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Jacobi inversion on strata of the Jacobian of the C_{rs} curve $y^r = f(x)$

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Abstract. By using the generalized sigma function of a C_{rs} curve $y^r = f(x)$, we give a solution to the Jacobi inversion problem over the stratification in the Jacobian given by the Abel image of the symmetric products of the curve. We show that determinants consisting of algebraic functions on the curve, whose zeros give the Abelian pre-image of the strata, are written by ratios of certain derivatives of the sigma function on the strata. We also discuss the order of vanishing of abelian functions on the strata in terms of intersection theory.

1. Introduction.

The addition law on the elliptic curve is given by the determinant:

$$\begin{vmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(w) & \wp'(w) \end{vmatrix} = 2 \cdot \frac{\sigma(u+v+w)\sigma(u-v)\sigma(v-w)\sigma(w-u)}{\sigma^3(u)\sigma^3(v)\sigma^3(w)}. \tag{1.1}$$

Many a Tripos problem given as an exercise in [**WW**] displays related matrices, which implement the profound relationship between linear series on a Riemann Surface and its theta function brought to light by Riemann. The primary type of matrix and result that we here generalize to higher genus is the following, which we call Frobenius-Stickelberger (FS, for short) and can be found in Miscellaneous Examples 21 [**WW**, Chapter 20], as well as the original paper [**FS**]:

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$$\begin{vmatrix} 1 & \wp(u_0) & \wp'(u_0) & \cdots & \wp^{(n-1)}(u_0) \\ 1 & \wp(u_1) & \wp'(u_1) & \cdots & \wp^{(n-1)}(u_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(u_n) & \wp'(u_n) & \cdots & \wp^{(n-1)}(u_n) \end{vmatrix}$$

$$= (-1)^{\frac{1}{2}n(n-1)} 1!2! \dots n! \frac{\sigma(u_0 + u_1 + \cdots + u_n) \prod_{i<j} \sigma(u_i - u_j)}{\sigma^{n+1}(u_0)\sigma^{n+1}(u_1) \cdots \sigma^{n+1}(u_n)}. \tag{1.2}$$

To generalize this result one needs two things: firstly, an analog of the algebraic properties of the Weierstrass $\wp$-function; secondly, an analog of the analytic properties of the σ function. The first ingredient is produced by a particular choice of the curve, or rather, two generators of its function field (corresponding to the $x = \wp$, $y = \wp'$ of the elliptic curve). While it is possible to write any curve as a plane curve up to birational equivalence, and it is possible to write any plane curve as a determinant (cf. e.g. [**G**]), we want to work with a smooth affine model of the curve to simplify the treatment (curves with the plane-model property, cf. [**MP**]), and also make here the more serious assumption that the curve admits a given symmetry. Still, the fledgling theory of the σ function for this type of curves [**BLE2**] is complicated enough, and this type of curves useful enough in applied mathematics as well as cryptography and coding theory [**Ar**], [**Mi**], that we believe our result gives a reasonable foundation for the theory. Having said that, we proceed to describe the curve. In a beautiful and overlooked chapter of differential algebra that was rediscovered in the 1970s in the context of integrable equations, Burchnall and Chaundy [**BC**] classified pairs of commuting ordinary differential operators of coprime orders, (n, s), say. These turn out to satisfy a polynomial equation, of the type that has recently been called (n, s)-curve [**BLE2**], except that we choose the notation $(n = r, s)$:

$$f(x, y) = y^r + x^s + f_{r-1,s-1}(x, y),$$

where r and s are positive integers such that the greatest common divisor $(r, s) = 1$, and if $f_{r-1,s-1}(x, y)$ contains a monomial $x^a y^b$, then $ar + bs < rs$.

If the polynomial $f(x, y)$ defines a smooth affine curve (this is precisely the definition of a C_{ab} curve [**Ar**], [**Mi**], if we let $a = r, b = s$), we can obtain a smooth complete curve X by adding just one point ∞; this curve has the property that $(2g - 2)\infty \sim K_X$, a canonical divisor (as usual $\sim$ denotes linear equivalence), or equivalently, the number $(2g - 1)$ is a Weierstrass gap. The point ∞ is a natural choice of base point for the Abel map, and this motivates the symmetry assumption: $y^r = x^s + \lambda_{s-1}x^{s-1} + \cdots + \lambda_1 x + \lambda_0$. $(r < s)$. The fact that the curve is now a cyclic Galois cover of the x-line, makes it possible to relate the algebraic and analytic properties of the Abelian images of the k-fold symmetric products of the curve:

$$\mathscr{W}^k := \kappa\left(\left\{\sum_{i=1}^{k}\int_{\infty}^{(x_i,y_i)}\begin{pmatrix}{\nu^{\mathrm{I}}}_1\\ \vdots\\ {\nu^{\mathrm{I}}}_g\end{pmatrix}\Bigg|(x_i,y_i)\in X\right\}\right)\subset\mathscr{J}. \tag{1.3}$$

where κ is the projection $\boldsymbol{C}^g\to\mathscr{J}=\boldsymbol{C}^g/\Lambda$, Λ is the period lattice of a canonical basis $\{{\nu^{\mathrm{I}}}_1,\dots,{\nu^{\mathrm{I}}}_g\}$ of $H^1(X,\mathscr{O}_X)$, and $\mathscr{J}$ is the Jacobian of X. We denote by w the Abel map from the k-fold symmetric product $S^k(X)$ of X to $\kappa^{-1}\mathscr{W}^k$ with base-point ∞, for any positive integer k. Note that there is a remaining Λ-ambiguity due to the choice of path of integration: our results below will be independent of such ambiguity, but they require a g-tuple of complex numbers to be stated, explicitly: $w:(P_1,\dots,P_k)\mapsto w(P_1,\dots,P_k)=\sum_{i=1}^{k}\int_{\infty}^{P_i}\nu^{\mathrm{I}}\in\boldsymbol{C}^g$, where we abbreviate by ν^{I} the g-vector of holomorphic differentials ${\nu^{\mathrm{I}}}_i$. When an analytic function, say, of g complex variables is evaluated on $u:=w(P_1,\dots,P_k)$, we view it as function of the coordinates $(u_1,\dots,u_g)$ of the (column) vector u, as the convention goes.

A first natural goal is to produce addition theorems that generalize the genus-1 formulae recalled above on these subvarieties, and then one has to reckon with the fact that the (appropriately defined and normalized) σ function will vanish (essentially by Riemann's theorem [**F1**, Theorem 1.1]) and take careful limits when a denominator equals zero. The second natural goal is to give Jacobi-inversion formulae[1], namely the algebraic coordinates for the divisors

$$\sum_{i=1}^{k}P_i\in w^{-1}\mathscr{W}^k.$$

Remarkably, both goals were achieved in genus 2, with partial results for higher genus, or for a general hyperelliptic curve (corresponding to the case $r=2$, with s then equal to $2g+1$ where g is the genus) by Baker [**B2**], [**B3**], whose work was recently furthered and refined by Ônishi [**O1**], [**O2**] and Buchstaber, Leykin, and Enolskii [**BEL**], [**BLE1**], [**BLE2**], [**BLE3**]. The relevant determinant, which we will study as well, has the form:

[1]Note that in the literature what is called the problem of "inversion" (of abelian integrals) is the much less naïve issue of determining the twist of the projective bundle $S^k(X)\to\mathscr{J}$, for k large enough (cf. e.g. [**K1**], [**K2**]).

$$\psi_n(P_1, \dots, P_n) := \begin{vmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \cdots & \phi_{n-1}(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \cdots & \phi_{n-1}(P_2) \\ 1 & \phi_1(P_3) & \phi_2(P_3) & \cdots & \phi_{n-1}(P_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_n) & \phi_2(P_n) & \cdots & \phi_{n-1}(P_n) \end{vmatrix}. \tag{1.4}$$

where the entries are monomials in the (x, y)-coordinates specified uniquely by their order of vanishing at ∞. We call the matrix in (1.4) an FS matrix, after Frobenius and Stickelberger. Not surprisingly, this matrix by now has appeared in much work related to integrable systems with spectral curve X, being one of the most suitable tools for achieving explicit linearization of the flows on the Jacobian of X.

As for the inversion formulae, we summarize the results known for the hyperelliptic case, which we generalize. In that case, $\phi_i = x^i$ for $i < g$, so that:

$$\begin{aligned} F_k(x) &:= \frac{\psi_{k+1}(P_1, P_2, \dots, P_k, P)}{\psi_k(P_1, P_2, \dots, P_k)} \\ &= (x - x_1)(x - x_2) \cdots (x - x_k) \\ &:= x^k + \sum_{i=0}^{k-1} (-)^i e_i^{(k)} x^i. \end{aligned} \tag{1.5}$$

The known inversion formulae look as follows (where subscripts denote certain derivatives of the σ function which will defined explicitly in Section 4 below):

Theorem 1.1 ([**EG**]). *For a hyperelliptic curve X of genus g whose affine part is given by $y^2 = f(x)$, the following relations hold,*

(1) $\mathscr{W}^g$ *case: For a point $(P_1, \dots, P_g)$ of the non-singular locus of $S^g X$ on $u := w(P_1, \dots, P_g) = \sum_{i=1}^g \int_\infty^{P_i} \nu^{\mathrm{I}}$,*

$$\frac{\sigma_i(u)\sigma_g(u) - \sigma_{gi}(u)\sigma(u)}{\sigma^2(u)} = (-)^{g-i+1} e_{g-i}^{(g)}.$$

(2) $\mathscr{W}^{g-1}$ *case: For a point $(P_1, \dots, P_{g-1})$ of the non-singular locus of $S^{g-1} X$ on $u := w(P_1, \dots, P_{g-1}) = \sum_{i=1}^{g-1} \int_\infty^{P_i} \nu^{\mathrm{I}}$,*

$$\frac{\sigma_i(u)}{\sigma_g(u)} = \begin{cases} (-1)^{g-i} e^{(g-1)}_{g-i-1} & \textit{for } i < g, \\ 1 & \textit{for } i = g. \end{cases}$$

(3) *$\mathscr{W}^k$ case, $k = 1, \dots g-2$: For a point $(P_1, \dots, P_k)$ of the non-singular locus of $S^k X$ on $u := w(P_1, \dots, P_k) = \sum_{i=1}^k \int_\infty^{P_i} \nu^{\mathrm{I}}$,*

$$\frac{\sigma_i(u)}{\sigma_{k+1}(u)} = \begin{cases} (-1)^{k-i+1} e^{(k)}_{k-i} & \textit{for } i < k+1, \\ 1 & \textit{for } i = k+1, \\ 0 & \textit{for } i > k+1. \end{cases}$$

We generalize this theorem to the (r, s) case by giving a dictionary between the FS matrix and the analytic expression of its entries.

This article is organized as follows. In Section 2 the FS-matrices are defined and investigated. In Section 3, we give explicit algebraic expressions for the several differentials which play an important role in Abelian function theory. Section 4 is devoted to the definition of the sigma function and its properties. Section 5 contains the main theorem.

