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Norm estimation of the harmonic Bergman projection on half-spaces

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Abstract. On the setting of the upper half-space $\boldsymbol{H}$ of the Euclidean n-spaces, we give a sharp norm estimate of the weighted harmonic Bergman projection on L^p_α for $1 < p < \infty$. Also, we obtain the norm estimate of the projection depending on $\alpha > -1$ when p is fixed.

1. Introduction.

For a fixed positive integer $n \geq 2$, let $\boldsymbol{H} = \boldsymbol{R}^{n-1} \times \boldsymbol{R}_+$ be the upper half-space where $\boldsymbol{R}_+$ denotes the set of all positive real numbers. We write point $z \in \boldsymbol{H}$ as $z = (z', z_n)$ where $z' \in \boldsymbol{R}^{n-1}$ and $z_n > 0$.

For $\alpha > -1$ and $1 \leq p < \infty$, let b^p_α denote *the weighted harmonic Bergman space* consisting of all real-valued harmonic functions u on $\boldsymbol{H}$ such that

$$\|u\|_{L^p_\alpha} := \left(\int_{\boldsymbol{H}} |u(z)|^p \, dV_\alpha(z) \right)^{1/p} < \infty,$$

where $dV_\alpha(z) = z_n^\alpha \, dz$ and dz is the Lebesgue measure on $\boldsymbol{R}^n$. The space b^p_α is a closed subspace of L^p_α. In particular b^2_α is a Hilbert space. Thus, there is a unique orthogonal projection Π_α of L^2_α onto b^2_α:

$$\Pi_\alpha f(z) = \int_{\boldsymbol{H}} f(w) R_\alpha(z, w) \, dV_\alpha(w)$$

for every $f \in L^2_\alpha$ and for each $z \in \boldsymbol{H}$ where $R_\alpha(z, w)$ is the weighted harmonic

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Bergman kernel. The projection Π_α is called the weighted harmonic Bergman projection.

Let $\mathscr{D}^s$ be the fractional differentiation of order $s \geq 0$ and let $P_z(w)$ be the extended Poisson kernel on $\boldsymbol{H}$, i.e.,

$$P_z(w) := P(z,w) = \frac{2}{n\sigma_n} \frac{z_n + w_n}{|z - \overline{w}|^n} \qquad z, w \in \boldsymbol{H} \tag{1.1}$$

where $\overline{w} = (w', -w_n)$ and σ_n is the volume of the unit ball in $\boldsymbol{R}^n$. Then the weighted harmonic Bergman kernel $R_\alpha(z,w)$ is given by

$$R_\alpha(z,w) = \frac{(-1)^{\lceil\alpha\rceil+1} 2^{\alpha+1}}{\Gamma(\alpha+1)} \mathscr{D}^{\alpha+1} P_z(w) \tag{1.2}$$

where Γ is the Gamma function and $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α. Also, it is well known

$$|R_\alpha(z,w)| \leq \frac{C_{n,\alpha}}{|z - \overline{w}|^{n+\alpha}} \tag{1.3}$$

for $z, w \in \boldsymbol{H}$. Thus, we have

$$\|R_\alpha(z,\cdot)\|_{L^q_\alpha} \lesssim z_n^{(n+\alpha)(1/q-1)} \tag{1.4}$$

for $1 < q < \infty$. Thus, the weighted harmonic Bergman projection Π_α is well defined whenever $f \in L^p_\alpha$ for $1 \leq p < \infty$. Moreover, Π_α is a bounded projection of L^p_α onto b^p_α for $1 < p < \infty$ (See [**1**] and [**2**] for details).

Recently, Zhu [**4**] gave a sharp norm estimate of the Bergman projection on L^p of the unit ball in $\boldsymbol{C}^n$. In this paper we show that the result of Zhu continues to hold on the setting of the *weighted harmonic* Bergman projection on L^p_α of the *unbounded domain* $\boldsymbol{H}$ (Theorem 3.1). Additionally, we obtain the norm estimate of Π_α depending on α when p is fixed (Theorem 3.2).

Throughout the paper we use the same letter C_a to denote various positive constants depending only on the constant a which may change at each occurrence. For nonnegative quantities A and B, we often write $A \lesssim B$ or $B \gtrsim A$ if A is dominated by B times some positive constant. Also, we write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2. Auxiliary estimates.

