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Asymptotic properties of solutions to dispersive equation of Schrödinger type

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Abstract. We find the asymptotic behavior for large time of solutions to the dispersive equations of Schrödinger type

$$u_t - \frac{i}{\rho}|\partial_x|^\rho u = 0, \quad (t,x) \in \boldsymbol{R} \times \boldsymbol{R},$$

where $\rho \geq 2$. We obtain some estimates of solutions of linear problem and apply them to nonlinear problems with power nonlinearities of order $p \geq 3$. The nonexistence of wave operator and existence of the modified wave operator for the critical nonlinearity $i\lambda|u|^2 u$ are studied.

1. Introduction.

We study the large time asymptotic behavior of solutions to the Cauchy problem for the following dispersive equation of the Schrödinger type

$$\begin{cases} u_t - \frac{i}{\rho}|\partial_x|^\rho u = 0, & (t,x) \in \boldsymbol{R} \times \boldsymbol{R}, \\ u(t_0,x) = u_0(x), & x \in \boldsymbol{R}, \end{cases} \tag{1.1}$$

where $\rho \geq 2$, $|\partial_x|^\rho = \mathscr{F}^{-1}|\xi|^\rho \mathscr{F}$, $\mathscr{F}\phi$ or $\hat{\phi}$ is the Fourier transform of ϕ defined by $\mathscr{F}\phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\boldsymbol{R}} e^{-ix\xi}\phi(x)dx$ and the inverse Fourier transformation $\mathscr{F}^{-1}$ is given by $\mathscr{F}^{-1}\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{\boldsymbol{R}} e^{ix\xi}\phi(\xi)d\xi$. Then we apply the estimates of solutions to problem (1.1) for studying the nonlinear final value problem

$$\begin{cases} u_t - \frac{i}{\rho}|\partial_x|^\rho u = \mathscr{N}(u,\overline{u}), & (t,x) \in \boldsymbol{R} \times \boldsymbol{R}, \\ \lim_{t\to\infty} U(-t)u(t) = u_+, & x \in \boldsymbol{R}, \end{cases} \tag{1.2}$$

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with a given final state u_+, where $U(t) = \mathscr{F}^{-1} \exp\left(\frac{i}{\rho} t|\xi|^\rho\right)\mathscr{F}$.

When $\rho = 2$, equations (1.1) and (1.2) are the linear Schrödinger equation and nonlinear Schrödinger equation respectively. Recently many works were devoted to the study of large time asymptotic behavior of solutions of these equations. In particular, in [**9**] it was proved the existence of the modified wave operator for (1.2) with $\rho = 2$, $\mathscr{N}(u, \overline{u}) = i\lambda|u|^2 u$, $\lambda \in \boldsymbol{R}$ under the condition that the final state $u_+ \in \boldsymbol{H}^{0,2}$ and the norm $\|\widehat{u}_+\|_{\boldsymbol{L}^\infty}$ is sufficiently small, where we denote the Lebesgue space $\boldsymbol{L}^p = \{\phi; \|\phi\|_{\boldsymbol{L}^p} < \infty\}$, with the norm $\|\phi\|_{\boldsymbol{L}^p} = (\int_{\boldsymbol{R}} |\phi(x)|^p dx)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_{\boldsymbol{L}^\infty} = \operatorname{ess.sup}_{x\in\boldsymbol{R}} |\phi(x)|$ if $p = \infty$. The weighted Sobolev space $\boldsymbol{H}_p^{m,s}$ is defined by $\boldsymbol{H}_p^{m,s} = \{\phi \in \boldsymbol{L}^p; \|\langle x\rangle^s \langle i\partial_x\rangle^m \phi\|_{\boldsymbol{L}^p} < \infty\}$, $m, s \in \boldsymbol{R}$ with $\langle x\rangle = \sqrt{1+|x|^2}$. The index 0 usually we omit if it does not cause a confusion. The homogeneous Sobolev space is defined by $\dot{\boldsymbol{H}}^s = \{\phi \in \boldsymbol{L}^2; \||\partial_x|^s \phi\|_{\boldsymbol{L}^2} < \infty\}$. In [**10**] by applying the method of paper [**9**], the modified wave operator for (1.2) with $\rho = 4$ in the case of the critical nonlinearity $\mathscr{N}(u, \overline{u}) = i\lambda|u|^2 u$, $\lambda \in \boldsymbol{R}$ was constructed under the condition that

$$\|\widehat{u_+}\|_{\boldsymbol{H}^{4,0}} + \sum_{k=0}^{4} \left\||\xi|^{k-12} \partial_\xi^k \widehat{u_+}\right\|_{\boldsymbol{L}^2}$$

is sufficiently small. They found some special asymptotic representation u_w for solutions and have checked that the function

$$R = \mathscr{L}_4 u_w - i\lambda|u_w|^2 u_w$$

is a remainder term in $\boldsymbol{L}^2$, where $\mathscr{L}_\rho = i\partial_t - \frac{1}{\rho}(-\partial_x^2)^{\frac{\rho}{2}}$ denotes a linear part of equation (1.2). Computation of $\mathscr{L}_4 u_w$ implies at least fourth differentiability $\|\widehat{u_+}\|_{\boldsymbol{H}^{4,0}} < \infty$. This method can be applied to higher order ρ if we assume that $\|\widehat{u_+}\|_{\boldsymbol{H}^{\rho,0}} < \infty$. However a longer computation is needed to obtain that R is remainder. By combining the method of papers [**9**] and [**12**], it was proved in [**11**] that

$$R = \mathscr{L}_4 u_W - i\lambda|u_W|^2 u_W$$

is a remainder term, when $\operatorname{Im}\lambda > 0$ under the condition that

$$\|\widehat{u_+}\|_{\boldsymbol{H}^{4,0}} + \sum_{k=0}^{4} \left\||\xi|^{-12+k} \partial_\xi^k \widehat{u_+}\right\|_{\boldsymbol{L}^2} < \infty.$$

The smallness condition on the data is removed due to the dissipation condition on the nonlinearity such that $\operatorname{Im}\lambda > 0$. Our results in the present paper improve those of the previous works [**9**], [**10**], [**11**] and [**12**]. Our method is also applicable for the case of non integer order ρ.

To construct the modified scattering operator it is important to study the Cauchy problem for nonlinear dispersive equation of Schrödinger type with critical nonlinearity $\mathscr{N}(u,\overline{u}) = i\lambda|u|^2u$, $\lambda \in \boldsymbol{R}$

$$\begin{cases} u_t - \dfrac{i}{\rho}|\partial_x|^\rho u = i\lambda|u|^2u, & (t,x) \in \boldsymbol{R}\times\boldsymbol{R}, \\ u(0,x) = u_0, & x \in \boldsymbol{R}. \end{cases} \tag{1.3}$$

In [**5**], the global existence and asymptotic behavior in time of small solutions to (1.3) with $\rho = 2$ was shown in $\boldsymbol{L}^2 \cap \boldsymbol{L}^\infty$ sense if the initial data $u_0 \in \boldsymbol{H}^\gamma \cap \boldsymbol{H}^{0,\gamma}$ with $\gamma > \frac{1}{2}$ have a sufficiently small norm $\|u_0\|_{\boldsymbol{H}^\gamma} + \|u_0\|_{\boldsymbol{H}^{0,\gamma}}$. This result yields the existence of the inverse modified wave operator. As far as we know the construction of the modified scattering operator is an open problem for higher order $\rho > 2$. Finally we note that some estimates for the linear dispersive equations similar to (1.1) were shown in [**7**] in the case of Benjamin-Ono type equation $u_t - \frac{1}{2}\partial_x|\partial_x|u = 0$ and in [**6**], [**8**] in the case of Kortweg-de Vries type equation $u_t - \frac{1}{3}\partial_x^3 u = 0$. Furthermore in [**7**], there were shown the global existence and large time asymptotic behavior of small solutions in $\boldsymbol{L}^2\cap\boldsymbol{L}^\infty$ sense for the Cauchy problem to the Benjamin-Ono type equation

$$\begin{cases} u_t - \dfrac{1}{2}\partial_x|\partial_x|u = \partial_x u^3, & (t,x) \in \boldsymbol{R}\times\boldsymbol{R}, \\ u(0,x) = u_0, & x \in \boldsymbol{R} \end{cases}$$

under the condition that the initial data $u_0 \in \boldsymbol{H}^{2,1} \cap \boldsymbol{H}^{1,2}$ are real valued and have a sufficiently small norm $\|u_0\|_{\boldsymbol{H}^{2,1}} + \|u_0\|_{\boldsymbol{H}^{1,2}}$. However the existence of the modified wave operator is an open problem due to the derivative of the unknown function u_x in the nonlinear term.

Applying the sharp large time asymptotics of solutions to nonlinear problems we can show the nonexistence of usual wave operator. Therefore it is impossible to apply our method to the case of subcritical nonlinearity such that $|u|^{p-1}u$ with $1 < p < 3$. We note here that in the case of the usual nonlinear Schrödinger equations with a gauge invariant nonlinearity $i\lambda|u|^{p-1}u$, $1 < p < 3$, $\lambda < 0$, the nonexistence of the usual wave operator was shown in [**1**] by making use of a sharp $\boldsymbol{L}^q$-time decay estimates of solutions for $q > 2$ which was shown in [**4**] through the pseudo-conformal conservation law.

