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A second main theorem of Nevanlinna theory for meromorphic functions on complete Kähler manifolds

Dedicated to Professor Yoichiro Takahashi on his sixtieth birthday

By Atsushi Atsuji

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Abstract. We show that a second main theorem of Nevanlinna theory holds for meromorphic functions on general complete Kähler manifolds. It is well-known in classical Nevanlinna theory that a meromorphic function whose image grows rapidly enough can omit at most two points. Our second main theorem implies this fact holds for meromorphic functions on general complete Kähler manifolds.

Introduction.

Let us recall the second main theorem in the classical Nevanlinna theory. Let f be a nonconstant meromorphic function on $\boldsymbol{C}$ or the unit disc in $\boldsymbol{C}$ and $a_1, \ldots, a_q$ be distinct points of $\boldsymbol{C} \cup \{\infty\}$. Suppose that $f(o) \neq a_k,\ k = 1, \ldots, q$.

In the case of $\boldsymbol{C}$, we have

$$\sum_{k=1}^{q} m(a_i, r) + N_1(r) \leq 2T(r) + O(\log T(r) + \log r)$$

holds for r outside an exceptional set of disjoint union of intervals of finite total length where $m(a, r)$, $T(r)$ and $N_1(r)$ mean proximity function of f, characteristic function of f and counting function of critical points of f, respectively(see [**9**] or [**10**] for their definitions). It is well-known that little Picard's theorem asserting that f can omit at most two points follows from this theorem. Note that $T(r)$ is unbounded and $O(\log T(r) + \log r)$ can be reduced to $o(T(r))$ in this case.

In the case of unit disc,

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$$\sum_{k=1}^{q} m(a_i, r) + N_1(r) \leq 2T(r) + O\left(\log \frac{1}{1-r}\right) + O(\log T(r))$$

holds for r outside an exceptional set of finite logarithmic length. Of course in this case little Picard's theorem does not hold. But we can see in both the cases that if $T(r)$ grows rapidly enough, f can omit at most two points. Our naive question is to ask if this holds even for meromorphic functions on general Kähler manifolds. Namely if the image of a meromorphic function on any Kähler manifold grows large enough, can it omit at most two points? Moreover can we have a second main theorem on general Kähler manifolds? Our result answers this question.

THEOREM 1. *Let f be a nonconstant meromorphic function on a complete Kähler manifold M and $a_1, \ldots, a_q$ distinct points in $\boldsymbol{C} \cup \{\infty\}$. Then*

$$\sum_{j=1}^{q} m(r, a_j) + N_1(r) \leq 2T(r) + S(r) + O(\log T(r))$$

holds except for r in an exceptional set of finite length where $S(r)$ is given by (3.1) *after Theorem* 13 *in the section* 3.

Here $m(a, r)$, $T(r)$, $N_1(r)$ are a proximity function, characteristic function and counting function of critical points of f, respectively. They can be defined similarly to the classical cases and are their natural generalization. We will give their definitions in the sections 1 and 2. We also note that the remainder term $S(r)$ is independent of f and determined by some quantities of M.

We can define a defect $\delta(a, f)$ by

$$\delta(a, f) = \liminf_{r \to \infty} \frac{m(r, a)}{T(r)}.$$

Since the first main theorem holds in our case as classical cases:

$$m(r, a) + N(r, a) = T(r) + O(1),$$

we have $\delta(a, f) = 1$ provided f omits a. From our second main theorem we see that if a meromorphic function f on M satisfies $\lim_{r \to \infty} S(r)/T(r) = 0$, then

$$\sum_{i=1}^{q} \delta(a_i, f) \leq 2.$$

This implies that f can omit at most two points under the assumption.