In this section, we recall the definition of the weighted harmonic Bergman kernel $R_\alpha(z,w)$ and then we get the estimate of it on some local area.

The fractional derivative of v of order $-s<0$ is defined by

$$\mathscr{D}^{-s}v(z)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}v(z',z_n+t)\,dt. \tag{2.1}$$

Let D be the differentiation with respect to the last component. Then the fractional derivative of $u\in b^p_\alpha$ of order $s>0$ is defined by

$$\mathscr{D}^s u=\mathscr{D}^{-(\lceil s\rceil-s)}D^{\lceil s\rceil}u.$$

Also, $\mathscr{D}^0$ is the identity operator. If α is not an integer, we have from (1.2)

$$R_\alpha(z,w)=\frac{(-1)^{\lceil\alpha\rceil+1}2^{\alpha+1}}{\Gamma(\alpha+1)\Gamma(\lceil\alpha\rceil-\alpha)}\int_0^\infty t^{\lceil\alpha\rceil-\alpha-1}D^{\lceil\alpha\rceil+1}P_z(w',w_n+t)\,dt \tag{2.2}$$

for $z,w\in\boldsymbol{H}$.

Note that for each $j=1,\dots,n-1$,

$$D_{z_j}P(z,w)=-D_{w_j}P(z,w)$$

and

$$D_{z_n}P(z,w)=D_{w_n}P(z,w).$$

Therefore, we can show that for multi-indices $\beta=(\beta_1,\dots,\beta_n)$ and $\gamma=(\gamma_1,\dots,\gamma_n)$,

$$\begin{aligned}D_z^\beta D_w^\gamma P(z,w)&=(-1)^{\gamma_1+\cdots+\gamma_{n-1}}D_{z_1}^{\beta_1+\gamma_1}\cdots D_{z_n}^{\beta_n+\gamma_n}P(z,w)\\&=(-1)^{\gamma_1+\cdots+\gamma_{n-1}}\frac{f_{\beta,\gamma}(z-\overline{w})}{|z-\overline{w}|^{n+2|\beta|+2|\gamma|}},\end{aligned} \tag{2.3}$$

where $f_{\beta,\gamma}$ is a homogeneous polynomial of degree $1+|\beta|+|\gamma|$.

Fix $\boldsymbol{e}=(0,1)$ for the rest of this paper. Then (2.3) gives that for each nonnegative integer k

$$f_{k+1}(\boldsymbol{e})=\frac{(-1)^{k+1}2\Gamma(n+k)}{n\sigma_n\Gamma(n-1)}\neq 0, \tag{2.4}$$

where $f_{k+1}(\boldsymbol{e}) = f_{(0,\cdots,0),(0,\cdots,0,k+1)}(\boldsymbol{e})$. Thus there exists $0 < \varepsilon < 1$ such that $C^{-1} \geq f_{k+1}(\boldsymbol{e}) f_{k+1}(z) \geq C$ for some constant $C > 0$ and every $z \in B(\boldsymbol{e}, \varepsilon)$ where $B(\boldsymbol{e}, \varepsilon)$ is the open ball in $\boldsymbol{R}^n$ centered at $\boldsymbol{e}$ with radius ε. For $w \in \boldsymbol{H} \cup \{0\}$, let $\Gamma_\varepsilon(w) = \{z \in \boldsymbol{H} \mid (z_n + w_n) > \varepsilon |z' - w'|\}$. Because f_{k+1} is a homogeneous polynomial of degree $k+2$, there exists $\epsilon_0 > 0$ such that

$$C^{-1} z_n^{k+2} \geq f_{k+1}(\boldsymbol{e}) f_{k+1}(z) \geq C z_n^{k+2} \tag{2.5}$$

for every $z \in \Gamma_{\varepsilon_0}(0)$. Note that $z - \overline{w} \in \Gamma_{\varepsilon_0}(0)$ for $w \in \boldsymbol{H}$ and $z \in \Gamma_{\varepsilon_0}(w)$. Thus (2.5) implies that

$$\begin{aligned} f_{k+1}(\boldsymbol{e}) D^{k+1} P_z(w) &= \frac{f_{k+1}(\boldsymbol{e}) f_{k+1}(z - \overline{w})}{|z - \overline{w}|^{n+2+2k}} \\ &\geq C \frac{(z_n + w_n)^{k+2}}{|z - \overline{w}|^{n+2+2k}} \\ &\geq \frac{C}{(z_n + w_n)^{n+k}}, \end{aligned} \tag{2.6}$$

for every $z \in \Gamma_{\varepsilon_0}(w)$. Then $R_\alpha(z, w)$ can be estimated as the following.

