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Ineffability of $\mathscr{P}_\kappa\lambda$ for λ with small cofinality

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Abstract. We study ineffability, the Shelah property, and indescribability of $\mathscr{P}_\kappa\lambda$ when $\mathrm{cf}(\lambda) < \kappa$. We prove that if λ is a strong limit cardinal with $\mathrm{cf}(\lambda) < \kappa$ then the ineffable ideal, the Shelah ideal, and the completely ineffable ideal over $\mathscr{P}_\kappa\lambda$ are the same, and that it can be precipitous. Furthermore we show that Π^1_1-indescribability of $\mathscr{P}_\kappa\lambda$ is much stronger than ineffability if $2^\lambda = \lambda^{<\kappa}$.

1. Introduction.

Combinatorial principles for a cardinal, ineffability, and weak compactness were studied thoroughly in Baumgartner [**4**]. First we review some definitions:

Definition 1.1. For a regular uncountable cardinal κ,

(1) κ is *weakly compact* if, for all $\langle a_\alpha : \alpha < \kappa\rangle$ with $a_\alpha \subseteq \alpha$, there exists $A \subseteq \kappa$ such that $\{\alpha < \kappa : A \cap \beta = a_\alpha \cap \beta\}$ is unbounded in κ for all $\beta < \kappa$,
(2) κ is *ineffable* (respectively *almost ineffable*) if, for all $\langle a_\alpha : \alpha < \kappa\rangle$ with $a_\alpha \subseteq \alpha$, there exists $A \subseteq \kappa$ such that $\{\alpha < \kappa : A \cap \alpha = a_\alpha\}$ is stationary in κ (respectively unbounded in κ).

The definition of ineffability and almost ineffability is due to Jensen and Kunen. Weak compactness originated from the study of compactness of infinitaly logic (see section 4 in Kanamori [**18**]). The above combinatorial definition (1) was found by Baumgartner [**4**]. Afterward ineffability was translated into $\mathscr{P}_\kappa\lambda$-structures by Jech [**13**], where κ is a regular uncountable cardinal, $\lambda \geq \kappa$ is a cardinal, and $\mathscr{P}_\kappa\lambda = \{x \subseteq \lambda : |x| < \kappa\}$. Carr [**8**] defined the Shelah property, mild ineffability, and indescribability of $\mathscr{P}_\kappa\lambda$ as a generalization of weak compactness of a cardinal. These properties of $\mathscr{P}_\kappa\lambda$ have been widely studied when $\mathrm{cf}(\lambda) \geq \kappa$, and it has been shown that ineffability, almost ineffability, and the Shelah property form a proper hierarchy. For instance, if κ is almost κ^+-ineffable then there are stationary many $\alpha < \kappa$ such that α is α^+-Shelah.

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On the other hand, Abe [**3**] showed that ineffability of $\mathscr{P}_\kappa\lambda$ coincides with almost ineffability if $2^\lambda = \lambda^{<\kappa}$. Hence the above mentioned hierarchy can be collapsed if $\mathrm{cf}(\lambda) < \kappa$. We will investigate ineffability, the Shelah property, and indescribability of $\mathscr{P}_\kappa\lambda$ when $\mathrm{cf}(\lambda) < \kappa$.

We know $\lambda^{<\kappa}$ is the size of $\mathscr{P}_\kappa\lambda$. We also try to decide the size of $\mathscr{P}_\kappa\lambda$ under weaker assumptions than before. Solovay [**20**] proved $\lambda^{<\kappa} = \lambda^+$ if κ is λ-(super)compact and $\mathrm{cf}(\lambda) < \kappa$, where λ^+ denotes the minimal cardinal greater than λ, and Johnson [**15**] showed that $\lambda^{<\kappa} = \lambda$ holds if κ is λ-Shelah and $\mathrm{cf}(\lambda) \geq \kappa$. We extend this to the following:

THEOREM 1.2.

(1) *If κ is mildly λ-ineffable and* $\mathrm{cf}(\lambda) \geq \kappa$*, then $\lambda^{<\kappa} = \lambda$, and*
(2) *if κ is λ-Shelah and* $\mathrm{cf}(\lambda) < \kappa$ *then $\lambda^{<\kappa} = \lambda^+$.*

The following theorem can be seen as an extension of a theorem of Abe in [**3**]. This shows that ineffability, the Shelah property, and complete ineffability of $\mathscr{P}_\kappa\lambda$ can be the same when $\mathrm{cf}(\lambda) < \kappa$, and the corresponding ideals can be precipitous. This contrasts with the fact that the completely ineffable ideal is not precipitous if $\mathrm{cf}(\lambda) \geq \kappa$.

THEOREM 1.3. *Assume λ is a strong limit cardinal with* $\mathrm{cf}(\lambda) < \kappa$*. Then*

(1) $\mathrm{NSh}_{\kappa\lambda} = \mathrm{NAIn}_{\kappa\lambda} = \mathrm{NIn}_{\kappa\lambda} = \mathrm{NCIn}_{\kappa\lambda}$*, and*
(2) *if κ is λ-ineffable and $\mu > \lambda$ is a Woodin cardinal, then, in $V^{\mathrm{Col}(\lambda^+,<\mu)}$, κ remains λ-ineffable and* $\mathrm{NSh}_{\kappa\lambda} = \mathrm{NAIn}_{\kappa\lambda} = \mathrm{NIn}_{\kappa\lambda} = \mathrm{NCIn}_{\kappa\lambda}$ *is precipitous.*

$\mathrm{NSh}_{\kappa\lambda}$, $\mathrm{NAIn}_{\kappa\lambda}$, $\mathrm{NIn}_{\kappa\lambda}$, and $\mathrm{NCIn}_{\kappa\lambda}$ are ideals corresponding to the Shelah property, almost ineffability, ineffability, and complete ineffability respectively. To prove Theorem 1.2, we give a simple characterization of $\mathrm{NIn}_{\kappa\lambda}$. Using this, we have the consistency of the statement that $\mathrm{cf}(\lambda) < \kappa$ and κ is completely λ-ineffable but not mildly $\lambda^{<\kappa}$-ineffable.

Baumgartner defined indescribability of $\mathscr{P}_\kappa\lambda$ and Carr [**8**] showed that Π^1_1-indescribability is equivalent to the Shelah property if $\mathrm{cf}(\lambda) \geq \kappa$. The next theorem shows that, if $\mathrm{cf}(\lambda) < \kappa$, this equivalence can be false. Moreover Π^1_1-indescribability can be much stronger than ineffability.

THEOREM 1.4. *Assume $2^\lambda = \lambda^{<\kappa}$. Then* $\mathrm{NIn}_{\kappa\lambda} \subseteq \Pi_{\kappa\lambda}$ *holds, and if κ is λ-ineffable then* $\mathrm{NIn}_{\kappa\lambda} \subsetneq \Pi_{\kappa\lambda}$.

$\Pi_{\kappa\lambda}$ is the ideal corresponding to Π^1_1-indescribability.

Part (2) of Theorem 1.2 and Theorem 1.4 are answers to questions of Abe in [**2**].

2. Preliminaries.

We refer the reader to Kanamori [**18**] for general background and basic notation. Throughout this paper, κ denotes an inaccessible cardinal and λ denotes a cardinal equal to or greater than κ. In fact, the properties mentioned in this paper imply the inaccessibility of κ.

Recall that $\mathscr{P}_\kappa\lambda = \{x \subseteq \lambda : |x| < \kappa\}$.

In this paper, an *ideal* (respectively a *filter*) over $\mathscr{P}_\kappa\lambda$ means a κ-complete fine ideal (respectively filter) over $\mathscr{P}_\kappa\lambda$. That is, $I \subseteq \mathscr{P}(\mathscr{P}_\kappa\lambda)$ is called an ideal over $\mathscr{P}_\kappa\lambda$ if the following hold:

(1) $\forall X \in I \forall Y \subseteq X\,(Y \in I)$,
(2) $\forall \gamma < \kappa \forall \{X_\xi : \xi < \gamma\} \subseteq I\ (\bigcup_{\xi<\gamma} X_\xi \in I)$,
(3) $\forall a \in \mathscr{P}_\kappa\lambda\,(\{x \in \mathscr{P}_\kappa\lambda : a \not\subseteq x\} \in I)$.

For an ideal I over $\mathscr{P}_\kappa\lambda$, I^* denotes the dual filter of I, and $I^+ = \mathscr{P}(\mathscr{P}_\kappa\lambda) \setminus I$. An element of I^+ is called an I-positive set. For $X \in I^+$, let $I|X = \{Y \subseteq \mathscr{P}_\kappa\lambda : Y \cap X \in I\}$. $I|X$ is the restriction of I to X.

An ideal I over $\mathscr{P}_\kappa\lambda$ is called *normal* if for every $X \in I^+$ and function $f : X \to \mathscr{P}_\kappa\lambda$ with $\forall x \in X\,(f(x) \in x)$, there exists $\alpha < \lambda$ such that $\{x \in X : f(x) = \alpha\} \in I^+$. In a trivial sense, the non-proper ideal is normal.

For a set $X \subseteq \mathscr{P}_\kappa\lambda$, X is *unbounded* if $\forall x \in \mathscr{P}_\kappa\lambda \exists y \in X\ (x \subseteq y)$. X is *closed* if for every $\gamma < \kappa$ and $\subseteq$-increasing sequence $\langle x_\xi : \xi < \gamma\rangle$ in X, $\bigcup_{\xi<\gamma} x_\xi \in X$. A closed and unbounded set is called *club*. A set $S \subseteq \mathscr{P}_\kappa\lambda$ is *stationary* if S intersects any club set.

The following fact is well-known:

Fact 2.1. For $X \subseteq \mathscr{P}_\kappa\lambda$, the following are equivalent:

(1) X is stationary in $\mathscr{P}_\kappa\lambda$,
(2) for every $f : \lambda \times \lambda \to \lambda$, there exists $x \in X$ such that $x \cap \kappa \in \kappa$ and $f``(x \times x) \subseteq x$, and
(3) for every $f : \lambda \times \lambda \to \mathscr{P}_\kappa\lambda$, there exists $x \in X$ such that $\bigcup f``(x \times x) \subseteq x$.

The non-stationary ideal over $\mathscr{P}_\kappa\lambda$, $\mathrm{NS}_{\kappa\lambda}$, is the set of all $X \subseteq \mathscr{P}_\kappa\lambda$ such that X is non-stationary in $\mathscr{P}_\kappa\lambda$.

Fact 2.2. $\mathrm{NS}_{\kappa\lambda}$ is the minimal normal ideal over $\mathscr{P}_\kappa\lambda$.

Definition 2.3. For $x, y \in \mathscr{P}_\kappa\lambda$, we define $x < y$ if $x \subseteq y$ and $|x| < |y \cap \kappa|$. For $X \subseteq \mathscr{P}_\kappa\lambda$, a function $f : X \to \mathscr{P}_\kappa\lambda$ is said to be $<$-*regressive* if $f(x) < x$ for every $x \in X$ with $x \cap \kappa \neq \emptyset$.