The proof of this result is based on a method developed in [**1**] and [**2**] where we used some stochastic tools. These tools are based on a diffusion process called Brownian motion or Kähler diffusion on M (cf. [**12**]) whose generator is one half of the Laplacian associated with the Kähler metric. If f is a nonconstant holomorphic map from M to $\boldsymbol{P}^1(\boldsymbol{C})$ and X_t is the Brownian motion on M, then $f(X_t)$ behaves as a Brownian motion on $\boldsymbol{P}^1(\boldsymbol{C})$ with a new clock. In the case that $M = \boldsymbol{C}$, this fact is called Lévy's conformal invariance of Brownian motion and applied to the studies of complex analysis. Especially B. Davis' proof of Picard's theorem ([**6**], [**7**]) is well-known (see also [**3**]). Remark that if M is assumed only to be Hermitian, this property does not necessarily hold. This property is fundamental also in our paper. We analyze these processes on the target $\boldsymbol{P}^1(\boldsymbol{C})$ in the section 2.1 and on the source manifolds in the section 2.2 independently. Remark that we essentially use Riemannian properties of the manifolds to obtain the results in these two sections. Kählerity of M implies the Lévy's invariance and thus we can synthesize these results to obtain our main results. For this analysis it is useful to write such Nevanlinna's quantities as the characteristic function, the proximity function and the counting function in terms of Brownian motion, while they can be defined in terms of the objects of analysis like Green's functions and harmonic measures. These probabilistic expressions are given in the section 1. The outline of the proofs of our main results presented here is similar to [**2**] where we considered the case of submanifolds in $\boldsymbol{C}^n$. To extend our method to general cases we need some other estimates of Green's functions discussed in the section 2.2.

Our method is also applicable to the case when the target manifolds are general compact Riemann surfaces. Let N be a compact Riemann surface without boundary. If a holomorphic map f from M as above to N satisfies $\lim_{r\to\infty} S(r)/T(r) = 0$, then we have a defect relation:

$$\sum_{i=1}^{q} \delta(a_i, f) \leq \chi(N),$$

where $\chi(N)$ is the Euler characteristic of N.

We would note that there are well-known works on generalization of Nevanlinna theory. H. Wu considered Nevanlinna theory for holomorphic maps between open Riemann surfaces and compact Riemann surfaces ([**22**]). He gave a natural generalization of the classical Nevanlinna theory. Though the method is slightly different from his, our result is in the same direction. W. Stoll considered Nevanlinna theory for meromorphic maps on parabolic manifolds ([**19**]). Note that the class of parabolic manifolds excludes some negatively curved manifolds.

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1. Proximity function, counting function and characteristic function in Nevanlinna theory.

We first give the setting of this paper. Let M be a smooth, complete Kähler manifold with a Kähler form $\omega = (\sqrt{-1}/2)g_{i\bar{j}}dz_i \wedge d\overline{z_j}$ with $m = \dim M$. Let Δ_M be the Laplacian with respect to the Kähler metric. We have a C^2-class, nonnegative exhaustion function u on M. Then $B(r)$ denotes $\{x \in M \mid u(x) < r\}$ and $\partial B(r) = \{x \in M \mid u(x) = r\}$. By Sard's theorem $\partial B(r)$ is a smooth submanifold in M for a.e.r. Then we can consider only smooth $\partial B(r)$ throughout this paper. Let $d\pi_r^x$ be a harmonic measure on $\partial B(r)$ with respect to $x \in B(r)$ and $g_r(x, y)$ be Green's function of $(1/2)\Delta_M$ on $B(r)$ with Dirichlet boundary condition. Let f be a nonconstant meromorphic function on M, namely a holomorphic map from M to $\boldsymbol{P}^1(\boldsymbol{C})$: 1-dimensional complex projective space.

We define proximity function, counting function and characteristic function for meromorphic function f on M.

Fix a reference point o in M with $u(o) < 1$. Let $a \in \boldsymbol{P}^1(\boldsymbol{C})$ such that $f(o) \neq a$. Define

$$m(r, a) = \int_{\partial B(r)} \log[f(z), a]^{-2} d\pi_r^o(z),$$

$$T(r) = c_m \int_{B(r)} g_r(o, z) f^*\omega_o \wedge \omega^{m-1}$$

where ω_o is the Fubini-Study metric on $\boldsymbol{P}^1(\boldsymbol{C})$, $c_m = 2\pi^m/(m-1)!$ and

$$[w, a] = \frac{|w - a|}{\sqrt{|w|^2 + 1}\sqrt{|a|^2 + 1}} \text{ (if } a \neq \infty),$$

$$= \frac{1}{\sqrt{|w|^2 + 1}} \text{ (if } a = \infty).$$

Remark that our definition of the characteristic function follows the manner of Ahlfors-Shimizu (cf. [**20**]) in the classical cases.