We denote by $\boldsymbol{C}(\boldsymbol{I};\boldsymbol{H})$ the space of continuous functions from a time interval $\boldsymbol{I}$ to a Banach space $\boldsymbol{H}$. Different positive constants might be denoted by the same letter C.

The rest of this paper is organized as follows. In Section 2 we obtain large time asymptotics and some $\boldsymbol{L}^\infty$-$\boldsymbol{L}^1$ estimates for the free evolution group of the Schrödinger type $U(t)=\mathscr{F}^{-1}\exp\left(\frac{i}{\rho}t|\xi|^\rho\right)\mathscr{F}$. Section 3 is devoted to the study of the final problem (1.2) in the supercritical case. Theorem 3.1 deals with the case of sufficiently general nonlinearity $\mathscr{N}(u,\overline{u})$, however its order is far from the critical value. In the case of a gauge invariant nonlinearity $i\lambda|u|^{p-1}u$, $\lambda\in\boldsymbol{R}$ we treat all powers $3<p<1+2\rho$ in Theorem 3.2. In Section 4 we consider the modified final problem for dispersive equation of Schrödinger type with a critical nonlinearity $i\lambda|u|^2u$, where $\operatorname{Im}\lambda\geq 0$. Finally in Section 5 we prove the nonexistence of the usual wave operator in the case of critical nonlinearity $i\lambda|u|^2u$, $\lambda\in\boldsymbol{R}$.

2. Estimates for the free evolution group.

In this section we study the large time asymptotic behavior for the free evolution group of the Schrödinger type $U(t)=\mathscr{F}^{-1}\exp\left(\frac{i}{\rho}t|\xi|^\rho\right)\mathscr{F}$, where $\rho\geq 2$.

Denote the norm

$$\|v\|_{\boldsymbol{Z}^\alpha}\equiv\left\|\{\xi\}^{-\alpha}v(\xi)\right\|_{\boldsymbol{L}^\infty}+\left\|\{\xi\}^{1-\alpha}v'(\xi)\right\|_{\boldsymbol{L}^\infty},\quad \{\xi\}=\frac{|\xi|}{\langle\xi\rangle}.$$

Theorem 2.1. *The estimate is true*

$$\|U(t)v\|_{\boldsymbol{L}^\infty}\leq Ct^{-\frac{1+\alpha}{\rho}}\left\||\partial_x|^{-\alpha}v\right\|_{\boldsymbol{L}^1}\tag{2.1}$$

for all $t>0$ provided that the right-hand side is finite, where $\alpha\in\left[0,\frac{\rho}{2}-1\right]$, $\rho\geq 2$. Furthermore the asymptotic formula for large time t holds

$$U(t)v=A(t,\chi)\hat{v}(\chi)+R(t,x),\tag{2.2}$$

where

$$A(t,\chi)=t^{-\frac{1}{2}}\sqrt{\frac{1}{i(\rho-1)}}|\chi|^{1-\frac{\rho}{2}}e^{-i(1-\frac{1}{\rho})|\chi|^\rho t}$$

and $\chi=-\frac{x}{|x|}\left(\frac{|x|}{t}\right)^{\frac{1}{\rho-1}}$. The reminder $R(t,x)$ satisfy the estimates

$$\|R(t)\|_{\boldsymbol{L}^\infty}\leq Ct^{-\frac{\alpha}{\rho}-\frac{1}{\rho}}\|\widehat{v}\|_{\boldsymbol{Z}^\alpha}$$

for $0 < \alpha < \rho - 1$ *and*

$$\|R(t)\|_{\boldsymbol{L}^2} \le Ct^{-\frac{\alpha}{\rho} - \frac{1}{2\rho}} \|\widehat{v}\|_{\boldsymbol{Z}^\alpha}$$

for $0 < \alpha < \frac{\rho-1}{2}$.

Remark 2.1. Note that the first estimate (2.1) was shown in [**2**] for $\rho = 4$, in [**6**] for $\rho = 3$ and in [**8**] for $\rho > 3$. We need estimate (2.1) for the proof of the Strichartz type estimates (see Lemma 2.2 below). Note that the $\boldsymbol{L}^2$ norm of the remainder $R(t,x)$ decay faster than $t^{\frac{1}{2\rho} - \frac{1}{2}}$ under the conditions $\alpha > \frac{\rho}{2} - 1$, $\rho \ge 2$.

Proof. We write

$$U(t)v = (2\pi)^{-1} t^{-\frac{1+\alpha}{\rho}} \int_{\boldsymbol{R}} G_\alpha\Big((x-y)t^{-\frac{1}{\rho}}\Big) |\partial_y|^{-\alpha} v(y) dy$$

with a kernel

$$G_\alpha(\eta) = \int_{\boldsymbol{R}} |\xi|^\alpha e^{i\xi\eta + \frac{i}{\rho}|\xi|^\rho} d\xi.$$

To estimate the $\boldsymbol{L}^\infty$-norm of $G_\alpha(\eta)$ we define

$$G_\alpha(\eta) = G_-(\eta) + G_+(\eta) \equiv \int_0^\infty \xi^\alpha e^{-i\xi\eta + \frac{i}{\rho}\xi^\rho} d\xi + \int_0^\infty \xi^\alpha e^{i\xi\eta + \frac{i}{\rho}\xi^\rho} d\xi.$$

We only consider $\eta > 0$ since the case of $\eta \le 0$ can be treated similarly. We denote $\mu = \eta^{\frac{1}{\rho-1}}$, therefore we have for all $\eta \ge 1$

$$\begin{aligned} G_-(\eta) = &\int_0^{2\mu} \frac{\xi^\alpha}{1 - i(\xi-\mu)(\mu^{\rho-1} - \xi^{\rho-1})} \partial_\xi \Big((\xi-\mu) e^{-i\xi\eta + \frac{i}{\rho}\xi^\rho}\Big) d\xi \\ &+ \int_{2\mu}^\infty \frac{i\xi^\alpha}{\mu^{\rho-1} - \xi^{\rho-1}} \partial_\xi e^{-i\xi\eta + \frac{i}{\rho}\xi^\rho} d\xi. \end{aligned}$$

Hence the integration by parts yields

$$\begin{aligned} |G_-(\eta)| &\le C\mu^\alpha \int_0^{2\mu} \frac{d\xi}{1 + (\xi-\mu)^2 \mu^{\rho-2}} + C\int_{2\mu}^\infty \xi^{\alpha-\rho} d\xi + C\mu^{\alpha+1-\rho} \\ &\le C\mu^{\alpha+1-\frac{\rho}{2}} \int_{\boldsymbol{R}} \langle y \rangle^{-2} dy + C\mu^{\alpha+1-\rho} \le C \end{aligned}$$

for all $\eta \geq 1$ if $\rho \geq 2$ and $0 \leq \alpha \leq \frac{\rho}{2} - 1$. In the same manner we obtain $|G_{+}(\eta)| \leq C$ for all $\eta \geq 1$. Therefore we get $\|G_{\alpha}\|_{\boldsymbol{L}^{\infty}} \leq C$. Now the application of the Young inequality implies estimate (2.1).

We now show the asymptotic formula (2.2). We only consider the case $x > 0$ and write the identity

$$\begin{aligned} U(t)v &\equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi\eta + \frac{i}{\rho}|\xi|^{\rho}} \hat{v}\Big(\xi t^{-\frac{1}{\rho}}\Big) d\xi \\ &= \frac{1}{\sqrt{2\pi} t^{\frac{1}{\rho}}} \bigg(\hat{v}(\chi) \int_{0}^{\infty} e^{-i\xi\eta + \frac{i}{\rho}\xi^{\rho}} d\xi + \int_{0}^{\infty} e^{i\xi\eta + \frac{i}{\rho}\xi^{\rho}} \hat{v}\Big(\xi t^{-\frac{1}{\rho}}\Big) d\xi \\ &\qquad + \int_{0}^{\infty} e^{-i\xi\eta + \frac{i}{\rho}\xi^{\rho}} \Big(\hat{v}\Big(-\xi t^{-\frac{1}{\rho}}\Big) - \hat{v}(\chi)\Big) d\xi \bigg), \end{aligned} \tag{2.3}$$

where $\eta = xt^{-\frac{1}{\rho}}$, $\chi = -\mu t^{-\frac{1}{\rho}} = -\frac{x}{|x|}(\frac{|x|}{t})^{\frac{1}{\rho-1}}$, and $\mu = \eta^{\frac{1}{\rho-1}} > 0$.