An ideal I over $\mathscr{P}_\kappa\lambda$ is *strongly normal* if the following condition is satisfied:

For every $X \in I^+$ and $<$-regressive function $f : X \to \mathscr{P}_\kappa\lambda$, there exists $y \in \mathscr{P}_\kappa\lambda$ such that $\{x \in X : f(x) = y\} \in I^+$.

The non-proper ideal is trivially strongly normal.

For $x \in \mathscr{P}_\kappa\lambda$, we denote the set $\{y \in \mathscr{P}_\kappa\lambda : y < x\}$ by $\mathscr{P}_{x\cap\kappa}x$. If $x \cap \kappa$ is a regular cardinal, then properties of $\mathscr{P}_\kappa\lambda$ correspond to the properties of $\mathscr{P}_{x\cap\kappa}x$. For example, $X \subseteq \mathscr{P}_{x\cap\kappa}x$ is stationary if for all $f : x \times x \to \mathscr{P}_{x\cap\kappa}x$ there exists $y \in X$ such that $\bigcup f``(y \times y) \subseteq y$.

For $f : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$, we let $C_f = \{x \in \mathscr{P}_\kappa\lambda : f``\mathscr{P}_{x\cap\kappa}x \subseteq \mathscr{P}_{x\cap\kappa}x\}$.

Definition 2.4. $\mathrm{WNS}_{\kappa\lambda} = \{X \subseteq \mathscr{P}_\kappa\lambda : \exists f : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda\,(C_f \cap X = \emptyset)\}$.

Fact 2.5 (Carr-Levinski-Pelletier [**10**]).

(1) $\mathrm{WNS}_{\kappa\lambda}$ is the minimal strongly normal ideal over $\mathscr{P}_\kappa\lambda$.
(2) $\mathrm{WNS}_{\kappa\lambda}$ is a proper ideal if and only if κ is Mahlo.
(3) $\{x \in \mathscr{P}_\kappa\lambda : x \cap \kappa$ is inaccessible and x is $< x \cap \kappa$-closed $\} \in \mathrm{WNS}^*_{\kappa\lambda}$.
(4) If $\pi : \mathscr{P}_\kappa\lambda \to \lambda$ is a bijection, then $\mathrm{WNS}_{\kappa\lambda} = \mathrm{NS}_{\kappa\lambda}|\{x \in \mathscr{P}_\kappa\lambda : \pi``\mathscr{P}_{x\cap\kappa}x = x\}$.

Fix a bijection $\pi : \mathscr{P}_\kappa\lambda \to \lambda^{<\kappa}$. We define $e : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda^{<\kappa}$ by $e(x) = \pi``\mathscr{P}_{x\cap\kappa}x$. We say that e is a *canonical map from $\mathscr{P}_\kappa\lambda$ to $\mathscr{P}_\kappa\lambda^{<\kappa}$*. Note that a canonical map does not depend on the choice of π in the following sense: Let π' be another bijection and e' a canonical map induced by π'. Then $\{x \in \mathscr{P}_\kappa\lambda : e(x) = e'(x)\} \in \mathrm{WNS}^*_{\kappa\lambda}$.

Fact 2.6 (Abe [**1**]).

(1) $\{x \in \mathscr{P}_\kappa\lambda : e(x) \cap \lambda = x\} \in \mathrm{WNS}^*_{\kappa\lambda}$.
(2) $\{x \in \mathscr{P}_\kappa\lambda^{<\kappa} : e(x \cap \lambda) = x\} \in \mathrm{WNS}^*_{\kappa\lambda^{<\kappa}}$.
(3) For $X \subseteq \mathscr{P}_\kappa\lambda$, $X \in \mathrm{WNS}_{\kappa\lambda}$ if and only if $e``X \in \mathrm{WNS}_{\kappa\lambda^{<\kappa}}$.

Ineffability and the Shelah property of $\mathscr{P}_\kappa\lambda$ are defined in the following.

Definition 2.7 (Carr [**8**], [**9**], Jech [**13**]). Let X be a subset of $\mathscr{P}_\kappa\lambda$.

(1) X is *ineffable* (respectively *almost ineffable*) if, for all $\langle a_x : x \in X\rangle$ with $a_x \subseteq x$, there exists $A \subseteq \lambda$ such that $\{x \in X : A \cap x = a_x\}$ is stationary (respectively unbounded).
(2) X has *the Shelah property*, or simply X is *Shelah* if, for all $\langle f_x : x \in X\rangle$ with $f_x : x \to x$, there exists $f : \lambda \to \lambda$ such that, for all $y \in \mathscr{P}_\kappa\lambda$, the set $\{x \in X : f|y = f_x|y\}$ is unbounded.
(3) X is *mildly ineffable* if, for all $\langle a_x : x \in X\rangle$ with $a_x \subseteq x$, there exists $A \subseteq \lambda$ such that, for all $y \in \mathscr{P}_\kappa\lambda$, the set $\{x \in X : A \cap y = a_x \cap y\}$ is unbounded.

We say that κ is *λ-ineffable* (*almost λ-ineffable, λ-Shelah, mildly λ-ineffable* respectively) if $\mathscr{P}_\kappa\lambda$ is ineffable (almost ineffable, Shelah, mildly ineffable respectively).

Notice that the Shelah property implies mildly ineffability,

$\mathrm{NIn}_{\kappa\lambda}$ (respectively $\mathrm{NAIn}_{\kappa\lambda}$, $\mathrm{NSh}_{\kappa\lambda}$) is the set of all $X \subseteq \mathscr{P}_\kappa\lambda$ such that X is not ineffable (respectively almost ineffable, Shelah).

FACT 2.8 (Carr [**8**], [**9**]).

(1) κ is weakly compact $\iff$ κ is κ-Shelah $\iff$ κ is mildly κ-ineffable.
(2) κ is ineffable (respectively almost ineffable) $\iff$ κ is κ-ineffable (respectively almost κ-ineffable).
(3) $\mathrm{NSh}_{\kappa\lambda}$, $\mathrm{NAIn}_{\kappa\lambda}$, and $\mathrm{NIn}_{\kappa\lambda}$ are normal ideals over $\mathscr{P}_\kappa\lambda$. Moreover these are strongly normal if $\mathrm{cf}(\lambda) \geq \kappa$.
(4) If κ is mildly λ-ineffable, then, for $X \subseteq \mathscr{P}_\kappa\lambda$, X is mildly ineffable if and only if X is unbounded.

FACT 2.9 (Carr [**9**]). For $X \subseteq \mathscr{P}_\kappa\lambda$, X is ineffable (almost ineffable) if and only if, for all $\langle f_x : x \in X\rangle$ with $f_x : x \to x$, there exists $f : \lambda \to \lambda$ such that $\{x \in X : f|x = f_x\}$ is stationary (unbounded). Hence $\mathrm{NSh}_{\kappa\lambda} \subseteq \mathrm{NAIn}_{\kappa\lambda} \subseteq \mathrm{NIn}_{\kappa\lambda}$ holds.

The next fact follows from the normality of $\mathrm{NSh}_{\kappa\lambda}$ and a standard coding argument.

FACT 2.10. For $X \subseteq \mathscr{P}_\kappa\lambda$, X is Shelah if and only if, for any $\langle f_x : x \in X\rangle$ with $f_x : x \to x$ and $\langle g_x : x \in X\rangle$ with $g_x : x \to x$, there exists $f : \lambda \to \lambda$ and $g : \lambda \to \lambda$ such that $\{x \in X : f|y = f_x|y,\ g|y = g_x|y\}$ is unbounded for all $y \in \mathscr{P}_\kappa\lambda$.

For an infinite set X, let $[X]^\omega$ be the set of all $x \subseteq X$ such that $|x| = \omega$. $F : [X]^\omega \to X$ is called an *ω-Jonsson function for X* if the following holds: There is no $Y \subsetneq X$ such that $F``[Y]^\omega \subseteq Y$ and $|Y| = |X|$. It is well-known that every infinite set X has an ω-Jonsson function for X (see Erdös-Hajnal [**11**]).

FACT 2.11 (Abe [**2**], Johnson [**16**]). Let μ be a cardinal with $\mu \leq \lambda$.

(1) If $F : [\mu]^\omega \to \mu$ is an ω-Jonsson function for μ, then $\{x \in \mathscr{P}_\kappa\lambda : F``[x \cap \mu]^\omega \subseteq x \cap \mu$ and $F|[x \cap \mu]^\omega$ is ω-Jonsson for $x \cap \mu\} \in \mathrm{NSh}^*_{\kappa\lambda}$.
(2) If μ is regular, then $\{x \in \mathscr{P}_\kappa\lambda : \mathrm{ot}(x \cap \mu)$ is regular$\} \in \mathrm{NSh}^*_{\kappa\lambda}$, where $\mathrm{ot}(x)$ denotes the order type of x.

3. Basic properties of ineffabilities.

In this section, we will show some basic properties of ineffabilities of $\mathscr{P}_\kappa\lambda$.

First we prove the strong normality of $\mathrm{NSh}_{\kappa\lambda}$, $\mathrm{NAIn}_{\kappa\lambda}$, and $\mathrm{NIn}_{\kappa\lambda}$ without the condition that $\mathrm{cf}(\lambda) \geq \kappa$.

Proposition 3.1. *$\mathrm{NSh}_{\kappa\lambda}$, $\mathrm{NAIn}_{\kappa\lambda}$, and $\mathrm{NIn}_{\kappa\lambda}$ are strongly normal ideals.*

Proof. We will only show the strong normality of $\mathrm{NSh}_{\kappa\lambda}$. The others can be verified by a similar argument. Let $X \in \mathrm{NSh}^+_{\kappa\lambda}$ and let $g : X \to \mathscr{P}_\kappa\lambda$ be a $<$-regressive function. By the normality of $\mathrm{NSh}_{\kappa\lambda}$, we may assume that there exists $\mu < \kappa$ such that $\mathrm{ot}(g(x)) = \mu$ for all $x \in X$. Furthermore we may assume $\mu \subseteq x$ for all $x \in X$. For each $x \in X$, let $h_x : \mu \to x$ be an increasing enumerating map of $g(x)$.

Let $X_a = \{x \in X : g(x) = a\}$. Suppose $X_a \in \mathrm{NSh}_{\kappa\lambda}$ for all $a \in \mathscr{P}_\kappa\lambda$. For each $a \in \mathscr{P}_\kappa\lambda$, let $\langle f^a_x : x \in X_a\rangle$ be a counterexample to the Shelah property of X_a. Consider the sequences $\langle f^{g(x)}_x : x \in X\rangle$ and $\langle h_x : x \in X\rangle$. By the Shelah property of X, there exist $f : \lambda \to \lambda$ and $h : \mu \to \lambda$ such that $\{x \in X : f|y = f^{g(x)}_x|y,\ h|y = h_x|y\}$ is unbounded for all $y \in \mathscr{P}_\kappa\lambda$. Let $b = h``\mu \in \mathscr{P}_\kappa\lambda$. We will prove that $\{x \in X_b : f|y = f^b_x|y\}$ is unbounded for all $y \in \mathscr{P}_\kappa\lambda$, which is a contradiction. Let $y \in \mathscr{P}_\kappa\lambda$. We may assume that $\mu \subseteq y$. Then $\{x \in X : f|y = f^{g(x)}_x|y,\ h|y = h_x|y\}$ is unbounded. Let $x \in X$ be such that $y \subseteq x$, $h|y = h_x|y$, and $f|y = f^{g(x)}_x|y$. Since $\mu \subseteq y$, we have $h = h|y = h_x|y = h_x$, and this means that $g(x) = b$. Therefore $f|y = f^{g(x)}_x|y = f^b_x|y$ holds. □

Next we show a variation of $(\mathrm{UP})_{\kappa\lambda X}$ in Carr [**8**] from mild ineffability. We will use this in the next section.