We are grateful to Victor Z. Enolski for posing the problem and for the references [**E**], [**EG**], [**BEL**], [**BLE1**], [**BLE2**], [**BLE3**], [**EEL**] which are essential to this article. This work is influenced by discussions at a meeting in Tokyo Metropolitan University 2005 and we also owe special thanks to the other team members, Chris Eilbeck and Yoshihiro Ônishi. The first-named author learned σ function theory and related theories from an unpublished note written by Y. Ônishi and used his constructions in this article without explicit mention. The first-named author thanks as well John Gibbons for alerting him to his work with Baldwin on the Benney equation [**BG1**], [**BG2**], to which this paper has potential applications. The second-named author is immensely grateful for the semester at the Mittag-Leffler Institute (Spring 2007), within the program *Moduli Spaces*: without the generous hospitality of the Organizers, the Director and the Staff, and the mission of peace for concentrated research of the Institute, her contribution would not have been possible.

2. Abelian structure for special C_{rs} curves.

We consider a plane curve over $\boldsymbol{C}$ given by an affine equation

$$y^r = f(x), \quad f(x) := x^s + \lambda_{s-1} x^{s-1} + \cdots + \lambda_1 x + \lambda_0, \tag{2.1}$$

for integers r and s such that $(r,s) = 1$ and $r < s$ and complex numbers $\lambda_0, \dots, \lambda_{s-1}$'s, such that the compactification

$$X := \{(x,y) \mid y^r = f(x)\} \cup \infty$$

is a Riemann surface of genus $g = (r-1)(s-1)/2$. This is a special case of what is called a C_{ab} curve for $a = r$, $b = s$, or (r,s) curve, as recalled in the Introduction.

Let $R := \boldsymbol{C}[x,y]/(y^r - f(x))$, $\mathscr{O}_X$ be the sheaf of holomorphic functions over X. We note that $R = \mathscr{O}_X(*\infty)$ is the ring of meromorphic functions on X with poles at most at the point ∞.

We define uniquely the (monic) monomial $\phi_n \in R$ for a non-negative integer n so that its order of pole $N(n)$ of the singularity at ∞ is the (increasing) sequence complementary to the Weierstrass gaps, e.g., $\phi_0 = 1$, $\phi_1 = x$, $\dots$; by letting t_∞ be a local parameter at ∞, the leading term of ϕ_n is proportional to $t_\infty^{-N(n)}$. We define the w-degree, w-deg : $R \to \boldsymbol{Z}$, which assigns to an element of R its order of pole at ∞, so that for example:

$$\text{w-deg}(x) = r, \quad \text{w-deg}(y) = s, \quad \text{w-deg}(\phi_n(P)) = N(n). \tag{2.2}$$

We also consider the ring $R_\lambda := \boldsymbol{Q}[x, y, \lambda_0, \dots, \lambda_{s-1}]/(y^r - f(x))$ by regarding λ's as indeterminates, and define a λ-degree, λ-deg : $R_\lambda \to \boldsymbol{Z}$ as an extension of the w-degree by assigning the degree $(s-i)r$ to each λ_i. This makes the equation of the curve $e = y^r - f(x)$ homogeneous with respect with the λ-degree, while if we compare the two degrees of an element e of R_λ by projection from R_λ to R, we see that:

$$\text{w-deg}_R e \geq \lambda\text{-deg}_{R_\lambda} e.$$

PROPOSITION 2.1. *For an (r,s)-curve X, the following holds*:

(1) $g = (s-1)(r-1)/2$,
(2) $N(g+i) = 2g+i = (s-1)(r-1)+i$ *for* $i \geq 0$. *Especially* $N(g) = sr-s-r+1$ *and* $N(g+r-1) = s(r-1)$.
(3) $N(g-1) = 2g-2 = (s-1)(r-1)-2$.
(4) $N(i) \leq 2g-2 = sr-r-s-1$ *for* $0 \leq i < g$.

PROOF. The proof in [**BC**], an elementary semi-lattice argument, is especially nice. See also Exercises E, Chapter I, in [**ACGH**]. $\square$

We denote a point $P \in X$ by its affine coordinates (x,y); we also loosely denote by a k-tuple $(P_1, \dots, P_k)$, or by a divisor $D = \sum_{i=1}^k P_i$, an element of

$S^k(X)$, the k-th symmetric product of the curve. For a given local parameter t at some P in X, by $d_>(t^\ell)$ (resp. $d_<(t^\ell)$) we denote the terms of a function on X in its t-expansion whose orders of zero at P are greater (resp. less) than ℓ; $d_\geq(t^\ell)$ (resp. $d_\leq(t^\ell)$) includes terms of equal order.

A basis of $H^1(X, \mathscr{O}_X)$, $\{\nu^{\mathrm{I}}{}_1, \ldots, \nu^{\mathrm{I}}{}_g\}$, is given in terms of the ϕ_i following [**B1**, Chapter VI, Section 91],

$$\nu^{\mathrm{I}}{}_i = \frac{\phi_{i-1}(P)dx}{ry^{r-1}}, \quad (i = 1, \ldots, g). \tag{2.3}$$

We extend the w-degree to one-forms, by fixing a local parameter t_∞: for a one-form $\nu = (t_\infty^n + d_>(t_\infty^n))dt_\infty$, w-deg$(\nu) = -n$, so that:

$$N(\nu^{\mathrm{I}}{}_i) = 2g - N(i-1) - 2.$$

The notation for the Abel map and its restriction to the k-fold symmetric product of the curve was defined in the Introduction. For later convenience, we also introduce

$$S^n_m(X) := \{D \in S^n(X) \mid \dim|D| \geq m\},$$

where $|D|$ is the complete linear system $w^{-1}(w(D))$ [**ACGH**, IV.1, p. 156]. The singular locus of $S^n(X)$ is $S^n_1(X)$ [**ACGH**, Chapter IV, Lemma 1.5].

The Brill-Noether matrix [**ACGH**, Section IV.1] is given in terms of the bases (2.3):

$$\frac{1}{r}\begin{pmatrix} 1 & 1 & 1 \cdots & 1 \\ \phi_1(P_1) & \phi_1(P_2) & \cdots & \phi_1(P_g) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{g-1}(P_1) & \phi_{g-1}(P_2) & \cdots & \phi_{g-1}(P_g) \end{pmatrix} \begin{pmatrix} dx_1/y_1^{r-1} & & \cdots & \\ & dx_2/y_2^{r-1} & \cdots & \\ & & \ddots & \\ & & \cdots & dx_g/y_g^{r-1} \end{pmatrix}. \tag{2.4}$$

For a number k between 0 and g, we call the upper-left $k \times k$ submatrix a *principal submatrix* of the Brill-Noether matrix:

$$\begin{pmatrix} {\nu^{\mathrm{I}}}_1(P_1) & \cdots & {\nu^{\mathrm{I}}}_1(P_k) \\ \vdots & \ddots & \vdots \\ {\nu^{\mathrm{I}}}_k(P_1) & \cdots & {\nu^{\mathrm{I}}}_k(P_k) \end{pmatrix}$$
$$= \frac{1}{r}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \phi_1(P_1) & \phi_1(P_2) & \cdots & \phi_1(P_k) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{k-1}(P_1) & \phi_{k-1}(P_2) & \cdots & \phi_{k-1}(P_k) \end{pmatrix}\begin{pmatrix} dx_1/y_1^{r-1} & & \cdots & \\ & dx_2/y_2^{r-1} & \cdots & \\ & & \ddots & \\ & & \cdots & dx_k/y_k^{r-1} \end{pmatrix}. \tag{2.5}$$

For the given k the Abel map in our notation is,

$$w : S^k(X) \to \kappa^{-1}\mathscr{W}^k, \quad w(P_1, \dots, P_k) = \sum_{i=1}^{k} w(P_i), \tag{2.6}$$

$w(P) := \int_\infty^P \nu^{\mathrm{I}}$ being the original Abel map on the curve. The (transpose of the) rectangular matrix

$$\frac{1}{r}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \phi_1(P_1) & \phi_1(P_2) & \cdots & \phi_1(P_k) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{g-1}(P_1) & \phi_{g-1}(P_2) & \cdots & \phi_{g-1}(P_k) \end{pmatrix}\begin{pmatrix} dx_1/y_1^{r-1} & & \cdots & \\ & dx_2/y_2^{r-1} & \cdots & \\ & & \ddots & \\ & & \cdots & dx_k/y_k^{r-1} \end{pmatrix}$$

gives a local coordinate representation of the homomorphism of locally-free sheaves that correspond to the (pull-back of the) tangent bundle under the divisor-sum map,

$$w_* : \boldsymbol{T}_{S^k(X)} \to j_k^{-1}\boldsymbol{T}_{\mathrm{Pic}^k(C)}. \tag{2.7}$$

Using this explicit representation, it can be proved that for $k < g$ the singular locus of $\mathscr{W}^k$ is $\mathscr{W}_1^k$ [**ACGH**, Chapter IV, Corollary 4.5].

We consider the "universal effective divisor Δ on X" [**ACGH**, Section IV.2], which is a divisor in $X \times S^n(X)$ consisting of (P_0, D) such that $P_0 \in \mathrm{supp}(D)$. Explicitly, Δ is the sum of divisors $\sum_{i=1}^n \Delta_i$, where $\Delta_i := \{(P_0, P_1, \dots, P_n) \mid P_0 = P_i\}$; by letting t_i be a local parameter near P_i, an equation for Δ is given by $\prod_{i=1}^n (t_0 - t_i)$. To study the universal divisor, we will introduce μ-functions in

Definition 2.3 as a generalization of the F_k given in (1.5) for hyperelliptic curves. Note that when $k = g$, F_k agrees with U of Mumford's (U, V, W) parameterization of a hyperelliptic Jacobian (which he attributes to Jacobi) [**Mu**]. We give a name to the natural projection,

$$\pi : X \times S^n(X) \to S^n(X). \tag{2.8}$$

For the definition of the μ's and explicit representation of (2.11), we introduce the Frobenius-Stickelberger (FS) matrix and its determinant, which appeared in [**O1**], [**O2**]; similar matrices appeared in several articles, e.g., [**B1**], [**B2**], [**B3**], [**BLE3**], [**ER**], [**F2**], [**P**]; note that for the $n < g$ case, these are the polynomial part of the principal submatrix of the Brill-Noether matrix and its determinant (but they do depend on the choice of functions x, y and of a local parameter; one can give a sheaf-theoretic interpretation, and view these objects as local sections).

Let n be a positive integer and $P_1, \ldots, P_n$ be in $X \backslash \infty$. We define the *ℓ-reduced Frobenius-Stickelberger* (FS) *matrix and its determinant* by:

$$\Psi_n^{(\check{\ell})}(P_1, P_2, \ldots, P_n) := \begin{pmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \cdots & \check{\phi}_\ell(P_1) & \cdots & \phi_n(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \cdots & \check{\phi}_\ell(P_2) & \cdots & \phi_n(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_n) & \phi_2(P_n) & \cdots & \check{\phi}_\ell(P_n) & \cdots & \phi_n(P_n) \end{pmatrix}.$$

and $\psi_n^{(\check{\ell})}(P_1, P_2, \ldots, P_n) := \det \Psi_n^{(\check{\ell})}(P_1, P_2, \ldots, P_n)$ (a check on top of a letter signifies deletion). It is also convenient to introduce the simpler notation:

$$\psi_n(P_1, \ldots, P_n) := \det(\Psi_n^{(\check{n})}(P_1, \ldots, P_n)),$$
$$\Psi_n(P_1, \ldots, P_n) := \Psi_n^{(\check{n})}(P_1, \ldots, P_n),$$

for the un-bordered matrix. We call this matrix *Frobenius-Stickelberger* (FS) *matrix* and its determinant *Frobenius-Stickelberger* (FS) *determinant.* These become singular for some tuples in $(X \backslash \infty)^n$.