Consider the asymptotic behavior with respect to $\mu \to \infty$ for the first summand in the right-hand side of (2.3)

$$\int_{0}^{\infty} e^{-i\xi\mu^{\rho-1} + \frac{i}{\rho}\xi^{\rho}} d\xi = e^{-i(1-\frac{1}{\rho})\mu^{\rho}} \int_{0}^{\infty} e^{iS(\xi,\mu)} d\xi$$

where $S(\xi,\mu) = \frac{1}{\rho}(\xi^{\rho} - \mu^{\rho} - \rho\mu^{\rho-1}(\xi - \mu))$. We can define a new variable

$$z(\xi,\mu) = \mu + \mu^{1-\frac{\rho}{2}} \sqrt{\frac{2}{\rho - 1} S(\xi,\mu)} \operatorname{sign}(\xi - \mu).$$

Note that $z_{\xi}(\mu,\mu) = 1$ and (see [**3**])

$$\int_{z(0,\mu)}^{\infty} e^{i\frac{\rho-1}{2}\mu^{\rho-2}(z-\mu)^{2}} dz = \sqrt{\frac{2\pi}{i(\rho-1)}} \mu^{1-\frac{\rho}{2}} + O(\mu^{1-\rho})$$

for $\mu \to \infty$, where $z(0,\mu) = \mu(1 - \sqrt{\frac{2}{\rho}})$. Then applying the identity

$$e^{iS(\xi,\mu)} = \frac{1}{1 + i(\xi - \mu)(\xi^{\rho-1} - \mu^{\rho-1})} \partial_{\xi}\big((\xi - \mu)e^{iS(\xi,\mu)}\big)$$

we integrate by parts in the second summand

$$\int_0^\infty e^{iS(\xi,\mu)}(1 - z_\xi(\xi,\mu))d\xi$$
$$= \int_0^\infty (\xi-\mu)e^{iS(\xi,\mu)}\partial_\xi\left(\frac{1 - z_\xi(\xi,\mu)}{1 + i(\xi-\mu)(\xi^{\rho-1} - \mu^{\rho-1})}\right)d\xi + O(\mu^{1-\rho}).$$

Hence

$$\left|\int_0^\infty e^{iS(\xi,\mu)}(1 - z_\xi(\xi,\mu))d\xi\right|$$
$$\le C\mu^{-1}\int_0^{2\mu}\frac{|\xi-\mu|d\xi}{1 + (\xi-\mu)^2\mu^{\rho-2}} + C\mu^{1-\frac{\rho}{2}}\int_{2\mu}^\infty \xi^{-1-\frac{\rho}{2}}d\xi + O(\mu^{1-\rho})$$
$$= O(\mu^{1-\rho}\ln\mu).$$

Therefore

$$\frac{1}{\sqrt{2\pi}t^{\frac{1}{\rho}}}\hat{v}(\chi)\int_0^\infty e^{-i\xi\eta + \frac{i}{\rho}\xi^\rho}d\xi = A(t,\chi)\hat{v}(\chi) + O\Big(t^{-\frac{1}{\rho}}\hat{v}(\chi)\mu^{1-\rho}\ln\mu\Big),$$

where the remainder term $O\big(t^{-\frac{1}{\rho}}\hat{v}(\chi)\mu^{1-\rho}\ln\mu\big)$ satisfies the estimates of the theorem.

Now we consider the second term in the right-hand side of (2.3). By applying the identity

$$e^{i\xi\eta + \frac{i}{\rho}\xi^\rho} = \frac{1}{1 + i\xi(\xi^{\rho-1} + \mu^{\rho-1})}\frac{\partial}{\partial\xi}\Big(\xi e^{i\xi\eta + \frac{i}{\rho}\xi^\rho}\Big)$$

the integration by parts with respect to ξ yields

$$\begin{aligned}\int_0^\infty e^{i\xi\eta + \frac{i}{\rho}\xi^\rho}\hat{v}\Big(\xi t^{-\frac{1}{\rho}}\Big)d\xi = &-t^{-\frac{1}{\rho}}\int_0^\infty \frac{\xi}{1 + i\xi(\xi^{\rho-1} + \mu^{\rho-1})}e^{i\xi\eta + \frac{i}{\rho}\xi^\rho}\hat{v}'\Big(\xi t^{-\frac{1}{\rho}}\Big)d\xi \\ &- i\int_0^\infty \frac{\xi\mu^{\rho-1} + \rho\xi^\rho}{(1 + i\xi(\xi^{\rho-1} + \mu^{\rho-1}))^2}e^{i\xi\eta + \frac{i}{\rho}\xi^\rho}\hat{v}\Big(\xi t^{-\frac{1}{\rho}}\Big)d\xi.\end{aligned}$$

Therefore we get

$$\left\|\int_0^\infty e^{i\xi\eta + \frac{i}{\rho}\xi^\rho}\hat{v}\Big(\xi t^{-\frac{1}{\rho}}\Big)d\xi\right\|_{\boldsymbol{L}_x^\infty} \le Ct^{-\frac{\alpha}{\rho}}\|\widehat{v}\|_{\boldsymbol{Z}^\alpha}\int_0^\infty \frac{\xi^\alpha d\xi}{1 + \xi^\rho} \le Ct^{-\frac{\alpha}{\rho}}\|\widehat{v}\|_{\boldsymbol{Z}^\alpha} \quad (2.4)$$

if $0 < \alpha < \rho - 1$ and

$$\begin{aligned}
&\left\| \int_0^\infty e^{i\xi\eta + \frac{i}{\rho}\xi^\rho} \hat{v}\Big(\xi t^{-\frac{1}{\rho}}\Big) d\xi \right\|_{\boldsymbol{L}^2_x} \\
&\le Ct^{-\frac{1}{2\rho}} \left\| \int_0^\infty \frac{\xi \mu^{\frac{\rho}{2}-1}}{1+\xi(\xi^{\rho-1}+\mu^{\rho-1})} \left| \hat{v}'\Big(\xi t^{-\frac{1}{\rho}}\Big)\right| d\xi \right\|_{\boldsymbol{L}^2_\mu} \\
&\quad + Ct^{\frac{1}{2\rho}} \left\| \int_0^\infty \frac{\mu^{\frac{\rho}{2}-1}}{1+\xi(\xi^{\rho-1}+\mu^{\rho-1})} \left| \hat{v}\Big(\xi t^{-\frac{1}{\rho}}\Big)\right| d\xi \right\|_{\boldsymbol{L}^2_\mu} \\
&\le Ct^{\frac{1}{2\rho}-\frac{\alpha}{\rho}} \|\widehat{v}\|_{\boldsymbol{Z}^\alpha} \left\| \int_0^\infty \frac{\xi^\alpha \mu^{\frac{\rho}{2}-1} d\xi}{1+\xi(\xi^{\rho-1}+\mu^{\rho-1})} \right\|_{\boldsymbol{L}^2_\mu} \\
&\le Ct^{\frac{1}{2\rho}-\frac{\alpha}{\rho}} \left\| \langle\mu\rangle^{\alpha-\frac{\rho}{2}} \right\|_{\boldsymbol{L}^2_\mu} \|\widehat{v}\|_{\boldsymbol{Z}^\alpha} \le Ct^{\frac{1}{2\rho}-\frac{\alpha}{\rho}} \|\widehat{v}\|_{\boldsymbol{Z}^\alpha} \qquad (2.5)
\end{aligned}$$

if $0 < \alpha < \frac{\rho-1}{2}$.

For the third summand in the right-hand side of (2.3) we apply the identity

$$e^{-i\xi\eta + \frac{i}{\rho}\xi^\rho} = \frac{1}{1+i(\xi-\mu)(\xi^{\rho-1}-\mu^{\rho-1})} \frac{\partial}{\partial\xi}\Big((\xi-\mu)e^{-i\xi\eta+\frac{i}{\rho}\xi^\rho}\Big),$$

then the integration by parts yields

$$\begin{aligned}
&\int_0^\infty e^{-i\xi\eta + \frac{i}{\rho}\xi^\rho} \Big(\hat{v}\Big(-\xi t^{-\frac{1}{\rho}}\Big) - \hat{v}(\chi)\Big) d\xi \\
&= \frac{\mu}{1+i\mu^\rho}\hat{v}(\chi) + t^{-\frac{1}{\rho}} \int_0^\infty \frac{(\xi-\mu)e^{-i\xi\eta+\frac{i}{\rho}\xi^\rho}\hat{v}'\Big(-\xi t^{-\frac{1}{\rho}}\Big)d\xi}{1+i(\xi-\mu)(\xi^{\rho-1}-\mu^{\rho-1})} \\
&\quad + i\int_0^\infty \frac{(\xi-\mu)(\xi^{\rho-1}-\mu^{\rho-1}) + (\rho-1)(\xi-\mu)^2\xi^{\rho-2}}{(1+i(\xi-\mu)(\xi^{\rho-1}-\mu^{\rho-1}))^2} \\
&\quad \times \Big(\hat{v}\Big(-\xi t^{-\frac{1}{\rho}}\Big) - \hat{v}\Big(-\mu t^{-\frac{1}{\rho}}\Big)\Big) e^{-i\xi\eta+\frac{i}{\rho}\xi^\rho} d\xi \equiv \sum_{j=1}^3 I_j(t,x).
\end{aligned}$$