Recall that a filter over $\mathscr{P}_\kappa\lambda$ means a κ-complete fine filter.

For a regular uncountable cardinal θ, H_θ denotes the set of all x such that $|TC(x)| < \theta$ where $TC(x)$ is the minimal transitive set containing x. It is known that H_θ is a model of ZFC$-$Power Set Axiom.

Proposition 3.2. *Let θ be a sufficiently large regular cardinal, and let N be any expansion of $\langle H_\theta, \in, \kappa, \lambda\rangle$. Let $X \subseteq \mathscr{P}_\kappa\lambda$ and $M \prec N$ be such that $X \in M$ and $|M| = \lambda \subseteq M$. Then X is mildly ineffable if and only if there exists a proper filter F over $\mathscr{P}_\kappa\lambda$ such that $X \in F$ and F is an M-ultrafilter. Here "F is an M-ultrafilter" means that, for all $X \in M \cap \mathscr{P}(\mathscr{P}_\kappa\lambda)$, either $X \in F$ or $\mathscr{P}_\kappa\lambda \setminus X \in F$.*

Proof. Assume X is mildly ineffable. We will construct an M-ultrafilter. Let $\langle X_\alpha : \alpha < \lambda\rangle$ be an enumeration of $\mathscr{P}(\mathscr{P}_\kappa\lambda) \cap M$. For each $x \in X$, let $a_x = \{\alpha \in x : x \in X_\alpha\}$. Then, by the mild ineffability of X, there exists $A \subseteq \lambda$ such that $\{x \in X : a_x \cap y = A \cap y\}$ is unbounded for all $y \in \mathscr{P}_\kappa\lambda$. Let F be the filter over $\mathscr{P}_\kappa\lambda$ generated by $\{X \cap \bigcap_{\alpha \in y} X_\alpha : y \in \mathscr{P}_\kappa A\}$, that is $Y \in F$ if and only if $X \cap \bigcap_{\alpha\in y} X_\alpha \subseteq Y$ for some $y \in \mathscr{P}_\kappa A$. It is clear that F is a κ-complete filter over $\mathscr{P}_\kappa\lambda$ and $X \in F$. Notice that $X_\alpha \in F$ for all $\alpha \in A$. We check that F is a proper fine filter and an M-ultrafilter.

Fineness. Let $\alpha < \lambda$. Since $\alpha \in \lambda \subseteq M$, there exists $\beta < \lambda$ such that $X_\beta = \{x \in \mathscr{P}_\kappa\lambda : \alpha \in x\}$. Take $x \in X$ such that $\alpha, \beta \in x$ and $A \cap \{\beta\} = a_x \cap \{\beta\}$. Since $\alpha \in x$, we have $x \in X_\beta$, so $\beta \in a_x$ and $\beta \in A$.

Properness. It is enough to show that $X \cap \bigcap_{\alpha \in y} X_\alpha \neq \emptyset$ for all $y \in \mathscr{P}_\kappa A$. For $y \in \mathscr{P}_\kappa A$, we can pick $x \in X$ such that $y \subseteq x$ and $a_x \cap y = A \cap y = y$. Then $x \in \bigcap_{\alpha \in a_x} X_\alpha \subseteq \bigcap_{\alpha \in y} X_\alpha$, thus $X \cap \bigcap_{\alpha \in y} X_\alpha \neq \emptyset$.

Now we check that F is an M-ultrafilter. Let $Y \in \mathscr{P}(\mathscr{P}_\kappa\lambda) \cap M$. Then there are $\alpha, \beta < \lambda$ such that $X_\alpha = Y$ and $X_\beta = \mathscr{P}_\kappa\lambda \setminus Y$. Take $x \in \mathscr{P}_\kappa\lambda$ such that $\alpha, \beta \in x$ and $A \cap \{\alpha, \beta\} = a_x \cap \{\alpha, \beta\}$. Then either $x \in X_\alpha$ or $x \in X_\beta$ hold, hence we have $\alpha \in a_x$ or $\beta \in a_x$. Thus $\alpha \in A$ or $\beta \in A$.

To show the converse, assume that there exists a proper M-ultrafilter F. By the elementarity of M, it is enough to show that, for all $\langle a_x : x \in \mathscr{P}_\kappa\lambda \rangle \in M$ with $a_x \subseteq x$, there exists $A \subseteq \lambda$ such that $\{x \in X : a_x \cap y = A \cap y\}$ is unbounded for all $y \in \mathscr{P}_\kappa\lambda$. Fix $\langle a_x : x \in \mathscr{P}_\kappa\lambda \rangle \in M$. Since $\lambda \subseteq M$ and F is an M-ultrafilter with $X \in F$, for each $\alpha < \lambda$, either $\{x \in X : \alpha \in a_x\} \in F$ or $\{x \in \mathscr{P}_\kappa\lambda : \alpha \notin a_x\} \in F$. Let $A = \{\alpha < \lambda : \{x \in X : \alpha \in a_x\} \in F\}$. Then it is not hard to see that $\{x \in X : a_x \cap y = A \cap y\} \in F$, so the set is unbounded for all $y \in \mathscr{P}_\kappa\lambda$. □

4. The Shelah property, mild ineffability, and the size of $\mathscr{P}_\kappa\lambda$.

Johnson [**16**] showed that $\lambda^{<\kappa} = \lambda$ holds if κ is λ-Shelah and $\mathrm{cf}(\lambda) \geq \kappa$. We see that the same result holds for mild ineffability, and moreover $\lambda^{<\kappa} = \lambda^+$ holds if κ is λ-Shelah and $\mathrm{cf}(\lambda) < \kappa$.

Proposition 4.1. *Assume κ is mildly λ-ineffable and* $\mathrm{cf}(\lambda) \geq \kappa$. *Then* $\lambda^{<\kappa} = \lambda$.

Proof. Mild ineffability is downward closed, that is, if $\mathscr{P}_\kappa\lambda$ is mildly ineffable and $\kappa \leq \lambda' < \lambda$ then $\mathscr{P}_\kappa\lambda'$ is mildly ineffable. Thus it is enough to prove the case when λ is regular. We will show that there exists an unbounded subset X of $\mathscr{P}_\kappa\lambda$ such that $|X| = \lambda$. If this can be shown, then $\mathscr{P}_\kappa\lambda = \bigcup\{\mathscr{P}(x) : x \in X\}$, which proves $\lambda^{<\kappa} \leq \lambda \cdot \kappa^{<\kappa} = \lambda$.

Let θ be a sufficiently large regular cardinal. Let $M \prec \langle H_\theta, \in, \kappa, \lambda \rangle$ be such that $\lambda \subseteq M$ and $|M| = \lambda$. Then, by Proposition 3.2, we can find a proper κ-complete fine M-ultrafilter F over $\mathscr{P}_\kappa\lambda$. M is not transitive, but we can take an ultrapower M by F in the usual way. Moreover it is not hard to see that Łoś's theorem holds between M and $\mathrm{Ult}(M, F)$: For any formula φ and $f_1, \dots f_n \in M \cap {}^{\mathscr{P}_\kappa\lambda}M$, $\{x \in \mathscr{P}_\kappa\lambda : M \models \varphi(f_1(x), \dots, f_n(x))\} \in F$ if and only if $\mathrm{Ult}(M, F)$

$\models \varphi([f_1], \dots, [f_n])$, where $[f]$ is an equivalence class of f. Since F is κ-complete in V, $\mathrm{Ult}(M, F)$ is well-founded. Let N be the transitive collapse of $\mathrm{Ult}(M, F)$. Now we identify N with $\mathrm{Ult}(M, F)$. Let $j : M \to N$ be the corresponding elementary embedding. Since F is fine, we have that $j``\lambda \subseteq [f_{\mathrm{id}}]$, where $f_{\mathrm{id}}$ is the identity map on $\mathscr{P}_\kappa\lambda$. Furthermore $F$ is $\kappa$-complete and $|[f_{\mathrm{id}}]|^N < j(\kappa)$, hence the critical point of $j$ is $\kappa$. Since $\sup(j``\lambda) \leq \sup([f_{\mathrm{id}}])$ and $\{x \in \mathscr{P}_\kappa\lambda : \sup(x) < \lambda\} \in F$, we have $\sup(j``\lambda) < j(\lambda)$. Notice that we do not require that $j``\lambda \in N$, but we have $j``x \in N$ for all $x \in \mathscr{P}_\kappa\lambda \cap M$.

We check that $j``\lambda$ is $<\kappa$-closed, that is, for all $c \subseteq j``\lambda$, $\sup(c) \in j``\lambda$ if $\mathrm{ot}(c) < \kappa$. Let $\alpha < \lambda$ be the minimal ordinal such that $\sup(c) \leq j(\alpha)$. Then $\sup(c) = \sup(j``\alpha)$. Hence $\mathrm{cf}(\alpha) < \kappa$. Take $d \in M$ such that $\mathrm{ot}(d) = \mathrm{cf}(\alpha)$ and d is unbounded in α. Then $j(\alpha) = \sup(j(d)) = \sup(j``d) = \sup(j``\alpha) = \sup(c)$. Therefore we have $\sup(c) \in j``\lambda$.

Now take an arbitrary stationary subset S of $\{\alpha < \lambda : \mathrm{cf}(\alpha) < \kappa\}$ with $S \in M$.

CLAIM 4.2. *$j(S) \cap \sup(j``\lambda)$ is stationary in $\sup(j``\lambda)$ in V.*

PROOF OF THE CLAIM 4.2. Let C be a $<\kappa$-club subset of $\sup(j``\lambda)$. Since $j``\lambda$ is also $<\kappa$-closed, we may assume that $C \subseteq j``\lambda$. Let $D = j^{-1}``C$. Then D is unbounded in λ. Thus there exists $\alpha \in S$ such that $D \cap \alpha$ is unbounded in α. Since $\alpha \in M$, we can take an unbounded subset b of α such that $b \in M$ and $\mathrm{ot}(b) = \mathrm{cf}(\alpha)$. Then $j(\alpha) = \sup j(b) = \sup(j``b) = \sup(j``\alpha)$. $D \cap \alpha$ is unbounded in α, hence $j``(D \cap \alpha) = j``D \cap j(\alpha)$ is unbounded in $j(\alpha)$. Since $j``D \subseteq C$, we have $j(\alpha) \in C$. Hence we have $j(\alpha) \in j(S) \cap C$. □

Now fix pairwise disjoint stationary subsets $\langle S_\alpha : \alpha < \lambda\rangle$ of $\{\beta < \lambda : \mathrm{cf}(\beta) < \kappa\}$ with $\langle S_\alpha : \alpha < \lambda\rangle \in M$. For $\beta < \lambda$ with $\omega < \mathrm{cf}(\beta) < \kappa$, let $c_\beta = \{\alpha < \beta : S_\alpha \cap \beta$ is stationary in $\beta\}$. Since the S_α's are pairwise disjoint, we have $|c_\beta| \leq \mathrm{cf}(\beta) < \kappa$. Now let $X = \{c_\beta : \beta < \lambda, \omega < \mathrm{cf}(\beta) < \kappa\}$. Then X is a subset of $\mathscr{P}_\kappa\lambda$ with $|X| = \lambda$. Finally we show that X is unbounded to complete the proof.