For the case of hyperelliptic curves, i.e., $(2, s)$-curves, and $n \leq g$, $\Psi_n^{(\check{n})}(P_1, \ldots, P_n)$ is in fact a Vandermonde matrix.

We now pursue a very naïve realization of the addition structure of Pic$X$ in terms of (2.8) and FS-matrices. For n points $(P_i)_{i=1,\ldots,n} \in X \backslash \infty$, we find an element of R associated with any point $P = (x, y)$ in $(X \backslash \infty)$, $\alpha_n(P) := \alpha_n(P; P_1, \ldots, P_n) = \sum_{i=0}^n a_i \phi_i(P)$, $a_i \in \boldsymbol{C}$ and $a_n = 1$, which has a zero at each point P_i (with multiplicity, if the P_i are repeated) and has smallest possible order of pole at ∞ with this property. Note that the divisor of α_n (viewed as a function of P) contains the universal effective divisor Δ over $(X\backslash\infty) \times S^n(X)$.

This procedure is similar to the one used by Jacobi for the hyperelliptic integral of genus two [**Ja**], [**B1**, Chapter VIII] (See also Remark 5.2), in order to give formulae for the Abelian sum in analogy to the elliptic function theory.

When the FS-matrix associated to the points P_i is invertible, the following linear system determines the coefficients of α_n:

$$\begin{pmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \cdots & \phi_n(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \cdots & \phi_n(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_n) & \phi_2(P_n) & \cdots & \phi_n(P_n) \end{pmatrix} \mathbf{a}_n = 0, \quad \mathbf{a}_n := \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix}. \tag{2.9}$$

We solve the equation by recalling the following elementary facts of linear algebra.

Proposition 2.2.

(1) *For matrices*

$$A := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad D := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & u_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & u_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & u_n \\ v_1 & v_2 & \ldots & v_n & w \end{pmatrix},$$

we have

$$\det D = [\det A]w - \sum_{p,q=1}^{n} A_{pq} u_p v_q,$$

where A_{pq} is the cofactor of A for a_{pq}, i.e., the signed determinant of the corresponding minor.

(2) *Assume* $\det A$ *does not vanish. If $h := \sum_i c_i v_i + w$, $c := {}^t(c_1, \ldots, c_n)$ and $u := {}^t(u_1, \ldots, u_n)$ satisfy $Ac = u$, then*

$$h = \frac{1}{\det A} \det D.$$

Definition 2.3. For $P, P_1, \dots, P_n \in (X \backslash \infty) \times S^n(X \backslash \infty)$, we define $\mu_n(P)$ by

$$\mu_n(P) := \mu_n(P; P_1, \dots, P_n) := \lim_{P_i' \to P_i} \frac{1}{\psi_n(P_1', \dots, P_n')} \psi_{n+1}(P_1', \dots, P_n', P),$$

where the P_i' are generic, the limit is taken (irrespective of the order) for each i; and $\mu_{n,k}(P_1, \dots, P_n)$ by

$$\mu_n(P) = \phi_n(P) + \sum_{k=0}^{n-1} (-1)^{n-k} \mu_{n,k}(P_1, \dots, P_n) \phi_k(P),$$

with the convention $\mu_{n,n}(P_1, \dots, P_n) \equiv 1$.

We extend the domains to $X \times S^n(X)$ to get possibly $\boldsymbol{P}$-valued functions.

Remark 2.4.

(1) The reason why we define μ by using a limit is the following. If P_2 is equal to P_1 for $(P_1, \dots, P_k)$ in $S^k(X \backslash \infty)$, with the other points generic, $\mu_k(P; P_1, P_2, \dots, P_k)$ becomes

$$\mu_k(P; P_1, P_2, \dots, P_k) = \frac{\begin{vmatrix} 1 & \phi_1(P_1) & \cdots & \phi_k(P_1) \\ 0 & \phi_{1,t_{P_1}}(P_1) & \cdots & \phi_{k,t_{P_1}}(P_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_k) & \cdots & \phi_k(P_k) \\ 1 & \phi_1(P) & \cdots & \phi_k(P) \end{vmatrix}}{\begin{vmatrix} 1 & \phi_1(P_1) & \cdots & \phi_{k-1}(P_1) \\ 0 & \phi_{1,t_{P_1}}(P_1) & \cdots & \phi_{k-1,t_{P_1}}(P_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_k) & \cdots & \phi_{k-1}(P_k) \end{vmatrix}},$$

where t_{P_1} is a local parameter at P_1 and $\phi_{i,t_{P_1}}(P_1)$ is the derivative of $\phi_i(P_1)$ with respect to t_{P_1}. All other cases in which $\psi_k(P_1, \dots, P_k)$ vanishes can be treated similarly.

(2) We view μ_n as a function over X^{n+1}.

(3) From the definition, for generic points, we get the explicit expression

$$\mu_{k,\ell}(P_1,\dots,P_k)=\frac{\psi_k^{(\check{\ell})}(P_1,\dots,P_k)}{\psi_k^{(\check{k})}(P_1,\dots,P_k)}. \tag{2.10}$$

This is a natural extension of the elementary symmetric function $e_i^{(k)}$ given in (1.5), with which $\mu_{k,\ell}$ agrees in the hyperelliptic-curve case [**B2**] (the ℓ index is absent in (1.5) because for $k<g$, ϕ_k is a function of x only). This $\mu_{g,\ell}(P_1,\dots,P_g)$ has often appeared in papers on integrable systems, e.g., [**ER**, p. 208], aside from the hyperelliptic case cited above [**Mu**].

(4) By the following Proposition, $S_1^n(X)$ is equal to

$$\big\{(P_1,\dots,P_n)\in S^n(X)\mid \exists\{i_1,\dots,i_k\}\subset\{1,\dots,n\}\\ \text{such that } \mu_k(P_{i_1};P_{i_2},\dots,P_{i_k})=0\big\}.$$

In the following Proposition and Lemma we show that $\mu_n(P)$ is associated with the addition structure of X from a classical viewpoint.

PROPOSITION 2.5. *For $(P_i)_{i=1,\dots,n}\in S^n(X\backslash\infty)\setminus S_1^n(X\backslash\infty)$, $\alpha_n(P;P_1,\dots,P_n)$ is equal to $\mu_n(P;P_1,\dots,P_n)$. When $(P_i)_{i=1,\dots,n}$ is a more general element of $S^n(X\backslash\infty)$, $\alpha_n(P;P_1,\dots,P_n)$ is a multiple of $\mu_n(P;P_1,\dots,P_n)$ by an element of R.*

PROOF. From the linear system (2.9) and under the assumption of invertibility, Proposition 2.2 yields the result. For more general $(P_i)_{i=1,\dots,n}\in S^n(X\backslash\infty)$, it is obvious that the zeros of $\mu_n(P;P_1,\dots,P_n)$ include $P_1,\dots,P_n$. Unless $(P_i)_{i=1,\dots,n}$ belongs to $S_1^n(X\backslash\infty)$, they include another set of $N(n)-n$ points in $X\backslash\infty$. As a consequence, there exist Q_i's in $X\backslash\infty$ such that

$$\mathrm{Div}(\mu_n(P))=\sum_{i=1}^n P_i+\sum_{i=1}^{N(n)-n}Q_i-N(n)\infty\sim 0.$$

The w-degree of $\mu_n(P)$ guarantees that it is the (monic) element of R with smallest order of pole at ∞ and with zero divisor that has $P_1+\cdots+P_n$ as a subdivisor.

When the tuple belongs to $S_1^n(X\backslash\infty)$, the minimality is not guaranteed in that some Q's are equal to ∞. Thus $\mu_n(P;P_1,\dots,P_n)$ is divisible by $\alpha_n(P;P_1,\dots,P_n)$ in R, viewed as a function of P. □

In the sequel, when we use α_n for explicit formulae, we make the convention that in the nongeneric case it denotes the corresponding μ_n (they may differ by multiplication by a polynomial but we need to secure a specific degree corresponding to the Weierstrass gaps at ∞, not necessarily the minimum).

LEMMA 2.6. *Let n be a positive integer. For $(P_i)_{i=1,\dots,n} \in S^n(X\backslash\infty)$, the function α_n over X induces the map (which we call by the same name):*

$$\alpha_n : S^n(X\backslash\infty) \to S^{N(n)-n}(X),$$

i.e., to $(P_i)_{i=1,\dots,n} \in S^n(X\backslash\infty)$ there corresponds an element $(Q_i)_{i=1,\dots,N(n)-n} \in S^{N(n)-n}(X)$, such that

$$\sum_{i=1}^{n} P_i - n\infty \sim - \sum_{i=1}^{N(n)-n} Q_i + (N(n)-n)\infty.$$

We want the preimage of α_n to include the base point ∞. For an effective divisor D in $S^n(X)$ of degree n, let D' be the maximal subdivisor of D which does not contain ∞, $D = D' + (n-m)\infty$ where $\deg D' = m (\leq n)$ and $D' \in S^m(X\backslash\infty)$. Then we extend the map to $\overline{\alpha}_n$ by defining $\overline{\alpha}_n(D) = \alpha_m(D') + [N(n)-n-(N(m)-m)]\infty$.

We see from the linear equivalence of Lemma 2.6:

PROPOSITION 2.7. *For a positive integer, the Abel map composed with α_n induces*

$$\iota_n : \mathscr{W}^n \to \mathscr{W}^{N(n)-n}, \quad \kappa \circ w \mapsto -\kappa \circ w.$$

Let image(ι_n) be denoted by $[-1]\mathscr{W}^n$.

REMARK 2.8. We recover the well-known results:

(1) The Serre involution, on $\mathrm{Pic}^{g-1}\mathscr{L} \mapsto \mathscr{O}_X(K_X)\mathscr{L}^{-1}$, is given by ι_{g-1},

$$\iota_{g-1} : \mathscr{W}^{g-1} \to [-1]\mathscr{W}^{g-1}.$$

(2) When $n \geq g$, $\iota_g \circ \iota_n$ gives the Abel sum

$$\mathscr{W}^n \xrightarrow{\iota_n} \mathscr{W}^g \xrightarrow{\iota_g} \mathscr{W}^g, \quad (w(P_1,\dots,P_n) \equiv w(Q_1,\dots,Q_g) \bmod \Lambda).$$

In particular, the addition law on the Jacobian is given by $\iota_g \circ \iota_{2g}$

$$\mathscr{W}^{2g} \xrightarrow{\iota_{2g}} \mathscr{W}^g \xrightarrow{\iota_g} \mathscr{W}^g, \quad (w(P_1,\dots,P_g,P'_1,\dots,P'_g) \equiv w(Q_1,\dots,Q_g) \bmod \Lambda).$$

Remark 2.9. The inclusion $i_n : S^n(X) \to S^{N(n)}(X)$, $(P_1, \dots, P_n) \mapsto (P_1, \dots, P_n, \overline{\alpha}_n(P_1, \dots, P_n))$ maps the stratification of symmetric products of (r, s)-curves,

$$S^0(X) \subset S^1(X) \subset \cdots \subset S^{g-2}(X) \subset S^{g-1}(X) \subset S^g(X),$$

to the stratification,

$$S^0(X) \subset S^{N(1)}(X) \subset \cdots \subset S^{N(g-2)}(X) \subset S^{2g-2}(X) \subset S^{2g}(X).$$

The fact that $2g-1$ is a gap is equivalent to $(2g-2)\infty \sim K_X$ [**ACGH**, Chapter 1 Exercises, E-2 (ii)].

