n=1}^{\infty}(1 + kp^n)$, $(k = 0, 1, \dots, p-1)$, over $\boldsymbol{Q}$ with its measure having the exponent $\mu < 2p$. Our proof is based on a modification of Maier's method which allows to construct explicit Padé type approximations (of the second kind) for certain q-hypergeometric series.

1. Introduction.

In the sequel we will investigate linear independence properties of the following q-series

$$D_{b,q}(z) = \sum_{n=0}^{\infty} (b)_n z^n, \quad E_{b,q}(z) = \sum_{n=0}^{\infty} \frac{1}{(b)_n} z^n,$$

where $(b)_0 = 1$ and $(b)_n = (1-b)(1-bq)\cdots(1-bq^{n-1})$, $n \in \boldsymbol{Z}^+$. By $\boldsymbol{N}$ and $\boldsymbol{Z}^+$ we denote the sets of non-negative and positive integers, respectively. In particular, we are interested in the functions $D_q(z) = D_{q,q}(z)$ and $E_q(z) = E_{q,q}(z)$ which may be regarded as q-analogues of Euler's divergent series $D(z) = \sum_{n=0}^{\infty} n! z^n$ and the usual exponential series, respectively. Let $p \in \{\infty\} \cup \boldsymbol{P}$, where $\boldsymbol{P}$ denotes the set of prime numbers. If $|q|_p < 1$, then $|z|_p < 1$ determines the disk of convergence for the series $D_b(z) = D_{b,q}(z)$ and $E_b(z) = E_{b,q}(z)$ but both series can be continued to meromorphic functions over the whole $\boldsymbol{C}_p$ having the possible poles at $q^{-\boldsymbol{N}}$. Here and in the sequel the notation $q^J = \{q^n \mid n \in J\}$ will be used for any $J \subseteq \boldsymbol{Z}$.

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Let $\boldsymbol{K}$ be a fixed number field of degree $\kappa = [\boldsymbol{K} : \boldsymbol{Q}]$, v a place of $\boldsymbol{K}$ and $|\ |_v$ the associated absolute value on the completion $\boldsymbol{K}_v$ with a local degree $\kappa_v = [\boldsymbol{K}_v : \boldsymbol{Q}_v]$. Take any $q \in \boldsymbol{K}$ satisfying $|q|_v < 1$. Let $\boldsymbol{K}^* = \boldsymbol{K} \setminus \{0\}$, and $f(t)$ denote each of the functions $E_q(t)$ and $D_t(a)$, where $a \in \boldsymbol{K}^* \setminus q^{-\boldsymbol{N}}$ and choose any m numbers $\alpha_1, \dots, \alpha_m \in \boldsymbol{K}^*$ satisfying the assumptions

$$\alpha_i \notin q^{-\boldsymbol{N}}, \quad \alpha_i \notin \alpha_j q^{\boldsymbol{Z}} \quad \text{for all} \quad i \neq j.$$

Further, let $\overline{k} = {}^t(k_0, k_1, \dots, k_m) \in \boldsymbol{K}^{m+1} \setminus \{\overline{0}\}$ be arbitrary. Then we shall establish lower bound estimates of the form

$$|k_0 + k_1 f(\alpha_1) + \dots + k_m f(\alpha_m)|_v > \frac{c}{H^{\mu\kappa/\kappa_v + d(\log H)^{-1/2}}}, \tag{1}$$

having a least possible exponent $\mu = \mu_q > 0$, where $H = \max(H(\overline{k}), H_0)$ and c, d and H_0 are certain positive constants. Here the notation $H(\overline{k})$ will be defined at the beginning of the next section.

In the following we give an overview of the existing linear independence results, which will be stated under certain simplified settings on $q \in \boldsymbol{K}$ for brevity. Let $\boldsymbol{I}$ denote an imaginary quadratic number field and $\boldsymbol{Z}_{\boldsymbol{K}}$ the ring of integers in $\boldsymbol{K}$. Then if $\boldsymbol{K} = \boldsymbol{I}$, the above mentioned condition reads as $1/q \in \boldsymbol{Z}_{\boldsymbol{K}}$. First we recall some pioneering results of qualitative nature. The irrationality of $E_q(\alpha)$, $\alpha \in \boldsymbol{K}^* = \boldsymbol{Q}^*$, was already proved in 1947 by Lototsky [**10**]. In the 1980th Stihl and Wallisser [**16**] derived a dimension estimate

$$\dim_{\boldsymbol{K}} \boldsymbol{K} E_q(\alpha_1) + \dots + \boldsymbol{K} E_q(\alpha_m) \geq 2m/7,$$

where $\boldsymbol{K} = \boldsymbol{I}$ and $\alpha_j = \alpha^j$, $\alpha \in \boldsymbol{K}^*$. Finally Bézivin's [**5**] considerations gave the linear independence of the numbres $1, E_q(\alpha_1), \dots, E_q(\alpha_m)$, $\boldsymbol{K} = \boldsymbol{I}$, in the archimedean case, see Andre [**2**] for a far more general situation. For reviewing quantitative aspects we first take the case of the q-exponential function $f(t) = E_q(t)$. Bundschuh [**6**] considered the case $m = 1$, $\boldsymbol{K} = \boldsymbol{I}$, $v = \infty$ and proved (1) with $\mu = 4/3$. In [**13**] the case $m = 2$, $k_0 = 0$ was considered while implying the value $\mu = \sqrt{7} + 2$, and further $\mu = 2.055$ in particular if $\alpha_2 = -\alpha_1$. In the situation $m = 2$ with $\alpha_2 = -\alpha_1$ Bundschuh and Väänänen [**7**] obtained $\mu = 8$, and in particular took $\alpha_1 = 1$ and $\alpha_2 = -1$ to achieve $\mu = 4.8901$. The case $m \in \boldsymbol{Z}^+$ (in full generality) of (1) was obtained by Väänänen [**17**] upon giving $\mu = O(m^3)$, while Väänänen and Zudilin [**18**] proved a Baker-Type estimate implying $\mu = 12m^2$. Note that in some papers the above mentioned results are given in terms of the function $E_{d,d}(z)$ with $|d|_v > 1$. But one may in fact travel between the cases $|q|_v > 1$ and $|q|_v < 1$ by the relation

$$E_{1/q,1/q}(t)E_{q,q}(qt) = 1. \tag{2}$$

Take now the case of q-divergent analogue $f(t) = D_t(a)$, where a few is known. The situation $\boldsymbol{K} = \boldsymbol{Q}$ and $m = 1$ is considered in [**12**] while giving $\mu = \log H$. For general $\boldsymbol{K}$ and $m = 1$, [**13**] gives $\mu = \sqrt{2} + 1$. Note also that most of the above mentioned results are in fact valid for a considerable extended class of functions.