Let f be a function on $\mathscr{P}_\kappa\lambda$ such that $f \in M$ and $[f] = \sup(j``\lambda)$. Since $j``\lambda \subseteq [f_{\mathrm{id}}]$, $[f_{\mathrm{id}}] \cap [f]$ is unbounded in $[f]$. Because $|[f_{\mathrm{id}}]|^N < j(\kappa)$, $\mathrm{cf}^N([f]) < j(\kappa)$ and so $\{x \in \mathscr{P}_\kappa\lambda : \mathrm{cf}(f(x)) < \kappa\} \in F$. Take an arbitrary $y \in \mathscr{P}_\kappa\lambda$. Let $\alpha \in y$. By Claim 4.2, $j(S_\alpha) \cap \sup(j``\lambda)$ is stationary. Hence $\{x \in \mathscr{P}_\kappa\lambda : S_\alpha \cap f(x)$ is stationary in $f(x)\} \in F$. By the κ-completeness of F, we have $\{x \in \mathscr{P}_\kappa\lambda : \forall\alpha \in y\,(S_\alpha \cap f(x)$ is stationary in $f(x))$, $\mathrm{cf}(f(x)) < \kappa\} \in F$. Therefore we can take $x \in \mathscr{P}_\kappa\lambda$ such that $\omega < \mathrm{cf}(f(x)) < \kappa$ and $y \subseteq c_{f(x)} \in X$. This shows X is unbounded. □

The proof of the above proposition shows that a simultaneous stationary reflection principle of $\{\alpha < \lambda : \mathrm{cf}(\alpha) < \kappa\}$ follows from mild λ-ineffability. The following is an extension of Johnson's result [**15**]:

PROPOSITION 4.3. *Assume λ is regular and κ is mildly λ-ineffable. Let $\delta < \kappa$ and $\langle S_\alpha : \alpha < \delta\rangle$ be stationary subsets of $\{\beta < \lambda : \mathrm{cf}(\beta) < \kappa\}$. Then, for every $\gamma < \kappa$, there exists $\beta < \lambda$ such that $\gamma < \mathrm{cf}(\beta)$ and $S_\alpha \cap \beta$ is stationary in β for all $\alpha < \delta$.*

Now we prove that the Shelah property of $\mathscr{P}_\kappa\lambda$ with $\mathrm{cf}(\lambda) < \kappa$ implies that $\lambda^{<\kappa} = \lambda^+$.

PROPOSITION 4.4. *Assume κ is λ-Shelah and $\mathrm{cf}(\lambda) < \kappa$. Then $\lambda^{<\kappa} = \lambda^+$.*

PROOF. This proof is based on an argument of Tryba [**21**]. First we introduce a notion of *scale*. Fix an increasing sequence of regular cardinals $\langle\lambda_i : i < \mathrm{cf}(\lambda)\rangle$ which converges to λ. We denote $\Pi_{i<\mathrm{cf}(\lambda)}\lambda_i$ by $\Pi\lambda_i$. For $f, g \in \Pi\lambda_i$, let $f <^* g$ if and only if $\{i < \mathrm{cf}(\lambda) : f(i) \geq g(i)\}$ is bounded in $\mathrm{cf}(\lambda)$. We say that $\langle f_\xi : \xi < \lambda^+\rangle$ is a *scale* for $\Pi\lambda_i$ if the following hold:

(1) $f_\xi \in \Pi\lambda_i$ for all $\xi < \lambda^+$,
(2) for $\xi < \eta < \lambda^+$, $f_\xi <^* f_\eta$, and
(3) for all $f \in \Pi\lambda_i$, there exists $\xi < \lambda^+$ such that $f <^* f_\xi$.

It is a basic fact of Shelah's PCF-theory that there exists a sequence of regular cardinals $\langle\lambda_i : i < \mathrm{cf}(\lambda)\rangle$ and a scale $\langle f_\xi : \xi < \lambda^+\rangle$ for $\Pi\lambda_i$ (see Burke-Magidor [**6**] or Shelah [**19**]).

Now fix an increasing sequence of regular cardinals $\langle\lambda_i : i < \mathrm{cf}(\lambda)\rangle$ which converges to λ and a scale $\langle f_\alpha : \alpha < \lambda^+\rangle$ for $\Pi\lambda_i$. For each λ_i, fix an ω-Jonsson function $h_i : [\lambda_i]^\omega \to \lambda_i$. Let $e : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda^{<\kappa}$ be a canonical map. Let $X \subseteq \mathscr{P}_\kappa\lambda$ be the set of all $x \in \mathscr{P}_\kappa\lambda$ such that:

- $x \cap \kappa$ is an inaccessible $> \mathrm{cf}(\lambda)$,
- $\mathrm{ot}(x \cap \lambda_i)$ is regular for all $i < \mathrm{cf}(\lambda)$,
- $h_i|[x \cap \lambda_i]^\omega$ is ω-Jonsson for $x \cap \lambda_i$, and
- $e(x) \cap \lambda = x$.

By Fact 2.6, 2.11, and Proposition 3.1, we have $X \in \mathrm{NSh}^*_{\kappa\lambda}$. We consider the set $e``X = \{e(x) : x \in X\}$. Note that this set is a $\mathrm{WNS}_{\kappa\lambda^{<\kappa}}$-positive set, so it is stationary in $\mathscr{P}_\kappa\lambda^{<\kappa}$. Fix a sufficiently large regular cardinal θ and let $C = \{M \cap \lambda^{<\kappa} : M \prec \langle H_\theta, \in\rangle,\ |M| < \kappa,\ M \cap \lambda^{<\kappa} \in e``X,\ \{\{\lambda_i : i < \mathrm{cf}(\lambda)\}, \langle f_\alpha : \alpha < \lambda^+\rangle, \pi, e\} \subseteq M$ and $M \cap \lambda^{<\kappa}$ is σ-closed $\}$. Then C is stationary in $\mathscr{P}_\kappa\lambda^{<\kappa}$. Note that if $M \cap \lambda^{<\kappa} \in C$ then $M \cap \lambda \in X$. Moreover by the definition of e, we have that $[M \cap \lambda]^{<M\cap\kappa} \subseteq M$.

The following claim assures that $\{x \cap \lambda^+ : x \in C\}$ is an unbounded subset of $\mathscr{P}_\kappa\lambda^+$ with size λ^+, which completes the proof.

Claim 4.5. *Let* $M \cap \lambda^{<\kappa} \in C$ *and* $M' \cap \lambda^{<\kappa} \in C$. *If* $\sup(M \cap \lambda^+) = \sup(M' \cap \lambda^+)$, *then* $M \cap \lambda^+ = M' \cap \lambda^+$.

Proof of the Claim 4.5. Let $M \cap \lambda^{<\kappa}$, $M' \cap \lambda^{<\kappa} \in C$ be such that $\sup(M \cap \lambda^+) = \sup(M' \cap \lambda^+)$. Let $N = M \cap M'$. Note that $\sup(N \cap \lambda^+) = \sup(M \cap \lambda^+)$ and $N \cap \lambda_i$ is closed under h_i.

Subclaim 4.6. *If* $M \cap \lambda = N \cap \lambda$, *then* $M \cap \lambda^+ = N \cap \lambda^+$.

Proof of the Subclaim 4.6. Choose any $\alpha \in (M \cap \lambda^+) \setminus \lambda$. We have $\beta \in N \cap \lambda^+$ such that $\alpha < \beta$. Let $\tau \in N$ be a bijection from λ to β. Since $\alpha < \beta$ and $\tau \in M$, there exists $\delta \in M \cap \lambda = N \cap \lambda$ such that $\pi(\delta) = \alpha$, hence $\alpha \in N \cap \lambda^+$. □

We show $M \cap \lambda = N \cap \lambda$. To show this, we need the following claim.

Subclaim 4.7. $\{i < \mathrm{cf}(\lambda) : \sup(N \cap \lambda_i) < \sup(M \cap \lambda_i)\}$ *is bounded in* $\mathrm{cf}(\lambda)$.

Proof of the Subclaim 4.7. Assume otherwise. Then define $f \in \Pi\lambda_i$ by $f(i) \in (M \cap \lambda_i) \setminus \sup(N \cap \lambda_i)$ if $\sup(N \cap \lambda_i) < \sup(M \cap \lambda_i)$ and $f(i) = 0$ otherwise. Then $f \in M$ since $M \cap \lambda$ is closed under $<(M \cap \kappa)$-sequences. Because $\langle f_\alpha : \alpha < \lambda \rangle$ is a scale for $\Pi\lambda_i$, there exists $\alpha \in M \cap \lambda^+$ such that $f <^* f_\alpha$, that is, $\{i < \mathrm{cf}(\lambda) : f(i) \geq f_\alpha(i)\}$ is bounded in $\mathrm{cf}(\lambda)$. Since $\sup(M \cap \lambda^+) = \sup(N \cap \lambda^+)$, there exists $\beta \in N \cap \lambda^+$ such that $\alpha < \beta$. $f < f_\alpha \leq^* f_\beta$, so we can take $i < \mathrm{cf}(\lambda)$ such that $f(i) \in (M \cap \lambda_i) \setminus \sup(N \cap \lambda_i)$ and $f(i) < f_\beta(i)$. However $f_\beta \in N$, hence $f_\beta(i) \in N \cap \lambda_i$. This is a contradiction. □

We return to the proof of the Claim. Let $i < \mathrm{cf}(\lambda)$ be such that $\sup(M \cap \lambda_i) = \sup(N \cap \lambda_i)$. Since $\mathrm{ot}(M \cap \lambda_i)$ is regular, $\mathrm{ot}(N \cap \lambda_i)$ is regular. Thus $|M \cap \lambda_i| = |N \cap \lambda_i|$. Since $h_i|[M \cap \lambda_i]^\omega$ is ω-Jonsson and $N \cap \lambda_i$ is closed under h_i, we have $M \cap \lambda_i = N \cap \lambda_i$. There are unboundedly many such i, hence $M \cap \lambda = N \cap \lambda$. We can show that $M' \cap \lambda = N \cap \lambda$ by the same argument. Thus $M \cap \lambda^+ = M' \cap \lambda^+$. □

The following question is natural, but the author cannot answer:

Question 1. Does $\lambda^{<\kappa} = \lambda^+$ follow from κ is mildly λ-ineffable and $\mathrm{cf}(\lambda) < \kappa$?