For later use, we prove the following relations.

Proposition 2.10. *For every* $(P_1, \dots, P_{k-1}; P_k) \in S^{k-1}(X \backslash \infty) \times X$ $(k = 1, \dots, g)$, *the following relations hold*:

$$\begin{aligned}
&\lim_{P_k \to \infty} \frac{\phi_{k-1}(P_k)}{\phi_k(P_k)} \mu_{k,\ell}(P_1, P_2, \dots, P_k) \\
&\quad = \begin{cases} -\mu_{k-1,\ell}(P_1, P_2, \dots, P_{k-1}), & \textit{for } 0 \le \ell \le k-2, \\ -1, & \textit{for } \ell = k-1, \\ 0, & \textit{for } \ell = k. \end{cases}
\end{aligned}$$

Proof. Noting (2.10), for the $\ell = k-1$ case,

$$\begin{aligned}
\frac{\phi_{k-1}(P_k)}{\phi_k(P_k)} \mu_{k,k-1} &= \frac{-\psi_{k-1}^{(\check{k-1})}(P_1, \dots, P_{k-1})\phi_k(P_k) + d_{\ge}\big(t_\infty^{-N(k-2)}\big)}{\psi_{k-1}^{(\check{k-1})}(P_1, \dots, P_{k-1})\phi_{k-1}(P_k) + d_{\ge}\big(t_\infty^{-N(k-2)}\big)} \frac{\phi_{k-1}(P_k)}{\phi_k(P_k)} \\
&\to -1, \quad (P_k \to \infty).
\end{aligned}$$

For the $\ell \le k-2$ case

$$\begin{aligned}
\frac{\phi_{k-1}(P_k)}{\phi_k(P_k)} \mu_{k,\ell} &= \frac{-\psi_{k-1}^{(\check{\ell})}(P_1, \dots, P_{k-1})\phi_k(P_k) + d_{\ge}\big(t_\infty^{-N(k-2)}\big)}{\psi_{k-1}^{(\check{k-1})}(P_1, \dots, P_{k-1})\phi_{k-1}(P_k) + d_{\ge}\big(t_\infty^{-N(k-2)}\big)} \frac{\phi_{k-1}(P_k)}{\phi_k(P_k)} \\
&\to -\mu_{k-1,\ell}, \quad (P_k \to \infty). \qquad \square
\end{aligned}$$

Algebraic expression of the Jacobian of a coordinate change.

While in this paper we do not make much use of vector fields on symmetric products of the curve, which are smooth manifolds, following the definition of the principal submatrix we give here an algebraic (local) expression for the corresponding differential operators: these are useful when taking limiting formulae, cf. [**MP**]. The technique was used by Weierstrass, Klein, Baker and others [**B1**], [**B2**], [**B3**], [**W**] to give algebraic coordinates on $S^k(X)$. Let k be a positive integer $\leq g$. By inverting the Jacobian determinant for coordinate change where the truncated map (defined only locally around the points, lest the paths of integration differ by homotopy):

$$k\text{-proj} \circ w : S^k(X) \to \boldsymbol{C}^g \to \boldsymbol{C}^k,\ (P_1, \ldots, P_k) \mapsto \left(\sum_{i=1}^{k} \int_{\infty}^{P_i} \nu_j^{\mathrm{I}} \right)_{j=1,\ldots,k}, \qquad (2.11)$$

is smooth, as in the Inverse Function Theorem of differential calculus, we give an algebraic expression for vector fields that correspond to the ‘partial’ differentials defined in (2.5). Thus, under the assumption that (2.11) be invertible over some open set $\mathscr{U} \subset S^k X$, and denoting, loosely, by $\partial/\partial u_1, \ldots, \partial/\partial u_k$ the coordinate vector field for projected coordinates $\boldsymbol{C}^g \to \boldsymbol{C}^k,\ (u_1, \ldots, u_g) \mapsto (u_1, \ldots, u_k)$,

$$\begin{pmatrix} \partial_{u_1} \\ \partial_{u_2} \\ \vdots \\ \partial_{u_k} \end{pmatrix} = r \begin{pmatrix} 1 & \phi_1(P_1) & \phi_1(P_2) & \cdots & \phi_1(P_k) \\ 1 & \phi_2(P_1) & \phi_2(P_2) & \cdots & \phi_2(P_k) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_{k-1}(P_1) & \phi_{k-1}(P_2) & \cdots & \phi_{k-1}(P_k) \end{pmatrix}^{-1} \begin{pmatrix} y_1^{r-1}\partial_{x_1} \\ y_2^{r-1}\partial_{x_2} \\ \vdots \\ y_k^{r-1}\partial_{x_k} \end{pmatrix}.$$

Applying Proposition 2.2 to this linear equation, we obtain:

PROPOSITION 2.11.

$$\sum_{i=1}^{k} \epsilon_i \frac{\partial}{\partial u_i} = \frac{r}{\psi_k(P_1, P_2, \ldots, P_k)} \begin{vmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \cdots & \phi_{k-1}(P_1) & y_1^{r-1}\partial_{x_1} \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \cdots & \phi_{k-1}(P_2) & y_2^{r-1}\partial_{x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \phi_1(P_k) & \phi_2(P_k) & \cdots & \phi_{k-1}(P_k) & y_k^{r-1}\partial_{x_k} \\ \epsilon_1 & \epsilon_2 & \epsilon_3 & \cdots & \epsilon_k & 0 \end{vmatrix},$$

where $(\epsilon_1, \ldots, \epsilon_k)$ is any k-tuple of numbers. By choosing $\epsilon_i = \delta_{ij}$, we get $\partial/\partial u_j$,

$$\frac{\partial}{\partial u_j} = \sum_{i=1}^{k} \frac{r\psi_{k-1}^{(\check{j})}(P_1, P_2, \dots, P_k)}{\psi_k(P_1, P_2, \dots, P_k)} y_i^{r-1} \frac{\partial}{\partial x_i}.$$

3. Differential forms.

We give an algebraic representation of a differential form, which up to a tensor of holomorphic one-forms equals the fundamental normalized differential of the second kind in [**F1**, Corollary 2.6]. We follow techniques of [**EEL**], [**BLE2**], which can be viewed as a natural generalization of methods in [**B1**, Chapter VII].

Definition 3.1. A two-form $\Omega(P_1, P_2)$ on $X \times X$ is called a *fundamental differential of the second kind* if it is symmetric,

$$\Omega(P_1, P_2) = \Omega(P_2, P_1), \tag{3.1}$$

it has its only pole (of second order) along the diagonal of $X \times X$, and in the vicinity of each point (P_1, P_2) is expanded in power series as

$$\Omega(P_1, P_2) = \left(\frac{1}{(t_{P_1} - t'_{P_2})^2} + d_{\geq}(1)\right) dt_{P_1} \otimes dt_{P_2} \quad (\text{as } P_1 \to P_2), \tag{3.2}$$

where t_P is a local coordinate at the point $P \in X$.

The w-degree (and the λ-degree) of R can be extended to $R \otimes R$, $\text{w-deg}_{R\otimes R} : R \otimes R \to \boldsymbol{Z}$, $\text{w-deg}_{R\otimes R}(f \otimes g) = \text{w-deg} f + \text{w-deg} g$ (similarly for $\lambda\text{-deg}_{R_\lambda \otimes R_\lambda}$). We refer to the first and the second components of $R \otimes R$ by R_1 and R_2 and sometimes write explicitly, e.g., w-deg_{R_1} for the w-degree with respect to R_1.

Let $\Sigma\big(P_1, P_2\big)$ be the following form,

$$\Sigma\big(P_1, P_2\big) := \frac{\sum_{k=1}^{r} y_1^{r-k} y_2^{k-1}}{(x_1 - x_2) r y_1^{r-1}} dx_1. \tag{3.3}$$

Proposition 3.2. *There exist differentials* $\nu^{\mathrm{II}}{}_j = \nu^{\mathrm{II}}{}_j(x, y)$ $(j = 1, 2, \dots, g)$ *of the second kind such that they have their only pole at* ∞ *and satisfy the relation,*

$$\begin{aligned} &d_{P_2}\Sigma(P_1, P_2) - d_{P_1}\Sigma(P_2, P_1) \\ &\quad = \sum_{i=1}^{g} \big(\nu^{\mathrm{I}}{}_i(P_2) \otimes \nu^{\mathrm{II}}{}_i(P_1) - \nu^{\mathrm{I}}{}_i(P_1) \otimes \nu^{\mathrm{II}}{}_i(P_2)\big), \end{aligned} \tag{3.4}$$

where

$$d_Q\Sigma(P,Q) := dx_P \otimes dx_Q \frac{\partial}{\partial x_Q} \frac{\sum_{k=1}^{r} y_P^{r-k} y_Q^{k-1}}{(x_P - x_Q) r y_P^{r-1}}. \tag{3.5}$$

The set of differentials $\{\nu^{\mathrm{II}}{}_1,\ \nu^{\mathrm{II}}{}_2,\ \ldots,\ \nu^{\mathrm{II}}{}_g\}$ *is determined modulo the* $\boldsymbol{C}$*-linear span* $\langle \nu^{\mathrm{I}}{}_j \rangle_{j=1,\ldots,g}$.

Proof. We let

$$\nu^{\mathrm{II}}{}_j(x,y) = \frac{h_j(x,y)}{r y^{r-1}} dx, \quad j = 1, 2, \ldots, g. \tag{3.6}$$

Since the requirement on the poles implies that $h_j(x,y)$ belongs to R, we have to solve for

$$h_a(x,y) = \sum_{i=0}^{N_a} c_{a,i} \phi_{g+i}$$

modulo the span $\langle \phi_i \rangle_{i=0,\ldots,g-1}$, coming from $\langle \phi_i dx / r y^{r-1} \rangle_{i=0,\ldots,g-1} = H^1(X, \mathscr{O}_X)$.

In order to show that $\nu^{\mathrm{II}}{}_a$ has the above form, we shall prove that the left-hand side of (3.4) does not have singularities except at ∞. Then, we will calculate the w-degree of $\nu^{\mathrm{II}}{}_a$ using the fact that the left-hand side of (3.4) has certain homogeneity properties.

