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Scattering for one dimensional perturbed Kirchhoff equations

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Abstract. The aim of this work is to show the existence of the wave operator and its inverse among Kirchhoff equations and free wave equations.

1. Introduction.

We consider the Cauchy problem for perturbed Kirchhoff equation in one dimensional space,

$$\partial_t^2 u(t,x) - \big(1+\varepsilon\|a(\cdot)u_x(t)\|_{L^2}^2\big)\partial_x(a(x)^2\partial_x u(t,x)) = 0,$$
$$t \in (-\infty,\infty),\ x \in R^1, \tag{1.1}$$

$$u(0,x) = f(x), \quad u_t(0,x) = g(x), \quad x \in R^1, \tag{1.2}$$

where $\varepsilon > 0$ is a small parameter and the coefficient $a(x) \in C^2(R^1)$ satisfies

$$0 < a_0 \le a(x) \le a_1, \quad x \in R^1 \tag{1.3}$$

and

$$|a^{(i)}(x)| \le \delta(1+|x|)^{-\sigma_0}, \quad x \in R^1,\ i = 1,2. \tag{1.4}$$

First of all we shall state the existence of the time global of solutions of the above Cauchy problem (1.1)–(1.2), under the assumption that the initial data $f \in C^2(R^1) \cap L^2(R^1)$, $g \in C^1(R^1)$ satisfy

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$$\left|\left(\frac{d}{dx}\right)^{i+1} f(x)\right| + \left|\left(\frac{d}{dx}\right)^{i} g(x)\right| \le C(1+|x|)^{-\sigma_1}, \quad x \in R^1,\ i = 0,1. \tag{1.5}$$

Namely we can prove the following theorem.

Theorem 1.1. *Assume that $a(x)$ satisfies (1.3)–(1.4) and the initial data $(f,g) \in (C^2(R^1) \cap L^2(R^1)) \times C^1(R^1)$ satisfies (1.5). Moreover assume $\sigma = \min\{\sigma_0, \sigma_1\} > 1$. Then there are $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that if $0 < \delta \le \delta_0$ is valid, for $0 < \varepsilon \le \varepsilon_0$ the Cauchy problem (1.1) and (1.2) has a unique solution u in $C^2(R^2) \cap C^0(R^1; L^2(R^1))$ such that $u_t(t,x), u_x(t,x) \in C^0(R^1; L^2(R^1))$.*

Next we mention the scattering for the equation (1.1).

Theorem 1.2. *Assume that $a(x)$ satisfies (1.3)–(1.4) and $\lim_{x\to\pm\infty} a(x) = a_\infty$ and that the initial data $(f_0^-, g_0^-) \in (C^2(R^1)\cap L^2(R^1)) \times C^1(R^1)$ satisfies (1.5). Moreover assume $\sigma = \min\{\sigma_0 - 1, \sigma_1\} > 1$. Then there are $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that if $0 < \delta \le \delta_0$ and $0 < \varepsilon \le \varepsilon_0$ are valid, there are $u \in C^2(R^2) \cap C^0(R^1; L^2(R^1))$ a unique solution of (1.1) for $t \in R^1, (f^+, g^+) \in C^2(R^1) \times C^1(R^1)$ and $c_\infty > 0$ such that*

$$\left\|u_t(t) - u_{0t}^{\pm}(c_\infty^{-1} S(t))\right\|_{L^2} + \left\|u_x(t) - u_{0x}^{\pm}(c_\infty^{-1} S(t))\right\|_{L^2} = O(|t|^{-\sigma+1}), \quad t \to \pm\infty \tag{1.6}$$

where $S(t) = \int_0^t (1 + \varepsilon\|a(\cdot)u_x(s)\|_{L^2}^2)^{\frac{1}{2}} ds$ and

$$\left(1 + \varepsilon\|a(\cdot)u_x(t)\|_{L^2}^2\right)^{\frac{1}{2}} - c_\infty = O(|t|^{-\sigma+1}), \quad t \to \pm\infty \tag{1.7}$$

where $u^{\pm}(t,x) \in C^2(R^2)$ denote solutions of the following equations

$$u_{0tt}^{\pm}(t,x) = a_\infty^2 c_\infty^2 u_{0xx}^{\pm}(t,x), \quad u_0^{\pm}(0,x) = f_0^{\pm}(x), \quad u_{0t}^{\pm}(0,x) = g_0^{\pm}(x) \tag{1.8}$$

and $\|\cdot\|_{L^2}$ stands for a norm of $L^2(R^1)$.

It should be remarked that in the case of the coefficient $a(x) = 1$ Theorem 1.1 is proved essentially by Greenberg and Hu in [**3**] under the assumption $\sigma_1 \ge 2$ and by D'Ancona and Spagnolo in [**1**] if $\sigma_1 > 6$ and by Yamazaki [**6**] in the case of $\sigma_1 > 1$. Rzmowski in [**5**] treated the Cauchy problem (1.1)–(1.2) in the L^1 framework. When $\sigma_1 > 2$, (1.6) in Theorem 1.2 is replaced by

$$\left\|u_t(t) - u_{0t}^{\pm}(t)\right\|_{L^2} + \left\|u_x(t) - u_{0x}^{\pm}(t)\right\|_{L^2} \to 0, \quad t \to \pm\infty, \tag{1.9}$$

because of $c_\infty^{-1}S(t) - t = O(|t|^{2-\sigma})$, $t \to \pm\infty$. When $a = 1$, Ghisi [**2**] gets (1.9) in the case of $t \to +\infty$ under the assumption $\sigma_1 > 6$ and Yamazaki [**6**] under the assumption $\sigma_1 > 2$ derived (1.9) in the both cases of $t \to \pm\infty$. On the other hand, Theorem 1 in Matsuyama [**4**] says that in general (1.9) in the case of $t \to +\infty$ does not holds if $\frac{1}{2} < \sigma_1 < 1$ and $a(x) = 1$. We can find many results for multi dimensional Kirchhoff type equations with constant coefficients. For example, see D'Ancona and Spagnolo [**1**], Yamazaki [**6**], Matsuyama [**4**] and their references.

We shall prove Theorem 1.1 and Theorem 1.2 by deriving the estimates of solutions of the equations (1.1) and (1.8) in L^∞ framework.

2. Linear equation.

In this section we transform our original equation into a two by two system of first order equations. We let $A(t,x) = u_t + a(x)c(t)u_x$ and $B(t,x) = u_t - a(x)c(t)u_x$, where $c(t)^2 = 1 + \varepsilon\|a(\cdot)u_x(t)\|_{L^2}^2$. We write $c' = \frac{dc(t)}{dt}$ and $a'(x) = \frac{da}{dx}(x)$. Then the equation (1.1) yields

$$\begin{aligned} A_t - a(x)c(t)A_x &= \frac{1}{2}\left(c(t)a'(x) + \frac{c'(t)}{c(t)}\right)(A-B), \\ B_t + a(x)c(t)B_x &= \frac{1}{2}\left(c(t)a'(x) - \frac{c'(t)}{c(t)}\right)(A-B). \end{aligned} \tag{2.1}$$

The initial conditions for A and B are computable in terms of f' and g. They are

$$A(0,x) = A_0(x); = g + a(x)c_0 f', \quad B(0,x) = B_0(x); = g - a(x)c_0 f', \tag{2.2}$$

where $c_0 = c(0) = (1 + \varepsilon\|a(\cdot)f'\|_{L^2}^2)^{\frac{1}{2}}$. The defining relation for $c(t)$ becomes

$$c(t)^2 = 1 + \frac{\varepsilon}{4c(t)^2}\|A(t,\cdot) - B(t,\cdot)\|_{L^2}^2. \tag{2.3}$$

We now introduce the change of variable $\tau = \int_0^t c(s)ds$. Clealy, τ is a strictly inceasing function of t. We denote its inverse function by $t = T(\tau)$ and regard A, B, c as functions of τ, that is, we write $A(\tau,x) = A(T(\tau),x)$, $B(\tau,x) = B(T(\tau),x)$, $c(\tau) = c(T(\tau))$ for simplicity of notation. Then by applying the change of variable to the equations (2.1), we get

$$A_\tau - a(x)A_x = \frac{1}{2}\left(a'(x) + \frac{c'}{c}\right)(A-B), \quad B_\tau + a(x)B_x = \frac{1}{2}\left(a'(x) - \frac{c'}{c}\right)(A-B), \tag{2.4}$$

and the initial condition is given by (2.2).

We introduce a functional space as follows

$$X_{\sigma,\delta,M} = \big\{c(\tau) \in C^1(R^1); c(0) = c_0,\ 1 \le c(\tau) \le M,\ |c'(\tau)| \le \delta(1+|\tau|)^{-\sigma},\ \tau \in R^1\big\}$$

with a norm $|c|_X = \sup|c(\tau)| + \sup(1+|\tau|)^\sigma |c'(\tau)|$. Let c be in $X_{\sigma,\delta,M}$ and consider the linear Cauchy problem (2.2)–(2.4). We denote its solution by (A_c, B_c). We define for $c \in X_{\sigma,\delta,M}$

$$\Phi(c)^2(\tau) = 1 + \frac{\varepsilon}{4c(\tau)^2}\|A_c(\tau,\cdot) - B_c(\tau,\cdot)\|^2_{L^2}. \tag{2.5}$$

Then we can prove the following theorem.

Theorem 2.1. *Assume that $a(x)$ satisfies* (1.3)–(1.4) *and A_0, $B_0 \in C^1(R^1)$ satisfy*

$$\big|A_0^{(i)}(x)\big| + \big|B_0^{(i)}(x)\big| \le C(1+|x|)^{-\sigma_1}, \quad x \in R^1,\ i = 0,1. \tag{2.6}$$

Then if $\sigma = \min\{\sigma_0, \sigma_1\} > 1$ is valid, there is $\varepsilon_0 > 0$ such that Φ is a contraction mapping in $X_{\sigma,\delta,M}$, that is,

$$|\Phi(c_1) - \Phi(c_2)|_X \le C\varepsilon|c_1 - c_2|_X, \tag{2.7}$$

for any c_1, $c_2 \in X_{\sigma,\delta,M}$ and $0 < \varepsilon \le \varepsilon_0$.

The proof of this theorem will be given in the Section 3.