Of course $\lambda^{<\kappa} = \lambda^+$ follows from mild ineffability of $\mathscr{P}_\kappa\lambda^{<\kappa}$ when $\mathrm{cf}(\lambda) < \kappa$. Unfortunately, however, mild ineffability of $\mathscr{P}_\kappa\lambda$ does not always lift up to that of $\mathscr{P}_\kappa\lambda^{<\kappa}$. (See the next section.)

5. The equivalence of the Shelah property and ineffability.

Abe [**3**] showed that ineffability and almost ineffability of $\mathscr{P}_\kappa\lambda$ are equivalent if $2^\lambda = \lambda^{<\kappa}$. We will see that if λ is strong limit and $\mathrm{cf}(\lambda) < \kappa$ then ineffability and the Shelah property are equivalent. First we will check that such equivalence is impossible if $\mathrm{cf}(\lambda) \geq \kappa$. Proposition 5.1 (3) was proved in Abe [**3**]. We present here a simple proof.

Proposition 5.1. *Let X be a subset of $\mathscr{P}_\kappa\lambda$.*

(1) *If X is Shelah, then $\{x \in X : X \cap \mathscr{P}_{x\cap\kappa}x$ is not Shelah in $\mathscr{P}_{x\cap\kappa}x\}$ has the Shelah property.*
(2) *If X is almost ineffable, then $\{x \in X : X \cap \mathscr{P}_{x\cap\kappa}x$ is not almost ineffable in $\mathscr{P}_{x\cap\kappa}x\}$ is almost ineffable.*
(3) *If X is ineffable, then $\{x \in X : X \cap \mathscr{P}_{x\cap\kappa}x$ is not ineffable in $\mathscr{P}_{x\cap\kappa}x\}$ is ineffable.*

Proof. We will only show (3). (1) and (2) can be proved by a similar argument. Suppose $X \subseteq \mathscr{P}_\kappa\lambda$ is ineffable. We may assume that $x\cap\kappa$ is inaccessible for all $x \in X$. Let $Y = \{x \in X : X \cap \mathscr{P}_{x\cap\kappa}x$ is not ineffable in $\mathscr{P}_{x\cap\kappa}x\}$.

Let $D = \mathscr{P}_\kappa\lambda \cup \{\lambda\}$. Then the relation $<$ on $\mathscr{P}_\kappa\lambda$ can be extended to D by identifying λ as the maximal element of D with respect to the relation $<$. We consider $\mathscr{P}_\kappa\lambda$ as $\mathscr{P}_{\lambda\cap\kappa}\lambda$. Note that the relation $<$ on D is well-founded. To show that Y is ineffable, we prove, by using induction on $<$, that, for any $x \in D \cap (X \cup \{\lambda\})$, $Y \cap \mathscr{P}_{x\cap\kappa}x$ is ineffable in $\mathscr{P}_{x\cap\kappa}x$ if $X \cap \mathscr{P}_{x\cap\kappa}x$ is ineffable. This is sufficient to show the proposition. Let $x \in X \cup \{\lambda\}$ and assume this claim is verified for all $y \in X$ with $y < x$. Suppose $X \cap \mathscr{P}_{x\cap\kappa}x$ is ineffable but $Y \cap \mathscr{P}_{x\cap\kappa}x$ is not ineffable. Let $\langle a_z : z \in Y \cap \mathscr{P}_{x\cap\kappa}x\rangle$ be a sequence which witnesses $Y \cap \mathscr{P}_{x\cap\kappa}x$ is not ineffable. Since $X \cap \mathscr{P}_{x\cap\kappa}x$ is ineffable but $Y \cap \mathscr{P}_{x\cap\kappa}x$ is not ineffable, $Z = (X \setminus Y) \cap \mathscr{P}_{x\cap\kappa}x$ is ineffable. For each $y \in Z$, $X \cap \mathscr{P}_{y\cap\kappa}y$ is ineffable in $\mathscr{P}_{y\cap\kappa}y$. Hence $Y \cap \mathscr{P}_{y\cap\kappa}y$ is ineffable by the induction hypothesis. Hence we can apply the ineffability of $Y \cap \mathscr{P}_{y\cap\kappa}y$ to $\langle a_z : z \in Y \cap \mathscr{P}_{y\cap\kappa}y\rangle$. So there exists $b_y \subseteq y$ such that $\{z \in Y \cap \mathscr{P}_{y\cap\kappa}y : b_y \cap z = a_z\}$ is stationary in $\mathscr{P}_{y\cap\kappa}y$. Since Z is ineffable, there exists $B \subseteq x$ such that $\{y \in Z : B \cap y = b_y\}$ is stationary in $\mathscr{P}_{x\cap\kappa}x$. We check that $\{z \in Y \cap \mathscr{P}_{x\cap\kappa}x : a_z = B \cap z\}$ is stationary, which is a contradiction. Take $f : x \times x \to x$. We want to find $z \in Y \cap \mathscr{P}_{x\cap\kappa}x$ such that $B \cap z = a_z$, $z \cap \kappa \in \kappa$, and $f``(z \times z) \subseteq z$. Since $\{y \in Z : B \cap y = b_y\}$ is stationary in $\mathscr{P}_{x\cap\kappa}x$, there exists $y \in Z$ such that $B \cap y = b_y$ and $f``(y \times y) \subseteq y$. Because $\{z \in Y \cap \mathscr{P}_{y\cap\kappa}y : b_y \cap z = a_z\}$ is stationary in $\mathscr{P}_{y\cap\kappa}y$, we can take $z \in Y \cap \mathscr{P}_{y\cap\kappa}y$ such that $a_z = b_y \cap z = B \cap z$, $z \cap \kappa \in \kappa$, and $f``(z \times z) \subseteq z$. This completes the proof. □

Corollary 5.2. *Assume* $\mathrm{cf}(\lambda) \geq \kappa$.

(1) *If* κ *is* λ*-Shelah, then* $\mathrm{NSh}_{\kappa\lambda} \subsetneq \mathrm{NAIn}_{\kappa\lambda}$.
(2) *If* κ *is almost* λ*-ineffable, then* $\mathrm{NAIn}_{\kappa\lambda} \subsetneq \mathrm{NIn}_{\kappa\lambda}$.

Proof.

(1). By Abe [**3**], $\{x \in \mathscr{P}_\kappa\lambda : x \cap \kappa \text{ is } x\text{-Shelah}\} \in \mathrm{NAIn}^*_{\kappa\lambda}$. Hence by Proposition 5.1, $\{x \in \mathscr{P}_\kappa\lambda : x \cap \kappa \text{ is not } x\text{-Shelah}\}$ is Shelah but not almost ineffable.

(2). By Kamo [**17**], $\{x \in \mathscr{P}_\kappa\lambda : x \cap \kappa \text{ is almost } x\text{-ineffable}\} \in \mathrm{NIn}^*_{\kappa\lambda}$. So $\{x \in \mathscr{P}_\kappa\lambda : x \cap \kappa \text{ is not almost } x\text{-ineffable}\}$ is almost ineffable but not ineffable. □

Proposition 5.3. *Assume* λ *is a strong limit cardinal with* $\mathrm{cf}(\lambda) < \kappa$ *(so* $2^\lambda = \lambda^{<\kappa}$ *holds). Let* $\langle A_x : x \in \mathscr{P}_\kappa\lambda\rangle$ *be an enumeration of* $\mathscr{P}(\lambda)$ *and* $X = \{x \in \mathscr{P}_\kappa\lambda : \forall a \subseteq x \exists y < x\,(a = A_y \cap x)\}$. *Then* $\mathrm{NSh}_{\kappa\lambda} = \mathrm{NIn}_{\kappa\lambda} = \mathrm{NAIn}_{\kappa\lambda} = \mathrm{WNS}_{\kappa\lambda}|X$. *In particular the following are equivalent*:

(1) κ *is* λ*-Shelah.*
(2) κ *is almost* λ*-ineffable.*
(3) κ *is* λ*-ineffable.*
(4) $X \in \mathrm{WNS}^+_{\kappa\lambda}$.

Proof. Since $\mathrm{WNS}_{\kappa\lambda} \subseteq \mathrm{NSh}_{\kappa\lambda} \subseteq \mathrm{NAIn}_{\kappa\lambda} \subseteq \mathrm{NIn}_{\kappa\lambda}$, it is enough to show that $X \in \mathrm{NSh}^*_{\kappa\lambda}$ and $\mathrm{NIn}_{\kappa\lambda} \subseteq \mathrm{WNS}_{\kappa\lambda}|X$. First we show that $X \in \mathrm{NSh}^*_{\kappa\lambda}$. Let $\langle B_\xi : \xi < \lambda\rangle$ be an enumeration of all bounded subsets of λ. First we claim that $Z = \{x \in \mathscr{P}_\kappa\lambda : \forall a \subseteq x\,(a \text{ is bounded in } \lambda \to \exists \xi \in x(a = B_\xi \cap x))\} \in \mathrm{NSh}^*_{\kappa\lambda}$. Assume otherwise, then by the normality of $\mathrm{NSh}_{\kappa\lambda}$, there exists $\alpha < \lambda$ such that $Y = \{x \in \mathscr{P}_\kappa\lambda : \exists a \subseteq x \cap \alpha \forall \xi \in x(a \neq B_\xi \cap x)\} \in \mathrm{NSh}^+_{\kappa\lambda}$. For each $x \in Y$, let $a_x \subseteq x \cap \alpha$ be a witness to $x \in Y$. Let $f_x : x \cap \alpha \to 2$ be the characteristic function of a_x and $g_x : x \to x$ a function such that $g_x(\beta) \in a_x \triangle (B_\beta \cap x)$ for each $\beta \in x$. By the Shelah property of Z, there exist $f : \alpha \to 2$ and $g : \lambda \to \lambda$ such that $\{x \in Y : f_x|y = f|y \text{ and } g_x|y = g|y\}$ is unbounded for all $y \in \mathscr{P}_\kappa\lambda$. Let $B = f^{-1}``\{1\}$. $B \subseteq \alpha$, so $B = B_\xi$ for some $\xi < \lambda$. Take $y \in \mathscr{P}_\kappa\lambda$ such that $\xi \in y$ and y is closed under g. Take $x \in Y$ such that $y < x$, $f_x|y = f|y$, and $g_x|y = g|y$. Then $B_\xi \cap x \neq a_x$ because $\xi \in y \subseteq x$. Since $f|x$ is the characteristic function of $B_\xi \cap x$, f_x is that of a_x, and $g_x(\xi) \in a_x \triangle (B_\xi \cap x)$, we have $f_x(g_x(\xi)) \neq f(g_x(\xi))$. Since $\xi \in y$ and y is closed under g, we have $g_x(\xi) = g(\xi) \in y$. But then $f_x(g(\xi)) = f(g(\xi))$, which is a contradiction.