Since $\log[f(z), a]^{-2}$ is a δ-subharmonic function, $\Delta_M \log[f(z), a]^{-2}$ can be regarded as a signed measure denoted by $d\mu$. This signed measure $d\mu$, which is called a Riesz charge of $\log[f(z), a]^{-2}$, has a unique Jordan decomposition $d\mu = d\mu_1 - d\mu_2$ (cf. [**11**]). We note that μ_2 is supported by $f^{-1}(a)$. We define counting function of the points $f^{-1}(a)$ by

$$N(r,a) = \frac{1}{2}\int_{B(r)\cap f^{-1}(a)} g_r(o,z)d\mu_2(z).$$

It is easy to see that these quantities coincide with ones defined in [**18**] up to constants in the case when $M = \boldsymbol{C}^n$ with the standard metric. We are seeing this here for the case of $M = \boldsymbol{C}$. In this case we have $g_r(o,z) = (1/\pi)\log(r/|z|)$. Set $A(t) = \int_{|z|<t} f^*\omega_o \wedge \omega^{m-1}$. By integration by parts

$$\begin{aligned} T(r) &= \int_{|z|<r} \frac{1}{\pi}\log\frac{r}{|z|} f^*\omega_o \wedge \omega^{m-1} \\ &= \int_0^r \frac{1}{\pi}\log\frac{r}{t}A'(t)dt \\ &= \frac{1}{\pi}\int_0^r \frac{1}{t}A(t)dt. \end{aligned}$$

Note that if ζ is the zero of $f - a$ with multiplicity $n_f(\zeta)$,

$$\Delta \log[f(z),a]^{-2} = -4\pi n_f(\zeta)\delta_\zeta + \text{positive part}$$

on a neighborhood of ζ. Then the negative part of the Riesz charge of $\log[f(z),a]^{-2}$, $d\mu_2 = \sum_{f(\zeta)=a} 4\pi n_f(\zeta)\delta_\zeta$. Set

$$n(t,a) = \int_{|\zeta|<t} \sum_{f(\zeta)=a} n_f(\zeta)\delta_\zeta.$$

Then

$$\begin{aligned} N(r,a) &= 2\pi \sum_{f(\zeta)=a,\ |\zeta|<r} g_r(o,\zeta)n_f(\zeta) \\ &= 2\sum_{f(\zeta)=a,\ |\zeta|<r} \log\frac{r}{|\zeta|}n_f(\zeta) \\ &= 2\int_0^r \frac{n(t,a)}{t}dt \end{aligned}$$

provided $f(o) \neq a$. Since the harmonic measure $d\pi_r^o(z)$ on $\{|z| = r\}$ is the normilized uniform measure, then

$$m(r,a) = \frac{1}{2\pi}\int_0^{2\pi} \log[f(re^{i\theta}),a]^{-2}d\theta.$$

We use the following probabilistic expression of these quantities since they are convenient for our calculus carried out later. We can define a diffusion process X whose generator is $(1/2)\Delta_M$. We call X a Kähler diffusion on M or Brownian motion on M (cf. [**12**]). Let $\tau_r = \inf\{t > 0 \mid X_t \notin B(r)\}$. Note that simple relationships between harmonic measure, Green's function and hitting distribution, occupation times of Brownian motion, respectively(cf. [**3**]). Let $\phi \in C_b(M)$. Then

$$\int_{\partial B(r)} \phi(z)d\pi_r^o(z) = E_o[\phi(X_{\tau_r})]$$

and

$$\int_{\partial B(r)} \phi(z)g_r(o,z)dV(z) = E_o\left[\int_0^{\tau_r} \phi(X_t)dt\right],$$

where E_o denotes the expectation by the measure P_o of X starting from o and dV is the volume measure defined from the Kähler metric. Let

$$e(x) = \frac{1}{2}\|df\|^2 = 2m\frac{f^*\omega_o \wedge \omega^{m-1}}{\omega^m}. \tag{1.1}$$

Then we have

$$m(r,a) = E_o[\log[f(X_{\tau_r}),a]^{-2}], \tag{1.2}$$

$$T(r) = E_o\left[\int_0^{\tau_r} e(X_s)ds\right]. \tag{1.3}$$

Locally

$$e(z) = \frac{2|df|^2}{(1+|f|^2)^2}(z),$$

where

$$|df|^2 = \sum_{i,j} g^{i\bar{j}}\frac{\partial f}{\partial z_i}\overline{\frac{\partial f}{\partial z_j}}$$

with $(g^{i\bar{j}}) = (g_{i\bar{j}})^{-1}$. Then the characterisitic function can be written as

$$T(r) = E_o\bigg[\int_0^{\tau_r} \frac{2|df|^2}{(1+|f|^2)^2}(X_s)ds\bigg].$$

We remark that the infinity of $\boldsymbol{P}^1(\boldsymbol{C})$ is of logarithmic capacity zero, namely polar for $f(X_t)$ unless f is constant. Thus the local expression of $T(r)$ as above is possible. Our counting function $N(r,a)$ has a probabilistic expression as follows ([**1**]).