Now we introduce again the change of variable with respect to x as follows. Let consider

$$\frac{dx}{d\tau} = \pm a(x), \quad x(0) = y \tag{2.8}$$

and we denote the solution by $x_\pm(\tau, y)$. Since $x_\pm(\tau, y)$ are strictly increasing functions with respect to y, we get the inverse fuction $y_\pm(\tau, x)$ as $x_\pm(\tau, y_\pm(\tau, x)) = x$. Hence we can define

$$\alpha_c(\tau, y) = A_c(\tau, x_-(\tau, y)), \quad \beta_c(\tau, y) = B_c(\tau, x_+(\tau, y)).$$

Then it holds

$$A_c(\tau, x) = \alpha_c(\tau, y_-(\tau, x)), \quad B_c(\tau, x) = \beta_c(\tau, y_+(\tau, x)). \tag{2.9}$$

Therefore we obtain the following integral equations from (2.2)–(2.4)

$$\alpha_c(\tau,y) = A_0(y) + \int_0^\tau F_c(s,y)ds, \quad \beta_c(\tau,y) = B_0(y) + \int_0^\tau G_c(s,y)ds \tag{2.10}$$

where the equation (2.1) and the relation (2.9) yield

$$F_c(s,y) = \frac{1}{2}\left(a'(x_-(s,y)) + \frac{c'(s)}{c(s)}\right)\big(\alpha_c(s,y) - \beta_c(s,y_+(s,x_-(s,y)))\big), \tag{2.11}$$

$$G_c(s,y) = \frac{1}{2}\left(a'(x_+(s,y)) - \frac{c'(s)}{c(s)}\right)\big(\alpha_c(s,y_-(s,x_+(s,y))) - \beta_c(s,y)\big). \tag{2.12}$$

To derive a priori estimates for (2.10), we introduce a norm in $C^i(R^1)$ as

$$|f|_i = \sup_{x\in R^1, 0\le k\le i} \langle x\rangle^\sigma |f^{(k)}(x)|, \quad i=0,1,\dots, \tag{2.13}$$

where $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$. Then we can prove the following proposition.

PROPOSITION 2.1. *Assume that the conditions of Theorem* 2.1 *are valid. Then we have*

$$|\alpha_c(\tau)|_i + |\beta_c(\tau)|_i \le C(|A_0|_i + |B_0|_i), \quad \tau\in R^1,\ i=0,1, \tag{2.14}$$

for $c\in X_{\sigma,\delta,M}$ *and*

$$|\alpha_{c_1}(\tau)-\alpha_{c_2}(\tau)|_1 + |\beta_{c_1}(\tau)-\beta_{c_2}(\tau)|_1 \le C(|A_0|_1+|B_0|_1)|c_1-c_2|_X, \quad \tau\in R^1, \tag{2.15}$$

for $c_1, c_2 \in X_{\sigma,\delta,M}$.

PROOF. Put

$$\gamma_i = \sup_{s\in R^1}\big\{|\alpha_c(s)|_i + |\beta_c(s)|_i\big\}, \quad i=0,1.$$

Then we can see easily that F_c, G_c satisfies

$$|F_c(s,y)| \le \delta\gamma_0\big(\langle x_-(s,y)\rangle^{-\sigma} + \langle s\rangle^{-\sigma}\big)\big\{\langle y_+(s,x_-(s,y))\rangle^{-\sigma} + \langle y\rangle^{-\sigma}\big\},$$

$$|G_c(s,y)| \le \delta\gamma_0\big(\langle x_+(s,y)\rangle^{-\sigma} + \langle s\rangle^{-\sigma}\big)\big\{\langle y_-(s,x_+(s,y))\rangle^{-\sigma} + \langle y\rangle^{-\sigma}\big\}.$$

Put

$$h(s,y) = \big(\langle x_-(s,y)\rangle^{-\sigma} + \langle s\rangle^{-\sigma}\big)\big\{\langle y_+(s,x_-(s,y))\rangle^{-\sigma} + \langle y\rangle^{-\sigma}\big\} + \big(\langle x_+(s,y)\rangle^{-\sigma} + \langle s\rangle^{-\sigma}\big)\big\{\langle y_-(s,x_+(s,y))\rangle^{-\sigma} + \langle y\rangle^{-\sigma}\big\}.$$

Taking account of $x_\pm(s,y) = y \pm \int_0^s a(x_\pm(\rho,y))d\rho$ and $y_\pm(s,x_\mp(s,y)) = x_\mp(s,y) \mp \int_0^s a(x_\pm(\rho, y_\pm(s,x_\mp(s,y))))d\rho$, we can see easily that

$$\left|\int_0^\tau h(s,y)ds\right| \le C\langle y\rangle^{-\sigma}, \quad \tau,\ y \in R^1 \tag{2.16}$$

holds. Hence we obtain from (2.10)

$$\gamma_0 \le \gamma_0\delta \sup_{\tau,y\in R^1}\langle y\rangle^\sigma\left|\int_0^\tau h(s,y)ds\right| + |A_0|_0 + |B_0|_0.$$

This yields (2.14) for $i = 0$ and for $0 < \delta \le \delta_0$ together with (2.16) if $\delta_0 > 0$ is sufficiently small. Next we shall prove (2.14) for $i = 1$. Differentiating (2.10) with respect to y

$$\alpha_{cy}(\tau,y) = A_0'(y) + \int_0^\tau F_{cy}(s,y)ds, \quad \beta_{cy}(\tau,y) = B_0'(y) + \int_0^\tau G_{cy}(s,y)ds, \tag{2.17}$$

where $F_{cy}(s,y)$ and $G_{cy}(s,y)$ are given by

$$\begin{aligned} F_{cy}(s,y) = {} & \frac{1}{2}\left(a'(x_-(s,y)) + \frac{c'(s)}{c(s)}\right) \\ & \times \big\{\alpha_{cy}(s,y) + \beta_{cy}(s,y_+(s,x_-(s,y)))y_{+x}(s,x_-(s,y))x_{-y}(s,y)\big\} \\ & + \frac{1}{2}a''(x_-(s,y))x_{-y}(s,y)\big\{\alpha_c(s,y) - \beta_c(s,y_+(s,x_-(s,y)))\big\}, \\ G_{cy}(s,y) = {} & \frac{1}{2}\left(a'(x_+(s,y)) - \frac{c'(s)}{c(s)}\right) \\ & \times \big\{\alpha_{cy}(s,y_-(s,x_+(s,y)))y_{-y}(s,x_+(s,y))x_{+y}(s,y) - \beta_{cy}(s,y)\big\} \\ & + \frac{1}{2}a''(x_+(s,y))x_{+y}(s,y)\big\{\alpha_c(s,y_-(s,x_+(s,y))) - \beta_c(s,y)\big\}. \end{aligned}$$

Taking account that $y_{\pm x}$ and $x_{\pm y}$ are bounded in R^2 we see from the assumption

(1.4) that it holds

$$|F_{cy}(s,y)| \le \delta\big(\langle x_-(s,y)\rangle^{-\sigma} + \langle s\rangle^{-\sigma}\big)\gamma_1\big\{\langle y_+(s,x_-(s,y))\rangle^{-\sigma} + \langle y\rangle^{-\sigma}\big\} \\ + C\gamma_0\langle x_-(s,y)\rangle^{-\sigma}\big\{\langle y\rangle^{-\sigma} + \langle y_+(s,x_-(s,y))\rangle^{-\sigma}\big\}$$

and

$$|G_{cy}(s,y)| \le \delta\big(\langle x_+(s,y)\rangle^{-\sigma} + \langle s\rangle^{-\sigma}\big)\gamma_1\big\{\langle y_-(s,x_+(s,y))\rangle^{-\sigma} + \langle y\rangle^{-\sigma}\big\} \\ + C\gamma_0\langle x_+(s,y)\rangle^{-\sigma}\big\{\langle y\rangle^{-\sigma} + \langle y_-(s,x_+(s,y))\rangle^{-\sigma}\big\}.$$

Therefore we get from(2.17) by use of (2.16)

$$\gamma_1 \le C(\delta\gamma_1 + \gamma_0)\sup_{y\in R^1}\langle y\rangle^{\sigma}\left|\int_0^{\tau} h(s,y)ds\right| + C(|A_0|_1 + |B_0|_1),$$

which implies (2.14) for $i = 1$ together with the fact $\gamma_0 \le C(|A_0|_0 + |B_0|_0)$, if δ is small. Next we shall prove that (2.15) holds. Put

$$\rho_1 = \sup_{\tau,y\in R^1,k\le 1}\langle y\rangle^{\sigma}\big(\big|\partial_y^k(\alpha_{c_1}(\tau,y) - \alpha_{c_2}(\tau,y))\big| + \big|\partial_y^k(\beta_{c_1}(\tau,y) - \beta_{c_2}(\tau,y))\big|\big).$$

Then α_{c_j}, β_{c_j}, $j = 1,2$ satisfy for $k = 0,1$

$$\begin{aligned} \partial_y^k(\alpha_{c_1} - \alpha_{c_2}) &= \int_0^{\tau} \partial_y^k(F_{c_1} - F_{c_2})(s,y)ds, \\ \partial_y^k(\beta_{c_1} - \beta_{c_2}) &= \int_0^{\tau} \partial_y^k(G_{c_1} - G_{c_2})(s,y)ds \end{aligned} \tag{2.18}$$

where

$$\partial_y^k(F_{c_1} - F_{c_2})(s,y) = \frac{1}{2}\left\{\frac{c_1'}{c_1} - \frac{c_2'}{c_2}\right\}\partial_y^k\{\alpha_{c_1}(s,y) - \beta_{c_1}(s,y_+(s,x_-(s.,y)))\} \\ + \frac{1}{2}\partial_y^k\left[\left(a'(x_-(s,y)) + \frac{c_2'}{c_2}\right)\right. \\ \left.\times\big\{(\alpha_{c_1} - \alpha_{c_2})(s,y) - (\beta_{c_1} - \beta_{c_2})(s,y_+(s,x_-(s,y)))\big\}\right]$$

and

$$\partial_y^k(G_{c_1}-G_{c_2})(s,y)=\frac{-1}{2}\left\{\frac{c_1'}{c_1}-\frac{c_2'}{c_2}\right\}\partial_y^k\{\alpha_{c_1}(s,y_-(s,x_+(s,y)))-\beta_{c_1}(s,y)\}$$
$$+\frac{1}{2}\partial_y^k\Big[\Big(a'(x_+(s,y))-\frac{c_2'}{c_2}\Big)$$
$$\times\big\{(\alpha_{c_1}-\alpha_{c_2})(s,y_-(s,x_+(s,y)))-(\beta_{c_1}-\beta_{c_2})(s,y)\big\}\Big]$$

hold. Since we can estimate for $k=0,1$

$$\big|\partial_y^k(F_{c_1}-F_{c_2})(s,y)\big|\le(\delta+M)|c_1-c_2|_X\gamma_1\langle s\rangle^{-\sigma}\big\{\langle y_+(s,x_-(s,y))\rangle^{-\sigma}+\langle y\rangle^{-\sigma}\big\}$$
$$+\delta\rho_1\big(\langle x_-(s,y)\rangle^{-\sigma}+\langle s\rangle^{-\sigma}\big)\big\{\langle y_+(s,x_-(s,y))\rangle^{-\sigma}+\langle y\rangle^{-\sigma}\big\}$$

and

$$\big|\partial_y^k(G_{c_1}-G_{c_2})(s,y)\big|\le(\delta+M)|c_1-c_2|_X\gamma_1\langle s\rangle^{-\sigma}\big\{\langle y_-(s,x_+(s,y))\rangle^{-\sigma}+\langle y\rangle^{-\sigma}\big\}$$
$$+\delta\rho_1\big(\langle x_+(s,y)\rangle^{-\sigma}+\langle s\rangle^{-\sigma}\big)\big\{\langle y_-(s,x_+(s,y))\rangle^{-\sigma}+\langle y\rangle^{-\sigma}\big\}$$

we obtain from (2.18) and (2.16)

$$\rho_1\le(\delta+M)|c_1-c_2|_X\sup_{\tau,y\in R^1}(1+|y|)^{-\sigma}\left(\gamma_1\left|\int_0^\tau h(s,y)ds\right|+\delta\rho_1\left|\int_0^\tau h(s,y)ds\right|\right).$$