Second we show that $X \in \mathrm{NSh}^*_{\kappa\lambda}$. Fix an increasing sequence $\langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ which converges to λ. Assume $X \notin \mathrm{NSh}^*_{\kappa\lambda}$. Then $Z' = \{x \in Z : \{\lambda_i : i < \mathrm{cf}(\lambda)\} \subseteq x \text{ and } \exists a \subseteq x \forall y < x\,(a \neq A_y \cap x)\} \in \mathrm{NSh}^+_{\kappa\lambda}$. For each $x \in Z'$, let $a_x \subseteq x$ be a witness to $x \in Z'$. For $x \in Z'$ and $i < \mathrm{cf}(\lambda)$, take $\xi_i^x \in x$ such that $a_x \cap \lambda_i = A_{\xi_i^x} \cap x$. Then, by the strong normality of $\mathrm{NSh}_{\kappa\lambda}$, there exists $\langle \xi_i : i < \mathrm{cf}(\lambda)\rangle$ such that $\{x \in Z' : \forall i < \mathrm{cf}(\lambda)\,(\xi_i^x = \xi_i)\} \in \mathrm{NSh}_{\kappa\lambda}^+$. Note that if $i < j < \mathrm{cf}(\lambda)$, then $A_{\xi_i} = A_{\xi_j} \cap \lambda_i$. Thus we can define $A \subseteq \lambda$ by $A \cap \lambda_i = A_{\xi_i}$ for all $i < \mathrm{cf}(\lambda)$. Take $y \in \mathscr{P}_\kappa\lambda$ such that $A = A_y$. It is easy to see that for $x \in Z'$, $a_x = A_y \cap x$ if $\xi_i^x = \xi_i$ for all $i < \mathrm{cf}(\lambda)$, which is a contradiction. Thus we have $X \in \mathrm{NSh}_{\kappa\lambda}^*$.

Last we show that $\mathrm{NIn}_{\kappa\lambda} \subseteq \mathrm{WNS}_{\kappa\lambda}|X$. Let $W \in (\mathrm{WNS}_{\kappa\lambda}|X)^+$. We may assume $W \subseteq X$. We claim that W is ineffable. To see this, take an arbitrary sequence $\langle a_x : x \in W\rangle$ such that $a_x \subseteq x$ for all $x \in W$. By the definition of X, for each $x \in W$ there exists $y_x < x$ such that $a_x = A_{y_x} \cap x$. Since $W \in \mathrm{WNS}_{\kappa\lambda}^+$, there exists $y \in \mathscr{P}_\kappa\lambda$ such that $W' = \{x \in W : y_x = y\} \in \mathrm{WNS}_{\kappa\lambda}^+$. Then W' is stationary and it is clear that $a_x = A_y \cap x$ for all $x \in W'$. □

Remark. If we replace "λ is strong limit" by "$2^\lambda = \lambda^{<\kappa}$" in the assumption of the previous proposition, then we can obtain that $\mathrm{NAIn}_{\kappa\lambda} = \mathrm{NIn}_{\kappa\lambda} = \mathrm{WNS}_{\kappa\lambda}|X$. The proof that $X \in \mathrm{NAIn}_{\kappa\lambda}^*$ is easy, so we omit it.

Corollary 5.4. *Assume $2^\lambda = \lambda^{<\kappa}$. For any $Y \in \mathrm{NIn}_{\kappa\lambda}^+$ ($= \mathrm{NAIn}_{\kappa\lambda}^+$) and $\langle a_x : x \in Y\rangle$ with $a_x \subseteq x$, there exists $A \subseteq \lambda$ such that $\{x \in Y : A \cap x = a_x\} \in \mathrm{NIn}_{\kappa\lambda}^+$.*

Proof. By the above remark, $\mathrm{NIn}_{\kappa\lambda} = \mathrm{NAIn}_{\kappa\lambda} = \mathrm{WNS}_{\kappa\lambda}|X$ holds, where X is as in Proposition 5.3. We can argue as in the proof of $\mathrm{NIn}_{\kappa\lambda} \subseteq \mathrm{WNS}_{\kappa\lambda}|X$ in Proposition 5.3. □

Next we turn to completely ineffability of $\mathscr{P}_\kappa\lambda$.

Definition 5.5. Let I be an ideal over $\mathscr{P}_\kappa\lambda$. $\mathscr{W}$ is called an *I-partition* if the following hold:

(1) $\mathscr{W} \subseteq I^+$,
(2) $\forall Y \in I^+ \exists Z \in \mathscr{W}\,(Y \cap Z \in I^+)$, and
(3) $\forall Y, Z \in \mathscr{W}\,(Y \neq Z \Rightarrow Y \cap Z \in I)$.

Let μ and ν be cardinals. An ideal I over $\mathscr{P}_\kappa\lambda$ is called *(μ,ν)-distributive* if, for every $X \in I^+$ and every $\langle \mathscr{W}_\alpha : \alpha < \mu\rangle$ where each $\mathscr{W}_\alpha$ is an I-partition with $|\mathscr{W}_\alpha| \leq \nu$, there exists $Y \in (I|X)^+$ such that Y satisfies the following:

For every $\alpha < \mu$, there exists $Z \in \mathscr{W}_\alpha$ such that $Y \setminus Z \in I$.

Fact 5.6 (Johnson [**16**]). Let I be an ideal over $\mathscr{P}_\kappa\lambda$. Then the following are equivalent:

(1) I is normal and (λ,λ)-distributive.

(2) For all $X \in I^+$ and $\langle a_x : x \in X \rangle$ with $a_x \subseteq x$, there exists $A \subseteq \lambda$ such that $\{x \in X : A \cap x = a_x\} \in I^+$.

We say that $X \subseteq \mathscr{P}_\kappa\lambda$ is *completely ineffable* if there exists a proper (λ, λ)-distributive normal ideal I such that $X \in I^+$, and that κ is *completely λ-ineffable* if $\mathscr{P}_\kappa\lambda$ is completely ineffable. Let $\mathrm{NCIn}_{\kappa\lambda} = \{X \subseteq \mathscr{P}_\kappa\lambda : X$ is not completely ineffable$\}$. Then $\mathrm{NCIn}_{\kappa\lambda}$ is the minimal normal (λ, λ)-distributive ideal and, equivalently, is the minimal normal ideal which satisfies (2) of the above fact. Clearly $\mathrm{NIn}_{\kappa\lambda} \subseteq \mathrm{NCIn}_{\kappa\lambda}$ holds.

PROPOSITION 5.7. *Assume* $\mathrm{cf}(\lambda) \geq \kappa$ *and* κ *is* λ*-ineffable. Then* $\mathrm{NIn}_{\kappa\lambda} \subsetneq \mathrm{NCIn}_{\kappa\lambda}$.

PROOF. By Kamo [**17**], $\{x \in \mathscr{P}_\kappa\lambda : x \cap \kappa$ is x-ineffable$\} \in \mathrm{NCIn}^*_{\kappa\lambda}$. Hence the assertion follows from Proposition 5.1. □

The next proposition can be easily verified by using Corollary 5.4 and Fact 5.6.

PROPOSITION 5.8. *Assume* $2^\lambda = \lambda^{<\kappa}$. *Then* $\mathrm{NIn}_{\kappa\lambda} = \mathrm{NCIn}_{\kappa\lambda}$ *holds. Thus* κ *is* λ*-ineffable if and only if* κ *is completely* λ*-ineffable under the assumption* $2^\lambda = \lambda^{<\kappa}$.

Now we show the preservation of ineffability under certain forcing methods. For a poset $\boldsymbol{P}$ and an ordinal α, $\Gamma_\alpha(\boldsymbol{P})$ denotes the following 2-player game:

Player I :	p_0		p_1		$\cdots$		$p_{\omega+1}$		$\cdots$
Player II :		q_0		q_1	$\cdots$	q_ω		$q_{\omega+1}$	$\cdots$

Player I and II choose elements of $\boldsymbol{P}$ alternately such that $p_0 \geq q_0 \geq p_1 \geq q_1 \geq \cdots$. At limit stage η, only Player II moves and Player II chooses a lower bound q_η of $\{q_\xi : \xi < \eta\}$. Player II wins if this game can be continued to length α, that is, Player II can choose q_β for all $\beta < \alpha$. A poset $\boldsymbol{P}$ is *α-strategically closed* if Player II has a winning strategy in $\Gamma_\alpha(\boldsymbol{P})$. It is well-known that α-strategically closed posets add no new $< \alpha$-sequences.

PROPOSITION 5.9. *Assume λ is a strong limit cardinal with* $\mathrm{cf}(\lambda) < \kappa$. *If κ is λ-ineffable* (*equivalently, λ-Shelah, almost λ-ineffable, or completely λ-ineffable*), *then* $\Vdash_{\boldsymbol{P}}$ *"κ is λ-ineffable" for every λ^+-strategically closed poset* $\boldsymbol{P}$.

PROOF. By Proposition 4.4, we have $2^\lambda = \lambda^{<\kappa} = \lambda^+$. Let $\langle A_x : x \in \mathscr{P}_\kappa\lambda \rangle$ be an enumeration of $\mathscr{P}(\lambda)$ and define X as in Proposition 5.3. Then $\mathrm{NIn}_{\kappa\lambda}$

$= \mathrm{WNS}_{\kappa\lambda}|X$.

Since λ^+-strategically closed forcing adds no new subsets of λ, $\langle A_x : x \in \mathscr{P}_\kappa\lambda\rangle$ remains an enumeration of $\mathscr{P}(\lambda)$ in $V^{\boldsymbol{P}}$. Thus it is enough to show that $X \in \mathrm{WNS}^+_{\kappa\lambda}$ in $V^{\boldsymbol{P}}$. Let $p \in \boldsymbol{P}$, and let $\dot{f}$ be a $\boldsymbol{P}$-name such that $p \Vdash$"$\dot{f} : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$". Let $\langle x_\alpha : \alpha < \lambda^+\rangle$ be an enumeration of $\mathscr{P}_\kappa\lambda$. Using the λ^+-strategic closedness of $\boldsymbol{P}$, we construct $\langle y_\alpha \in \mathscr{P}_\kappa\lambda : \alpha < \lambda^+\rangle$ and a descending sequence $\langle p_\alpha \in \boldsymbol{P} : \alpha < \lambda^+\rangle$ such that $p_0 \le p$ and $p_\alpha \Vdash$"$\dot{f}(x_\alpha) = y_\alpha$" for all $\alpha < \lambda^+$. Now define $g : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$ by $g(x_\alpha) = y_\alpha$. Since $X \in \mathrm{WNS}^+_{\kappa\lambda}$, there exists $x \in X$ such that g"$\mathscr{P}_{x\cap\kappa}x \subseteq \mathscr{P}_{x\cap\kappa}x$. Take a sufficiently large $\beta < \lambda^+$ such that $\mathscr{P}_{x\cap\kappa}x \subseteq \{x_\alpha : \alpha < \beta\}$. Then $p_\beta \Vdash$"$\dot{f}|\mathscr{P}_{x\cap\kappa}x = g|\mathscr{P}_{x\cap\kappa}x$". Hence we conclude that $p_\beta \Vdash$"$x \in X \cap C_{\dot{f}}$." □