$$N(r,a) = \lim_{\lambda\to\infty} \lambda P_o\Big(\sup_{0\le t\le\tau_r} \log[f(X_t),a]^{-2} > \lambda\Big).$$

By a direct calculation we have

$$\frac{1}{2}\Delta_M \log[f(z),a]^{-2} = \frac{2|df|^2}{(1+|f|^2)^2}(z)$$

on the set $\{z|f(z)\ne a\}$. Then Ito's formula or Dynkin's formula implies

Theorem 2 (First main theorem, [**1**], [**2**]).

$$m(r,a) + N(r,a) = T(r) + \log[f(o),a]^{-2} \ \textit{for } a \in \boldsymbol{P}^1(\boldsymbol{C}) \ \textit{with } f(o)\ne a, \quad 0<r<\infty.$$

2. Main lemmas.

We have two essential propositions about some properties of the target $\boldsymbol{P}^1(\boldsymbol{C})$ and the source manifold M. They play main roles for proving our second main theorem.

2.1. An estimate of additive functionals of Brownian motion on $\boldsymbol{P}^1(\boldsymbol{C})$.

Before stating the lemma we recall some facts about Brownian motion on $\boldsymbol{P}^1(\boldsymbol{C})$. We say a diffusion process Y_t whose generator is the half of Laplacian with respect to Fubini-Study metric is Brownian motion on $\boldsymbol{P}^1(\boldsymbol{C})$ specifically. If $f: M \to \boldsymbol{P}^1(\boldsymbol{C})$ is a holomorphic map,

$$f(X_t) = Y_{\rho_t},$$

for some Brownian motion Y_t on $\boldsymbol{P}^1(\boldsymbol{C})$ where

$$\rho_t = \int_0^t e(X_s)ds$$

and e is half of the energy density of f defined by (1.1). We need the following fact valid in general situation.

LEMMA 3 ([**1**]). *Let h be a nonnegative function on a Riemannian manifold N. Let X be Brownian motion on N. If h satisfies*

$$\sup_x E_x\left[\int_0^1 h(X_t)dt\right] < \infty,$$

then there exists a constant $c > 0$ such that

$$E\left[\int_0^T h(X_t)dt\right] \leq cE[T] + c,$$

for any stopping time T.

We use a local estimate of heat kernel for short time as follows:

LEMMA 4 (cf. [**5**]). *Let N be a d-dimensional compact Riemannian manifold without boundary and $p(t,x,y)$ the heat kernel with respect to $(1/2)\Delta_N$. Define*

$$e(t,x,y) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{d(x,y)^2}{2t}},$$

where $d(x,y)$ is the Riemannian distance of N. Then for any compact set K in N there exists a constant $C_K > 0$ such that

$$\lim_{t\to 0} \frac{p(t,x,y)}{e(t,x,y)} = 1 + C_K d(x,y) \qquad x, y \in K.$$

We wish to show

PROPOSITION 5. *Let*

$$k(w) = \frac{1}{\prod_{j=1}^q [w,a_j]^2(\log[w,a_j])^4},$$

for distinct points $a_1, a_2, \ldots, a_q$ *and* Y *the Brownian motion on* $\boldsymbol{P}^1(\boldsymbol{C})$ *with* $Y_0 = y \in \boldsymbol{P}^1(\boldsymbol{C}) \setminus \{a_1, \ldots, a_q\}$. *Then there exist constants* $c_1, c_2 > 0$ *such that for any stopping time* T

$$E_y\left[\int_0^T k(Y_t)dt\right] \leq c_1 E_y[T] + c_2$$

holds.