The first integral $I_1(t,x)$ is estimated for all $t \ge 1$ by

$$\|I_1(t)\|_{\boldsymbol{L}^\infty_x} \le \left\| \frac{\mu}{\langle\mu\rangle^\rho} \hat{v}\Big(-\mu t^{-\frac{1}{\rho}}\Big) \right\|_{\boldsymbol{L}^\infty_x} \le Ct^{-\frac{\alpha}{\rho}} \left\| \{\xi\}^{-\alpha} \hat{v}(\xi) \right\|_{\boldsymbol{L}^\infty} \qquad (2.6)$$

if $0 < \alpha < \rho - 1$ and

$$\|I_1(t)\|_{\mathbf{L}^2_x} \le Ct^{\frac{1}{2\rho}} \left\| \frac{\mu^{\frac{\rho}{2}}}{\langle\mu\rangle^{\rho}} \hat{v}\Big(-\mu t^{-\frac{1}{\rho}}\Big) \right\|_{\mathbf{L}^2_\mu}$$
$$\le Ct^{\frac{1}{2\rho}-\frac{\alpha}{\rho}} \left\|\{\xi\}^{-\alpha}\hat{v}(\xi)\right\|_{\mathbf{L}^\infty} \left\| \frac{\mu^{\alpha+\frac{\rho}{2}}}{\langle\mu\rangle^{\rho}} \right\|_{\mathbf{L}^2_\mu} \le Ct^{\frac{1}{2\rho}-\frac{\alpha}{\rho}} \|\widehat{v}\|_{\mathbf{Z}^\alpha} \tag{2.7}$$

if $0 < \alpha < \frac{\rho-1}{2}$. For the second term we find

$$\|I_2(t)\|_{\mathbf{L}^\infty_x} \le t^{-\frac{1}{\rho}} \left\| \int_0^\infty \frac{(\xi-\mu)e^{-i\xi\eta+\frac{i}{\rho}\xi^\rho}}{1+i(\xi-\mu)(\xi^{\rho-1}-\mu^{\rho-1})} \hat{v}'\Big(-\xi t^{-\frac{1}{\rho}}\Big) d\xi \right\|_{\mathbf{L}^\infty_\mu}$$
$$\le Ct^{-\frac{\alpha}{\rho}} \left\| |\xi|^{1-\alpha}\hat{v}'(\xi)\right\|_{\mathbf{L}^\infty_\xi} \left\| \int_0^\infty \frac{|\xi|^{\alpha-1}|\xi-\mu| d\xi}{1+|\xi-\mu||\xi^{\rho-1}-\mu^{\rho-1}|} \right\|_{\mathbf{L}^\infty_\mu}$$
$$\le Ct^{-\frac{\alpha}{\rho}} \|\widehat{v}\|_{\mathbf{Z}^\alpha} \tag{2.8}$$

since we have the estimate

$$\int_0^\infty \frac{|\xi|^{\alpha-1}|\xi-\mu| d\xi}{1+|\xi-\mu||\xi^{\rho-1}-\mu^{\rho-1}|}$$
$$\le C\langle\mu\rangle^{1-\rho} \int_0^{\frac{\mu}{2}} |\xi|^{\alpha-1} d\xi + C\mu^{\alpha-1} \int_{\frac{\mu}{2}}^{2\mu} \frac{|\xi-\mu| d\xi}{1+(\xi-\mu)^2\mu^{\rho-2}}$$
$$+ C \int_{2\mu}^\infty \frac{|\xi-\mu|^\alpha d\xi}{1+|\xi-\mu|^\rho} \le C \tag{2.9}$$

for all $\mu > 0$ if $0 < \alpha < \rho - 1$. Similarly to the proof of (2.7) we also obtain

$$\|I_2(t)\|_{\mathbf{L}^2_x} \le Ct^{-\frac{1}{2\rho}} \left\| \int_0^\infty \frac{\mu^{\frac{\rho}{2}-1}(\xi-\mu)e^{-i\xi\eta+\frac{i}{\rho}\xi^\rho}}{1+i(\xi-\mu)(\xi^{\rho-1}-\mu^{\rho-1})} \hat{v}'\Big(-\xi t^{-\frac{1}{\rho}}\Big) d\xi \right\|_{\mathbf{L}^2_\mu}$$
$$\le Ct^{\frac{1}{2\rho}-\frac{\alpha}{\rho}} \left\| |\xi|^{1-\alpha}\hat{v}'(\xi)\right\|_{\mathbf{L}^\infty} \left\| \int_0^\infty \frac{\mu^{\frac{\rho}{2}-1}|\xi|^{\alpha-1}|\xi-\mu| d\xi}{1+|\xi-\mu|\,|\xi^{\rho-1}-\mu^{\rho-1}|} \right\|_{\mathbf{L}^2_\mu}$$
$$\le Ct^{\frac{1}{2\rho}-\frac{\alpha}{\rho}} \|\widehat{v}\|_{\mathbf{Z}^\alpha} \tag{2.10}$$

if $0 < \alpha < \frac{\rho-1}{2}$, since as in (2.9) we have

$$\left\| \int_0^\infty \frac{\mu^{\frac{\rho}{2}-1}|\xi|^{\alpha-1}|\xi-\mu|d\xi}{1+|\xi-\mu|\,|\xi^{\rho-1}-\mu^{\rho-1}|}\right\|_{\boldsymbol{L}^2_\mu} \le C\left\|\langle\mu\rangle^{\alpha-\frac{\rho}{2}}\right\|_{\boldsymbol{L}^2_\mu} \le C.$$

The third integral I_3 is estimated as follows

$$\begin{aligned}\|I_3(t)\|_{\boldsymbol{L}^\infty} &\le C\int_0^\infty \frac{1}{1+|\xi-\mu|\,|\xi^{\rho-1}-\mu^{\rho-1}|}\left|\int_{-\mu t^{-\frac{1}{\rho}}}^{-\xi t^{-\frac{1}{\rho}}} \hat{v}'(y)dy\right|d\xi \\ &\le Ct^{-\frac{\alpha}{\rho}}\left\||\xi|^{1-\alpha}\hat{v}'(\xi)\right\|_{\boldsymbol{L}^\infty}\int_0^\infty \frac{|\xi^\alpha-\mu^\alpha|}{1+|\xi-\mu|\,|\xi^{\rho-1}-\mu^{\rho-1}|}d\xi \\ &\le Ct^{-\frac{\alpha}{\rho}}\|\widehat{v}\|_{\boldsymbol{Z}^\alpha} \end{aligned} \tag{2.11}$$

since

$$\begin{aligned}&\int_0^\infty \frac{|\xi^\alpha-\mu^\alpha|}{1+|\xi-\mu|\,|\xi^{\rho-1}-\mu^{\rho-1}|}d\xi \\ &\le C\langle\mu\rangle^{1+\alpha-\rho}+C\mu^{\alpha-1}\int_{\frac{\mu}{2}}^{2\mu}\frac{|\xi-\mu|d\xi}{1+(\xi-\mu)^2\mu^{\rho-2}}+C\int_{2\mu}^\infty\frac{|\xi-\mu|^\alpha d\xi}{1+|\xi-\mu|^\rho}\le C\end{aligned}$$

for all $\mu>0$ if $0<\alpha<\rho-1$. In the same way as in the proof of (2.10) we have

$$\begin{aligned}\|I_3(t)\|_{\boldsymbol{L}^2_x} &\le Ct^{\frac{1}{2\rho}-\frac{\alpha}{\rho}}\left\||\xi|^{1-\alpha}\hat{v}'(\xi)\right\|_{\boldsymbol{L}^\infty}\left\|\int_0^\infty\frac{\mu^{\frac{\rho}{2}-1}|\xi^\alpha-\mu^\alpha|}{1+|\xi-\mu|\,|\xi^{\rho-1}-\mu^{\rho-1}|}d\xi\right\|_{\boldsymbol{L}^2_\mu} \\ &\le Ct^{\frac{1}{2\rho}-\frac{\alpha}{\rho}}\left\|\langle\mu\rangle^{\alpha-\frac{\rho}{2}}\right\|_{\boldsymbol{L}^2_\mu}\|\widehat{v}\|_{\boldsymbol{Z}^\alpha}\le Ct^{\frac{1}{2\rho}-\frac{\alpha}{\rho}}\|\widehat{v}\|_{\boldsymbol{Z}^\alpha}\end{aligned} \tag{2.12}$$

if $0<\alpha<\frac{\rho-1}{2}$. Therefore by (2.1)–(2.12) we obtain the result of the theorem for the case of $x>0$. The case of $x<0$ is considered in the same way. This completes the proof of Theorem 2.1. □

Now we state the Strichartz estimate. It can be proved by the duality argument from [**13**] and by estimates of Theorem 2.1. Denote the ordering of the norms

$$\|\phi\|_{\boldsymbol{L}^q_t\boldsymbol{L}^r_x}=\left\|\|\phi(t)\|_{\boldsymbol{L}^r_x(\boldsymbol{R})}\right\|_{\boldsymbol{L}^q_t(\boldsymbol{I})},$$

where $\boldsymbol{I}$ is a bounded or unbounded time interval.