By Proposition 5.9, we have the following corollary:

Corollary 5.10. *Assume λ is a strong limit cardinal with* $\mathrm{cf}(\lambda) < \kappa$ *and κ is λ-ineffable. Then there exists a poset which preserves all cofinalities and forces that κ remains completely λ-ineffable and* $\{\alpha < \lambda^+ : \mathrm{cf}(\alpha) < \kappa\}$ *has a non-reflecting stationary subset.*

Proof. Let $\boldsymbol{P}$ be the standard forcing notion which adds a non-reflecting stationary subset of $\{\alpha < \lambda^+ : \mathrm{cf}(\alpha) < \kappa\}$ (see Burgess [**5**]). This poset is λ^+-strategically closed, hence, by Lemma 5.9, κ is completely λ-ineffable in $V^{\boldsymbol{P}}$. □

Abe [**3**] proved that λ-ineffability does not imply $\lambda^{<\kappa}$-ineffability if $\mathrm{cf}(\lambda) < \kappa$. We can improve Abe's result to the following:

Corollary 5.11. *Relative to a certain large cardinal assumption, it is consistent that κ is completely λ-ineffable with* $\mathrm{cf}(\lambda) < \kappa$*, but not mildly $\lambda^{<\kappa}$-ineffable.*

Proof. We suppose that κ is completely λ-ineffable with $\mathrm{cf}(\lambda) < \kappa$ and that $\{\alpha < \lambda^+ : \mathrm{cf}(\alpha) < \kappa\}$ has a non-reflecting stationary subset. This is consistent by Corollary 5.10. By Proposition 4.4, $\lambda^{<\kappa} = \lambda^+$ holds. By Proposition 4.3, κ is not mildly λ^+-ineffable. Hence, in this model, κ is completely λ-ineffable but not mildly $\lambda^{<\kappa}$-ineffable. □

Now we investigate the precipitousness of $\mathrm{NIn}_{\kappa\lambda}$.

Definition 5.12. For an ideal I over $\mathscr{P}_\kappa\lambda$, I is said to be *precipitous* if, for every $X \in I^+$ and for every I-partitions $\langle \mathscr{W}_n : n < \omega\rangle$ such that $\forall n \in \omega \forall Y \in \mathscr{W}_{n+1} \exists Z \in \mathscr{W}_n\,(Y \subseteq Z)$, there exists a sequence $\langle X_n : n < \omega\rangle$ such that $X_n \in \mathscr{W}_n$ for all $n < \omega$, $X \supseteq X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$, and $\bigcap_{n<\omega} X_n \neq \emptyset$.

For an information about precipitousness, see section 22 in Jech [**14**].

Fact 5.13 (Abe [**3**]). If $\mathrm{cf}(\lambda) \geq \kappa$, then $\mathrm{NCIn}_{\kappa\lambda}$ is not precipitous.

Now assume κ is a Mahlo cardinal. Let $e : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda^{<\kappa}$ be a canonical map and $X = \{x \in \mathscr{P}_\kappa\lambda : x \cap \kappa \text{ is inaccessible}, e(x) \cap \lambda = x\}$. Then $X \in \mathrm{WNS}^*_{\kappa\lambda}$ and $e``X \in \mathrm{WNS}^*_{\kappa\lambda^{<\kappa}}$. For each $Y \subseteq X$, $Y \in \mathrm{WNS}_{\kappa\lambda}$ if and only if $e``Y \in \mathrm{WNS}_{\kappa\lambda^{<\kappa}}$. Furthermore it is easy to see that $e|X$ is a bijection from X to $e``X$. Using this, we can easily verify the following lemma:

Lemma 5.14. *Let κ be a Mahlo cardinal and let $e : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda^{<\kappa}$ be a canonical map. Then, for $X \in \mathrm{WNS}^+_{\kappa\lambda}$, $\mathrm{WNS}_{\kappa\lambda}|X$ is precipitous if and only if $\mathrm{WNS}_{\kappa\lambda^{<\kappa}}|e``X$ is precipitous. In particular $\mathrm{WNS}_{\kappa\lambda}$ is precipitous if and only if $\mathrm{WNS}_{\kappa\lambda^{<\kappa}}$ is precipitous.*

Proposition 5.15. *Assume λ is a strong limit cardinal with $\mathrm{cf}(\lambda) < \kappa$ and κ is λ-Shelah (and so is λ-ineffable, etc.). Let μ be a Woodin cardinal greater than λ. Then $\Vdash_{\mathrm{Col}(\lambda^+,<\mu)}$ "$\mathrm{NSh}_{\kappa\lambda} = \mathrm{NAIn}_{\kappa\lambda} = \mathrm{NIn}_{\kappa\lambda} = \mathrm{NCIn}_{\kappa\lambda}$ is precipitous", where $\mathrm{Col}(\lambda^+, <\mu)$ is the standard λ^+-closed poset which collapses μ to λ^{++}.*

Proof. Let G be a $(V, \mathrm{Col}(\lambda^+, <\mu))$-generic filter and work in $V[G]$. $\mathrm{Col}(\lambda^+, <\mu)$ is λ^+-strategically closed. Hence κ is λ-Shelah, and $\mathrm{NSh}_{\kappa\lambda} = \mathrm{NAIn}_{\kappa\lambda} = \mathrm{NIn}_{\kappa\lambda} = \mathrm{NCIn}_{\kappa\lambda} = \mathrm{WNS}_{\kappa\lambda}|X$ for some X in $V[G]$. It is well-known that $\mathrm{NS}_{\kappa\lambda^+}$ is precipitous in $V[G]$ (see Goldring [**12**]). Since $(\lambda^+)^{<\kappa} = \lambda^+$, $\mathrm{WNS}_{\kappa\lambda^+} = \mathrm{NS}_{\kappa\lambda^+}|Y$ for some Y. Thus $\mathrm{WNS}_{\kappa\lambda^+}$ is also precipitous. By the previous lemma, we have that $\mathrm{WNS}_{\kappa\lambda}$ is precipitous. Hence $\mathrm{WNS}_{\kappa\lambda}|X = \mathrm{NSh}_{\kappa\lambda} = \mathrm{NAIn}_{\kappa\lambda} = \mathrm{NIn}_{\kappa\lambda} = \mathrm{NCIn}_{\kappa\lambda}$ is precipitous. □

Question 2. Can $\mathrm{NSh}_{\kappa\lambda}$, $\mathrm{NAIn}_{\kappa\lambda}$, and $\mathrm{NIn}_{\kappa\lambda}$ be precipitous even if $\mathrm{cf}(\lambda) \geq \kappa$? Furthermore can these ideals be λ^+-saturated?

6. Relationship between Π^1_1-indescribability and ineffability.

The indescribability of $\mathscr{P}_\kappa\lambda$ was introduced by Baumgartner and Carr [**8**] as a generalization of the indescribability of a cardinal. First we explain some basic notation. A sentence φ is a Π^1_1-*sentence* if φ is of the form $\forall X_0 \forall X_1 \cdots \forall X_n \psi(X_0, X_1, \ldots, X_n)$, where $X_0, X_1, \ldots, X_n$ are type 2 variables, and $\psi(X_0, X_1, \ldots, X_n)$ is a first order sentence with language $\{\in, =, X_0, \ldots, X_n\}$ where X_i is a unary predicate symbol. In the intended semantics, if D is the domain of a structure, type 2 variables will range over $\mathscr{P}(D)$.

Definition 6.1. An uncountable cardinal κ is Π^1_1-*indescribable* if, for any $R \subseteq V_\kappa$ and Π^1_1-sentence φ over the structure $\langle V_\kappa, \in, R\rangle$ (that is, φ is a Π^1_1-sentence with language $\{\in, =, R\}$),

$$\langle V_\kappa, \in, R\rangle \models \varphi \Rightarrow \exists\alpha < \kappa(\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi),$$

where V_α is the set of all sets with rank less than α.

FACT 6.2. An uncountable cardinal κ is weakly compact if and only if κ is Π^1_1-indescribable.

Baumgartner defined the following:

DEFINITION 6.3. Let S be a set with $\kappa \subseteq S$. Define $V_\alpha(\kappa, S)$ by induction on $\alpha \le \kappa$ in the following way:

- $V_0(\kappa, S) = S$,
- $V_{\alpha+1}(\kappa, S) = V_\alpha(\kappa, S) \cup \mathscr{P}_\kappa(V_\alpha(\kappa, S))$, and
- $V_\alpha(\kappa, S) = \bigcup_{\beta<\alpha} V_\beta(\kappa, S)$ if α is a limit ordinal.

For $X \subseteq \mathscr{P}_\kappa S$, we say that X is Π^1_1-*indescribable* if, for every $R \subseteq V_\kappa(\kappa, S)$ and Π^1_1-sentence φ over the structure $\langle V_\kappa(\kappa, S), \in, R\rangle$, the following holds:

If $\langle V_\kappa(\kappa, S), \in, R\rangle \models \varphi$, then there exists $x \in X$ such that $|x \cap \kappa| = x \cap \kappa$ and φ *reflects to* x, that is,

$$\langle V_{x\cap\kappa}(x \cap \kappa, x), \in, R \cap V_{x\cap\kappa}(x \cap \kappa, x)\rangle \models \varphi.$$

Let $\Pi_{\kappa\lambda}$ be the set of all $X \subseteq \mathscr{P}_\kappa\lambda$ such that X is not Π^1_1-indescribable.

FACT 6.4 (Abe [**2**], Carr [**8**]).

(1) $\Pi_{\kappa\lambda}$ is a strongly normal ideal over $\mathscr{P}_\kappa\lambda$.
(2) $\mathrm{NSh}_{\kappa\lambda} \subseteq \Pi_{\kappa\lambda}$.
(3) If $\mathrm{cf}(\lambda) \ge \kappa$, then $\mathrm{NSh}_{\kappa\lambda} = \Pi_{\kappa\lambda}$.

For further general background about indescribability of $\mathscr{P}_\kappa\lambda$, see Abe [**2**] and Carr [**8**].

We will use the following combinatorial characterization of Π^1_1-indescribability.

FACT 6.5 (Abe [**2**]). For $X \subseteq \mathscr{P}_\kappa\lambda$, the following are equivalent:

(1) X is Π^1_1-indescribable.
(2) $e``X$ is Shelah in $\mathscr{P}_\kappa\lambda^{<\kappa}$, where e is a canonical map from $\mathscr{P}_\kappa\lambda$ to $\mathscr{P}_\kappa\lambda^{<\kappa}$.
(3) For all $\langle f_x : x \in X\rangle$ with $f_x : \mathscr{P}_{x\cap\kappa}x \to \mathscr{P}_{x\cap\kappa}x$, there exists $f : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$ such that $\{x \in X : f|\mathscr{P}_{y\cap\kappa}y = f_x|\mathscr{P}_{y\cap\kappa}y\}$ is unbounded for all $y \in \mathscr{P}_\kappa\lambda$.

First we show that Π^1_1-indescribability implies a reflection principle for

$\mathrm{WNS}_{\kappa\lambda}$-positive sets.