In this paper we study the cases $f(t) = E_a(t)$ and $f(t) = D_t(a)$ in the most general situation of arbitrary $m \in \boldsymbol{Z}^+$. In both the cases we can show $\mu < 2m+2$ thus improving the earlier results on the q-exponential case, if $m \geq 2$ (except the situation $m = 2$, where $k_0 = 0$ or $\alpha_1 = -\alpha_2 = 1$), while in the q-divergent case $f(t) = D_t(a)$ our result seems to be new, if $m \geq 2$. Let p be a prime number and define the following sets of p-adic numbers

$$\prod_{n=1}^{\infty}(1+kp^n), \qquad k = 0, 1, \ldots, p-1,$$

$$\sum_{n=1}^{\infty}\prod_{i=1}^{n}(1+kp^i)p^n, \quad k = 0, 1, \ldots, p-1,$$

and real numbers

$$\prod_{n=1}^{\infty}(1+kp^{-n}), \qquad k = 0, 1, \ldots, p-1,$$

$$\sum_{n=1}^{\infty}\prod_{i=1}^{n}(1+kp^{-i})p^{-n}, \quad k = 0, 1, \ldots, p-1.$$

Then, in each of the four sets of p numbers above we can prove the linear independence of the p numbers over $\boldsymbol{Q}$ with a measure having an exponent $\mu < 2p$.

We next review the relevant methodology. The results in [**17**] and [**18**] are based on approximations constructed by Thue-Siegel lemma and an optimization process upon using the iterations of a corresponding q-difference equation. The exponent $\mu = O(m^2)$ seems to be an optimal bound as far as applying the known variants of Siegel-Shidlovskii's theory, due to Amou et al. [**1**]. On the other hand, the linear independence question about the q-exponential case has resisted the attacks from the methods of explicit (rational) Padé approximations. Namely, there are numerous works (which we mention, however, only a few) applying, say, Mahler's, Maier's and Skolem's methods [**8**], [**9**], [**14**], [**15**], [**16**] for studying arithmetic properties of other instances in a class of q-hypergeometric series. Stihl [**15**] applied Maier's [**11**] method in constructing explicit simultaneous Padé approximations (of the second kind) to a general q-hypergeometric series at m distinct points $\alpha_1, \ldots, \alpha_m \in \boldsymbol{C}$. In [**15**] also an optimization process on a free (non diagonal) parameter λ (see the appendix) is crucial. Maier-Stihl method has been refined by Katsurada [**9**] to include higher derivatives and has subsequently been applied (in several works) to the cases of non-archimedean and arbitrary algebraic number fields, see e.g. [**14**]. Thus we tress, that even the explicit Padé approximations are well-known (see, [**14**], [**15**]) for the q-exponential series, they have not so far yielded linear independence results.

Our strategy is to attack a slightly different case, namely, we shall modify Maier-Stihl method so as to construct simultaneous Padé type approximations (of the second kind) for the functions $D_{\alpha_i}(t)$ with m distinct α_i's. For that purpose we need to prove a new summation formula (in Lemma 1), which is crucial in the construction, and then we combine this with the known results from q-series. Our approximations are diagonal Padé type containing a free parameter ν, and an optimization process consequently becomes available similarly to the case of Stihl's method.

2. Linear independence results.

If the finite place v of $\boldsymbol{K}$ lies over the prime p, we write $v|p$, for an infinite place v of $\boldsymbol{K}$ we write $v|\infty$. We normalize the absolute value $| \ |_v$ of $\boldsymbol{K}$ so that

$$\begin{aligned} |p|_v &= p^{-1}, \quad \text{if} \quad v|p, \\ |x|_v &= |x|, \quad \text{if} \quad v|\infty, \end{aligned}$$

where $| \ |$ denotes the ordinary absolute value in $\boldsymbol{Q}$. Further, the notation

$$\|\alpha\|_v = |\alpha|_v^{\kappa_v/\kappa}, \quad \kappa_v = [\boldsymbol{K}_v : \boldsymbol{Q}_v],$$

will be used in the sequel. The height $H(\alpha)$ of α is defined by the formula

$$H(\alpha) = \prod_v \|\alpha\|_v^*, \quad \|\alpha\|_v^* = \max\{1, \|\alpha\|_v\}$$

and the height $H(\overline{\alpha})$ of the vector $\overline{\alpha} = {}^t(\alpha_1, \ldots, \alpha_m) \in \boldsymbol{K}^m$ is given by

$$H(\overline{\alpha}) = \prod_v \|\overline{\alpha}\|_v^*, \quad \|\overline{\alpha}\|_v^* = \max_{i=1,\ldots,m}\{1, \|\alpha_i\|_v\}.$$

Further, for any place v of $\boldsymbol{K}$, and $q \in \boldsymbol{K}^*$, $\|q\|_v \neq 1$, we define the characteristic λ by

$$\lambda = \lambda_q = \frac{\log H(q)}{\log \|q\|_v}.$$

To state our results we denote

$$\mu = \mu_q = \frac{u_0}{u_0 + \lambda_q s_0},$$

where

$$s_0 = m^2 + m + m\sqrt{m^2+m},$$
$$u_0 = m^2 + m + (m+1)\sqrt{m^2+m}.$$

Now we fix a place v of $\boldsymbol{K}$ throughout the following.