Therefore we obtain (2.15) analogously to the case of (2.14). □

3. Nonlinear equation.

In this section we shall prove Theorem 2.1 and Theorem 1.1. We can show that $\Phi(c)$ belongs to $X_{\sigma,\delta,M}$ for $c\in X_{\sigma,\delta,M}$. In fact we can see that for $c\in X_{\sigma,\delta,M}$, $1\le\Phi(c)^2\le 1+\varepsilon(\|A\|_{L^2}^2+\|B\|_{L^2}^2)/2\le 1+\varepsilon C(|A_0|_0^2+|B_0|_0^2)$ holds from (2.14). Hence if we take $M>0$, $\varepsilon>0$ suitablly, then we see $\Phi(c)\le M$. Besides $\Phi(c)(0)=1+(\varepsilon/4c_0^2)\|A_c(0)-B_c(0)\|^2=1+\varepsilon\|af'\|^2=c_0^2$. Here $\|\cdot\|$ stands for a norm of $L^2(R^1)$ and $(,)$ an inner product of $L^2(R^1)$. Next we shall prove that $|\Phi(c)'(\tau)|\le\delta\langle\tau\rangle^{-\sigma}$, $\tau\in R^1$. Differentiating $\Phi(c)^2$ with respect to τ,

$$2\Phi(c)\Phi(c)'(\tau)=\frac{-\varepsilon c'}{2c^3}\|A_c-B_c\|^2+\frac{\varepsilon}{2c^2}\Re(A_{c\tau}-B_{c\tau},A_c-B_c).\tag{3.1}$$

It follows from (2.14) that

$$\frac{\varepsilon|c'|}{2c^3}\|A_c - B_c\|^2 \le \varepsilon\delta\langle\tau\rangle^{-\sigma}C(|A_0|_0 + |B_0|_0)^2. \tag{3.2}$$

On the other hand, taking account that

$$\Re(aA_{cx}, A_c) = -\frac{1}{2}(a'A_c, A_c), \quad \Re(aB_{cx}, B_c) = -\frac{1}{2}(a'B_c, B_c)$$

are valid, we can see

$$\begin{aligned}\Re(A_{c\tau} - B_{c\tau}, A_c - B_c) &= \Re(aA_{cx} + F_c + aB_{cx} - G_c, A_c - B_c)\\ &= -\Re(aA_{cx}, B_c) + \Re(aB_{cx}, A_c) + \Re(F_c - G_c, A_c - B_c)\\ &\quad - \frac{1}{2}(a'A_c, A_c) - \frac{1}{2}(a'B_c, B_c).\end{aligned} \tag{3.3}$$

The assumption (1.4) and Proposition 2.1 imply

$$\begin{aligned}|(a'A_c, A_c)| &\le \int |a'(x)|\,|A_c(\tau,x)|^2 dx \le C|A_0|_0^2 \int \langle x\rangle^{-\sigma_0}\langle y_-(\tau,x)\rangle^{-2\sigma} dx\\ &\le C|A_0|_0^2\langle\tau\rangle^{-\sigma},\\ |(a'B_c, B_c)| &\le \int |a'(x)|\,|B_c(\tau,x)|^2 dx \le C|B_0|_0^2 \int \langle x\rangle^{-\sigma_0}\langle y_+(\tau,x)\rangle^{-2\sigma} dx\\ &\le C|B_0|_0^2\langle\tau\rangle^{-\sigma},\\ |(aA_{cx}, B_c)| &\le C|A_0|_1|B_0|_0 \int \langle y_-(\tau,x)\rangle^{-\sigma}\langle y_+(\tau,x)\rangle^{-\sigma} dx\\ &\le C|A_0|_1|B_0|_0\langle\tau\rangle^{-\sigma},\\ |(aB_{cx}, A_c)| &\le C|A_0|_0|B_0|_1 \int \langle y_-(\tau,x)\rangle^{-\sigma}\langle y_+(\tau,x)\rangle^{-\sigma} dx\\ &\le C|A_0|_0|B_0|_1\langle\tau\rangle^{-\sigma},\end{aligned}$$

and moreover

$$\begin{aligned}&|\Re(F_c, A_c - B_c)|\\ &\quad\le C\big(|A_0|_0^2 + |B_0|_0^2\big)\int (\langle x\rangle^{-\sigma} + \langle\tau\rangle^{-\sigma})\big(\langle y_-(\tau,x)\rangle^{-2\sigma} + \langle y_+(\tau,x)\rangle^{-2\sigma}\big) dx\\ &\quad\le C\big(|A_0|_0^2 + |B_0|_0^2\big)\langle\tau\rangle^{-\sigma}\end{aligned}$$

and analogously

$$|\Re(G_c, A_c - B_c)| \leq C\bigl(|A_0|_0^2 + |B_0|_0^2\bigr)\langle\tau\rangle^{-\sigma}.$$

Therefore we get

$$\frac{\varepsilon}{2c^2}|\Re(A_{c\tau} - B_{c\tau}, A_c - B_c)| \leq C\varepsilon\langle\tau\rangle^{-\sigma}$$

and consequently from (3.1)

$$|\Phi'(c)(\tau)| \leq C\varepsilon\bigl(|A_0|_0^2 + |B_0|_0^2\bigr)\langle\tau\rangle^{-\sigma} \leq \delta\langle\tau\rangle^{-\sigma}, \tag{3.4}$$

if $\varepsilon > 0$ is chosen suitably. Finally we shall prove (2.7). Let c_1, c_2 be in $X_{\sigma,\delta,M}$. We begin to prove

$$|\Phi(c_1)(\tau) - \Phi(c_2)(\tau)| \leq C\varepsilon|c_1 - c_2|_X, \quad \tau \in R^1. \tag{3.5}$$

The definition (2.5) of Φ gives

$$\begin{aligned}&\Phi(c_1)^2(\tau) - \Phi(c_2)^2(\tau)\\&\quad = \frac{\varepsilon}{4}\biggl\{\biggl(\frac{1}{c_1^2} - \frac{1}{c_2^2}\biggr)\|A_{c_1} - B_{c_1}\|^2 + \frac{\varepsilon}{4c_2^2}\bigl(\|A_{c_1} - B_{c_1}\|^2 - \|A_{c_2} - B_{c_2}\|^2\bigr)\biggr\}.\end{aligned}$$

Therefore noting that

$$\biggl|\frac{1}{c_1^2} - \frac{1}{c_2^2}\biggr| \leq 2|c_1 - c_2|_X$$

and

$$\begin{aligned}&\bigl|\|A_{c_1} - B_{c_1}\|^2 - \|A_{c_2} - B_{c_2}\|^2\bigr|\\&\quad \leq \bigl(\|A_{c_1} - A_{c_2}\| + \|B_{c_1} - B_{c_2}\|\bigr)\bigl(\|A_{c_1}\| + \|B_{c_1}\| + \|A_{c_2}\| + \|B_{c_2}\|\bigr)\end{aligned}$$

we can get (3.5) by use of Proposition 2.1. Next we shall prove

$$|\Phi(c_1)'(\tau) - \Phi(c_2)'(\tau)| \leq C\varepsilon|c_1 - c_2|_X\langle\tau\rangle^{-\sigma}, \quad \tau \in R^1, \tag{3.6}$$

for $c_1, c_2 \in X_{\sigma,\delta,M}$. It follows from (3.1)

$$
\begin{aligned}
&2\Phi(c_1)\Phi(c_1)'(\tau) - 2\Phi(c_2)\Phi(c_2)'(\tau) \\
&\quad = -\varepsilon\left(\frac{c_1'}{2c_1^3} - \frac{c_2'}{2c_2^3}\right)\|A_{c_1} - B_{c_1}\|^2 + \frac{\varepsilon c_2'}{2c_2^3}\big(\|A_{c_1} - B_{c_1}\|^2 - \|A_{c_2} - B_{c_2}\|^2\big) \\
&\qquad + \varepsilon\left(\frac{1}{2c_1^2} - \frac{1}{2c_2^2}\right)\big(\Re(A_{c_1\tau} - B_{c_1\tau}, A_{c_1} - B_{c_1})\big) \\
&\qquad + \frac{\varepsilon}{2c_2^2}\big(\Re(A_{c_1\tau} - B_{c_1\tau}, A_{c_1} - B_{c_1}) - \Re(A_{c_2\tau} - B_{c_2\tau}, A_{c_2} - B_{c_2})\big). \qquad (3.7)
\end{aligned}
$$

Besides, it follows from (3.3)

$$
\begin{aligned}
&\Re(A_{c_1\tau} - B_{c_1\tau}, A_{c_1} - B_{c_1}) - \Re(A_{c_2\tau} - B_{c_2\tau}, A_{c_2} - B_{c_2}) \\
&\quad = -\Re\big(a(A_{c_1x} - A_{c_2x}), B_{c_1}\big) - \Re\big(a(B_{c_1x} - B_{c_2x}), A_{c_1}\big) \\
&\qquad - \Re(aA_{c_2x}, B_{c_1} - B_{c_2}) - \Re(aB_{c_2x}, A_{c_1} - A_{c_2}) \\
&\qquad - \frac{1}{2}\big(\Re(a'A_{c_1}, A_{c_1}) - \Re(a'A_{c_2}, A_{c_2})\big) - \frac{1}{2}\big(\Re(a'B_{c_1}, B_{c_1}) - \Re(a'B_{c_2}, B_{c_2})\big) \\
&\qquad + \Re(F_{c_1} - F_{c_2} - G_{c_1} + G_{c_2}, A_{c_1} - B_{c_1}) \\
&\qquad + \Re\big(F_{c_2} - G_{c_2}, A_{c_1} - A_{c_2} - (B_{c_1} - B_{c_2})\big).
\end{aligned}
$$