Proof. By Lemma 3, we have only to check the following in our case.

$$\sup_x E_x\left[\int_0^1 k(Y_t)dt\right] < \infty. \tag{2.1}$$

To check the boundedness of the expectation in (2.1), we may consider only the integration over small neighborhoods of each a_j. Since a function $u \mapsto 1/(u^2(\log u)^4)$ is decreasing for $u \in (0, 1/2)$, we consider the integral

$$\int_0^1 \int_{d(x,y)<\delta} \frac{1}{2\pi t} e^{-\frac{d(x,y)^2}{2t}} \frac{1}{d(x,y)^2(\log d(x,y))^4} dv(y)dt$$

with small $\delta > 0$.
The convergence of this integral is equivalent to that of

$$\int_0^1 \int_0^\delta \frac{1}{2\pi t} e^{-\frac{r^2}{2t}} \frac{r}{r^2(\log r)^4} drdt.$$

Since

$$\int_0^1 \frac{1}{2\pi t} e^{-\frac{r^2}{2t}} dt = \frac{1}{2\pi} \int_{r^2/2}^\infty e^{-u} \frac{du}{u} \sim -\text{const.} \log r \quad (r \to 0),$$

the above integral is bounded by

$$C \int_0^\delta \frac{1}{r|\log r|^3} dr < \infty,$$

where $C > 0$ is a constant independent of x. □

We have the following corollary to Proposition 5.

Corollary 6. *If f is a nonconstant meromorphic function on M, then we have*

$$E_o\left[\int_0^{\tau_r} k\circ f(X_t)e(X_t)dt\right]\le c_1T(r)+c_2,$$

where $\tau_r=\{t>0\mid X_t\notin B(r)\}$ *and* $e(x)$ *is given by* (1.1).

Proof. There exists Brownian motion Y on $\boldsymbol{P}^1$ such that

$$f(X_t)=Y_{\rho_t},\ \text{with}\ \rho_t=\int_0^t e(X_s)ds.$$

Take this Y as one in Proposition 5 and $T=\rho_{\tau_r}$. By (1.1) and (1.3), $E_o[\rho_{\tau_r}]=T(r)$. Use time-change argument. □

2.2. Dirichlet Green's functions and harmonic measure on M.

The following inequality is a key point of our proof of the second main theorem.

Proposition 7. *Let X be a Kähler diffusion on M as above and $x\in B(r)$. Assume that K is a nonnegative locally integrable function on M and bounded on a neighborhood of x. For $\delta>0$ there exits $C(x,r,\delta)>0$ independent of K and $E_\delta\subset[0,\infty)$ with $|E_\delta|<\infty$ such that for $r\notin E_\delta$*

$$E_x[K(X_{\tau_r})]\le C(x,r,\delta)\left(E_x\left[\int_0^{\tau_r}K(X_t)dt\right]\right)^{(1+\delta)^2}$$

holds.

To get this we have to observe harmonic measures and Green's functions with Dirichlet boundary condition on $B(r)$. We first consider the harmonic measure on $\partial B(r)$. We assume that $\partial B(r)$ is a smooth submanifold of M. Let $P_r(x,y)$ be the Poisson kernel on $\partial B(r)$, namely

$$P_r(x,y)=\frac{d\pi_r^x(y)}{dA_r(y)},$$

where $d\pi_r^x(y)$ is the harmonic measure on $\partial B(r)$ with respect to x and dA_r is the induced measure on $\partial B(r)$.

To estimate $P_r(x,y)$ we use the following estimate originally due to Kasue ([**14**]).

LEMMA 8. *There exist constants $a_2(x,r)$, $a_3(r) > 0$ and $0 < \alpha < r$ such that*

$$g_r(x,y) \le a_2(x,r)\phi_1(y) \quad (y \in B(r) \setminus B(\alpha)), \tag{i}$$

$$\sup_{x\in\partial B(r)} \|\nabla\phi_1\|(x) \le a_3(r)\lambda_r \sup_{x\in B(r)} \phi_1(x), \tag{ii}$$

where ϕ_1 is the grand state corresponding to the first eigenvalue $\lambda_r > 0$ of the Dirichlet Laplacian on $B(r)$ with $\phi_1 > 0$ and $\int_{B(r)} \phi_1(x)^2 dv(x) = 1$.