THEOREM 1. *Let $m \in \boldsymbol{Z}^+$ be arbitrary, $a, q, \alpha_1, \dots, \alpha_m \in \boldsymbol{K}^*$, and $|q|_v < 1$. Denote by $f(t)$ each of the functions $D_t(a)$, $E_q(t)$ and $\prod_{n=0}^{\infty}(1-tq^n)$, and assume*

$$a \notin q^{-\boldsymbol{N}}, \quad \alpha_i \notin q^{-\boldsymbol{N}}, \quad \alpha_i \notin \alpha_j q^{\boldsymbol{Z}} \quad \text{for all} \quad i \neq j, \tag{3}$$

$$-\left(1 + \frac{1}{m+\sqrt{m^2+m}}\right) < \lambda_q \leq -1. \tag{4}$$

Then the numbers $1, f(\alpha_1), \dots, f(\alpha_m)$ belonging to $\boldsymbol{K}_v$ are linearly independent over $\boldsymbol{K}$. Further, there exist positive constants c, d, H_0 depending on a and α_i such that

$$\left|k_0 + k_1 f(\alpha_1) + \cdots + k_m f(\alpha_m)\right|_v > \frac{c}{H^{\mu\kappa/\kappa_v + d(\log H)^{-1/2}}} \tag{5}$$

for all $\overline{k} = {}^t(k_0, k_1, \dots, k_m) \in \boldsymbol{K}^{m+1} \setminus \{\overline{0}\}$ with $H = \max(H(\overline{k}), H_0)$.

COROLLARY 1. *Let $b, d, \alpha_1, \dots, \alpha_m \in \boldsymbol{K}^*$, $|d|_v > 1$. Put $f(t) = E_{b,d}(t)$ and assume*

$$b \notin d^{-\boldsymbol{N}}, \quad b\alpha_i \notin d^{\boldsymbol{N}}, \quad \alpha_i \notin \alpha_j d^{\boldsymbol{Z}} \quad \text{for all} \quad i \neq j, \tag{6}$$

$$-\left(1 + \frac{1}{m+\sqrt{m^2+m}}\right) < \lambda_{1/d} \leq -1. \tag{7}$$

Then the assertions of Theorem 1 *are also valid for the function* $f(t)$.

Define now

$$\rho(a,b) = \left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}} a^n b^{-n}}{(-aq)_n}.$$

Here we note that the function $\rho(a,b)$ satisfies an important reciprocity theorem

$$\rho(a,b)-\rho(b,a) = \left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b)_\infty (bq/a)_\infty (q)_\infty}{(-aq)_\infty(-bq)_\infty}, \quad a,b \notin -q^{\boldsymbol{Z}^-}$$

developed by Ramanujan, see [**4**].

COROLLARY 2. *Let* $a, q, \alpha_1, \ldots, \alpha_m \in \boldsymbol{K}^*$, $|q|_v < 1$ *and suppose that* q *satisfies* (4). *Put* $f(t) = \rho(a,t)$ *and assume*

$$a \notin -q^{-\boldsymbol{N}}, \quad \alpha_j \notin -q^{\boldsymbol{N}}, \quad \alpha_j \notin \alpha_i q^{\boldsymbol{Z}} \quad \textit{for all} \quad i \neq j. \tag{8}$$

Then the assertions of Theorem 1 *are also valid for the function* $f(t)$.

Here we note that $\lambda_q \leq -1$ always holds for $|q|_v < 1$, and the following cases in particular assert $\lambda_q = -1$:

1. $\boldsymbol{K} = \boldsymbol{I}$, v is the infinite place of $\boldsymbol{K}$, and $1/q \in \boldsymbol{Z}_{\boldsymbol{K}}$;
2. $\boldsymbol{K} = \boldsymbol{Q}$, $v = p \in \boldsymbol{P}$, and $q = p^l$, $l \in \boldsymbol{Z}^+$;
3. q is a negative power of a PV-number, for example in $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{5})$, $q_1 = (1+\sqrt{5})/2$, $q = q_1^l$, $l \in \boldsymbol{Z}^-$, where $\boldsymbol{Z}^-$ denotes the set of negative integers.

Now, if we take the value $\lambda = -1$, then we have

$$\mu = m + 1 + \sqrt{m^2+m} < 2m+2 \tag{9}$$

and in general we have $\mu = O(m)$, too. Hence, our Theorem 1 improves the results of [**1**], [**17**] and [**18**] for the q-exponential function, where the exponent μ in (5) takes respectively the forms $O(m^2)$, $O(m^3)$ and $12m^2$.

3. Padé type approximations of the second kind.

To prove Theorem 1 we start by constructing explicit simultaneous Padé type approximations (of the second kind) for the series

$$D_{-\beta_j}(z) = \sum_{k=0}^{\infty} (-\beta_j)_k z^k, \quad j = 1, \ldots, m.$$

In our construction some properties of the q-factorial polynomials

$$(x)_n = \prod_{h=0}^{n-1} (1 - xq^h), \quad n \in \boldsymbol{N}, \tag{10}$$

will be needed. First we write

$$(x)_n = \sum_{k=0}^{n} s(n,k) x^k, \quad n \in \boldsymbol{N}, \tag{11}$$

and set $s(n,k) = 0$, when $k < 0$ or $n < k$.

Lemma 1. *For any $n \in \boldsymbol{N}$ we have*

$$\sum_{k=0}^{n} s(n,k)(-x)_{n-k} = q^{\binom{n}{2}} x^n. \tag{12}$$

Proof. From the definition (10) we see that

$$(x)_n = (1 - xq^{n-1})(x)_{n-1}$$

for all $n \in \boldsymbol{Z}^+$, and thus the coefficients $s(n,k)$ satisfy the recurrence

$$s(n,k) = s(n-1,k) - q^{n-1} s(n-1,k-1) \tag{13}$$

for all $n \in \boldsymbol{Z}^+$, $0 \le k \le n$. Then, using (13), we get

$$\begin{aligned} &\sum_{k=0}^{n+1} s(n+1,k)(-x)_{n+1-k} \\ &\quad = -q^n \sum_{k=0}^{n} s(n,k)(-x)_{n-k} + (1+x) \sum_{k=0}^{n} s(n,k)(-xq)_{n-k} \\ &\quad = -q^n q^{\binom{n}{2}} x^n + (1+x) q^{\binom{n}{2}} (xq)^n = q^{\binom{n+1}{2}} x^{n+1}. \end{aligned} \tag{14}$$

Thus, by induction, (12) is valid for all $n \in \boldsymbol{N}$. □

In the sequel we shall use q-binomial coefficients defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}}$$

and also the following q-factorials

$$(b, a; q)_n = \prod_{h=1}^{n} (b - aq^{h-1})$$

generalizing the earlier notation $(a)_n = (1, a; q)_n$. The next well-known expansion of q-factorials $(b, a; q)_n$ is called the (finite) q-binomial theorem.