Lemma 2.1. *Given $\alpha > -1$ and $w \in \boldsymbol{H}$, there exists a constant $C = C_{n,\alpha} > 0$ such that*

$$R_\alpha(z, w) \geq \frac{C}{(z_n + w_n)^{n+\alpha}}$$

for every $z \in \Gamma_{\varepsilon_0}(w)$.

Proof. Let $w \in \boldsymbol{H}$. Note that $\Gamma_{\varepsilon_0}(w) \subset \Gamma_{\varepsilon_0}(w + (0, t))$ for every $t > 0$.
If α is not an integer, (2.2), (2.4) and (2.6) imply

$$\begin{aligned} &R_\alpha(z, w) \\ &= \frac{2^{\alpha+1}}{\Gamma(\alpha+1)\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^\infty t^{\lceil\alpha\rceil - \alpha - 1} (-1)^{\lceil\alpha\rceil + 1} D^{\lceil\alpha\rceil + 1} P_z(w', w_n + t)\, dt \\ &= \frac{n\sigma_n 2^\alpha \Gamma(n-1)}{\Gamma(\alpha+1)\Gamma(n + \lceil\alpha\rceil)\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^\infty t^{\lceil\alpha\rceil - \alpha - 1} f_{\lceil\alpha\rceil + 1}(\boldsymbol{e}) D^{\lceil\alpha\rceil + 1} P_z(w', w_n + t)\, dt \\ &\geq \frac{C_n 2^\alpha}{\Gamma(\alpha+1)\Gamma(n + \lceil\alpha\rceil)\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^\infty \frac{t^{\lceil\alpha\rceil - \alpha - 1}}{(z_n + w_n + t)^{n + \lceil\alpha\rceil}}\, dt \end{aligned}$$

for every $z \in \Gamma_{\varepsilon_0}(w)$. Since

$$\int_0^\infty \frac{t^{\lceil\alpha\rceil-\alpha-1}}{(z_n+w_n+t)^{n+\lceil\alpha\rceil}}dt = \frac{1}{(z_n+w_n)^{n+\alpha}}\int_0^1 s^{n+\alpha-1}(1-s)^{\lceil\alpha\rceil-\alpha-1}ds$$
$$= \frac{\Gamma(n+\alpha)\Gamma(\lceil\alpha\rceil-\alpha)}{\Gamma(n+\lceil\alpha\rceil)}\frac{1}{(z_n+w_n)^{n+\alpha}},$$

we have

$$R_\alpha(z,w) \geq \frac{C_{n,\alpha}}{(z_n+w_n)^{n+\alpha}}$$

for every $z \in \Gamma_{\varepsilon_0}(w)$.

If α is an integer, (1.2), (2.4) and (2.6) give us that

$$R_\alpha(z,w) = \frac{C_n 2^\alpha}{\Gamma(\alpha+1)\Gamma(n+\alpha)} f_{\alpha+1}(\boldsymbol{e})D^{\alpha+1}P_z(w)$$
$$\geq \frac{C_{n,\alpha}}{(z_n+w_n)^{n+\alpha}}$$

for every $z \in \Gamma_{\varepsilon_0}(w)$. The proof is complete. □

The following lemma is Proposition 3.1 in [**1**]. However we calculate the constant exactly to use it in this paper.

Lemma 2.2. *Given $a+b>-1$ and $b<0$, there exists a constant $C=C_n>0$ such that*

$$\int_{\boldsymbol{H}} \frac{w_n^{a+b}}{|z-\overline{w}|^{n+a}}dw = C\frac{\Gamma(\frac{a+1}{2})\Gamma(-b)\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(\frac{n+a}{2})}z_n^b \tag{2.7}$$

for every $z \in \boldsymbol{H}$.