Since at a branch point $(b_i, 0)$ satisfying $f(b_i) = 0$, $(i = 1, \ldots, s)$, a local parameter t is defined by $t^r = x - b_i$ and $dx/(r y^{r-1})$ behaves like $dt(1 + d_{\geq}(t))$, there is no singularity. For the $P_1 = P_2$ case, again there is no singularity as is easily seen from (3.3) and (3.4), taking derivatives. We also give an algebraic proof for later reference. After multiplying the left-hand side in (3.4) by $r^2 y_1^{r-1} y_2^{r-1} (x_1 - x_2)^2 / dx_1 \otimes dx_2$, we set:

$$\begin{aligned}
B_1(P_1,P_2) :=& \Bigg(r y_1^{r-1} y_2^{r-1} + r \sum_{k=2}^{r} y_1^{r-k} y_2^{k-2} f(x_2) \\
&\qquad + (x_1 - x_2) \sum_{k=2}^{r} y_1^{r-k} (k-1) y_2^{k-2} f'(x_2) \Bigg), \\
B(P_1,P_2) :=& B_1(P_1,P_2) - B_1(P_2,P_1) \\
=& \sum_{k=2}^{r} y_1^{r-k} y_2^{k-2} \big(r(f(x_2) - f(x_1)) \\
&\qquad + (x_1 - x_2)((k-1) f'(x_2) + (r-k+1) f'(x_1)) \big).
\end{aligned} \tag{3.7}$$

We have to show that $B(P_1, P_2)$ belongs to the ideal $((x_1 - x_2)^2)$ in $R \otimes R$. Since it is obvious that $(f(x_2) - f(x_1))/(x_1 - x_2)$ belongs to $R \otimes R$, the problem is to prove that

$$\frac{B(P_1, P_2)}{(x_1 - x_2)} = \sum_{k=2}^{r} y_1^{r-k} y_2^{k-2} \Bigg(r \frac{(f(x_2) - f(x_1))}{x_1 - x_2} \\ + ((k-1)f'(x_2) + (r-k+1)f'(x_1)) \Bigg) \tag{3.8}$$

belongs to $(x_1 - x_2)R \otimes R$. We have to show that

$$A_f(x_1, x_2) := \left(r \frac{(f(x_2) - f(x_1))}{x_1 - x_2} + ((k-1)f'(x_2) + (r-k+1)f'(x_1)) \right) \in R.$$

is divisible by $(x_1 - x_2)$. We expand in ϵ or $A_f(x_1, x_1 + \epsilon)$ is equal to

$$(r-k+1)f'(x_1) + r\frac{-df(x_1)}{dx_1} - r\frac{d^2 f(x_1)}{dx_1^2}\epsilon + (k-1)\left(f'(x_1) + \frac{d^2 f(x_1)}{dx_1^2}\epsilon \right) + d_>(\epsilon)$$
$$= (k-1-r)\frac{d^2 f(x_1)}{dx_1^2}\epsilon + d_>(\epsilon).$$

This means that $A_f(x_1, x_2)$ is divisible by $(x_1 - x_2)$ and we are done.

We consider the behavior around ∞ to show that ${\nu^{\mathrm{II}}}_i$ is of the second kind. From the definition, it is obvious that each term in $B(P_1, P_2)$ is homogeneous with respect to the λ-degree of $R_\lambda \otimes R_\lambda$,

$$\lambda\text{-}\deg_{R_\lambda \otimes R_\lambda} B(P_1, P_2) = 2s(r-1).$$

We also show that for the w-degree,

$$\text{w-}\deg_{R \otimes R} B(P_1, P_2) = 2s(r-1). \tag{3.9}$$

Notice that the w-degree and λ-degree of the terms which contain no λ coincide. When $f(x) = x^s + \lambda x^{s-1}$,

$$A_{x^s+\lambda x^{s-1}}(x_1,x_2)$$
$$= \Bigg(-r\bigg(\sum_{i=0}^{s-1} x_1^i x_2^{s-1-i} + \lambda \sum_{i=0}^{s-2} x_1^i x_2^{s-i-2}\bigg) + (k-1)\big(s x_2^{s-1} + \lambda(s-1)x_2^{s-2}\big)$$
$$+ (r-k+1)\big(s x_1^{s-1} + \lambda(s-1)x_1^{s-2}\big)\Bigg). \tag{3.10}$$

The terms containing $x_1^i y_1^j$ and $x_2^k y_2^\ell$ in $\sum y_1^{r-k} y_2^{k-2} A_{x^s+\lambda x^{s-1}}(x_1,x_2)/(x_1-x_2)$ take care of every $\phi_{i'}(P_1)$ and $\phi_{j'}(P_2)$ coming from numerators of ν^{I}'s and ν^{II}'s in (3.4). Every possible combination of them has w-degrees and w-$\deg_{R\otimes R}$ coming from terms of degree smaller than $2sr-2r-2s = 4g-2$ because of (2.2) and w-$\deg(y^{k-2}x^\ell) = r\ell + s(k-2)$ for $2 \le k \le r$ and $0 \le \ell \le s-2$.

For a general $f(x)$, $\{y^{k-2}x^\ell \mid 2 \le k \le r, 0 \le \ell \le s-2\}$ includes every element $\phi_n(P)$ for $0 \le n \le g-1$ and the $\phi_n(P)$'s for $n \in \{g \le n \le 3g-2\}$. Thus $B(P_1,P_2)/(x_1-x_2)^2$ contains one $\phi_n(P_1)$ for $(0 \le n \le g-1)$, and so satisfies (3.9). We return to the terms whose w-degree is smaller than $2g-1 = rs-r-s$ with respect to R_1 (resp. R_2) and assign them to the numerator of ${\nu^{\mathrm{I}}}_i(P_1)$ (resp. ${\nu^{\mathrm{I}}}_i(P_2)$), for which we are solving (3.4). Noting that by definition $B(P_1,P_2)$ is antisymmetric, we decompose $B(P_1,P_2)$ as

$$B(P_1,P_2) = (x_1-x_2)^2 \sum_{i=1}^{g} \big(\phi_{i-1}(P_2)h_i(P_1) - \phi_{i-1}(P_1)h_i(P_2)\big).$$

Due to homogeneity in λ-$\deg_{R_\lambda\otimes R_\lambda}$, $h_i(P_2)$ has λ-degree

$$\lambda\text{-}\deg_{R_\lambda} h_i(P) = 2(sr-s-r) - N(i-1).$$

But the expression (3.10) (see also the proof in Lemma 3.4) shows that w-$\deg_R h_i(P) = \lambda$-$\deg_{R_\lambda} h_i(P)$. Since w-$\deg_R dx/ry^{r-1} = r+s-sr+1$, by Proposition 2.1 $h_i(P)dx/ry^{r-1}$ has w-degree $sr-s-r+1-N(i-1) > 1$ for $1 \le i \le g$ and thus it is a differential of the second kind. □

Corollary 3.3.

(1) *The one-form*

$$\Pi_{P_1}^{P_2}(P) := \Sigma(P,P_1)dx - \Sigma(P,P_2)dx$$

is a differential of the third kind, whose only (first-order) poles are $P = P_1$

and $P = P_2$, and residues $+1$ and -1 respectively.

(2) *The fundamental differential of the second kind $\Omega(P_1, P_2)$ is given by*

$$\begin{aligned}\Omega(P_1, P_2) &= d_{P_2}\Sigma(P_1, P_2) + \sum_{i=1}^{g} {\nu^{\mathrm{I}}}_i(P_1) \otimes {\nu^{\mathrm{II}}}_i(P_2) \\ &= \frac{F(P_1, P_2)dx_1 \otimes dx_2}{(x_1 - x_2)^2 r^2 y_1^{r-1} y_2^{r-1}},\end{aligned} \tag{3.11}$$

where F is an element of $R \otimes R$.

Proof. (1) is obvious from the definition. In (2) it is clear that the left-hand side of (3.11) is symmetric and $F(P_1, P_2)$ belongs to $R \otimes R$. The proof of the previous proposition shows,

$$\text{w-deg}_{R\otimes R}\frac{F(P_1, P_2)}{(x_1 - x_2)^2} = 2rs - 2r - 2s.$$

However the terms in $d\Sigma/dx_2$ whose w-degrees (w-deg$_{R_a}$ $(a = 1, 2)$) are greater than $rs - s - r - 1$ are canceled by the additional terms (see Lemma 3.4) and we have

$$\text{w-deg}_{R_1}\frac{F(P_1, P_2)}{(x_1 - x_2)^2} = rs - s - r - 1.$$

Since w-deg$_R dx/ry^{r-1} = s - sr + r + 1$, $\Omega(P_1, P_2)$ has no singularity at infinity. It is clear that it satisfies the other defining properties of the fundamental differential of the second kind. □

Lemma 3.4. *We have*

$$\lim_{P_1\to\infty}\frac{F(P_1, P_2)}{\phi_{g-1}(P_1)(x_1 - x_2)^2} = \phi_g(P_2). \tag{3.12}$$

Proof. From Proposition 2.1 (2) and (3),

$$\begin{aligned}\text{w-deg}_R\phi_{g-1}(P) &= 2g - 2 = rs - r - s - 1,\\ \text{w-deg}_R\phi_g(P) &= 2g = rs - r - s + 1.\end{aligned}$$

We also note that λ-deg$_{R_\lambda\otimes R_\lambda} F(P_1, P_2)/(x_1 - x_2)^2$ is equal to w-deg$_{R\otimes R}F(P_1,$

$P_2)/(x_1-x_2)^2$, which is the same as w-deg$_{R\otimes R}(\phi_{g-1}(P_2)\phi_g(P_1)) = 2sr-2r-2s$. We are only concerned with monomials whose w-degree agrees with their λ-degree in the left-hand side of (3.12). Because of uniqueness, we find that $\phi_{g-1}(P)\phi_g(P) = x^{s-2}y^{r-2}$, and $\phi_{g-1}(P_1)\phi_g(P_2)$ is identified with $x_1^{i_x}x_2^{s-i_x-2}y_1^{i_y}y_2^{r-i_y-2}$ for suitable i_x and i_y coming from the leading term in $A_f(x_1,x_2)$, expanded as in (3.10), as seen by the definition of F in (3.11). We check that the coefficient of this monomial is 1 in (3.8). Here we note that i_x and i_y satisfy the relation,

$$si_y+ri_x = sr-s-r-1, \quad \text{i.e.,} \quad s(r-i_y-2)+r(s-i_x-2) = sr-s-r+1. \tag{3.13}$$

Using the identity,

$$\sum_{i=0}^{m} a_i x_1^i x_2^{n-i} = (x_1-x_2)\sum_{i=0}^{m-1}\Big(\sum_{j=0}^{i} a_j\Big)x_1^i x_2^{n-i-1} + \Big(\sum_{j=0}^{m} a_j\Big)x_2^m,$$

we calculate the term in $B(P_1,P_2)/(x_1-x_2)^2$ of (3.8) which corresponds to a given monomial. (*N.B.* In Proposition 3.2, we proved that $B(P_1,P_2)/(x_1-x_2)^2$ belongs to $R\otimes R$ by using an ϵ expansion. This identity provides another proof, which more precisely determines $h_i(x,y)$.) For a certain k and i in the sum, we find a monomial $(y_1^{r-k}y_2^{k-2}x_1^i x_2^{s-i-2})$ in $B(P_1,P_2)/(x_1-x_2)^2$ of (3.8), which has the properties,

$$\begin{aligned}
\text{w-deg}_{R_1}\big(y_1^{r-k}y_2^{k-2}x_1^i x_2^{s-i-2}\big) &= s(r-k)+ri,\\
\text{w-deg}_{R_2}\big(y_1^{r-k}y_2^{k-2}x_1^i x_2^{s-i-2}\big) &= s(k-2)+r(s-i-2)
\end{aligned}$$

and the coefficient $(k-1)s-(i+1)r$. By using (3.13), i.e., $k=r-i_y$ and $i=i_x$, we see that the coefficient is 1. It is not difficult to prove that the other terms vanish in the limit by watching their w-deg$_{R_1}$. □

Hereafter we expand $h_i(P) = ry^{r-1}\nu^{\mathrm{II}}{}_i(P)/dx$, $1\le i\le g$, in the monomial basis of R, to avoid carrying holomorphic one-forms.