Lemma 2 ([**3**, p. 490, Corollary 10.2.2(c)]). *For any $n \in \boldsymbol{N}$ we have*

$$(b, a; q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} b^{n-k} (-a)^k. \tag{15}$$

Note that by Lemma 2 we get an explicit expression for the coefficients $s(n, k)$, namely

$$s(n, k) = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}}. \tag{16}$$

The following application of the q-binomial theorem (15) is a cornerstone of Maier-Stihl method.

Lemma 3 ([**15**]). *Let $\overline{\beta} = {}^t(\beta_1, \ldots, \beta_m)$ be given and define $\sigma_{i,l} = \sigma_{i,l}(\overline{\beta})$ by*

$$\prod_{t=1}^{m} (\beta_t, w; q^{-1})_l = \sum_{i=0}^{ml} \sigma_{i,l} w^i. \tag{17}$$

Then

$$\sum_{i=0}^{ml} \sigma_{i,l} w^i = 0 \tag{18}$$

if and only if $w = \beta_j q^k$ with $j \in \{1, \ldots, m\}$; $k \in \{0, \ldots, l-1\}$. Moreover the equality

$$\sigma_{i,l} = (-1)^i q^{-m\binom{l}{2}} \Sigma_{ml-i}, \tag{19}$$

holds with

$$\Sigma_h = \Sigma_h(\overline{\beta}) = \sum_{i_1+\cdots+i_m=h} \begin{bmatrix} l \\ i_1 \end{bmatrix} \cdots \begin{bmatrix} l \\ i_m \end{bmatrix} q^{\binom{i_1}{2}+\cdots+\binom{i_m}{2}} \beta_1^{i_1} \cdots \beta_m^{i_m}. \tag{20}$$

Proof. All we need to note is that from Lemma 2 we get the expansion

$$\begin{aligned}(\beta, w; q^{-1})_l &= (-1)^l q^{-\binom{l}{2}} (w, \beta; q)_l \\ &= (-1)^l q^{-\binom{l}{2}} \sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix} q^{\binom{i}{2}} w^{l-i} (-\beta)^i.\end{aligned}$$ □

In order to construct our approximations we now define the following (polynomial) coefficients

$$b_{l,\nu,h}(z) = (-1)^{ml-h} q^{\binom{ml+\nu}{2}-\binom{ml+\nu-h}{2}} (z)_{ml+\nu-h} \Sigma_h, \tag{21}$$

$$b_{l,\nu,H} = q^{m\binom{l}{2}} \sum_{\substack{ml-i+f=H \\ 0\le f\le i+\nu\le ml+\nu}} q^{\binom{ml+\nu}{2}-\binom{i+\nu}{2}} s(i+\nu, f)\sigma_i, \tag{22}$$

$$a_{l,\nu,j,N} = -\sum_{H+n=N} b_{l,\nu,H} (-\beta_j)_n, \tag{23}$$

$$s_{l,\nu,j,k} = q^{k\nu} (-\beta_j q^l)_k \prod_{t=1}^{m} (\beta_t, \beta_j q^{k+1}; q)_l. \tag{24}$$

Theorem 2. *For any $l, \nu \in \boldsymbol{N}$, $j = 1, \ldots, m$ set*

$$\begin{aligned} B_{l,\nu}(z) &= \sum_{h=0}^{ml} b_{l,\nu,h}(z) z^h, && (25) \\ A_{l,\nu,j}(z) &= \sum_{N=0}^{ml+\nu-1} a_{l,\nu,j,N} z^N, \\ S_{l,\nu,j}(z) &= \sum_{k=0}^{\infty} s_{l,\nu,j,k} z^k, \\ L_{l,\nu,j}(z) &= z^{(m+1)l+\nu} q^{\binom{ml+\nu}{2}+m\binom{l}{2}+\nu l} (-\beta_j)_l \beta_j^{\nu} S_{l,\nu,j}(z). && (26) \end{aligned}$$

Then we have the relations

$$B_{l,\nu}(z) = \sum_{H=0}^{ml+\nu} b_{l,\nu,H} z^H \tag{27}$$

and

$$B_{l,\nu}(z) D_{-\beta_j}(z) + A_{l,\nu,j}(z) = L_{l,\nu,j}(z). \tag{28}$$

Furthermore the properties

$$\deg_z B_{l,\nu}(z) = ml + \nu, \quad \deg_z A_{l,\nu,j}(z) \le ml + \nu - 1, \tag{29}$$

$$\operatorname*{ord}_{z=0} L_{l,\nu,j}(z) = (m+1)l + \nu \tag{30}$$

show that (28) gives a diagonal type Padé approximation (of the second kind) with the free parameter ν.

Proof. First, by using (19) and (11), we rewrite the polynomial in (25) as follows

$$\begin{aligned} B_{l,\nu}(z) &= q^{m\binom{l}{2}} \sum_{i=0}^{ml} z^{ml-i} q^{\binom{ml+\nu}{2} - \binom{i+\nu}{2}} (z)_{i+\nu} \sigma_i \\ &= q^{m\binom{l}{2}} \sum_{H=0}^{ml+\nu} z^H \sum_{\substack{ml-i+f=H \\ 0 \le f \le i+\nu \le ml+\nu}} q^{\binom{ml+\nu}{2} - \binom{i+\nu}{2}} s(i+\nu, f) \sigma_i \\ &= \sum_{H=0}^{ml+\nu} b_{l,\nu,H} z^H, \end{aligned}$$

which proves (27).