Since

$$
\begin{aligned}
&F_{c_1} - F_{c_2} - G_{c_1} + G_{c_2} \\
&\quad = \left(\frac{c_1'}{c_1^2} - \frac{c_2'}{c_2^2}\right)(A_{c_1} - B_{c_1}) + \frac{c_2'}{c_2^2}(A_{c_1} - A_{c_2} - B_{c_1} + B_{c_2})
\end{aligned}
$$

holds, it follows from Proposition 2.1 that we can show

$$
\begin{aligned}
&\big|\Re(A_{c_1\tau} - B_{c_1\tau}, A_{c_1} - B_{c_1}) - \Re(A_{c_2\tau} - B_{c_2\tau}, A_{c_2} - B_{c_2})\big| \\
&\quad \le C(|A_0|_1 + |B_0|_1)|c_1 - c_2|_X.
\end{aligned}
$$

Moreover we can show using again Proposition 2.1

$$
\left|\frac{c_1'}{2c_1^3} - \frac{c_2'}{2c_2^3}\right|\|A_{c_1\tau} - B_{c_1\tau}\|^2 \le C(|A_0|_1 + |B_0|_1)^2|c_1 - c_2|_X\langle\tau\rangle^{-\sigma},
$$

$$
\left|\frac{c_2'}{2c_2^3}\big(\|A_{c_1} - B_{c_1}\|^2 - \|A_{c_2} - B_{c_2}\|^2\big)\right| \le C(|A_0|_0 + |B_0|_0)|c_1 - c_2|_X\langle\tau\rangle^{-\sigma}.
$$

Therefore, taking account of the equality

$$\Phi'(c_1) - \Phi'(c_2) = \frac{(\Phi'(c_1)\Phi(c_1) - \Phi'(c_2)\Phi(c_2))}{\Phi(c_1)} + \frac{\Phi'(c_2)(\Phi(c_1) - \Phi(c_2))}{\Phi(c_2)}$$

we can obtain (3.6) from (3.5) and (3.7). Thus we have completed the proof of Theorem 2.1.

Proof of Theorem 1.1. Theorem 2.1 assures the existence of solutions A, B, c of the equations (2.1)–(2.2) and (2.3). Put $P = (A+B)/2$ and $Q = (A-B)/2ac$. Then we can find u such that $u_t = P$ and $u_x = Q$, since (P, Q) is complete, that is, $P_x = Q_t$. In deed, we see

$$P_x = \frac{A_x + B_x}{2} = \frac{A_t - F - B_t + G}{2ac} = \frac{(A-B)_t - \frac{c'}{c}(A-B)}{2ac} = Q_t.$$

Put

$$u(t,x) = f(x) + \int_0^t \frac{(A+B)(s,x)}{2} ds$$

which solves (1.1) uniquely in $C^0([0,\infty); L^2(R^1))$.

4. Scattering for Kirchhoff equations and perturbed linear equations.

In this section we shall show the existence of wave operators among Kirchhoff equation (1.1) and the following linear equations

$$u^{\pm}_{tt}(t,x) = c^2_{\infty}(a(x)^2 u^{\pm}_x(t,x))_x, \quad u^{\pm}(0,x) = f^{\pm}(x), \quad u^{\pm}_t(0,x) = g^{\pm}(x),$$
$$\pm t \geq 0,\ x \in R^1. \tag{4.1}$$

Theorem 4.1. *Assume that $a(x)$ satisfies (1.3)–(1.4) and the initial data $f^- \in C^2(R^1) \cap L^2(R^1)$ and $g^- \in C^1(R^1)$ satisfy (1.5). Moreover assume $\sigma = \min\{\sigma_0, \sigma_1\} > 1$. Then there are $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that if $0 < \delta \leq \delta_0$ and $0 < \varepsilon \leq \varepsilon_0$ are valid, there are $u \in C^2(R^2)$ a solution of (1.1), $c_\infty > 0$ and $(f^+, g^+) \in C^2(R^1) \cap L^2(R^1) \times C^1(R^1)$ satisfying (1.5) such that*

$$\left\|u_t(t) - u^{\pm}_t(c^{-1}_{\infty} S(t))\right\| + \left\|u_x(t) - u^{\pm}_x(c^{-1}_{\infty} S(t))\right\| = O(|t|^{-\sigma+1}), \quad t \to \pm\infty \tag{4.2}$$

and

$$\left(1+\varepsilon\|a(\cdot)u_x(t)\|^2\right)^{\frac{1}{2}} - c_\infty = O(|t|^{-\sigma+1}), \quad t \to \pm\infty \tag{4.3}$$

where $u^{\pm}(t,x) \in C^2(R^2)$ *are solutions of* (4.1) *and* $S(t) = \int_0^t (1+\varepsilon\|a(\cdot)u_x(s)\|^2)^{\frac{1}{2}} ds$.

Proof. We let $A_1(t,x) = u_t + a(x)c(t)u_x$ and $B_1(t,x) = u_t - a(x)c(t)u_x$, where $c(t)^2 = 1 + \varepsilon\|a(\cdot)u_x(t)\|^2$ and $A_1^-(t,x) = u_t^- + a(x)c_\infty u_x^-$ and $B_1^-(t,x) = u_t^- - a(x)c_\infty u_x^-$. Then the equation (1.1) yields

$$\begin{aligned} A_{1t} - a(x)c(t)A_{1x} &= \frac{1}{2}\left(c(t)a'(x) + \frac{c'(t)}{c(t)}\right)(A_1 - B_1), \\ B_{1t} + a(x)c(t)B_{1x} &= \frac{1}{2}\left(c(t)a'(x) - \frac{c'(t)}{c(t)}\right)(A_1 - B_1) \end{aligned} \tag{4.4}$$

and the equation (4.1) gives

$$\begin{aligned} A_{1t}^- - a(x)c_\infty A_{1x}^- &= \frac{1}{2}c_\infty a'(x)(A_1^- - B_1^-), \\ B_{1t}^- + a(x)c_\infty B_{1x}^- &= \frac{1}{2}c_\infty a'(x)(A_1^- - B_1^-). \end{aligned} \tag{4.5}$$

The initial data is given by

$$\begin{aligned} A_1^-(0,x) &= A_0^-(x); = g^-(x) + a(x)c_\infty (f^-)'(x), \\ B_1^-(0,x) &= B_0^-(x); = g^-(x) - a(x)c_\infty (f^-)'(x). \end{aligned} \tag{4.6}$$

Let $T(\tau)$ be the inverse function of $\tau = S(t)$. Put $A(\tau,x) = A_1(T(\tau),x)$, $B(\tau,x) = B_1(T(\tau),x)$, $A^-(\tau,x) = A_1^-(c_\infty^{-1}\tau,x)$, $B^-(\tau,x) = B_1^-(c_\infty^{-1}\tau,x)$ and $\gamma(\tau) = c(T(\tau))$. Then (4.4) and (4.5) yield

$$\begin{aligned} A_\tau - a(x)A_x &= \frac{1}{2}\left(a'(x) + \frac{\gamma'(\tau)}{\gamma(\tau)}\right)(A - B), \\ B_\tau + a(x)B_x &= \frac{1}{2}\left(a'(x) - \frac{\gamma'(\tau)}{\gamma(\tau)}\right)(A - B) \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} A^-_\tau - a(x)A^-_x &= \frac{1}{2}a'(x)(A^- - B^-), \\ B^-_\tau + a(x)B^-_x &= \frac{1}{2}a'(x)(A^- - B^-), \quad \tau,\ x \in R^1, \end{aligned} \tag{4.8}$$

respectively. Here we pose the condition below to solve (4.8)

$$\|A(\tau) - A^-(\tau)\| + \|B(\tau) - B^-(\tau)\| = O(|\tau|^{-\sigma+1}), \quad \tau \to -\infty, \tag{4.9}$$

which is equivalent to (4.2). γ is given by

$$\gamma(\tau)^2 = 1 + \frac{\varepsilon}{4\gamma(\tau)^2}\|A(\tau) - B(\tau)\|^2. \tag{4.10}$$

Then we note that (4.3) is equivalent to

$$\gamma(\tau)^2 - c_\infty^2 = 1 + \frac{\varepsilon}{4\gamma(\tau)^2}\|A(\tau) - B(\tau)\|^2 - c_\infty^2 = O(|\tau|^{-\sigma+1}), \quad \tau \to -\infty. \tag{4.11}$$

Denote by $x_\pm(\tau, y)$ the solutions of the ordinary equations of (2.8) and by $y_\pm(\tau, x)$ the inverse function of $x_\pm(\tau, y) = x$. If we put $\alpha(\tau, y) = A^-(\tau, x_-(\tau, y))$, $\beta(\tau, y) = B^-(\tau, x_+(\tau, y))$, then we can prove analogously to the proof of Proposition 2.1 that $\alpha(\tau, y)$ and $\beta(\tau, y)$ satisfy (2.14). Therefore we can see

Lemma 4.1. *Assume that a satisfies* (1.3) *and* (1.4) *and $A_0^-(x)$, $B_0^-(x)$ satisfies*

$$\left|A_0^{(i)}(x)\right| + \left|B_0^{(i)}(x)\right| \le C(1 + |x|)^{-\sigma_1}, \quad x \in R^1,\ i = 0, 1. \tag{4.12}$$

Then if $\sigma = \min\{\sigma_0, \sigma_1\} > 1$, the solution A^-, B^- satisfies

$$\begin{gathered} \left|\partial_y^i A^-(\tau, x)\right| \le C(1 + |y_-(\tau, x)|)^{-\sigma}, \quad \left|\partial_y^i B^-(\tau, x)\right| \le C(1 + |y_+(\tau, x)|)^{-\sigma}, \\ \tau \le 0,\ x \in R^1,\ i = 0, 1. \end{gathered} \tag{4.13}$$

We continue to prove Theorem 4.1. First of all we define c_∞ as a positive root of the following equation

$$c_\infty^2 = 1 + \frac{\varepsilon}{4c_\infty^2}\left(\|g^-\|^2 + c_\infty^2\|a(f^-)'\|^2\right), \tag{4.14}$$

which satisfies

$$c_\infty^2 = 1 + \frac{\varepsilon}{4c_\infty^2}\big(\|A_0^-\|^2 + \|B_0^-\|^2\big) \tag{4.15}$$

because of $\|A_0^-\|^2 + \|B_0^-\|^2 = (\|g^-\|^2 + c_\infty^2\|a(f^-)'\|^2)$. On the other hand, noting that Lemma 4.1 implies

$$|(A^-(\tau), B^-(\tau))| \leq C\int_{-\infty}^{\infty}(1+|y_+(\tau,x)|)^{-\sigma}(1+|y_-(\tau,x)|)^{-\sigma}dx \leq C(1+|\tau|)^{-\sigma},$$
$$\tau \leq 0$$

and taking account of the relation $\|A^-(\tau)\|^2 + \|B^-(\tau)\|^2 = \|A_0^-\|^2 + \|B_0^-\|^2$ we can estimate

$$\big|\|A^-(\tau) - B^-(\tau)\|^2 - \|A_0^-\|^2 - \|B_0^-\|^2\big| = 2|\Re(A^-(\tau), B^-(\tau))| \leq C(1+|\tau|)^{-\sigma},$$
$$\tau \leq 0.$$

Therefore if $(A(\tau), B(\tau))$, γ satisfies (4.7), (4.9) and (4.11) we get

$$\begin{aligned}
\big|\gamma(\tau)^2 - c_\infty^2\big| &= \left|\frac{\sqrt{1+\varepsilon\|A(\tau)-B(\tau)\|^2} - \sqrt{1+\varepsilon(\|A_0^-\|^2 + \|B_0^-\|^2)}}{2}\right| \\
&\leq \frac{\varepsilon}{2}\big\{\|A(\tau)-B(\tau)\|^2 - \|A_0^-\|^2 - \|B_0^-\|^2\big\} \\
&\leq \varepsilon\big|\|A(\tau)-B(\tau)\|^2 - \|A^-(\tau)-B^-(\tau)\|^2\big| + \varepsilon\big|\Re(A^-(\tau), B^-(\tau))\big| \\
&\leq C\varepsilon\big\{(1+|\tau|)^{-\sigma+1} + (1+|\tau|)^{-\sigma}\big\}, \quad \tau \leq 0
\end{aligned}$$

which implies (4.11).