Proof. Integration in polar coordinates and change of variable yield

$$\int_{\boldsymbol{H}} \frac{w_n^{a+b}}{|z-\overline{w}|^{n+a}}dw = \int_0^\infty \int_{R^{n-1}} \frac{w_n^{a+b}}{(|z'-w'|^2+(z_n+w_n)^2)^{(n+a)/2}}dw'dw_n$$
$$= (n-1)\sigma_{n-1}\int_0^\infty\int_0^\infty \frac{r^{n-2}w_n^{a+b}}{(r^2+(z_n+w_n)^2)^{(n+a)/2}}dr\,dw_n$$

$$
\begin{aligned}
&= (n-1)\sigma_{n-1}\left(\int_0^\infty \frac{t^{a+b}}{(z_n+t)^{a+1}}dt\right)\left(\int_0^\infty \frac{r^{n-2}}{(r^2+1)^{(n+a)/2}}dr\right)\\
&:= (n-1)\sigma_{n-1} I_1 \cdot I_2.
\end{aligned}
$$

Since $-b > 0$ and $a+b+1>0$, using change of variable again, we have

$$
I_1 = z_n^b \int_0^1 s^{-b-1}(1-s)^{a+b} ds = \frac{\Gamma(-b)\Gamma(a+b+1)}{\Gamma(a+1)} z_n^b.
$$

Similarly, we get

$$
I_2 = \frac{1}{2}\int_0^1 s^{\frac{a-1}{2}}(1-s)^{\frac{n-3}{2}} ds = \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{a+1}{2}\right)}{2\Gamma\left(\frac{n+a}{2}\right)}
$$

because $(a+1)/2 > 0$. Combining these estimates, we obtain (2.7) as desired. □

3. Main result.

Now we are ready to prove the main theorem.

THEOREM 3.1. *For any $\alpha > -1$, the weighted harmonic Bergman projection has the following norm estimation*:

$$
\|\Pi_\alpha\| \approx \max\left\{\frac{1}{p-1}, p\right\},
$$

where $\|\Pi_\alpha\|$ is the norm of Π_α on L^p_α for $1 < p < \infty$.

PROOF. We first estimate the lower bound. For some $0 < c_1 < \varepsilon_0$ satisfying $1 - c_1 - c_1\varepsilon_0 > 0$, let $f_0 = \chi_{B(\boldsymbol{e},c_1)}$. Note that

$$
\|f_0\|^p_{L^p_\alpha} = V_\alpha(B(\boldsymbol{e},c_1)) \geq C_{n,\alpha}(1+c_1)^{(n+\alpha)}. \tag{3.1}
$$

Take some constant ε_1 satisfying $\varepsilon_1 > \varepsilon_0/(1-c_1-c_1\varepsilon_0)$. Since

$$
\frac{1+c_1\varepsilon_1}{\varepsilon_1(1-c_1)} < \frac{1}{\varepsilon_0}
$$

and

$$\frac{|w'|}{c_1} < 1 < \frac{w_n}{1-c_1} \qquad \text{for} \quad w \in B(\boldsymbol{e}, c_1),$$

we see that for $z \in \Gamma_{\varepsilon_1}(\boldsymbol{e})$ and $w \in B(\boldsymbol{e}, c_1)$,

$$\begin{aligned} |z' - w'| &< \frac{z_n + 1}{\varepsilon_1} + \frac{c_1}{1-c_1} w_n \\ &< \frac{z_n}{\varepsilon_1} + \frac{1}{\varepsilon_1} \frac{w_n}{1-c_1} + \frac{c_1}{1-c_1} w_n \\ &= \frac{z_n}{\varepsilon_1} + \frac{1 + c_1 \varepsilon_1}{\varepsilon_1(1-c_1)} w_n \\ &< \frac{z_n + w_n}{\varepsilon_0}. \end{aligned}$$

Therefore, we obtain $\Gamma_{\varepsilon_1}(\boldsymbol{e}) \subset \Gamma_{\varepsilon_0}(w)$ for $w \in B(\boldsymbol{e}, c_1)$. Thus, applying Lemma 2.1, we have

$$\int_{B(\boldsymbol{e},c_1)} R_\alpha(z,w)\, dV_\alpha(w) \geq C_{n,\alpha} \int_{B(\boldsymbol{e},c_1)} \frac{dV_\alpha(w)}{(z_n + w_n)^{n+\alpha}} \tag{3.2}$$

for $z \in \Gamma_{\varepsilon_1}(\boldsymbol{e})$. Since $z_n + w_n \leq 2(|z|+1)$ for $w \in B(\boldsymbol{e}, c_1)$, we have by (3.2)