PROOF. For $x \in B(r)$ there exists $\alpha > 0$ such that $u(x) < \alpha$. Set

$$a_2(x,r) = \frac{\sup_{y\in\partial B(\alpha)} g_r(x,y)}{\inf_{\partial B(\alpha)} \phi_1(x)}.$$

Then since ϕ_1 is a superharmonic function on $B(r)$ and $g_r(x,y)$ is harmonic function on $B(r)\setminus B(\alpha)$, maximum principle implies the assertion (i). As for (ii), Kasue showed in [**14**] (Corollary 3.1) that

$$\sup_{\partial B(r)} \|\nabla\phi_1\|(x) \le \int_0^{\mathscr{I}_{B(r)}} \lambda_r \sup_{x\in\partial B(t)} \phi_1(x) h_{R,\Lambda}^{m-1}(t)dt,$$

where $h_{R,\Lambda}(t)$ is the solution of

$$h''(t) + Rh(t) = 0, \quad \text{with } h(0) = 1, \quad \text{and } h'(0) = \Lambda,$$

R is the lower bound of Ricci curvature of $B(r)$, Λ is the upper bound of the trace of second fundamental form of $\partial B(r)$ and $\mathscr{I}_{B(r)} = \sup_{B(r)} d(x, \partial B(r))$. Hence we can take

$$a_3(r) = \int_0^{\mathscr{I}_{B(r)}} h_{R,\Lambda}^{m-1}(t)dt. \qquad \square$$

Define

$$C_1(x,r) = a_2(x,r)a_3(r)\lambda_r \sup_{B(r)} \phi_1(x). \tag{2.2}$$

Then we have

Lemma 9. *For each* $x \in B(r)$

$$P_r(x,y) \le C_1(x,r).$$

Proof. Take the normal derivative of both sides of (i) in Lemma 8. □

Remark 10. If we have a subharmonic exhaustion function u, then we can give the constant $C_1(x,r)$ defined by (2.2) in a simpler form. We remark that there always exists a subharmonic exhaustion function on any complete Riemannian manifold ([**8**]).

Suppose that u is a subharmonic exhaustion function and u is smooth. If $u(x) < \alpha < r$, for $y \in \partial B(r)$

$$P_r(x,y) \le \frac{\sup_{y\in\partial B(\alpha)} g_r(x,y)}{r-\alpha}\|\nabla u\|(y).$$

Proof. Set

$$v(y) = \sup_{z\in\partial B(\alpha)} g_r(x,z)\frac{r-u(y)}{r-\alpha}.$$

Then v is superharmonic on $B(r)\setminus\overline{B(\alpha)}$ and $v(y)=0$ on $\partial B(r)$. Then $v(y) \ge g_r(x,y)$ on $B(r)\setminus B(\alpha)$. Take normal derivatives of both sides. □

We also need a lower estimate of the Green's function. Define $\sigma_r = \inf\{t > 0 : X_t \in B(r)\}$.

Lemma 11. *Assume that* $\|\nabla u\|(x) \neq 0$ *for* $x \in B(r_2)\setminus B(r_1)$. *If* $\mu^{(0)}(t)$ *is a Lipschitz continuous function on* $(0,\infty)$ *such that*

$$\frac{1}{2}\sup_{x\in\partial B(t)} \frac{\Delta_M u}{\|\nabla u\|^2}(x) \le \mu^{(0)}(t) \tag{2.3}$$

for $t \in [r_1, r_2]$, *then*

$$P_y(\sigma_{r_1} < \tau_{r_2}) \ge \frac{\int_{u(y)}^{r_2} \exp\big(-\int_{r_1}^{t} 2\mu^{(0)}(z)dz\big)dt}{\int_{r_1}^{r_2} \exp\big(-\int_{r_1}^{t} 2\mu^{(0)}(z)dz\big)dt}$$

for $y \in B(r_2)\setminus B(r_1)$.

PROOF. By Ito's formula

$$u(X_t) - u(X_0) = B\biggl(\int_0^t \|\nabla u\|^2(X_s)ds\biggr) + \frac{1}{2}\int_0^t \Delta_M u(X_s)ds,$$

where $B(t)$ is a one dimensional standard Brownian motion. Set $\eta_t = \int_0^t \|\nabla u\|^2(X_s)ds$. Define a new process $\tilde{X}_t$ by $\tilde{X}_{\eta_t} = X_t$. Then

$$u(\tilde{X}_t) - u(X_0) = B(t) + \frac{1}{2}\int_0^t \frac{\Delta_M u}{\|\nabla u\|^2}(\tilde{X}_s)ds.$$