Now we shall find the solution (A, B) and γ satisfying (4.7), (4.9) and (4.10) by the simillar way of the proof of Theorem 2.1. Let $\sigma > 0$, $\delta > 0$ and $M > 0$ and introduce

$$X_{\sigma,\delta,M} = \big\{\gamma(\tau) \in C^1((-\infty, 0]); 1 \leq \gamma(\tau) \leq M, |\gamma'(\tau)| \leq \delta(1+|\tau|)^{-\sigma}\big\}.$$

For $\gamma \in X_{\sigma,\delta,M}$ we consider the linear equation of (4.7) and (4.9). We change a unkown function (A, B) of (4.7) to (U, V) as $U = A - A^-$, $V = B - B^-$ which satisfies

$$U_\tau - a(x)U_x = \frac{1}{2}\left(a'(x) + \frac{\gamma'(\tau)}{\gamma(\tau)}\right)(U-V) + \frac{\gamma'(\tau)}{2\gamma(\tau)}W, \quad \tau \le 0,\ x \in R^1, \quad (4.16)$$

$$V_\tau + a(x)V_x = \frac{1}{2}\left(a'(x) - \frac{\gamma'(\tau)}{\gamma(\tau)}\right)(U-V) - \frac{\gamma'(\tau)}{2\gamma(\tau)}W, \quad \tau \le 0,\ x \in R^1, \quad (4.17)$$

where $W = A^- - B^-$. Moreover (4.9) gives

$$\|U(\tau)\| + \|V(\tau)\| \le C(1+|\tau|)^{-\sigma+1} \to 0, \quad \tau \to -\infty. \quad (4.18)$$

In stead of (A, B) we shall find (U, V) satisfying (4.16), (4.17) and (4.18). To do so, we need the following lemma in the argument below.

Lemma 4.2. *Let $\sigma = \min\{\sigma_0, \sigma_1\} > 1$. Then there is a positive function $\varphi_\mp(\tau, y)$ such that*

$$\begin{aligned}\int_{I_\mp(\tau)} &\big\{(1+|x_\mp(s,y)|)^{-\sigma_0} + (1+|s|)^{-\sigma_0}\big\} \\ &\times \big\{(1+|y_\pm(s, x_\mp(s,y))|)^{-\sigma_1} + (1+|y|)^{-\sigma_1}\big\}ds \\ &\le C\varphi_\mp(\tau,y)(1+|y|)^{-\sigma}, \end{aligned} \quad (4.19)$$

and

$$\int_{R^1} \varphi_\mp(\tau,y)^2(1+|y|)^{-2\sigma}dy \le C(1+|\tau|)^{-2(\sigma-1)}, \quad \mp\tau \ge 0, \quad (4.20)$$

where $I_-(\tau) = (-\infty, \tau)$, $I_+(\tau) = (\tau, \infty)$ and $\varphi_\mp(\tau, y)$ are bounded in R^2.

Proof. Put

$$\varphi_\mp(\tau,y) = \int_{I_\mp} \big\{(1+|x_\mp(s,y)|)^{-\sigma_0} + (1+|y_\mp(s,x_\pm(s,y))|)^{-\sigma_1}\big\}ds + (1+|\tau|)^{1-\sigma_0}.$$

We can see easily that $\varphi_\mp(\tau, y) \le C$. To show (4.19) it suffices to check

$$\begin{aligned}&(1+|x_\mp(s,y)|)^{-\sigma}(1+|y_\pm(s,x_\mp(s,y))|)^{-\sigma} \\ &\quad \le C(1+|y|)^{-\sigma}\big\{(1+|x_\mp(s,y)|)^{-\sigma_0} + (1+|y_\pm(s,x_\mp(s,y))|)^{-\sigma}\big\},\end{aligned}$$

which can be showed easily. Next we can show, for example

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\tau}\left(1+|x_-(s,y)|\right)^{-\sigma}ds\right)^2(1+|y|)^{-2\sigma}dy\le C(1+|\tau|)^{-2(\sigma-1)},\quad \tau\le 0. \tag{4.21}$$

In fact, in the case of $x_-(\tau,y)\ge 0$ we can see easily

$$\int_{-\infty}^{\tau}(1+|x_-(s,y)|)^{-\sigma}ds\le C(1+|x_-(\tau,y)|)^{-\sigma+1},\quad \tau\le 0.$$

Hence taking account of the inequality $(1+|x_-(\tau,y)|)\ge c_0(1+|\tau|)(1+|y|)^{-1}$ we get

$$\begin{aligned}\int_{x_-(\tau,y)\ge 0}&\left(\int_{-\infty}^{\tau}(1+|x_-(s,y)|)^{-\sigma}ds\right)^2(1+|y|)^{-2\sigma}dy\\ &\le(1+|\tau|)^{-2(\sigma-1)}\int_{x_-(\tau,y)\ge 0}(1+|y|)^{-2}dy\\ &\le C(1+|\tau|)^{-2(\sigma-1)},\quad \tau\le 0.\end{aligned}$$

In the case of $x_-(\tau,y)\le 0$, noting that $|y|\ge c_0|\tau|$ if $\tau\le 0$, we see

$$\begin{aligned}\int_{x_-(\tau,y)\le 0}(1+|y|)^{-2\sigma}dy&\le C\int_{|y|\ge c_0|\tau|}(1+|y|)^{-2\sigma}dy\\ &\le C(1+|\tau|)^{-2\sigma+1},\quad \tau\le 0.\end{aligned}$$

Thus we get (4.21). Besides we can estimate the other terms by the same way. □

Now we can prove the following proposition.

PROPOSITION 4.1. *Let $\sigma=\min\{\sigma_0,\sigma_1\}>1$ and γ be in $X_{\sigma,\delta,M}$. Assume that a satisfies* (1.3) *and* (1.4) *and that (A_0^-,B_0^-) satisfies* (4.12). *Then there is $\delta_0>0$ such that if $\delta_0\ge\delta>0$,* (4.16)–(4.18) *has a unique solution (U,V) satisfying*

$$\begin{aligned}\left|\partial_x^i U(\tau,x)\right|&\le C(|A_0^-|_i+|B_0^-|_i)(1+|y_-(\tau,x)|)^{-\sigma},\\ \left|\partial_x^i V(\tau,x)\right|&\le C(|A_0^-|_i+|B_0^-|_i)(1+|y_+(\tau,x)|)^{-\sigma},\end{aligned} \tag{4.22}$$

for $\tau\le 0$ and $i=0,1$, where we denote $|A|_i=\sup_{x\in R^1,k\le i}(1+|x|)^\sigma|\partial_x^k A(x)|$.

PROOF. Define $\alpha(\tau,y)=U(\tau,x_-(\tau,y))$, $\beta(\tau,y)=V(\tau,x_+(\tau,y))$ and put

$$e_i = \sup_{\tau \le 0, x \in R^1} (1+|y|)^\sigma \big(|\partial_y^i \alpha(\tau,y)| + |\partial_y^i \beta(\tau,y)|\big), \quad i=0,1.$$

Let (α,β) be the solution of the following integral equation

$$\alpha(\tau,y) = \int_{-\infty}^{\tau} \left\{ \frac{1}{2}\left(a'(x_-(s,y)) + \frac{\gamma'(s)}{\gamma(s)} \right) \big(\alpha(s,y) - \beta(s, y_+(s, x_-(s,y)))\big) + \frac{\gamma'(s)}{2\gamma(s)} W(s, x_-(s,y)) \right\} ds, \tag{4.23}$$

$$\beta(\tau,y) = \int_{-\infty}^{\tau} \left\{ \frac{1}{2}\left(a'(x_+(s,y)) - \frac{\gamma'(s)}{\gamma(s)} \right) \big(\alpha(s, y_-(s, x_+(s,y))) - \beta(s,y)\big) - \frac{\gamma'(s)}{2\gamma(s)} W(s, x_+(s,y)) \right\} ds \tag{4.24}$$

solves. Then $U(\tau,x) = \alpha(\tau, y_-(\tau,x))$ and $V(\tau,x) = \beta(\tau, y_+(\tau,x))$ solves (4.16)–(4.17). Taking account that $V(s, x_-(s,y)) = \beta(s, y_+(s, x_-(s,y)))$ and that (4.13) gives

$$\begin{aligned} \big|\partial_y^i W(s, x_-(s,y))\big| &= \big|\partial_y^i (B^- - A^-)(s, x_-(s,y))\big| \\ &\le \big(|A_0^-|_i + |B_0^-|_i\big)\big\{(1+|y_+(s,x_-(s,y))|)^{-\sigma} + (1+|y|)^{-\sigma}\big\}, \\ &\qquad i=0,1, \end{aligned} \tag{4.25}$$

we get from (4.23) by use of (4.19) with $-$,

$$\begin{aligned} \big|\partial_y^i \alpha(\tau,y)\big| \le \int_{-\infty}^{\tau} \Big[&\frac{\delta}{2}\big\{(1+|x_-(s,y)|)^{-\sigma_0} + (1+|s|)^{-\sigma}\big\} \\ &\times e_i \big\{(1+|y_+(s,x_-(s,y))|)^{-\sigma} + (1+|y|)^{-\sigma}\big\} \\ &+ \delta(1+|s|)^{-\sigma}(|A_0^-|_i + |B_0^-|_i) \\ &\times \big\{(1+|y_+(s,x_-(s,y))|)^{-\sigma} + (1+|y|)^{-\sigma}\big\}\Big] ds \\ \le C\big\{\delta e_i + (|A_0^-|_i &+ |B_0^-|_i)\big\}(1+|y|)^{-\sigma}, \quad i=0,1. \end{aligned}$$

Analogously

$$
\begin{aligned}
\left|\partial_y^i\beta(\tau,y)\right| \le \int_{-\infty}^{\tau} \Big[& \frac{\delta}{2}\big\{(1+|x_+(s,y)|)^{-\sigma_0}+(1+|s|)^{-\sigma}\big\} \\
& \times e_i\big\{(1+|y_-(s,x_+(s,y))|)^{-\sigma}+(1+|y|)^{-\sigma}\big\} \\
& + \delta(1+|s|)^{-\sigma}(|A_0^-|_i+|B_0^-|_i) \\
& \times \big\{(1+|y_-(s,x_+(s,y))|)^{-\sigma}+(1+|y|)^{-\sigma}\big\}\Big] ds \\
\le C(\delta e_i + |A_0^-|_i & + |B_0^-|_i)(1+|y|)^{-\sigma}, \quad i=0,1.
\end{aligned}
$$

Thus we get

$$e_i \le C\delta e_i + C(|A_0^-|_i+|B_0^-|_i),$$

which implies (4.22), if we take $C\delta < 1$. Next we prove (4.18) holds. In fact, we see from (4.23) and (4.24) by use of (4.22) and (4.20)

$$
\begin{aligned}
\|U(\tau)\|^2+\|V(\tau)\|^2 &\le C(\|\alpha(\tau)\|^2+\|\beta(\tau)\|^2) \\
&\le C\int_{-\infty}^{\infty}\varphi(\tau,y)^2(1+|y|)^{-2\sigma}dy \\
&\le C(1+|\tau|)^{-2(\sigma-1)} \to 0, \quad \tau\to-\infty,
\end{aligned}
$$

which implies (4.18).