We next study the expansion of the product

$$B_{l,\nu}(z) D_{-\beta_j}(z) = \sum_{N=0}^{\infty} r_N z^N, \tag{31}$$

where

$$r_N = \sum_{H+n=N} b_{l,\nu,H} (-\beta_j)_n.$$

Set $N = ml + \nu + a$, where $a \in \boldsymbol{N}$. First we consider the case $0 \le a \le l-1$. It follows for the summation indices in (22) and (23) that $n = i + \nu - f + a$ and thus

$$\begin{aligned} r_N &= q^{m\binom{l}{2}} \sum_{i=0}^{ml} \sum_{f=0}^{i+\nu} q^{\binom{ml+\nu}{2} - \binom{i+\nu}{2}} s(i+\nu, f) \sigma_i(-\beta_j)_{i+\nu-f+a} \\ &= q^{m\binom{l}{2}} \sum_{i=0}^{ml} \sigma_i(-\beta_j)_a q^{\binom{ml+\nu}{2} - \binom{i+\nu}{2}} \sum_{f=0}^{i+\nu} s(i+\nu, f)(-\beta_j q^a)_{i+\nu-f}. \end{aligned} \tag{32}$$

Here the inner f-sum is evaluated by Lemma 1, and then the resulting expression can be computed by Lemma 3, which gives

$$\begin{aligned} r_{ml+\nu+a} &= q^{m\binom{l}{2} + \binom{ml+\nu}{2}} (-\beta_j)_a \sum_{i=0}^{ml} \sigma_i(\beta_j q^a)^{i+\nu} \\ &= q^{m\binom{l}{2} + \binom{ml+\nu}{2} + a\nu} (-\beta_j)_a \beta_j^\nu \prod_{t=1}^{m} (\beta_t, \beta_j q^a; q^{-1})_l = 0 \end{aligned} \tag{33}$$

for any $0 \le a \le l-1$.

Next we consider the case $a = l + k$, $k \in \boldsymbol{N}$. Then

$$\begin{aligned} r_N &= r_{(m+1)l+\nu+k} \\ &= q^{m\binom{l}{2} + \binom{ml+\nu}{2} + (l+k)\nu} (-\beta_j)_{l+k} \beta_j^\nu \prod_{t=1}^{m} (\beta_t, \beta_j q^{k+1}; q)_l. \end{aligned} \tag{34}$$

Consequently (33) and (34) imply the assertions (26) and (28). □

4. Determinant.

We define

$$\Delta_{l,\nu}(z) = \begin{vmatrix} -B_{l,\nu}(z) & -B_{l,\nu+1}(z) & \cdots & -B_{l,\nu+m}(z) \\ A_{l,\nu,1}(z) & A_{l,\nu+1,1}(z) & \cdots & A_{l,\nu+m,1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ A_{l,\nu,m}(z) & A_{l,\nu+1,m}(z) & \cdots & A_{l,\nu+m,m}(z) \end{vmatrix} \tag{35}$$

and denote

$$E_i = m(m+1)l + m\nu - i + \binom{m+1}{2}, \qquad i = 0, \dots, m,$$

$$H_i = m\nu - i + m^2\binom{l}{2} + \binom{m+1}{2} + \sum_{h=\nu, h\neq i}^{\nu+m} \binom{ml+h}{2}, \quad i = 0, \dots, m.$$

Our determinant argument proceeds in a slightly different way from Stihl's [**15**], and it is significant that the explicit evaluation of $\Delta_{l,\nu}(z)$ above is in fact possible. In contrast with our case, Stihl showed the nonvanishing of the corresponding determinant by extracting its dominant order as $l \to +\infty$.

Lemma 4. *If*

$$\beta_i \notin -q^{-\boldsymbol{N}}, \quad \beta_i \notin \beta_j q^{\boldsymbol{Z}} \quad \textit{for all} \quad i \neq j, \tag{36}$$

then for any $l, \nu \in \boldsymbol{N}$ we have

$$\Delta_{l,\nu}(z) = (-1)^{m+1}(z)_\nu z^{E_m} q^{H_m} \cdot$$
$$\prod_{j=1}^{m}(-\beta_j)_l \prod_{j=1}^{m}\prod_{t=1}^{m}(\beta_t, \beta_j q; q)_l \prod_{1\le i<j\le m}(\beta_j - \beta_i). \tag{37}$$

Further,

$$\Delta_{l,\nu}(a) \neq 0, \tag{38}$$

for all $a \in \boldsymbol{C}_p^* \setminus q^{-\boldsymbol{N}}$.

Proof. First we get

$$B_{l,\nu}(0) = b_{l,\nu,0}(0) = (0)_{ml+\nu}\Sigma_0 = 1,$$

which gives

$$\operatorname*{ord}_{z=0} B_{l,\nu}(z) = 0 \tag{39}$$

for all $l, \nu \in \boldsymbol{N}$. Next we have

$$S_{l,\nu,j}(0) = s_{l,\nu,j,0} = \prod_{t=1}^{m}(\beta_t, \beta_j q; q)_l \neq 0$$

by the assumption (36) and thus

$$\operatorname*{ord}_{z=0} S_{l,\nu,j}(z) = 0 \tag{40}$$

for all $l,\nu \in \boldsymbol{N}$, $j = 1,\dots,m$. Now we apply (28) to modify the determinant in (35), getting

$$\Delta_{l,\nu}(z) = \begin{vmatrix} -B_{l,\nu}(z) & -B_{l,\nu+1}(z) & \cdots & -B_{l,\nu+m}(z) \\ L_{l,\nu,1}(z) & L_{l,\nu+1,1}(z) & \cdots & L_{l,\nu+m,1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ L_{l,\nu,m}(z) & L_{l,\nu+1,m}(z) & \cdots & L_{l,\nu+m,m}(z) \end{vmatrix}. \tag{41}$$

Then the right side of (41) is expanded with respect to the first row

$$\begin{aligned} \Delta_{l,\nu}(z) = & -B_{l,\nu}(z) z^{E_0} q^{H_0} \prod_{j=1}^{m} (-\beta_j)_l F_0(z) + \cdots + \\ & -(-1)^m B_{l,\nu+m}(z) z^{E_m} q^{H_m} \prod_{j=1}^{m} (-\beta_j)_l F_m(z), \end{aligned} \tag{42}$$

where $F_i(z)$, $i = 0,1,\dots,m$ are defined by eliminating the common factors from the corresponding minors. Noting that

$$E_0 > \cdots > E_m \tag{43}$$

we will consider the order at $z = 0$ of the last minor

$$F_m(z) = \begin{vmatrix} S_{l,\nu,1}(z) & \beta_1 S_{l,\nu+1,1}(z) & \cdots & \beta_1^{m-1} S_{l,\nu+m-1,1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ S_{l,\nu,m}(z) & \beta_m S_{l,\nu+1,m}(z) & \cdots & \beta_m^{m-1} S_{l,\nu+m-1,m}(z) \end{vmatrix}.$$