$$\begin{aligned} \|\Pi_\alpha f_0\|_{L^p_\alpha}^p &= \int_{\boldsymbol{H}} \left| \int_{B(\boldsymbol{e},c_1)} R_\alpha(z,w)\, dV_\alpha(w) \right|^p dV_\alpha(z) \\ &\geq C_{n,\alpha}^p \int_{\Gamma_{\varepsilon_1}(\boldsymbol{e})} \left(\int_{B(\boldsymbol{e},c_1)} \frac{dV_\alpha(w)}{(z_n + w_n)^{n+\alpha}} \right)^p dV_\alpha(z) \\ &\geq C_{n,\alpha}^p V_\alpha(B(\boldsymbol{e}, c_1))^p \int_{\Gamma_{\varepsilon_1}(\boldsymbol{e})} \frac{dV_\alpha(z)}{(1 + |z|^{n+\alpha})^p}. \end{aligned} \tag{3.3}$$

Using integration in polar coordinates, we get

$$\begin{aligned} \int_{\Gamma_{\varepsilon_1}(\boldsymbol{e})} \frac{dV_\alpha(z)}{(1 + |z|^{n+\alpha})^p} &\geq \int_{\Gamma_{\varepsilon_1}(0) \cap \{z \in \boldsymbol{H} | |z|=1\}} \zeta_n^\alpha \, d\sigma(\zeta) \int_{c_1}^\infty r^{n-1+\alpha}(1 + r^{n+\alpha})^{-p}\, dr \\ &= C_{n,\alpha} \frac{(1 + c_1^{n+\alpha})^{1-p}}{(n+\alpha)(p-1)}. \end{aligned}$$

Thus, by (3.3) and (3.1), we have

$$\|\Pi_\alpha f_0\|_{L^p_\alpha}^p \geq \frac{C_{n,\alpha}^p}{(n+\alpha)(p-1)} \|f_0\|_{L^p_\alpha}^p.$$

Note that for $1 < p \leq 2$,

$$\left(\frac{1}{p-1}\right)^{\frac{1}{p}} = \left(\frac{1}{p-1}\right)\left(\frac{1}{p-1}\right)^{\frac{1}{p}-1} \geq \frac{C}{p-1}.$$

Consequently, we have

$$\sup_{\|f\|_{L^p_\alpha}=1} \|\Pi_\alpha f\|_{L^p_\alpha} \geq \frac{\|\Pi_\alpha f_0\|_{L^p_\alpha}}{\|f_0\|_{L^p_\alpha}} \geq \frac{C_{n,\alpha}}{p-1} \qquad \text{for} \quad 1 < p \leq 2. \tag{3.4}$$

Also, by the duality and (3.4), we have

$$\sup_{\|f\|_{L^p_\alpha}=1} \|\Pi_\alpha f\|_{L^p_\alpha} \geq C_{n,\alpha} p \qquad \text{for} \quad p > 2. \tag{3.5}$$

In fact, with $1/p+1/q = 1$, the duality property of the weighted harmonic Bergman space(See [**2**]) and (3.4) imply

$$\begin{aligned}
\sup_{\|f\|_{L^p_\alpha}=1} \|\Pi_\alpha f\|_{L^p_\alpha} &\geq \sup_{\|f\|_{L^p_\alpha}=1} \left|\left\langle \Pi_\alpha f, \frac{f_0}{\|f_0\|_{L^q_\alpha}}\right\rangle\right| \\
&= \sup_{\|f\|_{L^p_\alpha}=1} \left|\left\langle f, \Pi_\alpha \frac{f_0}{\|f_0\|_{L^q_\alpha}}\right\rangle\right| \\
&= \frac{\|\Pi_\alpha f_0\|_{L^q_\alpha}}{\|f_0\|_{L^q_\alpha}} \\
&\geq \frac{C_{n,\alpha}}{q-1} \\
&\geq C_{n,\alpha}\, p
\end{aligned}$$