LEMMA 2.2. *For any time interval $\boldsymbol{I}$ and for any $s \in \overline{\boldsymbol{I}}$ the Strichartz estimate is true*

$$\left\| \int_s^t U(t-\tau)\phi(\tau)d\tau \right\|_{\boldsymbol{L}_t^q \boldsymbol{L}_x^r} \le C\|\phi\|_{\boldsymbol{L}_t^{\overline{q}'} \boldsymbol{L}_x^{\overline{r}'}}$$

with a constant C independent of $\boldsymbol{I}$ and s, where $0 \le \frac{\rho}{q} = \frac{1}{2} - \frac{1}{r} < 1$ and $0 \le \frac{\rho}{q'} = \frac{1}{2} - \frac{1}{r'} < 1$, $\frac{1}{r'} + \frac{1}{\overline{r}'} = 1$, and $\frac{1}{q'} + \frac{1}{\overline{q}'} = 1$.

3. Supercritical case.

In this section we study the final problem in the supercritical case

$$\begin{cases} u_t - \dfrac{i}{\rho}|\partial_x|^\rho u = \mathscr{N}(u,\overline{u}), & (t,x) \in \boldsymbol{R} \times \boldsymbol{R}, \\ \lim\limits_{t\to\infty} U(-t)u(t) = u_+, & x \in \boldsymbol{R}, \end{cases} \tag{3.1}$$

where the free evolution group $U(t) = \mathscr{F}^{-1} \exp\left(\frac{i}{\rho}t|\xi|^\rho\right)\mathscr{F}$. We can write problem (3.1) in the form of the integral equation

$$U(-t)u = u_+ + \int_t^\infty U(-\tau)\mathscr{N}(u,\overline{u})(\tau)d\tau. \tag{3.2}$$

As above we use the norm $\|v\|_{\boldsymbol{Z}^\alpha} \equiv \|\{\xi\}^{-\alpha}v(\xi)\|_{\boldsymbol{L}^\infty} + \|\{\xi\}^{1-\alpha}v'(\xi)\|_{\boldsymbol{L}^\infty}$.

We prove the following result.

THEOREM 3.1. *Let the final data $u_+ \in \boldsymbol{L}^2$ satisfy the estimate $\|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha} < \infty$ with $\frac{\rho-2}{2} \le \alpha < \frac{\rho-1}{2}$. Assume that the nonlinearity $\mathscr{N}(u,\overline{u})$ satisfies the growth condition*

$$\left|\mathscr{N}(v,\overline{v}) - \mathscr{N}(u,\overline{u})\right| \le C|v-u|^p + C|v-u||u|^{p-1}$$

with $\frac{1+8\rho}{1+2\rho} < p < 1+2\rho$. Then there exists a time $T > 1$ and a unique global solution $u \in \boldsymbol{C}([T,\infty);\boldsymbol{L}^2)$ of the final problem (3.1) such that

$$\|u(t) - U(t)u_+\|_{\boldsymbol{L}^2} + \|u(\cdot) - U(\cdot)u_+\|_{\boldsymbol{L}_t^{2\rho}(t,\infty)\boldsymbol{L}_x^\infty} \le Ct^{-b}$$

for all $t > T$, where $\frac{2\rho+1-p}{4\rho} < b < \frac{p-3}{2}$.

In the case $p > 2+\rho$, the $\boldsymbol{L}^\infty$-$\boldsymbol{L}^1$ estimates for the linear problem obtained in Theorem 2.1 (2.1) yields the global existence of solutions of (1.2) in the usual Sobolev spaces. So we concentrated on the case $1+2\rho > 2+\rho \geq p$ in the above theorem.

Proof. We define the following function space

$$\boldsymbol{X} = \{\varphi \in \boldsymbol{C}([T,\infty);\boldsymbol{L}^2); \|\varphi\|_{\boldsymbol{X}} < \infty\},$$

where

$$\|\varphi\|_{\boldsymbol{X}} = \sup_{t\in[T,\infty)} t^b \big(\|\varphi\|_{\boldsymbol{L}_t^\infty(t,\infty)\boldsymbol{L}_x^2} + \|\varphi\|_{\boldsymbol{L}_t^{2\rho}(t,\infty)\boldsymbol{L}_x^\infty}\big)$$

and $\frac{2\rho+1-p}{4\rho} < b < \frac{p-3}{2}$. This implies the condition on $p > \frac{1+8\rho}{1+2\rho}$. Denote the first approximation $u_1(t) = U(t)u_+$, and the second approximation

$$u_2(t) = \int_t^\infty U(t-\tau)\mathscr{N}(u_1,\overline{u_1})d\tau.$$

Let $\boldsymbol{X}_r$ be a closed ball in $\boldsymbol{X}$ with a radius $r > \|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha}$ and a center u_1. Let $v \in \boldsymbol{X}_r$ and define the mapping $\mathscr{M}$ by

$$\mathscr{M}(v) = u_1(t) + u_2(t) + \int_t^\infty U(t-\tau)(\mathscr{N}(v,\overline{v}) - \mathscr{N}(u_1,\overline{u_1}))d\tau. \tag{3.3}$$

Since $\|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha} < \infty$ with $\frac{\rho-2}{2} \leq \alpha < \frac{\rho-1}{2}$ by Theorem 2.1

$$\|u_1(t)\|_{\boldsymbol{L}_x^\infty} \leq Crt^{-\frac{1}{2}}.$$

We also get by the Strichartz estimate (see Lemma 2.2) if $b < \frac{p-3}{2}$

$$\|u_2\|_{\boldsymbol{X}} \leq \sup_{t\in[T,\infty)} t^b \big\||u_1|^p\big\|_{\boldsymbol{L}_t^1(t,\infty)\boldsymbol{L}_x^2} \leq Cr^pT^{b-\frac{p-3}{2}}. \tag{3.4}$$

Again by the Strichartz estimate we obtain

$$\begin{aligned}&\Big\|\int_t^\infty U(t-\tau)\big(\mathscr{N}(v,\overline{v}) - \mathscr{N}(u_1,\overline{u_1})\big)d\tau\Big\|_{\boldsymbol{X}} \\ &\leq C\sup_{t\in[T,\infty)} t^b\Big(\big\||v-u_1|^p\big\|_{\boldsymbol{L}_t^{\frac{2\rho}{2\rho-1}}(t,\infty)\boldsymbol{L}_x^1} + \big\||v-u_1||u_1|^{p-1}\big\|_{\boldsymbol{L}_t^1(t,\infty)\boldsymbol{L}_x^2}\Big).\end{aligned} \tag{3.5}$$

By the Hölder inequality we get

$$
\begin{aligned}
\left\| |\phi|^p \right\|_{\boldsymbol{L}_t^{\frac{2\rho}{2\rho-1}}(t,\infty)\boldsymbol{L}_x^1} &\le C \left\| \|\phi\|_{\boldsymbol{L}_x^2}^{\frac{4\rho}{2\rho-1}} \|\phi\|_{\boldsymbol{L}_x^\infty}^{\frac{2\rho(p-2)}{2\rho-1}} \right\|_{\boldsymbol{L}_t^1(t,\infty)}^{\frac{2\rho-1}{2\rho}} \\
&\le C \left\| \|\phi\|_{\boldsymbol{L}_x^2}^{\frac{4\rho}{2\rho+1-p}} \right\|_{\boldsymbol{L}_t^1(t,\infty)}^{\frac{2\rho+1-p}{2\rho}} \|\phi\|_{\boldsymbol{L}_t^{2\rho}(t,\infty)\boldsymbol{L}_x^\infty}^{p-2}.
\end{aligned}
$$

Hence for the first summand in (3.5) we find

$$
\begin{aligned}
&\sup_{t\in[T,\infty)} t^b \left\| |v-u_1|^p \right\|_{\boldsymbol{L}_t^{\frac{2\rho}{2\rho-1}}(t,\infty)\boldsymbol{L}_x^1} \\
&\le \sup_{t\in[T,\infty)} t^b \left\| \|v-u_1\|_{\boldsymbol{L}_x^2}^{\frac{4\rho}{2\rho+1-p}} \right\|_{\boldsymbol{L}_t^1(t,\infty)}^{\frac{2\rho+1-p}{2\rho}} \|v-u_1\|_{\boldsymbol{L}_t^{2\rho}(t,\infty)\boldsymbol{L}_x^\infty}^{p-2} \\
&\le Cr^p \sup_{t\in[T,\infty)} t^{-b(p-3)} \left(\int_t^\infty \tau^{-\frac{4\rho b}{2\rho+1-p}}\, d\tau \right)^{\frac{2\rho+1-p}{2\rho}} \le Cr^p T^{-b(p-3)} \qquad (3.6)
\end{aligned}
$$

since $\frac{2\rho+1-p}{4\rho} < b < \frac{p-3}{2}$. Then for the second summand in (3.5) we obtain

$$
\sup_{t\in[T,\infty)} t^b \left\| |v-u_1|\, |u_1|^{p-1} \right\|_{\boldsymbol{L}_t^1(t,\infty)\boldsymbol{L}_x^2} \le Cr^p T^{-\frac{p-3}{2}}. \qquad (3.7)
$$

By (3.4), (3.5)–(3.7) we get

$$
\|\mathscr{M}(v) - u_1\|_{\boldsymbol{X}} \le Cr^p T^{-\varepsilon} \qquad (3.8)
$$

with some $\varepsilon > 0$. Via (3.8) we see that there exists a time T such that $\mathscr{M}(v) \in \boldsymbol{X}_r$. We now consider $v_1,\ v_2 \in \boldsymbol{X}_r$, then in the same way as in the proof of (3.8) it follows that there exists T such that

$$
\|\mathscr{M}(v_1) - \mathscr{M}(v_2)\|_{\boldsymbol{X}} \le \frac{1}{2}\|v_1 - v_2\|_{\boldsymbol{X}}.
$$

Therefore $\mathscr{M}$ is a contraction mapping in $\boldsymbol{X}_r$, hence we have the result of Theorem 3.1. □

In the case of the nonlinearity $\mathscr{N}(u,\overline{u}) = i\lambda|u|^{p-1}u$, $\lambda \in \boldsymbol{R}$ we can consider final problem (3.1) with any powers $3 < p < 1 + 2\rho$.