Finally we shall show the existence of solutions of the integral equation (4.23)–(4.24). We seek a solution $(\alpha,\beta)(\tau,y)$ as

$$\alpha(\tau,y)=\sum_{n=0}^{\infty}\alpha_n(\tau,y), \quad \beta(\tau,y)=\sum_{n=0}^{\infty}\beta_n(\tau,y),$$

where

$$
\begin{aligned}
\alpha_0(\tau,y) &= \int_{-\infty}^{\tau}\frac{\gamma(s)}{2\gamma'(s)}(A^- - B^-)(s,x_-(s,y))ds, \\
\beta_0(\tau,y) &= -\int_{-\infty}^{\tau}\frac{\gamma(s)}{2\gamma'(s)}(A^- - B^-)(s,x_+(s,y))ds,
\end{aligned}
$$

and for $n \ge 1$

$$\alpha_n(\tau,y)=\int_{-\infty}^{\tau}\frac{1}{2}\biggl(a'(x_-(s,y))+\frac{\gamma'(s)}{\gamma(s)}\biggr)\bigl(\alpha_{n-1}(s,y)-\beta_{n-1}(s,y_+(s,x_-(s,y)))\bigr)ds$$

and

$$\beta_n(\tau,y)=\int_{-\infty}^{\tau}\frac{1}{2}\biggl(a'(x_+(s,y))-\frac{\gamma'(s)}{\gamma(s)}\biggr)\bigl(\alpha_{n-1}(s,y_-(s,x_+(s,y)))-\beta_{n-1}(s,y)\bigr)ds.$$

We can show easily by induction

$$|\alpha_n(\tau,y)|+|\beta_n(\tau,y)|\le C_1(|A_0^-|_0+|B_0^-|_0)(C_2\delta)^n(1+|y|)^{-\sigma},$$

for $n=0,1,\dots$. $U(\tau,x)=\alpha(\tau,y_-(\tau,x))$ and $V(\tau,x)=\beta(\tau,y_+(\tau,x))$ solves (4.16)–(4.18). Thus we have proved Proposition 4.1. □

The solution (U,V) of (4.16)–(4.18) depends on $\gamma\in X_{\sigma,\delta,M}$. So we denote it by (U_γ,V_γ).

Proposition 4.2. *Let $\sigma=\min\{\sigma_0,\sigma_1\}>1$ and γ_1,γ_2 be in $X_{\sigma,\delta,M}$. Assume that (A_0^-,B_0^-) satisfies* (4.12). *Then there is $\delta_0>0$ such that if $\delta_0\ge\delta>0$, $(U_{\gamma_1},V_{\gamma_1})$ and $(U_{\gamma_2},V_{\gamma_2})$ satisfy*

$$\begin{aligned}&\bigl\|\partial_x^i(U_{\gamma_1}(\tau,\cdot)-U_{\gamma_2}(\tau,\cdot))\bigr\|+\bigl\|\partial_x^i(V_{\gamma_1}(\tau,\cdot)-V_{\gamma_2}(\tau,\cdot))\bigr\|\\&\quad\le C(|A_0^-|_1+|B_0^-|_1)|\gamma_1-\gamma_2|_X,\quad i=0,1.\end{aligned}\tag{4.26}$$

Proof. Put

$$\alpha(\tau,y)=(U_{\gamma_1}-U_{\gamma_2})(\tau,x_-(\tau,y)),\quad \beta(\tau,x)=(V_{\gamma_1}-V_{\gamma_2})(\tau,x_+(\tau,y)).$$

Then (α,β) satisfies

$$\begin{aligned}\alpha(\tau,y)&=\int_{-\infty}^{\tau}(F_{\gamma_1}-F_{\gamma_2})(s,x_-(s,y))ds,\\ \beta(\tau,y)&=\int_{-\infty}^{\tau}(G_{\gamma_1}-G_{\gamma_2})(s,x_+(s,y))ds,\end{aligned}\tag{4.27}$$

where

$$F_\gamma(\tau,x)=\frac{1}{2}\biggl(a'(x)+\frac{\gamma'(\tau)}{\gamma(\tau)}\biggr)(U_\gamma-V_\gamma)(\tau,x)+\frac{\gamma'(\tau)}{2\gamma(\tau)}(A^--B^-)(\tau,x)\tag{4.28}$$

and

$$G_\gamma(\tau,x) = \frac{1}{2}\left(a'(x) - \frac{\gamma'(\tau)}{\gamma(\tau)}\right)(U_\gamma - V_\gamma)(\tau,x) - \frac{\gamma'(\tau)}{2\gamma(\tau)}(A^- - B^-)(\tau,x). \quad (4.29)$$

Hence we see

$$\begin{aligned}
&(F_{\gamma_1} - F_{\gamma_2})(s, x_-(s,y)) \\
&\quad = \frac{1}{2}\left(\frac{\gamma_1'(s)}{\gamma_1(s)} - \frac{\gamma_2'(s)}{\gamma_2(s)}\right)(U_{\gamma_1} - V_{\gamma_1} - W)(s, x_-(s,y)) \\
&\qquad + \frac{1}{2}\left(a'(x_-(s,y)) + \frac{\gamma_2'(s)}{\gamma_2(s)}\right)(U_{\gamma_1} - U_{\gamma_2} - V_{\gamma_1} + V_{\gamma_2})(s, x_-(s,y))
\end{aligned}$$

and

$$\begin{aligned}
&(G_{\gamma_1} - G_{\gamma_2})(s, x_+(s,y)) \\
&\quad = \frac{1}{2}\left(-\frac{\gamma_1'(s)}{\gamma_1(s)} + \frac{\gamma_2'(s)}{\gamma_2(s)}\right)(U_{\gamma_1} - V_{\gamma_1} - W)(s, x_+(s,y)) \\
&\qquad + \frac{1}{2}\left(a'(x_+(s,y)) - \frac{\gamma_2'(s)}{\gamma_2(s)}\right)(U_{\gamma_1} - U_{\gamma_2} - V_{\gamma_1} + V_{\gamma_2})(s, x_+(s,y)).
\end{aligned}$$

Define

$$e_i = \sup_{s\le 0, y\in R^1} (1+|y|)^\sigma \big(|\partial_x^i \alpha(s,y)| + |\partial_x^i \beta(s,y)|\big), \quad i = 0, 1.$$

Noting that $(V_{\gamma_1} - V_{\gamma_2})(s, x_-(s,y)) = \beta(s, y_+(s, x_-(s,y)))$ and $(U_{\gamma_1} - U_{\gamma_2})(s, x_+(s,y)) = \alpha(s, y_-(s, x_+(s,y)))$ and taking account of Lemma 4.1, Proposition 4.1 and (4.19), we get from (4.27)

$$\begin{aligned}
\big|\partial_y^i \alpha(\tau,y)\big| &\le \int_{-\infty}^{\tau} \big\{(1+|x_-(s,y)|)^{-\sigma_0} + (1+|s|)^{-\sigma}\big\} \\
&\qquad \times \big((1+|y_+(s,x_-(s,y))|)^{-\sigma} + (1+|y|)^{-\sigma}\big) ds \\
&\qquad \times \big(\delta e_i + (|A_0^-|_0 + |B_0^-|_0)|\gamma_1 - \gamma_2|_X\big) \\
&\le C\big\{\delta e_i + (|A_0^-|_i + |B_0^-|_i)|\gamma_1 - \gamma_2|_X\big\}(1+|y|)^{-\sigma}
\end{aligned}$$

and analogously

$$\left|\partial_y^i\beta(\tau,y)\right| \le C\big\{\delta e_i + (|A_0^-|_1 + |B_0^-|_1)|\gamma_1-\gamma_2|_X\big\}(1+|y|)^{-\sigma}$$

which imply that $e_i \le C|\gamma_1-\gamma_2|_X$ if δ is sufficiently small, that is, we get $|\partial_y^i\alpha(\tau,y)| + |\partial_y^i\beta(\tau,y)| \le C(|A_0^-|_1+|B_0^-|_1)|\gamma_1-\gamma_2|_X(1+|y|)^{-\sigma}$, $i=0,1$ which yields (4.26). □

We continue to prove Theorem 4.1. For $\gamma \in X_{\sigma,\delta,M}$ we define

$$\Phi(\gamma)(\tau)^2 = 1 + \frac{\varepsilon}{4\gamma(\tau)^2}\|U_\gamma(\tau) - V_\gamma(\tau) + W(\tau)\|^2,$$

where (U_γ, V_γ) denotes the solution of (4.16)–(4.18) and $W(\tau,x) = (A^- - B^-)(\tau,x)$. We shall prove that $\Phi(\gamma)$ is in $X_{\sigma,\delta,M}$ by the similar way as that of the proof of Theorem 2.1. It is trivial that $1 \le \Phi(\gamma)(\tau)^2 \le 1 + C(M)\varepsilon \le M^2$, if ε is small, because U_γ, V_γ, and W are bounded in $L^2(R^1)$ from Proposition 4.1. Next we shall prove that $|\Phi(\gamma)'(\tau)| \le \delta(1+|\tau|)^{-\sigma}$. Differentiating $\Phi^2(\gamma)(\tau)$ with respect to τ

$$\begin{aligned}2\Phi(\gamma)(\tau)\Phi(\gamma)'(\tau) &= \frac{-\varepsilon\gamma'(\tau)}{2\gamma(\tau)^3}\|U_\gamma(\tau) - V_\gamma(\tau) + W(\tau)\|^2\\ &\quad + \frac{\varepsilon}{2\gamma(\tau)^2}\Re\big((U_\gamma(\tau) - V_\gamma(\tau) + W(\tau))_\tau,\, U_\gamma(\tau) - V_\gamma(\tau) + W(\tau)\big).\end{aligned}$$