For later convenience we introduce the notation:

$$\begin{aligned}
\Omega_{Q_1,Q_2}^{P_1,P_2} &:= \int_{P_2}^{P_1}\int_{Q_2}^{Q_1}\Omega(P,Q)\\
&= \int_{P_2}^{P_1}(\Sigma(P,Q_1)-\Sigma(P,Q_2)) + \sum_{i=1}^{g}\int_{P_2}^{P_1}\nu^{\mathrm{I}}{}_i(P)\int_{Q_2}^{Q_1}\nu^{\mathrm{II}}{}_i(P).
\end{aligned} \tag{3.14}$$

4. The sigma function.

As customary, we choose a basis α_i, β_j ($1 \leqq i, j \leqq g$) of $H_1(X, \boldsymbol{Z})$ such that their intersection numbers are $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$ and $\alpha_i \cdot \beta_j = \delta_{ij}$, and we denote the period matrices by

$$\begin{aligned}
[\omega' \ \omega''] &= \frac{1}{2}\left[\int_{\alpha_i} {\nu^{\mathrm{I}}}_j \quad \int_{\beta_i} {\nu^{\mathrm{I}}}_j\right]_{i,j=1,2,\dots,g}, \\
[\eta' \ \eta''] &= \frac{1}{2}\left[\int_{\alpha_i} {\nu^{\mathrm{II}}}_j \quad \int_{\beta_i} {\nu^{\mathrm{II}}}_j\right]_{i,j=1,2,\dots,g}.
\end{aligned} \tag{4.1}$$

The following Lemma corresponds to Corollary 2.6 (ii) in [**F1**].

Lemma 4.1.

$$\Omega_{Q_1,Q_2}^{P_1,P_2} = \int_{P_2}^{P_1} \tau_{Q_1,Q_2} + \sum_{i,j=1}^{g} \gamma_{ij} \int_{P_2}^{P_1} {\nu^{\mathrm{I}}}_i \int_{Q_2}^{Q_1} {\nu^{\mathrm{I}}}_j,$$

where τ has residues $+1, -1$ at Q_1, Q_2, is regular everywhere else, and is normalized i.e., $\int_{\alpha_i} \tau_{P,Q} = 0$ and $\gamma = \omega'^{-1}\eta'$.

Proof. This is obtained by choosing an appropriate path Γ from Q_1 to Q_2, homotopic to α_k, and taking the differential of Ω. □

The following Proposition provides a symplectic structure in the Jacobian, known as *generalized Legendre relation* [**B1**], [**BLE1**].

Proposition 4.2. *The matrix,*

$$M := \begin{bmatrix} 2\omega' & 2\omega'' \\ 2\eta' & 2\eta'' \end{bmatrix}, \tag{4.2}$$

satisfies

$$M \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} {}^t M = 2\pi\sqrt{-1} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}. \tag{4.3}$$

Proof. By comparing Lemma 4.1 and (3.11) in Corollary 3.3, we choose appropriately $(2g)^2$ paths and take the integrals along the paths. □

By the Riemann relations [**F1**], it is known that $\mathrm{Im}\,({\omega'}^{-1}\omega'')$ is positive definite. Referring to Theorem 1.1 in [**F1**], let

$$\delta := \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} \in \left(\frac{1}{2}\boldsymbol{Z}\right)^{2g} \tag{4.4}$$

be the theta characteristic which gives the Riemann constant with respect to the base point ∞ and the period matrix $[2\omega'\ 2\omega'']$.

We define an entire function of (a column-vector) $u = {}^t(u_1, u_2, \ldots, u_g) \in \boldsymbol{C}^g$,

$$\begin{aligned}\sigma(u) &= \sigma(u; M) = \sigma(u_1, u_2, \ldots, u_g; M)\\ &= c\exp\left(-\frac{1}{2}\,{}^t u\eta'{\omega'}^{-1}u\right)\vartheta[\delta]\left(\frac{1}{2}{\omega'}^{-1}u;\ {\omega'}^{-1}\omega''\right)\\ &= c\exp\left(-\frac{1}{2}\,{}^t u\eta'{\omega'}^{-1}\,u\right)\\ &\quad\times\sum_{n\in\boldsymbol{Z}^g}\exp\left[\pi\sqrt{-1}\left\{{}^t(n+\delta'){\omega'}^{-1}\omega''(n+\delta') + {}^t(n+\delta')({\omega'}^{-1}u+\delta'')\right\}\right],\end{aligned} \tag{4.5}$$

where c is a constant. In this article, the constant c is not important because we deal only with ratios of σ functions, so we say nothing more about it.

For a given $u \in \boldsymbol{C}^g$, we introduce u' and u'' in $\boldsymbol{R}^g$ so that

$$u = 2\omega' u' + 2\omega'' u''.$$

PROPOSITION 4.3. *For u, $v \in \boldsymbol{C}^g$, and ℓ $(= 2\omega'\ell' + 2\omega''\ell'') \in \Lambda$, we define*

$$\begin{aligned}L(u, v) &:= 2\ {}^t u(\eta' v' + \eta'' v''),\\ \chi(\ell) &:= \exp\left[\pi\sqrt{-1}\left(2({}^t\ell'\delta'' - {}^t\ell''\delta') + {}^t\ell'\ell''\right)\right]\ (\in \{1, -1\}).\end{aligned}$$

The following holds

$$\sigma(u+\ell) = \sigma(u)\exp\left(L\left(u + \frac{1}{2}\ell, \ell\right)\right)\chi(\ell). \tag{4.6}$$

PROOF. The proof is standard; note that σ is essentially the same as the normalized theta function in Chapter VI of [**L**]. □

The vanishing locus of σ is:

$$\Theta^{g-1} = \left(\mathscr{W}^{g-1} \cup [-1]\mathscr{W}^{g-1}\right) = \mathscr{W}^{g-1}. \tag{4.7}$$

The last equality is due to Proposition (4.3), which shows that σ is an even or odd function under the action of $[-1]$; the reason for introducing $\mathscr{W}^{g-1} \cup [-1]\mathscr{W}^{g-1}$ is that the analogous loci when $g-1$ is replaced by k play an important role and $\mathscr{W}^k$ is not $[-1]$-invariant in general.

We review a relation which we call the Riemann fundamental relation [**R**], [**B1**, Section 195]:

Proposition 4.4. *For* $(P, Q, P_i, P'_i) \in X^2 \times (S^g(X) \setminus S^g_1(X)) \times (S^g(X) \setminus S^g_1(X))$,

$$u := \sum_{i=1}^{g} w(P_i), \quad v := \sum_{i=1}^{g} w(P'_i),$$

$$\begin{aligned}\exp\left(\sum_{i,j=1}^{g} \Omega^{P,Q}_{P_i,P'_j}\right) &= \frac{\sigma(w(P)-u)\sigma(w(Q)-v)}{\sigma(w(Q)-u)\sigma(w(P)-v)} \\ &= \frac{\sigma(w(P)-w(P_1,\dots,P_g))\sigma(w(Q)-w(P'_1,\dots,P'_g))}{\sigma((w(Q)-w(P_1,\dots,P_g))\sigma(w(P)-w(P'_1,\dots,P'_g))}.\end{aligned}$$

Proof. The right-hand side can be expressed as

$$\exp(\text{bilinear term in } w\text{'s})\frac{\theta(\omega'^{-1}(w(P)-u)+\xi)\theta(\omega'^{-1}(w(Q)-v)+\xi)}{\theta(\omega'^{-1}(w(Q)-u)+\xi)\theta(\omega'^{-1}(w(P)-v))+\xi)},$$

where ξ is the Riemann vector. By Riemann's theorem for theta functions [**F1**, p. 23], the above becomes

$$\exp(\text{bilinear term in } w\text{'s}) \exp\left(\sum_{j=1}^{g}\int_Q^P \tau_{P_j,P'_j}\right).$$

The exponential part of the bilinear term turns out to be

$$(u-v)^t\gamma(w(P)-w(Q)) = \sum_{i,j,k}\gamma_{ij}\int_{P'_k}^{P_k}\nu^{\mathrm{I}}{}_i\int_Q^P \nu^{\mathrm{I}}{}_j,$$

which equals (Lemma (4.1))

$$\sum_{i=1}^{g} \Omega_{P_i,P_i'}^{P,Q} - \sum_{i=1}^{g} \int_Q^P \tau_{P_i,P_i'}.$$

The above integrals depend upon the paths we choose, but (4.6) shows that such dependence cancels. Thus the right-hand side coincides with the left-hand side. □

PROPOSITION 4.5. *For $(P, P_1, \dots, P_g) \in X \times S^g(X) \setminus S_1^g(X)$ and $u := w(P_1, \dots, P_g)$, the equality*

$$\sum_{i,j=1}^{g} \wp_{i,j}\Big(w(P) - u\Big)\phi_{i-1}(P)\phi_{j-1}(P_a) = \frac{F(P, P_a)}{(x - x_a)^2},$$

holds for every $a = 1, 2, \dots, g$, where we set

$$\wp_{ij}(u) := -\frac{\sigma_i(u)\sigma_j(u) - \sigma(u)\sigma_{ij}(u)}{\sigma(u)^2} \equiv -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u).$$

We remark that the first equation in this proposition,

$$-\sum_{i,j=1}^{g} \frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(w(P) - u) \nu^{\mathrm{I}}{}_i(P) \nu^{\mathrm{I}}{}_j(P_a) = \Omega(P, P_a), \tag{4.8}$$

corresponds to (29) of Corollary 2.6 in [**F1**]. In our case, $\Omega(P, P_a)$ has an explicit expression in terms of the affine coordinates on the curve.

PROOF. Using the relation

$$\sum_{i,j=1}^{g} \phi_{i-1}(P_1)\phi_{j-1}(P_2) \frac{\partial^2}{\partial w_i(P_1) \partial w_j(P_2)} = r^2 y_1^{r-1} y_2^{r-1} \frac{\partial^2}{\partial x_1 \partial x_2},$$

taking logarithm of both sides and differentiating along $P_1 = P$ and $P_2 = P_a$, we obtain the claim. □

PROPOSITION 4.6. *For $(P, P_1, \dots, P_g) \in X \times S^g(X) \setminus S_1^g(X)$,*

(1) $\mu_g(P; P_1, \dots, P_g) = \phi_g(P) - \sum_{j=1}^{g} \wp_{gj}(w(P_1, \dots, P_g))\phi_{j-1}(P)$.
(2) $\wp_{g,k+1}(w(P_1, \dots, P_g)) = (-1)^{g-k-1} \mu_{g,k}(P_1, \dots, P_g)$, $\quad (k = 0, \dots, g-1)$.

Proof. In Proposition 4.5, we let $P \to \infty$ after dividing both sides by $\phi_{g-1}(P)$ and we obtain

$$\sum_{j=1}^{g} \wp_{g,j}(u)\phi_{j-1}(P_i) = \phi_g(P_i).$$

for every i. Let Q_i be such that $u = \sum_{i=1}^{g} w(P_i) = -\sum_{i=1}^{g} w(Q_i)$. Note that $\wp_{ij}(u)$ is an even function. Thus both $\mu_g(P; P_1, \ldots, P_g)$ and

$$\phi_g(P) - \sum_{j=1}^{g} \wp_{g,j}(u)\phi_{j-1}(P), \tag{4.9}$$

are elements of R vanishing only at P_i and Q_i, viewed as functions of P. As in Lemma 2.6 and Proposition 2.7, we have the identity due to the uniqueness of $\mu_g(P; P_1, \ldots, P_g)$. $\square$

5. Jacobi inversion formulae over Θ^k.

We introduce

$$\Theta^k := \mathscr{W}^k \cup [-1]\mathscr{W}^k. \tag{5.1}$$

For $(r = 2, s = 2g + 1)$ (hyperelliptic) curves and ∞ a branch point, Θ^k equals $\mathscr{W}^k$ for every positive integer k but in general it does not.