We have

$$F_m(0) = \prod_{j=1}^{m} \prod_{t=1}^{m} (\beta_t, \beta_j q; q)_l \begin{vmatrix} 1 & \beta_1 & \cdots & \beta_1^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \beta_m & \cdots & \beta_m^{m-1} \end{vmatrix}$$

$$= \prod_{j=1}^{m} \prod_{t=1}^{m} (\beta_t, \beta_j q; q)_l \prod_{1 \le i < j \le m} (\beta_j - \beta_i) \ne 0 \tag{44}$$

by the assumptions in (36). Thus

$$\operatorname*{ord}_{z=0} F_m(z) = 0 \tag{45}$$

for all $l, \nu \in \boldsymbol{Z}^+$, and consequently

$$\operatorname*{ord}_{z=0} \Delta_{l,\nu}(z) = E_m = m(m+1)l + m\nu + \binom{m}{2}. \tag{46}$$

Further, we note that

$$\deg \Delta_{l,\nu}(z) \le m(m+1)l + (m+1)\nu + \binom{m}{2}. \tag{47}$$

Next, from (21) we see that the polynomials $B_{\nu+j,l}(z)$, $j = 0, 1, \ldots, m$, have a common factor $(z)_\nu$, which implies by the order (46) and by the degree estimate (47) that

$$\Delta_{l,\nu}(z) = c_1 z^{E_m} (z)_\nu \tag{48}$$

for some constant c_1, which may be evaluated by considering

$$\frac{\Delta_{l,\nu}(z)}{z^{E_m}} \tag{49}$$

at $z = 0$. Using the expansion (42) we get

$$\begin{aligned} c_1 = \left. \frac{\Delta_{l,\nu}(z)}{z^{E_m}} \right|_{z=0} &= (-1)^{m+1} B_{l,\nu+m}(0) q^{H_m} \prod_{j=1}^{m} (-\beta_j)_l F_m(0) \\ &= (-1)^{m+1} q^{H_m} \prod_{j=1}^{m} (-\beta_j)_l \prod_{j=1}^{m} \prod_{t=1}^{m} (\beta_t, \beta_j q; q)_l \prod_{1 \le i < j \le m} (\beta_j - \beta_i). \end{aligned} \tag{50}$$

□

5. Estimates.

Lemma 5. *For all $l, \nu \in \boldsymbol{N}$ and $j = 1, \ldots, m$ we have*

$$B_{l,\nu}(z),\ A_{l,\nu,j}(z) \in \boldsymbol{K}[z,q], \tag{51}$$

$$\deg_q B_{l,\nu}(z) \le \frac{m^2+m}{2}l^2 + ml\nu + \frac{\nu^2}{2}, \tag{52}$$

$$\deg_q A_{l,\nu,j}(z) \le \frac{m^2+m}{2}l^2 + ml\nu + \frac{\nu^2}{2}. \tag{53}$$

Proof. It is well-known that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] \in \boldsymbol{Z}[q]$ and this shows (51). Further, the q-binomial coefficients have the property that (cf. [**3**, p. 490, Corollary 10.2.2(d)])

$$\deg_q \begin{bmatrix} n \\ k \end{bmatrix} = k(n-k), \quad 0 \le k \le n. \tag{54}$$

Thus

$$\deg_q s(i+\nu, f) = \deg_q \begin{bmatrix} i+\nu \\ f \end{bmatrix} q^{\binom{f}{2}} = f(i+\nu-f) + f^2/2 \tag{55}$$

and

$$\begin{aligned}\deg_q \Sigma_h &= \deg_q \left\{ \sum_{i_1+\cdots+i_m=h} \begin{bmatrix} l \\ i_1 \end{bmatrix} \cdots \begin{bmatrix} l \\ i_m \end{bmatrix} q^{\binom{i_1}{2}+\cdots+\binom{i_m}{2}} \right\} \\ &\le \max_{i_1+\cdots+i_m=h} \left\{ i_1(l-i_1) + \cdots + i_m(l-i_m) + i_1^2/2 + \cdots + i_m^2/2 \right\} \\ &\le hl - h^2/(2m). \end{aligned} \tag{56}$$

By (19) and (22) we have

$$\begin{aligned}\deg_q b_{l,\nu,H} &\le \deg_q \left\{ \sum_{\substack{ml-i+f=H \\ 0 \le f \le i+\nu \le ml+\nu}} q^{\binom{ml+\nu}{2} - \binom{i+\nu}{2}} s(i+\nu, f) \Sigma_{ml-i} \right\} \\ &\le \max_{\substack{ml-i+f=H \\ 0 \le f \le i+\nu \le ml+\nu}} \left\{ (ml+\nu)^2/2 - (i+\nu)^2/2 + f(i+\nu-f) \right. \\ &\qquad \left. + f^2/2 + (ml-i)l - (ml-i)^2/(2m) \right\} \end{aligned}$$

$$
\begin{aligned}
&= \max_{\max\{0,H-ml\}\le f\le H} \big\{(ml+\nu)^2/2-(ml+f+\nu-H)^2/2 \\
&\qquad\qquad + f(\nu+ml-H)+f^2/2+(H-f)l-(H-f)^2/(2m)\big\} \\
&= \max_{\max\{0,H-ml\}\le f\le H} \big\{(ml+\nu)^2/2-(ml+\nu-H)^2/2 \\
&\qquad\qquad + (H-f)l-(H-f)^2/(2m)\big\} \\
&\le \begin{cases} (ml+\nu)^2/2-(ml+\nu-H)^2/2+H(l-H/(2m)), & \text{if} \quad 0\le H\le ml; \\ (ml+\nu)^2/2-(ml+\nu-H)^2/2+ml^2/2, & \text{if} \quad ml\le H\le ml+\nu; \end{cases}
\end{aligned} \tag{57}
$$

and hence the above maximum is bounded as

$$\le ml^2/2+(ml+\nu)H-H^2/2, \quad \text{for} \quad 0\le H\le ml+\nu. \tag{58}$$