It follows from (4.16), (4.17)

$$\begin{aligned}&\Re\big((U_\gamma(\tau) - V_\gamma(\tau) + W(\tau))_\tau,\, U_\gamma(\tau) - V_\gamma(\tau) + W(\tau)\big)\\ &\quad = \Re\big(a(x)(U_\gamma + V_\gamma)_x(\tau) + W(\tau)_\tau + F_\gamma - G_\gamma,\, U_\gamma(\tau) - V_\gamma(\tau) + W(\tau)\big)\\ &\quad = \frac{1}{2}\big\{\Re(a'(x)U_\gamma(\tau), U_\gamma(\tau)) - \Re(a'(x)V_\gamma(\tau), V_\gamma(\tau))\big\}\\ &\qquad - \Re(a(x)U_{\gamma x}(\tau), V_\gamma(\tau)) + \Re(a(x)V_{\gamma x}(\tau), U_\gamma(\tau))\\ &\qquad + \Re\big(W(\tau)_\tau + F_\gamma - G_\gamma,\, U_\gamma(\tau) - V_\gamma(\tau) + W(\tau)\big)\end{aligned}$$

where F_γ, G_γ is given by (4.28), (4.29). Using Proposition 4.1 and 4.2 we can estimate from (1.4)

$$\begin{aligned}&|F_\gamma(\tau,x)| + |G_\gamma(\tau,x)|\\ &\quad \le C\big\{(1+|x|)^{-\sigma} + (1+|\tau|)^{-\sigma}\big\}\big\{(1+|y_-(\tau,x)|)^{-\sigma} + (1+|y_+(\tau,x)|)^{-\sigma}\big\}.\end{aligned}$$

Therefore we can show $|\Phi(\gamma)'(\tau)| \le \delta(1+|\tau|)^{-\sigma}$ analogously to (3.4), if we take $\varepsilon > 0$ small. Moreover we can show similarly to (4.3)–(4.5) by use of Proposition 4.1 and Proposition 4.2,

$$|\Phi(\gamma_1) - \Phi(\gamma_2)|_X \le C\varepsilon|\gamma_1 - \gamma_2|_X \tag{4.30}$$

for any γ_1, $\gamma_2 \in X_{\sigma,\delta,M}$, which implies that Φ is a contraction mapping in $X_{\sigma,\delta,M}$, if ε is small. Denote by $\gamma(\tau) \in X_{\sigma,\delta,M}$ the fixed point Φ and by $(U_\gamma, V_\gamma)(\tau, x)$ the solution of (4.16)–(4.18).

Define $T(\tau) = \int_0^\tau \gamma(s)^{-1} ds$ and denote by $S(t)$ the inverse function of $t = T(\tau)$. Put $c(t) = \gamma(S(t))$. Then we get the relation $S(t) = \int_0^t c(s)ds$. Moreover $A(\tau, x) = U_\gamma + A^-(\tau, x)$ and $B(\tau, x) = V_\gamma(\tau, x) + B^-(\tau, x)$ solve (4.7) and (4.9). Therefore $A_1(t, x) = A(S(t), x)$, $B_1(t, x) = B(S(t), x)$ solves (4.4) and (4.9) implies

$$\left\|A_1(t) - A_1^-(c_\infty^{-1}S(t))\right\| + \left\|B_1(t) - B_1^-(c_\infty^{-1}S(t))\right\| = O(|t|^{-\sigma+1}) \to 0, \quad t \to -\infty. \tag{4.31}$$

We define

$$u(t,x) = \int_{-\infty}^t \frac{A_1(s,x) + B_1(s,x)}{2} ds, \quad t \le 0 \tag{4.32}$$

which solves (1.1) for $t \le 0$ and satisfies (4.2) and (4.3) for $t \le 0$ from (4.9) and (4.11) respectively. Moreover we can extend $u(t, x)$ to $t > 0$ by use of Theorem 1.1 as a solution of (1.1) for $t \ge 0$, because $(u(0, x), u_t(0, x))$ satisfies the decay condition (1.5) from Lemma 4.1 and Proposition 4.1.

Next we shall prove that there is $(f^+, g^+) \in C^2(R^1) \times C^1(R^1)$, that is, $u^+(t, x)$ a solution of (4.1) and (4.2). Let $A_1 = u_t + acu_x$, $B_1 = u_t - acu_x$, $A_1^+ = u_t^+ + ac_\infty u_x^+$ and $B_1^+ = u_t^+ - ac_\infty u_x^+$ as above and also define $A(\tau, x) = A_1(T(\tau), x)$, $B(\tau, x) = B_1(T(\tau), x)$, $A^+(\tau, x) = A_1^+(c_\infty^{-1}\tau, x)$, $B^+(\tau, x) = B_1^+(c_\infty^{-1}\tau, x)$, $U = A^+ - A$, and $V = B^+ - B$. Then (U, V) satisfies like (4.16) and (4.17)

$$U_\tau - a(x)U_x = \frac{1}{2}\left(a'(x) + \frac{\gamma'(\tau)}{\gamma(\tau)}\right)(U - V) - \frac{\gamma'(\tau)}{\gamma(\tau)}W, \quad \tau \ge 0,\ x \in R^1, \tag{4.33}$$

$$V_\tau + a(x)V_x = \frac{1}{2}\left(a'(x) - \frac{\gamma'(\tau)}{\gamma(\tau)}\right)(U - V) + \frac{\gamma'(\tau)}{\gamma(\tau)}W, \quad \tau \ge 0,\ x \in R^1, \tag{4.34}$$

where $W = A - B$. Moreover (4.2) is equivalent to

$$\|U(\tau)\| + \|V(\tau)\| \le c(1 + |\tau|)^{-\sigma+1} \to 0, \quad \tau \to \infty. \tag{4.35}$$

Set $U(\tau,x)=\alpha(\tau,y_-(\tau,x))$ and $V(\tau,x_+(\tau,y))=\beta(\tau,y_+(\tau,x))$, where (α,β) satisfies the following integral equation

$$\begin{aligned}\alpha(\tau,y)=-\int_\tau^\infty\Big\{&\frac{1}{2}\Big(a'(x_-(s,y))+\frac{\gamma'(s)}{\gamma(s)}\Big)\big(\alpha(s,y)-\beta(s,y_+(s,x_-(s,y)))\big)\\&+\frac{\gamma'(s)}{2\gamma(s)}W(s,x_-(s,y))\Big\}ds,\end{aligned}\tag{4.36}$$

$$\begin{aligned}\beta(\tau,y)=-\int_\tau^\infty\Big\{&\frac{1}{2}\Big(a'(x_+(s,y))-\frac{\gamma'(s)}{\gamma(s)}\Big)\big(\alpha(s,y_-(s,x_+(s,y)))-\beta(s,y)\big)\\&-\frac{\gamma'(s)}{2\gamma(s)}W(s,x_+(s,y))\Big\}ds.\end{aligned}\tag{4.37}$$

Since $W=A-B$ satisfies the estimate (4.25) from (4.22), we can find similarly to the argument in proof of Proposition 4.1 (α,β) satisfying (4.36) and (4.37) and consequently we get (U,V) the solution of (4.33)–(4.34) satisfying (4.35). Then

$$u^+(t,x)=-\int_t^\infty\frac{A_1^+(s,x)+B_1^+(s,x)}{2}ds$$

solves (4.1) and moreover we can prove similarly that u and u^+ satisfy (4.2) and (4.3) for $t\geq 0$. Thus we finished the proof of Theorem 4.1.

5. Wave operators among linear perturbed equations and free equations.

In this section we shall prove the existence of wave operators among the following linear equation

$$w_{tt}-c_\infty^2(a(x)^2w_x)_x=0,\quad t,x\in R^1,\tag{5.1}$$

and the free equation (1.8). Let $u_0^-(t,x)$ a solution of (1.8) with $-$ and assume (f_0^-,g_0^-) satisfies (1.5). Then we shall show that there are $w(t,x)\in C^2(R^2)\cap C^0((-\infty,\infty);L^2(R^1))$ a solution of (5.1) and $u_0^+(t,x)\in C^2(R^2)\cap C^0([0,\infty);L^2(R^1))$ satisfying (1.8) such that

$$\|w_t(t)-u_{0t}^\pm(t)\|+\|w_x(t)-u_{0x}^\pm(t)\|=O(|t|^{-\sigma+1}),\quad \pm t\to\infty.\tag{5.2}$$

Let $A^-=w_t+c_\infty a(x)w_x$, $B^-=w_t-c_\infty a(x)w_x$ be a solution of the following equations

$$\begin{aligned} A_t^- - c_\infty a(x) A_x^- &= \frac{1}{2} a'(x)(A^- - B^-), \\ B_t^- + c_\infty a(x) B_x^- &= \frac{1}{2} a'(x)(A^- - B^-) \end{aligned} \tag{5.3}$$

for $t \leq 0$ and $A_0^-(t,x) = u_{0t}^- + c_\infty a_\infty u_{0x}^-$ and $B_0^-(t,x) = u_{0t}^- - c_\infty a_\infty u_{0x}^-$ which satisfy the following equations,

$$\begin{aligned} A_{0t}^- - a_\infty c_\infty A_{0x}^- = 0, &\quad A_0^-(0,x) = \big(g_0^- + a_\infty c_\infty f_0^{-'}\big)(x), \\ B_{0t}^- + a_\infty c_\infty B_{0x}^- = 0, &\quad B_0^-(0,x) = \big(g_0^- - a_\infty c_\infty f_0^{-'}\big)(x). \end{aligned}$$

Put $U = A^- - A_0^-$, $V = B^- - B_0^-$. Then (5.2) is equivalent to

$$\|U(t)\| + \|V(t)\| = O(|t|^{-\sigma+1}), \quad t \to -\infty, \tag{5.4}$$

and (U,V) solves

$$U_t - a(x) c_\infty U_x = \frac{1}{2} a'(x) c_\infty (U - V + A_0^- - B_0^-) + c_\infty (a(x) - a_\infty) A_{0x}^-, \qquad \tau \leq 0,\ x \in R^1, \tag{5.5}$$

$$V_t + a(x) c_\infty V_x = \frac{1}{2} a'(x) c_\infty (U - V + A_0^- - B_0^-) + c_\infty (a(x) - a_\infty) B_{0x}^-, \qquad \tau \leq 0,\ x \in R^1. \tag{5.6}$$