for $p > 2$. Therefore, we have from (3.4) and (3.5),

$$\|\Pi_\alpha\| \geq C_{n,\alpha} \max\left\{\frac{1}{p-1}, p\right\}.$$

Now, let's estimate upper bound. (1.3) and Hölder's inequality imply

$$\begin{aligned}
|\Pi_\alpha f(z)|^p &\le C_{n,\alpha}^p \Big(\int_{\boldsymbol{H}} \frac{|f(w)| w_n^\alpha}{|z-\overline{w}|^{n+\alpha}}\, dw \Big)^p \\
&\le C_{n,\alpha}^p \Big(\int_{\boldsymbol{H}} \frac{w_n^{\alpha - \frac{\alpha+1}{p}}}{|z-\overline{w}|^{n+\alpha}}\, dw \Big)^{p/q} \int_{\boldsymbol{H}} \frac{|f(w)|^p w_n^{\alpha+\frac{\alpha+1}{q}}}{|z-\overline{w}|^{n+\alpha}}\, dw \\
&= C_{n,\alpha}^p \Big(C_{n,\alpha} \Gamma\Big(\frac{\alpha+1}{p}\Big) \Gamma\Big(\frac{\alpha+1}{q}\Big)\Big)^{p/q} z_n^{-\frac{\alpha+1}{q}} \int_{\boldsymbol{H}} \frac{|f(w)|^p w_n^{\alpha+\frac{\alpha+1}{q}}}{|z-\overline{w}|^{n+\alpha}}\, dw
\end{aligned}$$

where the last equality holds by Lemma 2.2. Thus by Fubini's theorem and Lemma 2.2, we have

$$\begin{aligned}
&\|\Pi_\alpha f\|_{L^p_\alpha}^p \\
&\quad \le C_{n,\alpha}^p \Big(C_{n,\alpha} \Gamma\Big(\frac{\alpha+1}{p}\Big) \Gamma\Big(\frac{\alpha+1}{q}\Big)\Big)^{p/q} \int_{\boldsymbol{H}} |f(w)|^p w_n^{\alpha+\frac{\alpha+1}{q}} \int_{\boldsymbol{H}} \frac{z_n^{\alpha-\frac{\alpha+1}{q}}\, dz}{|z-\overline{w}|^{n+\alpha}}\, dw \\
&\quad = C_{n,\alpha}^p \Big(C_{n,\alpha} \Gamma\Big(\frac{\alpha+1}{p}\Big) \Gamma\Big(\frac{\alpha+1}{q}\Big)\Big)^{p/q+1} \int_{\boldsymbol{H}} |f(w)|^p \, dV_\alpha(w).
\end{aligned}$$

So we obtain

$$\|\Pi_\alpha f\|_{L^p_\alpha} \le C_{n,\alpha} \Gamma\Big(\frac{\alpha+1}{p}\Big) \Gamma\Big(\frac{\alpha+1}{q}\Big) \|f\|_{L^p_\alpha}. \tag{3.6}$$

In case $p \to \infty$, we know $\Gamma\big(\frac{\alpha+1}{q}\big) \approx C_\alpha$ and $\Gamma\big(\frac{\alpha+1}{p}\big) \approx p$. Also, if $p \to 1$, then $\Gamma\big(\frac{\alpha+1}{p}\big) \approx C_\alpha$ and $\Gamma\big(\frac{\alpha+1}{q}\big) \approx \frac{1}{p-1}$. Thus (3.6) means that

$$\|\Pi_\alpha\| \le C_{n,\alpha} \max\Big\{ \frac{1}{p-1}, p \Big\}$$

as desired. The proof is complete. □

When p is fixed, the same argument of the proof of Theorem 3.1 gives us the following.

THEOREM 3.2. *Let $1 < p < \infty$. Then there exists a constant $C = C_{n,p} > 0$ such that*

$$\|\Pi_\alpha\| \le \frac{C}{\alpha+1}$$

as $\alpha \to -1^+$.

Proof. Note that $\|\Pi_\alpha\| \ge 1$.
If α is an integer, we have form (1.2)

$$|R_\alpha(z,w)| \le \frac{C_n 2^\alpha \Gamma(n+\alpha)}{\Gamma(\alpha+1)} \frac{1}{|z-\overline{w}|^{n+\alpha}}.$$

If α is not an integer, we have from (2.2),

$$\begin{aligned}
|R_\alpha(z,w)| &\le \frac{2^{\alpha+1}}{\Gamma(\alpha+1)\Gamma(\lceil\alpha\rceil-\alpha)} \int_0^\infty t^{\lceil\alpha\rceil-\alpha-1} |D^{\lceil\alpha\rceil+1} P_z(w', w_n+t)|\, dt \\
&\le \frac{C_n 2^\alpha \Gamma(n+\lceil\alpha\rceil)}{\Gamma(\alpha+1)\Gamma(\lceil\alpha\rceil-\alpha)} \int_0^\infty \frac{t^{\lceil\alpha\rceil-\alpha-1}}{(|z-\overline{w}|+t)^{n+\lceil\alpha\rceil}}\, dt \\
&= \frac{C_n 2^\alpha \Gamma(n+\lceil\alpha\rceil)}{\Gamma(\alpha+1)\Gamma(\lceil\alpha\rceil-\alpha)} \frac{1}{|z-\overline{w}|^{n+\alpha}} \int_0^\infty \frac{t^{\lceil\alpha\rceil-\alpha-1}}{(1+t)^{n+\lceil\alpha\rceil}}\, dt \\
&= \frac{C_n 2^\alpha \Gamma(n+\alpha)}{\Gamma(\alpha+1)} \frac{1}{|z-\overline{w}|^{n+\alpha}}.
\end{aligned}$$