Thus

$$\deg_q b_{l,\nu,H} \le (ml+\nu)^2/2+ml^2/2, \quad \text{for} \quad 0\le H\le ml+\nu, \tag{59}$$

which gives (52). Consider then

$$
\begin{aligned}
\deg_q a_{l,\nu,j,N} &= \deg_q \left\{ \sum_{H=0}^{N} b_{l,\nu,H}(-\beta_j)_{N-H} \right\} \\
&\le \max_{0\le H\le N} \left\{(ml+\nu)H-H^2/2+ml^2/2+(N-H)^2/2\right\} \\
&\le \max_{0\le H\le N} \left\{N^2/2+(ml+\nu-N)H+ml^2/2\right\} \\
&\le -N^2/2+(ml+\nu)N+ml^2/2 \\
&\le \frac{m^2+m}{2}l^2+ml\nu+\frac{\nu^2}{2}, \quad \text{for} \quad 0\le N\le ml+\nu.
\end{aligned} \tag{60}
$$

which proves (53). □

Put

$$\Theta_j = D_{-\beta_j}(a), \quad B_{l,\nu}=B_{l,\nu}(a), \quad A_{l,\nu,j}=A_{l,\nu,j}(a), \quad L_{l,\nu,j}=L_{l,\nu,j}(a)$$

for brevity, and define $\delta(w)$ to be 1 and 0 according to $w|\infty$ or $w\nmid\infty$. Then by using Lemma 5, (24), (26) and (28) we get the following approximation forms for

Θ_j $(j = 1, \ldots, m)$ with the asymptotic bounds.

Lemma 6. *Let $a, q, \beta_1, \ldots, \beta_m \in \boldsymbol{K}^*$, $|q|_v < 1$, $|a|_v < 1$ and set $\nu = \lfloor \tau l \rfloor$, $\tau \geq 0$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Then for all $l, \nu \in \boldsymbol{N}$ and $j = 1, \ldots, m$ we have approximation forms*

$$B_{l,\nu}\Theta_j + A_{l,\nu,j} = L_{l,\nu,j} \tag{61}$$

satisfying

$$\max(\|B_{l,\nu}\|_w, \|A_{l,\nu,j}\|_w) \leq 2^{\delta(w)O(l)}\|q\|_w^{*\,((m^2+m)/2+m\tau+\tau^2/2)l^2} \tag{62}$$

for any place w of $\boldsymbol{K}$, and

$$\|L_{l,\nu,j}\|_v \leq 2^{O(l)}\|q\|_v^{((m^2+m)/2+(m+1)\tau+\tau^2/2)l^2}. \tag{63}$$

The implied O-constant here (and in the next section) depend on a and β_j.

6. Proof of Theorem 1.

We now quote a general result from [**1**] which we shall apply to establish the linear independence results in Theorem 1. Assume that we have a sequence of (binary) approximation forms

$$\overline{L}_{n,T} = B_{n,T}\overline{\Theta} + \overline{A}_{n,T} \tag{64}$$

for $\overline{\Theta} = {}^t(\Theta_1, \ldots, \Theta_m) \in \boldsymbol{K}_v^m$, where $B_{n,T} \in \boldsymbol{K}$, $\overline{A}_{n,T} = {}^t(A_{n,T,1}, \ldots, A_{n,T,m}) \in \boldsymbol{K}^m$ and $\overline{L}_{n,T} = {}^t(L_{n,T,1}, \ldots, L_{n,T,m})$. Let

$$\max\{\|B_{n,T}\|_w^*, \|\overline{A}_{n,T}\|_w^*\} \leq P_w(n,T) \tag{65}$$

for any place w of $\boldsymbol{K}$,

$$\|\overline{L}_{n,T}\|_v \leq R_v(n,T), \tag{66}$$

and let ρ_1, ρ_2 with $\rho_1 < \rho_2$ and c_2 be positive constants independent of n such that

$$\Delta_{n,T} = \begin{vmatrix} -B_{n,T} & -B_{n,T+1} & \cdots & -B_{n,T+m} \\ A_{n,T,1} & A_{n,T+1,1} & \cdots & A_{n,T+m,1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,T,m} & A_{n,T+1,m} & \cdots & A_{n,T+m,m} \end{vmatrix} \neq 0 \tag{67}$$

with some integer $T \in [\rho_1 n, \rho_2 n - m]$ for all $n \geq c_2$. (Here and in the sequel c_i's denote positive constants independent of n.)

Now we suppose that the assumptions (65)–(66) are valid with

$$\prod_w P_w(n, \tau n) \leq c_3^n H(q)^{s(\tau)n^2}, \quad c_3 \geq 1, \tag{68}$$

and

$$R_v(n, \tau n) \leq c_4^n \|q\|_v{}^{u(\tau)n^2}, \quad c_4 \geq 1, \tag{69}$$

for all $\rho_1 \leq \tau \leq \rho_2$. In (68) and (69) we further suppose that $s(\tau)$ and $u(\tau)$ are bounded positive valued functions on the interval $\rho_1 \leq \tau \leq \rho_2$ satisfying

$$u(\tau) + \lambda s(\tau) \geq c_5 \tag{70}$$

with some $c_5 > 0$. Moreover we put

$$\mu(\tau) = \frac{u(\tau)}{u(\tau) + \lambda s(\tau)}, \quad \mu = \sup_{\rho_1 \leq \tau \leq \rho_2} \mu(\tau).$$

Theorem A ([**1**]). *If the above assumptions* (65)–(70) *are valid, then there exist positive constants* c, d *and* H_0 *depending on the numbers* $\Theta_1, \ldots, \Theta_m$ *and* $\rho_1, \rho_2, c_2, c_3, c_4$ *and* c_5 *such that*

$$|k_0 + k_1\Theta_1 + \cdots + k_m\Theta_m|_v > \frac{c}{H^{\mu\kappa/\kappa_v} H^{d(\log H)^{-1/2}}} \tag{71}$$

for all $\overline{k} = {}^t(k_0, k_1, \ldots, k_m) \in \boldsymbol{K}^{m+1} \setminus \{\overline{0}\}$ *with* $H = \max\{H(\overline{k}), H_0\}$.