Put $\alpha(t,y) = U(t, x_{-,\infty}(t,y))$ and $\beta(t,y) = V(t, x_{+,\infty}(t,y))$, where $x_{\pm,\infty}$ is a solution of $\frac{dx}{dt} = \pm c_\infty a(x)$, $x(0) = y$. (5.4)–(5.6) yields

$$\begin{aligned} \alpha(t,y) = \int_{-\infty}^{t} \Big\{ &\frac{1}{2} a'(x_{-,\infty}(s,y)) c_\infty \big(\alpha(s,y) - \beta(s, y_+(s, x_{-,\infty}(s,y))) \\ &+ A_0^-(s, x_{-,\infty}(s,y)) - B_0^-(s, x_{-,\infty}(s,y))\big) \\ &+ c_\infty (a(x_{-,\infty}(s,y)) - a_\infty) A_0^-(s, x_{-,\infty}(s,y)) \Big\} ds, \end{aligned}$$

$$\beta(t,y) = \int_{-\infty}^{t} \Big\{ \frac{1}{2} a'(x_{+,\infty}(s,y)) c_\infty \big(\alpha(s, y_-(s, x_{+,\infty}(s,y))) - \beta(s,y) + A_0^-(s, x_{+,\infty}(s,y)) - B_0^-(s, x_{+,\infty}(s,y))\big) + c_\infty (a(x_{+,\infty}(s,y)) - a_\infty) B_0^-(s, x_{+,\infty}(s,y)) \Big\} ds.$$

Denote $e_0 = \sup_{s \le 0, y \in R^1} (1+|y|)^\sigma (|\alpha(s,y)| + |\beta(s,y)|)$. Taking account that it follows from the assumptions (1.4) and $\lim_{x \to \pm\infty} a(x) = a_\infty$ that we have $|a(x) - a_\infty| \le C(1+|x|)^{-\sigma_0+1}$ for $x \in R^1$, we see

$$|\alpha(t,y)| \le C \int_{-\infty}^{t} \big[\delta (1+|x_{-,\infty}(s,y)|)^{-\sigma_0} \big\{ e_0 (1+|y_+(s, x_{-,\infty}(s,y))|)^{-\sigma} + (1+|y|)^{-\sigma} + C_0 (1+|x_{-,\infty}(s,y) - a_\infty s|)^{-\sigma} + (1+|x_{-,\infty}(s,y) + a_\infty s|)^{-\sigma} \big\} + C_0 (1+|x_{-,\infty}(s,y)|)^{-\sigma_0+1} (1+|x_{-,\infty}(s,y) - a_\infty s|)^{-\sigma} \big] ds, \tag{5.7}$$

where $C_0 = C(|A_0^-|_0 + |B_0^-|_0)$. Put

$$\tilde{\varphi}_-(t,y) = \int_{-\infty}^{t} \big\{ (1+|x_{-,\infty}(s,y)|)^{-\sigma_0+1} + (1+|x_{-,\infty}(s,y) - a_\infty s|)^{-\sigma} + (1+|y_+(s, x_{-,\infty}(s,y))|)^{-\sigma} \big\} ds + (1+|t|)^{-\sigma+1}.$$

Noting that it holds analogously to Lemma 4.2

$$\begin{aligned} &\int_{-\infty}^{t} \big[(1+|x_{-,\infty}(s,y)|)^{-\sigma_0} \big\{ (1+|y_+(s, x_{-,\infty}(s,y))|)^{-\sigma} + (1+|y|)^{-\sigma} \\ &\quad + (1+|x_{-,\infty}(s,y) - a_\infty s|)^{-\sigma} + (1+|x_{-,\infty}(s,y) + a_\infty s|)^{-\sigma} \big\} \\ &\quad + (1+|x_{-,\infty}(s,y)|)^{-\sigma_0+1} (1+|x_{-,\infty}(s,y) - a_\infty s|)^{-\sigma} \big] ds \\ &\le C \tilde{\varphi}_-(t,y) (1+|y|)^{-\sigma} \le C(1+|y|)^{-\sigma}, \end{aligned} \tag{5.8}$$

because of $\sigma_0 - 1 \ge \sigma > 1$ and $|x_{-,\infty}(s,y) + a_\infty s| \ge c_0 |y| - c_1$ for $s \le 0$, we get

$$|\alpha(t,y)| \le C(\delta e_0 + C_0)(1+|y|)^{-\sigma}, \quad t \le 0.$$

Similarly we have

$$|\beta(t,y)| \leq C(\delta e_0 + C_0)(1+|y|)^{-\sigma}, \quad t \leq 0.$$

Therefore we obtain $e_0 \leq C$ for $t \leq 0$, if $\delta > 0$ is small. Moreover using again (5.7) and (5.8) we can show

$$\int_{-\infty}^{\infty} |\alpha(t,y)|^2 dy \leq C \int_{-\infty}^{\infty} \tilde{\varphi}_-(t,y)^2 (1+|y|)^{-2\sigma} dy \leq c(1+|t|)^{-2(\sigma-1)}, \\ t \to -\infty$$

and β has also the same property as α. Thus we showed (5.4) and therefore we obtain $w(t,x) = \int_{-\infty}^{t} \frac{1}{2}(A^- + B^-)(s,x)ds \in C^2((-\infty,0] \times R^1) \cap C^0((-\infty,0]; L^2(R^1))$ satisfying (5.1) for $t \leq 0$. Now we can define the wave operator $W_-(f^-, g^-) = (w(0), w_t(0))$. We can define W_+ analogously. Moreover it follows from Theorem 1.1 that we can extend w to $[0,\infty)$ as a solution of (5.2), because $(w(0), w_t(0))$ satisfies the decay condition (1.5). Thus we obtain $w \in C^2(R^2) \cap C^0((-\infty,\infty); L^2(R^1))$ satisfying (5.1).

Next conversely we shall prove the existence of the inverse of the wave operator W_+. Let $w(t,x) \in C^2(R^2) \cap C^0((-\infty,\infty); L^2(R^1))$ a solution of (5.1) such that $(w(0,x), w_t(0,x))$ satisfies the decay condition (1.5). Then we shall show that there is $u_0^+(t,x)$ a solution of (1.8) satisfying (5.2) instead of initial data. Let $A_0^+ = u_{0t}^+ + c_\infty a_\infty u_{0x}^+$, $B^+ = u_{0t}^+ - c_\infty a_\infty u_{0x}^+$ which satisfies

$$A_{0t}^+ - a_\infty c_\infty A_{0x}^+ = 0, \quad B_{0t}^+ + a_\infty c_\infty B_{0x}^+ = 0 \tag{5.9}$$

and denote by $(A^+(t,x),\, B^+(t,x))$ a solution of the following equation,

$$A_t^+ - a(x)c_\infty A_x^+ = \frac{1}{2}c_\infty a'(x)(A^+ - B^+), \quad A^+(0,x) = (g^+ + a_\infty c_\infty f^{+'})(x),$$

$$B_t^+ + a(x)c_\infty B_x^+ = \frac{1}{2}c_\infty a'(x)(A^+ - B^+), \quad B^+(0,x) = (g^+ - a_\infty c_\infty f^{+'})(x).$$

Put $U = A_0^+ - A^+, V = B_0^+ - B^-$. We can show the existence of $(A_0^+(t,x), B_0^+(t,x))$ satisfying

$$\left\|A^+ - A_0^+\right\| + \left\|B^+ - B_0^+\right\| = \|U(t)\| + \|V(t)\| = O(|t|^{-\sigma+1}), \quad t \to \infty, \tag{5.10}$$

if (f^+, g^+) satisfies (1.5). In fact, (U,V) solves

$$U_t - a_\infty c_\infty U_x = -\frac{1}{2}a'(x)c_\infty(U - V + A^+ - B^+) + c_\infty(a(x) - a_\infty)A_x^+,$$
$$t \geq 0,\ x \in R^1, \quad (5.11)$$

$$V_t + a_\infty c_\infty V_x = -\frac{1}{2}a'(x)c_\infty(U - V + A^+ - B^+) + c_\infty(a(x) - a_\infty)B_x^+,$$
$$t \geq 0,\ x \in R^1. \quad (5.12)$$

Taking account that $|a(x) - a_\infty| \leq C(1 + |x|)^{-\sigma_0+1}$, $|\partial_x^i A^+(t, x)| \leq C(1 + |y_-(t, x)|)^{-\sigma}$ and $|\partial_x^i B^+(t, x)| \leq C(1 + |y_+(t, x)|)^{-\sigma}$ hold for $i = 0, 1$, we can show the existence of (U, V) satisfying (5.10), (5.11) and (5.12) analogously to the above argument and consequently we have $A_0^+ = U + A^+$, $B_0^+ = V + B_0^+$ the solution of (5.9)–(5.10). We define $u_0^+(t, x) = -\int_t^\infty 1/2(A_0^+ + B_0^+)(s, x)ds$ which is in $C^2([0, \infty) \times R^1) \cap C^0([0, \infty]; L^2(R^1))$ satisfying (1.8)–(5.2) with $+$. Therefore we can define the inverse $W_+^{-1}(f^+, g^+) = (u_0^+(0), u_{0t}^+(0))$. Thus we have proved the following theorem.

THEOREM 5.1. *Assume that a satisfies* (1.3), (1.4) *and* $\lim_{x\to\pm\infty} a(x) = a_\infty$. *Moreover suppose that* (f_0^-, g_0^-) *satisfies* (1.5) *and* $\sigma = \min\{\sigma_0 - 1, \sigma_1\} > 1$ *is valid. Let* $u_0^- \in C^2((-\infty, 0] \times R^1) \cap C^0((-\infty, 0]; L^2(R^1))$ *the solution of* (1.8) *with* $-$. *Then there are* $w \in C^2(R^2) \cap C^0((-\infty, \infty); L^2(R^1))$ *a solution of* (5.1) *and* $u_0^+ \in C^2([0, \infty) \times R^1) \cap C^0([0, \infty); L^2(R^1))$ *a solution of* (1.8) *with* $+$ *satisfying* (5.2).

PROOF OF THEOREM 1.2. Theorem 4.1 and Theorem 5.1 imply Theorem 1.2 directly.

References

[1] P. D'Ancona and S. Spagnolo, A nonlinear hyperbolic problems with global solutions, Arch. Ration. Mech. Anal., **124** (1993), 201–219.

[2] M. Ghisi, Asymptotic behavior for the Kirchhoff equation, Ann. Math. Pure Appl., **171** (1996), 293–312.

[3] J. M. Greenberg and S. H. Hu, The initial-value problem for a streached string, Quart. Appl. Math., **5** (1980), 289–311.

[4] T. Matsuyama, Asymptotic profiles for Kirchhoff equation, Rend. Lincei Mat. Appl., **17** (2006), 377–395.

[5] W. Rzymowski, One-dimensional Kirchhoff equation, Nonlinear Anal., **48** (2002), 209–221.

[6] T. Yamazaki, Scattering for a quasilinear hyperbolic equation of Kirchhoff type, J. Differ. Equ., **143** (1998), 1–59.

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