Thus we have

$$|R_\alpha(z,w)| \le \frac{C_n 2^\alpha \Gamma(n+\alpha)}{\Gamma(\alpha+1)} \frac{1}{|z-\overline{w}|^{n+\alpha}} \tag{3.7}$$

for $\alpha > -1$. Then Hölder's inequality give us that

$$\begin{aligned}
|\Pi_\alpha f(z)|^p &\le \left(\int_{\boldsymbol{H}} |f(w)||R_\alpha(z,w)|\, dV_\alpha(w) \right)^p \\
&\le \left(\frac{C_n 2^\alpha \Gamma(n+\alpha)}{\Gamma(\alpha+1)} \right)^p \left(\int_{\boldsymbol{H}} \frac{w_n^{\alpha-\frac{\alpha+1}{p}}}{|z-\overline{w}|^{n+\alpha}}\, dw \right)^{p/q} \int_{\boldsymbol{H}} \frac{|f(w)|^p w_n^{\alpha+\frac{\alpha+1}{q}}}{|z-\overline{w}|^{n+\alpha}}\, dw \\
&= \left(\frac{C_n 2^\alpha \Gamma(n+\alpha)}{\Gamma(\alpha+1)} \right)^p \left(\frac{C_n \Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha+1}{p}\right)\Gamma\left(\frac{\alpha+1}{q}\right)}{\Gamma(\alpha+1)\Gamma\left(\frac{n+\alpha}{2}\right)} \right)^{p/q} \\
&\qquad \times z_n^{-\frac{\alpha+1}{q}} \int_{\boldsymbol{H}} \frac{|f(w)|^p w_n^{\alpha+\frac{\alpha+1}{q}}}{|z-\overline{w}|^{n+\alpha}}\, dw
\end{aligned}$$

where the last equality holds by Lemma 2.2. So we have

$$\begin{aligned}\|\Pi_\alpha f\|_{L^p_\alpha}^p &\le \left(\frac{C_n 2^\alpha \Gamma(n+\alpha)}{\Gamma(\alpha+1)}\right)^p \left(\frac{C_n \Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha+1}{p}\right)\Gamma\left(\frac{\alpha+1}{q}\right)}{\Gamma(\alpha+1)\Gamma\left(\frac{n+\alpha}{2}\right)}\right)^{p/q} \\ &\quad \times \int_{\boldsymbol{H}} |f(w)|^p w_n^{\alpha+\frac{\alpha+1}{q}} \int_{\boldsymbol{H}} \frac{z_n^{\alpha-\frac{\alpha+1}{q}}}{|z-\overline{w}|^{n+\alpha}}\, dz\, dw \\ &= \left(\frac{C_n 2^\alpha \Gamma(n+\alpha)}{\Gamma(\alpha+1)}\right)^p \left(\frac{C_n \Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha+1}{p}\right)\Gamma\left(\frac{\alpha+1}{q}\right)}{\Gamma(\alpha+1)\Gamma\left(\frac{n+\alpha}{2}\right)}\right)^{p/q+1} \|f\|_{L^p_\alpha}^p.\end{aligned}$$

Consequently,

$$\begin{aligned}\|\Pi_\alpha f\|_{L^p_\alpha} &\le \frac{C_n 2^\alpha \Gamma(n+\alpha)\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha+1}{p}\right)\Gamma\left(\frac{\alpha+1}{q}\right)}{\Gamma(\alpha+1)^2\Gamma\left(\frac{n+\alpha}{2}\right)} \|f\|_{L^p_\alpha} \\ &\le \frac{C_{n,p}}{\alpha+1} \|f\|_{L^p_\alpha}\end{aligned}$$

as $\alpha \to -1^+$. The proof is complete. □

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