LEMMA 6.6. *Assume $\mathscr{P}_\kappa\lambda$ is Π^1_1-indescribable. Then, for each $X \in \mathrm{WNS}^+_{\kappa\lambda}$, $\{x \in \mathscr{P}_\kappa\lambda : x \cap \kappa$ is regular, $X \cap \mathscr{P}_{x\cap\kappa}x \in \mathrm{WNS}^+_{x\cap\kappa,x}\} \in \Pi^*_{\kappa\lambda}$.*

PROOF. Assume otherwise. Then $Y = \{x \in \mathscr{P}_\kappa\lambda : x \cap \kappa$ is regular and $X \cap \mathscr{P}_{x\cap\kappa}x \in \mathrm{WNS}_{x\cap\kappa,x}\} \in \Pi^+_{\kappa\lambda}$. For each $x \in Y$, let $f_x : \mathscr{P}_{x\cap\kappa}x \to \mathscr{P}_{x\cap\kappa}x$ be a function which witnesses $X \cap \mathscr{P}_{x\cap\kappa}x \in \mathrm{WNS}_{x\cap\kappa,x}$. By Fact 6.5, we can take $f : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$ such that, for all $y \in \mathscr{P}_\kappa\lambda$, $\{x \in Y : f|\mathscr{P}_{y\cap\kappa}y = f_x|\mathscr{P}_{y\cap\kappa}y\}$ is unbounded. Since $X \in \mathrm{WNS}^+_{\kappa\lambda}$, there exists $y \in X$ such that $f``\mathscr{P}_{y\cap\kappa}y \subseteq \mathscr{P}_{y\cap\kappa}y$. Take $x \in Y$ such that $y < x$ and $f|\mathscr{P}_{y\cap\kappa}y = f_x|\mathscr{P}_{y\cap\kappa}y$. Then $y \in X \cap \mathscr{P}_{x\cap\kappa}x$ and $f_x``\mathscr{P}_{y\cap\kappa}y = f``\mathscr{P}_{y\cap\kappa}y \subseteq \mathscr{P}_{y\cap\kappa}y$, thus $y \in (X \cap \mathscr{P}_{x\cap\kappa}x) \cap C_{f_x}$. This is a contradiction. □

We have another proof since "$X \in \mathrm{WNS}^+_{\kappa\lambda}$" can be stated in a Π^1_1-sentence over $\langle V_\kappa(\kappa,\lambda), \in, X\rangle$. Also note that, for every $X \in \mathrm{NS}^+_{\kappa\lambda}$, $\{x \in \mathscr{P}_\kappa\lambda : x \cap \kappa$ is regular, $X \cap \mathscr{P}_{x\cap\kappa}x \in \mathrm{NS}^+_{x\cap\kappa,x}\} \in \mathrm{NSh}^*_{\kappa\lambda}$.

The next proposition shows that Π^1_1-indescribability of $\mathscr{P}_\kappa\lambda$ can be much stronger than ineffability if $\mathrm{cf}(\lambda) < \kappa$.

PROPOSITION 6.7. *Assume $2^\lambda = \lambda^{<\kappa}$. Then the following hold:*

(1) $\mathrm{NIn}_{\kappa\lambda} \subseteq \Pi_{\kappa\lambda}$. *Hence κ is λ-ineffable if $\mathscr{P}_\kappa\lambda$ is Π^1_1-indescribable.*
(2) *If $Y \subseteq \mathscr{P}_\kappa\lambda$ is ineffable, then $\{x \in \mathscr{P}_\kappa\lambda : Y \cap \mathscr{P}_{x\cap\kappa}x$ is ineffable$\} \in \Pi^*_{\kappa\lambda}$.*
(3) *If κ is λ-ineffable, then* $\mathrm{NIn}_{\kappa\lambda} \subsetneq \Pi_{\kappa\lambda}$.

PROOF. Take X and $\langle A_x : x \in \mathscr{P}_\kappa\lambda\rangle$ as in Proposition 5.3.

(1). By the remark after Proposition 5.3, $\mathrm{NIn}_{\kappa\lambda} = \mathrm{WNS}_{\kappa\lambda}|X$ holds. Since $\mathrm{WNS}_{\kappa\lambda} \subseteq \mathrm{NSh}_{\kappa\lambda} \subseteq \Pi_{\kappa\lambda}$, it is enough to show that $X \in \Pi^*_{\kappa\lambda}$. Assume otherwise. Then $Y = \{x \in \mathscr{P}_\kappa\lambda : \exists a_x \subseteq x \forall y < x\,(a_x \neq A_y \cap x)\} \in \Pi^+_{\kappa\lambda}$. For each $x \in Y$, let $a_x \subseteq x$ be a witness to $x \in Y$. Now define $f_x : x \to 2$ and $g_x : \mathscr{P}_{x\cap\kappa}x \to x$ as follows: f_x is the characteristic function of a_x and $g_x(y) \in a_x \triangle (A_y \cap x)$. Then there exist $f : \lambda \to 2$ and $g : \mathscr{P}_\kappa\lambda \to \lambda$ such that $\{x \in Y : f_x|y = f|y, g_x|\mathscr{P}_{y\cap\kappa}y = g|\mathscr{P}_{y\cap\kappa}y\}$ is unbounded for all $y \in \mathscr{P}_\kappa\lambda$. Let $A = f^{-1}``\{1\}$. Then $A = A_z$ for some $z \in \mathscr{P}_\kappa\lambda$. Take $y \in \mathscr{P}_\kappa\lambda$ such that $z < y$ and $g``\mathscr{P}_{y\cap\kappa}y \subseteq y$. Then we can find $x \in Y$ such that $y < x$, $f|y = f_x|y$, and $g_x|\mathscr{P}_{y\cap\kappa}y = g|\mathscr{P}_{y\cap\kappa}y$. Since $z < y < x$, $a_x \neq A_z \cap x$. Since $g_x(z) = g(z)$, we have that $g(z) \in a_x \triangle (A_z \cap x)$. However $g(z) \in y$, thus $f(g(z)) = f_x(g(z))$, which contradicts to $g(z) \in a_x \triangle (A_x \cap x)$.

(2). Let $Z \subseteq \mathscr{P}_\kappa\lambda$ be ineffable. Since $\mathrm{NIn}_{\kappa\lambda} = \mathrm{WNS}_{\kappa\lambda}|X$, we may assume that $Z \subseteq X$. Let $x \in X$ such that $x \cap \kappa$ is regular. By the definition of X, $\langle A_y \cap x : y < x\rangle$ can be seen as an enumeration of $\mathscr{P}(x)$ which is indexed by elements of $\mathscr{P}_{x\cap\kappa}x$. Let $X' = \{y \in \mathscr{P}_{x\cap\kappa}x : \forall a \subseteq y \exists z < y\,(a = A_z \cap y)\}$.

Then $X' = X \cap \mathscr{P}_{x\cap\kappa}x$. By the proof of Proposition 5.3, we see that, for $x \in X$ such that $x \cap \kappa$ is regular, $Z \cap \mathscr{P}_{x\cap\kappa}x$ is ineffable if $Z \cap \mathscr{P}_{x\cap\kappa}x \in \mathrm{WNS}^+_{x\cap\kappa,x}$. It is clear that $\{x \in X : x \cap \kappa$ is regular, $Z \cap \mathscr{P}_{x\cap\kappa}x \in \mathrm{WNS}^+_{x\cap\kappa,x}\} \in \Pi^*_{\kappa\lambda}$ by Lemma 6.6.

(3). By (2), it is enough to show that $\{x \in \mathscr{P}_\kappa\lambda : x \cap \kappa$ is not x-ineffable$\}$ $\in \mathrm{NIn}^+_{\kappa\lambda}$. This follows from Proposition 5.1. □

Assume $\lambda = \kappa^{+\omega}$, $2^\lambda = \lambda^{<\kappa}$, and $\mathscr{P}_\kappa\lambda$ is Π^1_1-indescribable. Then $\{x \in \mathscr{P}_\kappa\lambda : \mathrm{ot}(x) = (x \cap \kappa)^{+\omega}\} \in \Pi^*_{\kappa\lambda}$. By the above proposition, we have $\{x \in \mathscr{P}_\kappa\lambda : \mathrm{ot}(x) = x \cap \kappa^{+\omega}$ and $x \cap \kappa$ is x-ineffable$\} \in \Pi^*_{\kappa\lambda}$, thus we can show that $\{\alpha < \kappa : \alpha$ is $\alpha^{+\omega}$-ineffable$\}$ is stationary in κ. In particular, under GCH, if $\kappa = \min\{\alpha : \alpha$ is $\alpha^{+\omega}$-ineffable$\}$, then $\mathscr{P}_\kappa\kappa^{+\omega}$ is not Π^1_1-indescribable. Hence, the assumption that $\mathrm{cf}(\lambda) \geq \kappa$ in (3) of Fact 6.4 cannot be dropped.

Lemma 6.8. *Let $X \subseteq \mathscr{P}_\kappa\lambda$ be Π^1_1-indescribable. Then $\{x \in X : X \cap \mathscr{P}_{x\cap\kappa}x$ is not Π^1_1-indescribable$\}$ is Π^1_1-indescribable.*

Proof. Let $Y = \{x \in X : X \cap \mathscr{P}_{x\cap\kappa}x$ is not Π^1_1-indescribable$\}$, $R \subseteq V_\kappa(\kappa,\lambda)$, and φ be a Π^1_1-sentence such that $\langle V_\kappa(\kappa,\lambda), \in, R\rangle \vDash \varphi$. We show that there exists $x \in Y$ such that φ reflects to x. Take $x \in X$ such that x is a $<$-minimal element of $\{y \in X : \varphi$ reflects to $y\}$. Then φ holds in $\langle V_{x\cap\kappa}(x\cap\kappa, x), \in, R \cap V_{x\cap\kappa}(x\cap\kappa,x)\rangle$ but there is no $y \in X \cap \mathscr{P}_{x\cap\kappa}x$ such that φ reflects to y by the minimality of x. Hence x is an element of Y. □

As an immediate corollary, we have the following:

Corollary 6.9. *Assume $2^\lambda = \lambda^{<\kappa}$ and $\mathscr{P}_\kappa\lambda$ is Π^1_1-indescribable. Then $\{x \in \mathscr{P}_\kappa\lambda : x \cap \kappa$ is x-ineffable but $\mathscr{P}_{x\cap\kappa}x$ is not Π^1_1-indescribable$\} \in \Pi^+_{\kappa\lambda}$.*

Thus, for instance, $\{\alpha < \kappa : \alpha$ is $\alpha^{+\omega}$-ineffable but $\mathscr{P}_\alpha\alpha^{+\omega}$ is not Π^1_1-indescribable$\}$ is stationary in κ if $\mathscr{P}_\kappa\kappa^{+\omega}$ is Π^1_1-indescribable.

Question 3. In this paper, we frequently used the assumptions that "λ is a strong limit cardinal" or "$2^\lambda = \lambda^{<\kappa}$". Can we eliminate these assumptions?

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