Theorem 5.1. *The following relations hold*

(1) Θ^g *case: for* $(P_1, \ldots, P_g) \in S^g(X) \setminus S^g_1(X)$ *and* $u = \pm w(P_1, \ldots, P_g) \in \kappa^{-1}(\Theta^g)$,

$$\frac{\sigma_i(u)\sigma_g(u) - \sigma_{gi}(u)\sigma(u)}{\sigma^2(u)} = (-1)^{g-i+1}\mu_{g,i-1}(P_1, \ldots, P_g), \quad \textit{for } 0 < i \le g.$$

(2) Θ^{g-1} *case: for* $(P_1, \ldots, P_{g-1}) \in S^{g-1}(X) \setminus S^{g-1}_1(X)$ *and* $u = \pm w (P_1, \ldots, P_{g-1}) \in \kappa^{-1}(\Theta^{g-1})$,

$$\frac{\sigma_i(u)}{\sigma_g(u)} = \begin{cases} (-1)^{g-i}\mu_{g-1,i-1}(P_1, \ldots, P_{g-1}) & \textit{for } 0 < i \le g, \\ 1 & \textit{for } i = g. \end{cases}$$

(3) Θ^k *case*: *for* $(P_1,\dots,P_k) \in S^k(X) \setminus S^k_1(X)$ *and* $u = \pm w(P_1,\dots,P_k) \in \kappa^{-1}(\Theta^k)$,

$$\frac{\sigma_i(u)}{\sigma_{k+1}(u)} = \begin{cases} (-1)^{k-i+1}\mu_{k,i-1}(P_1,\dots,P_k) & \textit{for } 0 < i \le k, \\ 1 & \textit{for } i = k+1, \\ 0 & \textit{for } k+1 < i \le g. \end{cases}$$

We provide some context before proving the theorem. The theorem is a natural generalization of a formula of Grant [**Gr**] and Jorgenson [**Jo**]; the first generalizations of it, for C_{rs} curves of a particular type or genus, were given in [**E**], [**EG**], [**BG1**], [**BG2**].

Remark 5.2. Jacobi [**Ja**] investigated the Jacobian of a curve of genus two using ${\nu^{\mathrm{I}}}_1$ and ${\nu^{\mathrm{I}}}_2$ and Abel's theorem, to generalize the addition formulae on an elliptic curve. He showed that the addition structure is given by solving a degree-2 polynomial $A + Bt + Ct^2 = 0$ to obtain (x, y) coordinates of the sum, a divisor of degree two. Noting $\phi_i(P) = x^i$ $(i < g + 1)$ for the hyperelliptic case, (4.9) is exactly the statement of Jacobi. He also posed the 'Jacobi inversion problem' of expressing A/C, B/C in terms of $w(P)+w(Q)$. Theorem 5.1 is a natural extension of Jacobi's formulae (and Fay's as show in Remark 5.10 below).

Remark 5.3. If we retain only the leading term of each element of the FS-matrix in terms of a local parameter at ∞, which is what is left if the curve is $y^r = x^s$, $\mu_{i,k}$ is a ratio of Schur polynomials. On the other hand, the leading term of the σ-function is explicitly given by a Schur polynomial as shown in [**BLE1**], [**BLE2**].

Remark 5.4.

(1) The case $k = 1$ of (3) means that for $(x, y) = P \in X$ and $u := w(P) \in \kappa^{-1}\Theta^1$, we have

$$-\frac{\sigma_1(u)}{\sigma_2(u)} = x. \tag{5.2}$$

For a genus-two curve this is Grant's [**Gr**] and Jorgenson's [**Jo**] formula. For every (r, s)-curve and a positive integer i_y such that $\phi_{i_y}(P) = y^{r-2}$ and $i_y < g$, then ${\nu^{\mathrm{I}}}_{i_r}(P) = dx/ry$. By differentiating along the curve, we obtain

$$\frac{1}{r}\frac{d}{du_{i_y}}\frac{\sigma_1(u)}{\sigma_2(u)} = y.$$

Thus both are the simplest Jacobi inversion formulae, and these functions generate R. For example, to express the FS-matrix in terms of σ's over Θ^1's, for $P_1, P_2 \in X$, e.g.,

$$\frac{1}{\sigma_2(w(P_1))\sigma_2(w(P_2))}\begin{vmatrix} \sigma_1(w(P_2)) & \sigma_2(w(P_2)) \\ \sigma_1(w(P_1)) & \sigma_2(w(P_1)) \end{vmatrix} = -\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}.$$

(2) As in [**BG1**], [**BG2**], our results might have applications to the solution of the Benney equation, which requires Jacobi inversion. To mention another application to integrable PDEs, for $u := w((x, y))$, since $du_1 = dx/ry^{r-1}$, $du_{i_n} = x^n dx/ry^{r-1}$ for a suitable i_n,

$$\frac{d}{du_1}\frac{\sigma_1(u)}{\sigma_2(u)} = \left(-\frac{\sigma_1(u)}{\sigma_2(u)}\right)^n \frac{d}{du_{i_n}}\frac{\sigma_1(u)}{\sigma_2(u)}.$$

This is an equation of the Burgers-Hopf hierarchy [**KK**]: a solution is thus associated to any (r, s) curve.

(3) Theorem 5.1 (2) can be rewritten:

$$\sigma_i(u) = (-1)^{g-i}\mu_{g-1,i-1}(P_1, \dots, P_{g-1})\sigma_g(u),$$

or

$$\psi_{g-1}^{(i\check{-}1)}(P_1, \dots, P_{g-1})\frac{\partial}{\partial u_i}\sigma(u) = (-1)^{g-i}\psi_{g-1}^{(g\check{-}1)}(P_1, \dots, P_{g-1})\frac{\partial}{\partial u_g}\sigma(u),$$

which should be compared with Proposition 2.11 for $k = g$.

(4) We should note that Grant's formula (5.2) for genus two also appeared in [**GT**, (7.1)] where it was used to give explicit Hamiltonians for the Hitchin system [**vGP**].

For the proof of Theorem 5.1, let a decomposition of $u \in \kappa^{-1}\mathscr{W}^k$, whose preimage is given by $(P_1, \dots, P_{k-1}, P_k) \in S^k(X)$, be denoted by $u = u^{[k-1]} + v$, where $u^{[k-1]} := w(P_1, \dots, P_{k-1})$ and $v := w(P_k)$.

PROOF OF THEOREM 5.1 (1), (2). Since $\mu_{k,\ell}$ is invariant under the action of $[-1]$ (cf. Proposition 2.7), we can replace Θ^k by $\mathscr{W}^k$. Proposition 4.6 gives (1). To obtain (2) we will use Lemma 5.5 below and 2.10. Let $u := w(P_1, \dots, P_g)$ for $(P_1, \dots, P_g) \in S^g(X)$. By multiplying both sides of (1) by $\sigma(u)^2/\sigma_g(u)^2$ and taking the limit $P_g \to \infty$, we have

$$\lim_{P_g\to\infty}\left(\frac{\sigma(u)}{\sigma_g(u)}\right)^2\left(\frac{\sigma_g(u)\sigma_i(u)-\sigma(u)\sigma_{gi}(u)}{\sigma(u)^2}\right)=\lim_{P_g\to\infty}\left(\frac{\sigma_i(u)}{\sigma_g(u)}-\frac{\sigma(u)\sigma_{gi}(u)}{\sigma_g(u)^2}\right)$$
$$=\lim_{P_g\to\infty}\left(\frac{\sigma_i(u)}{\sigma_g(u)}\right). \tag{5.3}$$

The right-hand side is a function on $\mathscr{W}^{g-1}$ as shown in Lemma 5.5 (1). Lemma 5.5 (2) allows us to write the left-hand side:

$$(-1)^{g-i}\lim_{P_g\to\infty}\left(\frac{\phi_{g-1}(P_g)}{\phi_g(P_g)}\right)\mu_{g,i-1}(P_1,\dots,P_g)$$

and Proposition 2.1 (2) and (3) allow us to see that this limit gives (2), using Proposition 2.10. □

Lemma 5.5. *Let $u^{[g-1]}\in\kappa^{-1}(\mathscr{W}^{g-1})$ be a non-singular point.*

(1)
$$\sigma_i(u^{[g-1]}+\ell)=\sigma_i(u^{[g-1]})\exp\left(L\left(u^{[g-1]}+\frac{1}{2}\ell,\ell\right)\right)\chi(\ell). \tag{5.4}$$

(2) *For $v:=w(P)$, $u:=u^{[g-1]}+v\in\kappa^{-1}(\mathscr{W}^g)$, $\sigma(u)^2/\sigma_g(u)^2$ is expanded around $v=0$ by*

$$\frac{\sigma(u)^2}{\sigma_g(u)^2}=(v_g)^2+d_{\geq}((v_g)^3).$$

Proof.

(1): After differentiating both sides of the relation (4.6) in Proposition 4.3 with respect to u_i, we restrict the domain to $\kappa^{-1}\mathscr{W}^{g-1}$ noting that σ vanishes simply on $\kappa^{-1}\mathscr{W}^{g-1}$. The equation follows.

(2): The quasi-periodic properties (1) show that the quotient in (2) is well-defined over $\kappa^{-1}(\mathscr{W}^g)$. We know that σ vanishes on $\kappa^{-1}\mathscr{W}^{g-1}$; $\sigma(u)$ is expanded as

$$\sigma(u)=\sigma(u^{[g-1]})+\frac{\partial\sigma(u^{[g-1]})}{\partial u_1}v_1+\frac{\partial\sigma(u^{[g-1]})}{\partial u_2}v_2+\cdots+\frac{\partial\sigma(u^{[g-1]})}{\partial u_g}v_g+\cdots,$$

$$\frac{\partial}{\partial u_g}\sigma(u)=\frac{\partial}{\partial u_g}\sigma(u^{[g-1]})+\frac{\partial^2\sigma(u^{[g-1]})}{\partial u_1\partial u_g}v_1+\frac{\partial^2\sigma(u^{[g-1]})}{\partial u_2\partial u_g}v_2+\cdots$$
$$+\frac{\partial^2\sigma(u^{[g-1]})}{\partial u_g\partial u_g}v_g+\cdots.$$

By using a local parameter t_∞ on the curve around ∞, v_i can be expanded as function of v_g, $v_i = -(-v_g)^{2g-N(i-1)-1}/(2g-N(i-1)-1)+d_>((v_g)^{2g-N(i-1)-1})$; $v_g = -t_\infty + d_{>0}(t_\infty)$ up to a constant factor. Hence (2) holds:

$$\frac{\sigma(u)}{\sigma_g(u)} = v_g + d_\geq((v_g)^2). \qquad \square$$

The following Lemma will provide an induction step to prove the theorem.