Proofs of Theorem 1 and corollaries. First we note that the series $D_b(z)$ defines an analytic function in the unit disk $|z|_p < 1$. Then using the q-difference equation

$$bzD_b(qz) = (z-1)D_b(z) + 1 \tag{72}$$

we get a meromorphic continuation to the whole $\boldsymbol{C}_p$ except the poles at $q^{-\boldsymbol{N}}$. Put now $f(t) = D_t(a)$ and $\overline{\beta} = -\overline{\alpha}$. If $|a|_v \geq 1$, then by using (72) repeatedly we may return to apply the estimates in Lemma 6. Thus it follows from Lemmas 4 and 6 that Theorem A can be applied with any constants $\rho > 0$ and $1 > \omega > 0$, upon taking

$$[\rho_1, \rho_2] = [\rho, \rho + \omega], \tag{73}$$

if l is sufficiently large, say $l \geq (m+1)/\omega$. Now

$$s(\tau) = (m^2 + m)/2 + m\tau + \tau^2/2,$$
$$u(\tau) = (m^2 + m)/2 + (m+1)\tau + \tau^2/2,$$

and the function $s(\tau)/u(\tau)$, $\tau \geq 0$, attains its minimum at

$$\tau_0 = \sqrt{m^2 + m}.$$

Hence the optimal value of

$$\mu = \max_{\rho \leq \tau \leq \rho + \omega} \frac{u(\tau)}{u(\tau) + \lambda s(\tau)} = \max\{\mu(\rho), \mu(\rho + \omega)\}$$

will be obtained if

$$s(\rho_0)/u(\rho_0) = s(\rho_0 + \omega)/u(\rho_0 + \omega), \quad \rho_0 \leq \tau_0 \leq \rho_0 + \omega. \tag{74}$$

The unique positive solution of (74) is

$$\rho_0 = \frac{\sqrt{4m^2 + 4m + \omega^2} - \omega}{2}.$$

Now, we may choose $\omega > 0$ arbitrary small and consequently we may put $\mu = \mu(\tau_0)$ which at the same time gives the exact values of $s_0 = s(\tau_0)$ and $u_0 = u(\tau_0)$. Hence (5) is valid for the function $D_t(a)$.

Next we start from the identity

$$(U)_{m+1} - (U)_m = -U(U)_m q^m, \tag{75}$$

which by telescoping gives

$$(U)_\infty = 1 - U \sum_{m=0}^{\infty} (U)_m q^m. \tag{76}$$

On the other hand the series

$$E_q(z) = \sum_{n=0}^{\infty} \frac{1}{(q)_n} z^n, \quad \hat{E}_q(z) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q)_n} z^n$$

satisfy the q-difference equations

$$E_q(qz) = (1-z)E_q(z), \quad \hat{E}_q(z) = (1+z)\hat{E}_q(qz). \tag{77}$$

By using (77) one gets the well-known Euler formulae

$$E_q(z) = \frac{1}{(z)_\infty}, \quad \hat{E}_q(z) = (-z)_\infty. \tag{78}$$

Hence, by (76) and (78) we have

$$E_q(z) = \frac{1}{1 - zD_z(q)}, \tag{79}$$

and thus (5) is valid for the function $f(t) = E_q(t)$, too. From (78) we get

$$\hat{E}_q(t)E_q(-t) = 1 \tag{80}$$

which implies

$$E_{1/q,1/q}(t)E_{q,q}(qt) = 1$$

and proves (2).

Finally we note that both the functions

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}\alpha^n}{(qz)_n} z^n, \quad 1 + \alpha z D_{-\alpha}(qz)$$

are unique analytic solutions of the q-difference equation

$$\alpha z G(qz) = (1-qz)G(z) - (1-qz), \quad G(0) = 1. \tag{81}$$

Hence we have the relation

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}\alpha^n}{(qz)_n} z^n = 1 + \alpha z D_{-\alpha,q}(qz), \tag{82}$$

and this implies

$$\sum_{n=0}^{\infty} \frac{t^n}{(b;d)_n} = 1 - \frac{t}{b} D_{bt,1/d}(1/b) \tag{83}$$

from which Theorem 1 yields Corollary 1. Together (72) and (82) give

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}\alpha^n}{(qz)_n} z^n = (1-z) D_{-\alpha,q}(z). \tag{84}$$

Thus we get a connecting relation between $D_{-\alpha,q}(z)$ and Ramanujan's reciprocity function $\rho(a,b)$ (cf. [**4**]), namely

$$\rho(a,b) = \frac{(1+a)(1+b)}{b} D_{-q/b,q}(-a). \tag{85}$$

□

7. Appendix.

Let $P(x)$ and $Q(x)$ be polynomials and define a q-hypergeometric series

$$F(z) = \sum_{n=0}^{\infty} \frac{\prod_{k=0}^{n-1} P(q^k)}{\prod_{k=0}^{n-1} Q(q^k)} z^n.$$

Stihl [**15**] constructed the following simultaneous Padé approximations (of the second kind) for the series $F(z)$.

Theorem B ([**15**]). *Let $l, m, \lambda \in \boldsymbol{N}$, $d = \max\{\deg P(x), \deg Q(x)\}$, $\rho = \lfloor l/d \rfloor + ml + \lambda - 1$ and choose m numbers $\alpha_1, \ldots, \alpha_m$. Put $\sigma_{i,l} = \sigma_{i,l}(\overline{\alpha})$ and*

$$A_{l,\lambda}(z) = \sum_{i=0}^{ml} z^{ml-i} \sigma_{i,l} \frac{\prod_{k=0}^{i+\rho-ml-1} Q(q^k)}{\prod_{k=0}^{i+\lambda-1} P(q^k)}.$$

Then

$$A_{l,\lambda}(z)F(\alpha_t z) - B_{t,l,\lambda}(z) = R_{t,l,\lambda}(z),$$

where

$$\deg B_{t,l,\lambda}(z) \le ml + \lambda - 1,$$

$$R_{t,l,\lambda}(z) = \sum_{k=\rho+1}^{\infty} r_{t,k} z^k.$$

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Tapani Matala-aho
Matemaattisten tieteiden laitos
PL 3000, 90014 Oulun Yliopisto
Finland
E-mail: tma@cc.oulu.fi