Theorem 3.2. *Let the final data* $u_+ \in \boldsymbol{L}^2$ *satisfy the estimate* $\|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha} < \infty$ *with* $\frac{\rho-2}{2} \le \alpha < \frac{\rho-1}{2}$. *Assume that the nonlinearity* $\mathscr{N}(u,\overline{u}) = i\lambda|u|^{p-1}u$, *with* $\lambda \in \boldsymbol{R}$, $3 < p < 1+2\rho$. *Then there exists a time* $T > 1$ *and a unique global solution* $u \in \boldsymbol{C}([T,\infty);\boldsymbol{L}^2)$ *of the final problem* (3.1) *such that for some positive constants* C_1, C_2

$$C_2 t^{-\frac{p-3}{2}} \le \|u(t) - U(t)u_+\|_{\boldsymbol{L}^2} \le C_1 t^{-\frac{p-3}{2}}$$

for all $t > T$.

Proof. We define

$$\widehat{w}(t,\xi) = \widehat{u_+}(\xi)\exp\left(\frac{2i\lambda}{p-3}t^{-\frac{p-3}{2}}|B(\xi)\widehat{u_+}(\xi)|^{p-1}\right),$$

where $B(\xi) = \sqrt{\frac{1}{i(\rho-1)}}|\xi|^{1-\frac{\rho}{2}}$. Then by equation (3.1) we have

$$\begin{aligned}
&(\mathscr{F}U(-t)u - \widehat{w}(t,\xi))_t \\
&\quad = i\lambda\mathscr{F}U(-t)|u|^{p-1}u - i\lambda t^{-\frac{p-1}{2}}|B(\xi)\widehat{u_+}(\xi)|^{p-1}\widehat{w}(t,\xi) \\
&\quad = i\lambda\mathscr{F}U(-t)\Big(|u|^{p-1}u - U(t)\mathscr{F}^{-1}\Big|t^{-\frac{1}{2}}B(\xi)\widehat{u_+}(\xi)\Big|^{p-1}\widehat{w}(t,\xi)\Big). \qquad (3.9)
\end{aligned}$$

By Theorem 2.1 we can write

$$\begin{aligned}
&U(t)\mathscr{F}^{-1}\Big|t^{-\frac{1}{2}}B(\xi)\widehat{u_+}(\xi)\Big|^{p-1}\widehat{w}(t,\xi) \\
&\quad = A(t,\chi)\Big|t^{-\frac{1}{2}}B(\chi)\widehat{u_+}(\chi)\Big|^{p-1}\widehat{w}(t,\chi) + R_1 = |h|^{p-1}h + R_1, \qquad (3.10)
\end{aligned}$$

where $h(t,x) = A(t,\chi)\widehat{w}(t,\chi)$, $\chi = -\frac{x}{|x|}\big(\frac{|x|}{t}\big)^{\frac{1}{\rho-1}}$ and

$$A(t,\chi) = t^{-\frac{1}{2}}B(\chi)e^{-i(1-\frac{1}{\rho})|\chi|^\rho t}.$$

The remainder R_1 satisfies the estimates of Theorem 2.1

$$\|R_1(t)\|_{\boldsymbol{L}^2} \le Ct^{-\frac{\alpha}{\rho}-\frac{1}{2\rho}}\Big\|\Big|t^{-\frac{1}{2}}B(\xi)\widehat{u_+}(\xi)\Big|^{p-1}\widehat{w}(t,\xi)\Big\|_{\boldsymbol{Z}^\alpha}$$

for $0 < \alpha < \frac{\rho-1}{2}$. We have

$$\begin{aligned}
&\left\| \left| t^{-\frac{1}{2}} B(\xi) \widehat{u_+}(\xi) \right|^{p-1} \widehat{w}(t,\xi) \right\|_{\boldsymbol{Z}^\alpha} \\
&\quad = \left\| \{\xi\}^{-\alpha} \left| t^{-\frac{1}{2}} B(\xi) \widehat{u_+}(\xi) \right|^{p-1} \widehat{u_+}(\xi) \right\|_{\boldsymbol{L}^\infty} \\
&\qquad + \left\| \{\xi\}^{1-\alpha} \partial_\xi \left(\left| t^{-\frac{1}{2}} B(\xi) \widehat{u_+}(\xi) \right|^{p-1} \widehat{w}(t,\xi) \right) \right\|_{\boldsymbol{L}^\infty} \\
&\quad \le C t^{-\frac{p-1}{2}} \|\widehat{u_+}\|^{p-1}_{\boldsymbol{Z}^{\frac{\rho}{2}-1}} \|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha} + C t^{2-p} \|\widehat{u_+}\|^{2p-2}_{\boldsymbol{Z}^{\frac{\rho}{2}-1}} \|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha}
\end{aligned}$$

if $\frac{\rho}{2} - 1 \le \alpha$. Therefore

$$\|R_1(t)\|_{\boldsymbol{L}^2} \le C t^{-\frac{\alpha}{\rho} - \frac{1}{2\rho} - \frac{p-1}{2}} \left(\|\widehat{u_+}\|^p_{\boldsymbol{Z}^\alpha} + \|\widehat{u_+}\|^{2p-1}_{\boldsymbol{Z}^\alpha} \right).$$

Substitution of (3.10) into (3.9) yields

$$(\mathscr{F} U(-t) u - \widehat{w}(t,\xi))_t = i\lambda \mathscr{F} U(-t) \left(|u|^{p-1} u - |h|^{p-1} h - R_1 \right). \tag{3.11}$$

Note that by Theorem 2.1

$$U(t) \mathscr{F}^{-1} \widehat{w}(t,\xi) = h(t,x) + R_2,$$

hence we get from (3.11)

$$\partial_t (\mathscr{F} U(-t)(u - h - R_2)) = i\lambda \mathscr{F} U(-t) \left(|u|^{p-1} u - |h|^{p-1} h - R_1 \right), \tag{3.12}$$

where the reminder $R_2(t,x)$ satisfy the estimates

$$\begin{aligned}
\|R_2(t)\|_{\boldsymbol{L}^2} + t^{\frac{1}{2\rho}} \|R_2(t)\|_{\boldsymbol{L}^\infty} &\le C t^{-\frac{\alpha}{\rho} - \frac{1}{2\rho}} \|\widehat{w}(t)\|_{\boldsymbol{Z}^\alpha} \\
&\le C t^{-\frac{\alpha}{\rho} - \frac{1}{2\rho}} \left(\|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha} + t^{-\frac{p-3}{2}} \|\widehat{u_+}\|^p_{\boldsymbol{Z}^\alpha} \right)
\end{aligned} \tag{3.13}$$

for $\frac{\rho-2}{2} \le \alpha < \frac{\rho-1}{2}$ since

$$\begin{aligned}
\|\widehat{w}(t)\|_{\boldsymbol{Z}^\alpha} &= \left\| \{\xi\}^{-\alpha} \widehat{u_+}(\xi) \right\|_{\boldsymbol{L}^\infty} + \left\| \{\xi\}^{1-\alpha} \widehat{w}_\xi(t,\xi) \right\|_{\boldsymbol{L}^\infty} \\
&\le C \|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha} + C t^{-\frac{p-3}{2}} \|\widehat{u_+}\|^{p-1}_{\boldsymbol{Z}^{\frac{\rho}{2}-1}} \|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha}.
\end{aligned}$$

As in the proof of Theorem 3.1 we define the function space

$$\boldsymbol{X} = \{\varphi \in \boldsymbol{C}([T,\infty);\boldsymbol{L}^2); \|\varphi\|_{\boldsymbol{X}} < \infty\},$$

where

$$\|\varphi\|_{\boldsymbol{X}} = \sup_{t\in[T,\infty)} t^b \big(\|\varphi\|_{\boldsymbol{L}_t^\infty(t,\infty)\boldsymbol{L}_x^2} + \|\varphi\|_{\boldsymbol{L}_t^{2\rho}(t,\infty)\boldsymbol{L}_x^\infty}\big)$$

with $b = \frac{\alpha}{\rho} + \frac{1}{2\rho} \in \big[\frac{\rho-1}{2\rho}, \frac{1}{2}\big)$. Let $\boldsymbol{X}_r$ be a closed ball in $\boldsymbol{X}$ with a radius $r > C\|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha}$ and a center $h(t)$. We write (3.12) as the integral equation

$$u - h - R_2 = i\lambda \int_t^\infty U(t-\tau)\big(|u|^{p-1}u - |h|^{p-1}h - R_1\big)d\tau,$$

and solve it by the contraction mapping principle in $\boldsymbol{X}_r$. Define the mapping for $v \in \boldsymbol{X}_r$

$$\mathscr{M}(v) = h + R_2 + i\lambda \int_t^\infty U(t-\tau)\big(|v|^{p-1}v - |h|^{p-1}h - R_1\big)d\tau.$$