Lemma 5.6. *For $k < g-1$, a non-singular point $u^{[k-1]} \in \kappa^{-1}(\mathscr{W}^{k-1})$, and $u := u^{[k-1]} + w(P_k)$ whose preimage is given by $(P_1, \dots, P_k) \in S^{k-1}(X) \times X$, if the relation,*

$$\frac{\sigma_i(u)}{\sigma_{k+1}(u)} = \begin{cases} (-1)^{k-i+1}\mu_{k,i-1} & \text{for } 0 < i \leq k, \\ 1 & \text{for } i = k+1, \\ 0 & \text{for } k+1 < i \leq g, \end{cases}$$

holds, we have the relations,

$$\frac{\sigma_{k+1}(u)}{\sigma_k(u)} = (w(P_k)_g)^{N(k)-N(k-1)} + d_\geq((w(P_k)_g)^{N(k)-N(k-1)+1})$$

and

$$\frac{\sigma_{k+1}(u)}{\sigma_k(u)} = \frac{\phi_{k-1}(P_k)}{\phi_k(P_k)} + d_\geq((w(P_k)_g)^{N(k)-N(k-1)+1}).$$

Proof. From the assumptions, we have

$$\frac{\sigma_{k+1}(u)}{\sigma_k(u)} = -\frac{1}{\mu_{k,k-1}} = -\frac{\psi_k^{(\check{k})}(P_1, \dots, P_k)}{\psi_k^{(\check{k-1})}(P_1, \dots, P_k)}.$$

Let us consider the behavior around $P_k = \infty$. By a similar argument to the one used in Proposition 2.10, the right-hand side behaves like

$$\frac{1}{\mu_{k,k-1}} = \frac{-\phi_{k-1}(P_k)\psi_{k-1}^{(\check{k-1})}(1, \dots, P_{k-1}) + d_\geq((w(P_k)_g)^{-N(k)+1})}{\phi_k(P_k)\psi_{k-1}^{(\check{k-1})}(1, \dots, P_{k-1}) + d_\geq((w(P_k)_g)^{-N(k)+1})}$$
$$+ d_\geq\big((w(P_k)_g)^{-N(k)+N(k-1)-1}\big)$$

$$= -\frac{\phi_{k-1}(P_k)}{\phi_k(P_k)} + d_{\geq}\big((w(P_k)_g)^{N(k)-N(k-1)+1}\big). \qquad \square$$

From the assumptions of the Lemma, we therefore also derive:

$$\frac{\sigma_i(u)}{\sigma_j(u)} = (-1)^{i-j}\frac{\mu_{k,i-1}}{\mu_{k,j-1}},$$

and may take limits of both sides.

Proof of Theorem 5.1 (3). We prove (3) using descending induction with respect to k; the base step is (2), which was proved above. Again, we work over $\mathscr{W}^k$ by an argument similar to the one used in the proof of (1), (2). Assume the relation over $\mathscr{W}^k$. Under the same assumption of Lemma 5.6, we multiply both sides of the relation by $\sigma_{k+1}(u)/\sigma_k(u)$ and obtain by Lemma 5.6:

$$\lim_{P_k\to\infty}\frac{\sigma_{k+1}(u)}{\sigma_k(u)}\frac{\sigma_i(u)}{\sigma_{k+1}(u)} = -\lim_{P_k\to\infty}\frac{\phi_{k-1}(P)}{\phi_k(P)}\mu_{k,i-1}.$$

Proposition 2.10 now yields (3). $\square$

From the algebraic expressions for quotients of σ's we deduce the following facts about the order of vanishing:

Corollary 5.7.

(1) *Let k be $1,\ldots,g-1$ and $\mathscr{W}_1^k = w(S_1^k(X))$. For $u \in \kappa^{-1}(\mathscr{W}^k \setminus \mathscr{W}_1^k \cup \iota(\mathscr{W}^k \setminus \mathscr{W}_1^k))$ and $i \leq k$,*

$$\mathrm{ord}_{\kappa^{-1}(\Theta^{k-1})}\sigma_{k+1}(u) = \mathrm{ord}_{\kappa^{-1}(\Theta^{k-1})}\sigma_i(u) + N(k) - N(k-1),$$

where $i = 1,\ldots,k$.

(2) *$\sigma_i(u)/\sigma_{k+1}(u)$ $(i = 1,\ldots,k)$ belongs to $H^0(\Theta^k, \mathscr{O}((N(k)-N(k-1))\Theta^{k-1}))$.*

Proof. Both parts follow immediately from the theorem. For example, when $k = g-1$ in (1), we use

$$\begin{aligned}&\sigma_i(w(P_1,\ldots,P_{g-1}) + w(P))\\ &\quad = (-1)^{g-i}\mu_{g-1,i}(P,P_1,\ldots,P_{g-1})\sigma_g(w(P_1,\ldots,P_{g-1}) + w(P))\end{aligned}$$

and take the limit as $P \to \infty$. $\square$

REMARK 5.8. We comment on the interpretation of Theorem 5.1 in the context of Riemann's singularity theorem [**ACGH**, Chapter VI Section 1] and more precise vanishing theorems given in [**F2**], revisited in the context of Sato's τ function on an infinite Grassmann manifold [**BV**], [**SW**]. Note however that the latter papers only deal with derivatives with respect to the "Sato coordinate" u_g (the derivative along the curve, embedded in its Jacobian, at the point ∞). These hold for any curve.

(1) Riemann' singularity theorem says the following: Let D belong to $S^k(X) \setminus S^k_1(X)$, $(k < g)$, $u := \int_{k\infty}^{D} \nu^{\mathrm{I}}$, and

$$n_k := h^0(D + (g - k + 1)\infty) = \#\{\ell \mid 0 \le \ell, N(\ell) < k\}.$$

a) For every multiple index $(\alpha_1, \dots, \alpha_m)$ with $\alpha_i \in \{1, \dots, g\}$ and $m < n_k$,

$$\frac{\partial^s}{\partial u_{\alpha_1} \dots \partial u_{\alpha_s}} \sigma(u) = 0.$$

b) There exists a multiple index $(\beta_1, \dots, \beta_{n_k})$ such that

$$\frac{\partial^{n_k}}{\partial u_{\beta_1} \dots \partial u_{\beta_{n_k}}} \sigma(u) \neq 0. \tag{5.5}$$

Fay's results [**F2**, Theorem 1.2] allow us to make the order of vanishing more precise; we adapt Fay's notation to the situation in which $(2g-2)\infty$ is a canonical divisor.

Let ν_i^+ $(0 \le \nu_1^+ < \nu_2^+ < \cdots < \nu_{n_k}^+ \le g - 1)$ such that

$$\begin{aligned} h^0(D + (g - k - \ell - 1)\infty) &= n_k - i + 1 \quad &&\text{for } \ell = \nu_i^+, \\ h^0(D + (g - k - \ell + 1)\infty) &\le n_k - i &&\text{for } \ell > \nu_i^+; \end{aligned}$$

let ν_i^- be defined the same way, but replacing D by $-D$, and

$$N_k := n_k + \sum_{i=1}^{n_k} (\nu_i^+ + \nu_i^-)$$

For $u := u^{[k]} + v \in \kappa^{-1}\mathscr{W}^{k+1}$ and $v = w(P)$ of $P \in X$,

$$\frac{\partial^{N_k-1}}{\partial {v_g}^{N_k-1}}\sigma(u)\Big|_{v=0}=0,\quad\text{and}\quad \frac{\partial^{N_k}}{\partial {v_g}^{N_k}}\sigma(u)\Big|_{v=0}\neq 0. \tag{5.6}$$

(2) For our special curves, if besides (5.5), we impose another condition on the multiple index $(\beta_1,\dots,\beta_{n_k})$:

$$\lim_{u\to\Theta^{k-1}}\frac{\partial^{n_k}}{\partial u_{\beta_1}\dots\partial u_{\beta_{n_k}}}\sigma(u)=0, \tag{5.7}$$

then Corollary 5.7 (1) gives $N_k=\sum_{i=1}^{n_k}\mathrm{ord}(u_{\beta_i})$, where $\mathrm{ord}(u_{\beta_i})$ signifies the degree of u_{β_i} in a local parameter at ∞, and $\{\beta_i\}$ in (5.5) contains $k+1$ with notation as in (1) for $D\in\Theta^k$. Note that on the points of $\mathscr{W}^k$ for which $n_k\neq 1$, the relations in Theorem 5.1 should be regarded as zero over zero and interpreted using L'Hospital's theorem, to be expressed algebraically as well. Enolskii ([**E**], and private communication to the first-named author) proposed for the first time an expression for the quotient $\sigma_{gg\dots gi}/\sigma_{gg\dots gk}$ when it is finite.

Also immediate from Theorem 5.1 are the following formulae (recall Proposition 4.6 for (1) and Proposition 2.2 and (2.10) for (3)):

Corollary 5.9.

(1) *For* $(P_1,P_2,\dots,P_g)\in S^g(X)\setminus S^g_1(X)$, $u:=w(P_1,P_2,\dots,P_g)$ *and* $(x,y)=P\in X$,

$$\mu_g(P;P_1,\dots,P_g)\frac{dx}{ry^{r-1}}=-\sum_{i=1}^{g}\wp_{ig}(u){\nu^{\mathrm{I}}}_i(P)+{\nu^{\mathrm{I}}}_{g+1}(P).$$

(2) *For* $(P_1,P_2,\dots,P_k)\in S^k(X)\setminus S^k_1(X)$ $(k<g)$, $u:=w(P_1,P_2,\dots,P_k)$ *and* $(x,y)=P\in X$,

$$\mu_k(P;P_1,\dots,P_k)\frac{dx}{ry^{r-1}}=\sum_{i=1}^{k+1}\frac{\sigma_i(u){\nu^{\mathrm{I}}}_i(P)}{\sigma_{k+1}(u)}.$$

(3) *For* $(P_1,P_2,\dots,P_k)\in S^k(X)\setminus S^k_1(X)$ $(k<g)$ *and* $u:=w(P_1,P_2,\dots,P_k)$,

$$\frac{\begin{vmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \dots & \phi_k(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \dots & \phi_k(P_2) \\ 1 & \phi_1(P_3) & \phi_2(P_3) & \dots & \phi_k(P_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_k) & \phi_2(P_k) & \dots & \phi_k(P_k) \\ a_1 & a_2 & a_3 & \dots & a_{k+1} \end{vmatrix}}{\begin{vmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \dots & \phi_k(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \dots & \phi_k(P_2) \\ 1 & \phi_1(P_3) & \phi_2(P_3) & \dots & \phi_k(P_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_k) & \phi_2(P_k) & \dots & \phi_k(P_k) \\ b_1 & b_2 & b_3 & \dots & b_{k+1} \end{vmatrix}} = \frac{\sum_{i=1}^{k+1} \sigma_i(u) a_i}{\sum_{i=1}^{k+1} \sigma_i(u) b_i}.$$

Remark 5.10. We connect Corollary 5.9 to results of Fay and Jorgenson. Part (1) appears as Corollary 2.12 of [**F1**] because (4.8) is the form in (28), (29) of [**F1**] and by using formulae (37), (37)′ in [**F1**]. The form $\sum_{i=1}^{g} \sigma_i(u) {\nu^{\mathrm{I}}}_i(P)$ is the H_f in Corollary 1.4 of [**F1**] and the X in [**B1**, Chapter XIV Section 273], used for the definition of the prime form. Thus (2) in the $k = g-1$ case is a factor in the right-hand side of the second equality proved in Corollary 2.17 of [**F1**], which provides an addition formula.

The case $k = g-1$ of (3) is the specific algebraic expression (for (r, s) curves) of the formula obtained by Jorgenson in Theorem 1 of [**Jo**].

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