By (3.13) we have $\|R_2\|_{\boldsymbol{X}} \le C\|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha}$. Then as in (3.5)–(3.7) by the Strichartz estimate we obtain

$$\begin{aligned}
\|\mathscr{M}(v) - h\|_{\boldsymbol{X}} &\le \left\| R_2 + i\lambda \int_t^\infty U(t-\tau)\big(|v|^{p-1}v - |h|^{p-1}h - R_1\big)d\tau \right\|_{\boldsymbol{X}} \\
&\le \|R_2\|_{\boldsymbol{X}} + C \sup_{t\in[T,\infty)} t^b \big\| |v-h|^p \big\|_{\boldsymbol{L}_t^{\frac{2\rho}{2\rho-1}}(t,\infty)\boldsymbol{L}_x^1} \\
&\quad + C \sup_{t\in[T,\infty)} t^b \big\| |v-h|\,|h|^{p-1} \big\|_{\boldsymbol{L}_t^1(t,\infty)\boldsymbol{L}_x^2} + C \sup_{t\in[T,\infty)} t^b \|R_1\|_{\boldsymbol{L}_t^1(t,\infty)\boldsymbol{L}_x^2} \\
&\le C\|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha} + Cr^pT^{-b(p-3)} + Cr^pT^{-\frac{p-3}{2}} \le r
\end{aligned} \tag{3.14}$$

since $\frac{2\rho+1-p}{4\rho} < b$, if T is sufficiently large. As in the proof of (3.14) we get

$$\|\mathscr{M}(v_1) - \mathscr{M}(v_2)\|_{\boldsymbol{X}} \le \frac{1}{2}\|v_1 - v_2\|_{\boldsymbol{X}}$$

for any $v_1, v_2 \in \boldsymbol{X}_r$, which shows that $\mathscr{M}$ is a contraction mapping in $\boldsymbol{X}_r$. Theorem 3.2 is proved. □

4. Critical case.

In this section we study the modified final problem for dispersive equation of Schrödinger type with critical nonlinearity $\mathscr{N}(u,\overline{u}) = i\lambda|u|^2u$

$$\begin{cases} u_t - \dfrac{i}{\rho}|\partial_x|^\rho u = i\lambda|u|^2u, & (t,x) \in \boldsymbol{R}\times\boldsymbol{R}, \\ \lim\limits_{t\to\infty}(\mathscr{F}U(-t)u(t) - \widehat{w}(t)) = 0, & x\in\boldsymbol{R}, \end{cases} \tag{4.1}$$

where we define the modified final state

$$\widehat{w}(t,\xi) = \widehat{u_+}(\xi)\exp\left(\frac{i\lambda}{1-\rho}|\xi|^{2-\rho}|\widehat{u_+}(\xi)|^2\log t\right)$$

if $\lambda \in \boldsymbol{R}$ and in the case of the dissipative nonlinearity $\mathscr{N}(u,\overline{u}) = i\lambda|u|^2u$ with $\operatorname{Im}\lambda > 0$, we define a modified final state

$$\widehat{w}(t,\xi) = \widehat{u_+}(\xi)\exp\left(\frac{i\lambda}{\operatorname{Im}\lambda}\varphi(t,\xi)\right),$$

where

$$\varphi(t,\xi) = \frac{1}{2}\log\left(1 + \frac{2\operatorname{Im}\lambda}{\rho-1}|\xi|^{2-\rho}|\widehat{u_+}(\xi)|^2\log t\right).$$

Note that

$$\widehat{w}_t(t,\xi) = i\lambda\left|t^{-\frac{1}{2}}B(\xi)\widehat{w}(t,\xi)\right|^2\widehat{w}(t,\xi).$$

As above we use the norm $\|v\|_{\boldsymbol{Z}^\alpha} \equiv \|\{\xi\}^{-\alpha}v(\xi)\|_{\boldsymbol{L}^\infty} + \|\{\xi\}^{1-\alpha}v'(\xi)\|_{\boldsymbol{L}^\infty}$.

We first prove the following result.

Theorem 4.1. *Let the final data $u_+ \in \boldsymbol{L}^2$ satisfy the estimate $\|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha} < \varepsilon$, where $\varepsilon > 0$ is sufficiently small and $\frac{\rho-2}{2} < \alpha < \frac{\rho-1}{2}$. Assume that* $\operatorname{Im}\lambda \geq 0$. *Then for some $T \geq 1$ there exists a unique global solution $u \in \boldsymbol{C}([T,\infty);\boldsymbol{L}^2)$ of the modified final problem* (4.1) *such that*

$$\|u-h\|_{\boldsymbol{L}_t^\infty(t,\infty)\boldsymbol{L}_x^2} + \|u-h\|_{\boldsymbol{L}_t^{2\rho}(t,\infty)\boldsymbol{L}_x^\infty(\boldsymbol{R})} \leq Ct^{-b},$$

where $\frac{\rho-1}{2\rho} < b < \frac{\alpha}{\rho} + \frac{1}{2\rho} < \frac{1}{2}$ and $h(t,x) = A(t,\chi)\widehat{w}(t,\chi)$, $\chi = -\frac{x}{|x|}\left(\frac{|x|}{t}\right)^{\frac{1}{\rho-1}}$ and

$$A(t,\chi) = t^{-\frac{1}{2}}\sqrt{\frac{1}{i(\rho-1)}}|\chi|^{1-\frac{\rho}{2}}e^{-i(1-\frac{1}{\rho})|\chi|^{\rho}t}.$$

Proof. As in the proof of Theorem 3.2 by virtue of Theorem 2.1 we obtain from (4.1)

$$\partial_t(\mathscr{F}U(-t)u - \widehat{w}(t,\xi)) = i\lambda\mathscr{F}U(-t)\big(|u|^2u - |h|^2h - R_1\big) \tag{4.2}$$

where R_1 satisfies the estimates

$$\begin{aligned}\|R_1(t)\|_{\boldsymbol{L}^2} &\leq Ct^{-\frac{\alpha}{\rho}-\frac{1}{2\rho}}\Big\|\Big|t^{-\frac{1}{2}}B(\xi)\widehat{w}(t,\xi)\Big|^2\widehat{w}(t,\xi)\Big\|_{\boldsymbol{Z}^\alpha}\\ &\leq Ct^{-\frac{\alpha}{\rho}-\frac{1}{2\rho}-1}\big\|\{\xi\}^{-\alpha}|\xi|^{2-\rho}|\widehat{w}(t,\xi)|^2\widehat{u_+}(\xi)\big\|_{\boldsymbol{L}^\infty}\\ &\quad + Ct^{-\frac{\alpha}{\rho}-\frac{1}{2\rho}-1}\big\|\{\xi\}^{1-\alpha}\partial_\xi\big(|\xi|^{2-\rho}|\widehat{w}(t,\xi)|^2\widehat{w}(t,\xi)\big)\big\|_{\boldsymbol{L}^\infty}\\ &\leq Ct^{-\frac{\alpha}{\rho}-\frac{1}{2\rho}-1}\log t\big(\|\widehat{u_+}\|^3_{\boldsymbol{Z}^\alpha} + \|\widehat{u_+}\|^5_{\boldsymbol{Z}^\alpha}\big)\end{aligned}$$

since $\frac{\rho-2}{2} < \alpha < \frac{\rho-1}{2}$. By Theorem 2.1

$$U(t)\mathscr{F}^{-1}\widehat{w}(t,\xi) = h(t,x) + R_2.$$

Hence we get from (4.2)

$$\partial_t(\mathscr{F}U(-t)(u-h-R_2)) = i\lambda\mathscr{F}U(-t)\big(|u|^2u - |h|^2h - R_1\big), \tag{4.3}$$

where the reminder $R_2(t,x)$ satisfies the estimates

$$\begin{aligned}\|R_2(t)\|_{\boldsymbol{L}^2} + t^{\frac{1}{2\rho}}\|R_2(t)\|_{\boldsymbol{L}^\infty} &\leq Ct^{-\frac{\alpha}{\rho}-\frac{1}{2\rho}}\|\widehat{w}(t)\|_{\boldsymbol{Z}^\alpha}\\ &\leq Ct^{-\frac{\alpha}{\rho}-\frac{1}{2\rho}}\big(\|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha} + \log t\|\widehat{u_+}\|^3_{\boldsymbol{Z}^\alpha}\big)\end{aligned} \tag{4.4}$$

for $\frac{\rho-2}{2} < \alpha < \frac{\rho-1}{2}$. As in the previous section we define the following set

$$\boldsymbol{X} = \big\{\varphi \in \boldsymbol{C}([T,\infty);\boldsymbol{L}^2); \|\varphi\|_{\boldsymbol{X}} < \infty\big\},$$

where

$$\|\varphi\|_{\boldsymbol{X}} = \sup_{t\in[T,\infty)} t^b\big(\|\varphi\|_{\boldsymbol{L}^\infty_t(t,\infty)\boldsymbol{L}^2_x(\boldsymbol{R})} + \|\varphi\|_{\boldsymbol{L}^{2\rho}_t(t,\infty)\boldsymbol{L}^\infty_x(\boldsymbol{R})}\big)$$

with $\frac{\rho-1}{2\rho} < b < \frac{\alpha}{\rho} + \frac{1}{2\rho} < \frac{1}{2}$. Let $\boldsymbol{X}_r$ be a closed ball in $\boldsymbol{X}$ with a small radius $r > 0$ and a center $h(t)$. We solve (4.3) by the contraction mapping principle in $\boldsymbol{X}_r$. Define the mapping for $v \in \boldsymbol{X}_r$

$$\mathscr{M}(v) = h + R_2 + i\lambda \int_t^\infty U(t-\tau)\big(|v|^2 v - |h|^2 h - R_1\big)d\tau.$$

By (4.4) we have $\|R_2\|_{\boldsymbol{X}} \le C\|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha}$ since the data u_+ are sufficiently small. Then by the Strichartz estimate we obtain

$$\begin{aligned}
\|\mathscr{M}(v) - h\|_{\boldsymbol{X}} &\le \left\| R_2 + i\lambda \int_t^\infty U(t-\tau)\big(|v|^2 v - |h|^2 h - R_1\big)d\tau \right\|_{\boldsymbol{X}} \\
&\le \|R_2\|_{\boldsymbol{X}} + C \sup_{t\in[T,\infty)} t^b \big\| |v-h|^3 \big\|_{\boldsymbol{L}_t^{\frac{2\rho}{2\rho-1}}(t,\infty)\boldsymbol{L}_x^1} \\
&\quad + C \sup_{t\in[T,\infty)} t^b \big\| |v-h|\,|h|^2 \big\|_{\boldsymbol{L}_t^1(t,\infty)\boldsymbol{L}_x^2} + C \sup_{t\in[T,\infty)} t^b \|R_1\|_{\boldsymbol{L}_t^1(t,\infty)\boldsymbol{L}_x^2} \\
&\le C\|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha} + Cr^3 + Cr^3 \le r
\end{aligned} \tag{4.5}$$

since $\frac{\rho-1}{2\rho} < b$ and the data u_+ are sufficiently small. Similarly to (4.5) we get

$$\|\mathscr{M}(v_1) - \mathscr{M}(v_2)\|_{\boldsymbol{X}} \le \frac{1}{2}\|v_1 - v_2\|_{\boldsymbol{X}}$$

for $v_1, v_2 \in \boldsymbol{X}_r$, which shows that $\mathscr{M}$ is a contraction mapping in $\boldsymbol{X}_r$. Thus there exists a unique global solution $u \in \boldsymbol{C}([1,\infty); \boldsymbol{L}^2)$ of the modified final problem (4.1) such that $u \in \boldsymbol{X}_r$. This completes the proof of Theorem 4.1. □

REMARK 4.1. When $\operatorname{Im}\lambda > 0$ we do not need the smallness condition on the data since we can easily see that the approximate solution has an additional time decay

$$\|U(t)w\|_\infty \le Ct^{-\frac{1}{2}}(\log t)^{-\frac{1}{2}}$$

from which the estimate (4.5) can be written as

$$\|\mathscr{M}(v) - h\|_{\boldsymbol{X}} \le C(\log t)^{-\frac{1}{2}} \le r$$

for all $t \ge T$, if $T \ge 1$ is sufficiently large.

5. Nonexistence of the wave operator.

In this section we prove the nonexistence of the usual wave operator for the final problem for dispersive equation of Schrödinger type with a critical nonlinearity

$$\begin{cases} u_t - \dfrac{i}{\rho}|\partial_x|^\rho u = i\lambda |u|^2 u, & (t,x) \in \boldsymbol{R} \times \boldsymbol{R}, \\ \lim_{t\to\infty} U(-t)u(t) = u_+, & x \in \boldsymbol{R}, \end{cases} \tag{5.1}$$

where $\lambda \in \boldsymbol{R}$.

Theorem 5.1. *Let the final data $u_+ \in \boldsymbol{L}^2$ satisfy the estimate $\|\widehat{u_+}\|_{\boldsymbol{Z}^\alpha} < \infty$ and $\frac{\rho-2}{2} < \alpha < \frac{\rho-1}{2}$. Assume that $\lambda \in \boldsymbol{R}$. We also assume that there exists a solution $u \in \boldsymbol{C}([T,\infty); \boldsymbol{L}^2)$ to final problem* (5.1) *such that*

$$\lim_{t\to\infty} \|u(t) - U(t)u_+\|_{\boldsymbol{L}^2} = 0,$$

then $u = 0$.

Proof. Multiplying equation (5.1) by $U(-t)$ and integrating with respect to time we find

$$\begin{aligned} &U(-t)u(t) - U(-s)u(s) \\ &\quad = i\lambda \int_s^t U(-\tau)\big(|u|^2 u - \tau^{-1} U(\tau)\mathscr{F}^{-1}|\xi|^{2-\rho}|\widehat{u_+}|^2 \widehat{u_+}(\xi)\big) d\tau \\ &\qquad + i\lambda \mathscr{F}^{-1}|\xi|^{2-\rho}|\widehat{u_+}|^2\widehat{u_+}(\xi) \int_s^t \tau^{-1} d\tau. \end{aligned}$$

Hence by Theorem 4.1

$$\begin{aligned} &\|U(-t)u(t) - U(-s)u(s)\|_{\boldsymbol{L}^2} \\ &\quad \geq |\lambda| \big\| |\xi|^{2-\rho}|\widehat{u_+}|^2 \widehat{u_+}(\xi)\big\|_{\boldsymbol{L}^2} \int_s^t \tau^{-1} d\tau - C\int_s^t \|R_1\|_{\boldsymbol{L}^2} d\tau \\ &\qquad - C\int_s^t \|u(\tau) - h(\tau)\|_{\boldsymbol{L}^\infty} \|u(\tau) - h(\tau)\|_{\boldsymbol{L}^2} \|A(\tau)\widehat{u_+}(\tau)\|_{\boldsymbol{L}^\infty} d\tau \\ &\qquad - C\int_s^t \|u(\tau) - A(\tau)\widehat{u_+}(\tau)\|_{\boldsymbol{L}^2} \|h(\tau)\|_{\boldsymbol{L}^\infty}^2 \, d\tau. \end{aligned}$$

Then as in (3.6) by the Hölder inequality we get

$$\begin{aligned}
&\int_s^t \|u(\tau)-h(\tau)\|_{\mathbf{L}^\infty}\|u(\tau)-h(\tau)\|_{\mathbf{L}^2}\|A(\tau)\widehat{u_+}(\tau)\|_{\mathbf{L}^\infty}d\tau \\
&\quad \le C\Big\|\tau^{-b-\frac12}\|u(\tau)-h(\tau)\|_{\mathbf{L}^\infty}\Big\|_{\mathbf{L}^1_\tau(s,t)} \\
&\quad \le C\|\tau^{-b-\frac12}\|_{\mathbf{L}_t^{\frac{2\rho}{2\rho-1}}(t,\infty)}\|u(\tau)-h(\tau)\|_{\mathbf{L}^{2\rho}_\tau(s,t)\mathbf{L}^\infty_x} \\
&\quad \le Cs^{-b}\bigg(\int_s^t \tau^{-(b+\frac12)\frac{2\rho}{2\rho-1}}d\tau\bigg)^{\frac{2\rho-1}{2\rho}} \le Cs^{-b}
\end{aligned}$$

since $\frac{\rho-1}{2\rho}<b$. Hence we have

$$\begin{aligned}
&\|U(-t)u(t)-U(-s)u(s)\|_{\mathbf{L}^2} \\
&\quad \ge |\lambda|\big\||\xi|^{2-\rho}|\widehat{u_+}|^2\widehat{u_+}(\xi)\big\|_{\mathbf{L}^2}\int_s^t\frac{d\tau}{\tau}-Cs^{-b}-C\int_s^t\|u(\tau)-U(\tau)u_+\|_{\mathbf{L}^2}\frac{d\tau}{\tau}.
\end{aligned} \tag{5.2}$$

Estimate (5.2) along with the condition of the theorem imply that for any $\varepsilon>0$ there exists $T(\varepsilon)$ such that for any $t>s>T(\varepsilon)$

$$\|U(-t)u(t)-U(-s)u(s)\|_{\mathbf{L}^2}\ge\big(|\lambda|\big\||\xi|^{2-\rho}|\widehat{u_+}|^2\widehat{u_+}(\xi)\big\|_{\mathbf{L}^2}-\varepsilon\big)\int_s^t\frac{d\tau}{\tau}.$$

Consequently $u_+=0$. Since the solution satisfies the conservation of $\mathbf{L}^2$-norm, we have $u\equiv 0$. Theorem 5.1 is proved. □

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