Define another diffusion process x_t by the solution of the following stochastic differential equation:

$$dx_t = dB(t) + \mu_0(x_t)dt, \quad x_0 = u(X_0).$$

Set $\tilde{\tau}_r = \inf\{t > 0 : x_t > r\}$ and $\tilde{\sigma}_r = \inf\{t > 0 : x_t < r\}$. Similarly to [**21**] (see also Chapter 6.4 of [**12**]), by the comparison theorem of stochastic differential equations we have

$$x_t \geq \tilde{X}_t \text{ for } t \leq \eta_{\sigma_{r_1}} \wedge \tilde{\tau}_{r_2}.$$

Hence

$$P_y(\sigma_{r_1} < \tau_{r_2}) = P_y(\eta_{\sigma_{r_1}} < \eta_{\tau_{r_2}}) \geq P_{u(y)}(\tilde{\sigma}_{r_1} < \tilde{\tau}_{r_2}).$$

By the formula of the scale functions of one dimensional diffusion processes (cf. [**13**]) we have

$$P_{u(y)}(\tilde{\sigma}_{r_1} < \tilde{\tau}_{r_2}) = \frac{\int_{u(y)}^{r_2} \exp\bigl(-\int_{r_1}^t 2\mu^{(0)}(z)dz\bigr)dt}{\int_{r_1}^{r_2} \exp\bigl(-\int_{r_1}^t 2\mu^{(0)}(z)dz\bigr)dt}. \qquad \square$$

LEMMA 12. *There exists r' with $r > r' > 0$ such that*

$$g_r(x,y) \geq C_2(x,r)\int_{u(y)}^{r} \exp\biggl(-\int_{u(x)}^t 2\mu(z)dz\biggr)dt$$

with $u(x) < u(y)$ and $u(x) \leq r'$. Here $C_2(x,r)$ is defined by

$$C_2(x,r) = \inf_{y\in\partial B(r')} g_r(x,y)\Big(\int_{u(x)}^r \exp\Big(-\int_{u(x)}^t 2\mu(z)dz\Big)dt\Big)^{-1},$$

where $\mu(t)$ is defined by

$$\mu(t) = 0 \ \ \textit{for } 0 \le t < r', \qquad \mu(t) = \mu^{(0)}(t) \ \ \textit{for } r' \le t < r$$

and $\mu^{(0)}(t)$ is a Lipschitz continuous function satisfying (2.3) *in Lemma* 11 *on* $[r', r]$.

Remark that if u is smooth enough (e.g. C^3-class), we can take

$$\mu^{(0)}(z) = \frac{1}{2}\sup_{x\in\partial B(z)} \Delta_M u/\|\nabla u\|^2(x).$$

PROOF. Recall that we assume $\partial B(r)$ is smooth and $\|\nabla u\| \neq 0$ there. Hence there exists $r > r' > 0$ such that $\|\nabla u\| \neq 0$ on $B(r)\setminus B(r')$.

Let $y \in B(r)\setminus B(r')$. By maximum principle, we have

$$g_r(x,y) \ge \inf_{z\in\partial B(r')} g_r(x,z)P_y(\sigma_{r'} < \tau_r).$$

By Lemma 11

$$P_y(\sigma_{r'} < \tau_r) \ge \frac{\int_{u(y)}^r \exp\big(-\int_{r'}^t 2\mu^{(0)}(z)dz\big)dt}{\int_{r'}^r \exp\big(-\int_{r'}^t 2\mu^{(0)}(z)dz\big)dt}.$$

For $y \in B(r')$ we have $g_r(x,y) \ge \inf_{z\in\partial B(r')} g_r(x,z)$, since $g_r(x,y)$ is a superharmonic function in y. □

PROOF OF PROPOSITION 7. Set

$$\gamma_r(s) = \int_s^r \exp\Big(-\int_{u(x)}^t 2\mu(z)dz\Big)dt.$$

$$\begin{aligned} E_x\Big[\int_0^{\tau_r} K(X_t)dt\Big] &= \int_{B(r)} K(y)g_r(x,y)dv(y) \\ &\ge C_2(x,r)\int_{B(r)} K(y)\gamma_r(u(y))dv(y) \\ &= C_2(x,r)\int_0^r \gamma_r(t)\int_{\partial B(t)} K(y)\frac{dA_t(y)}{\|\nabla u\|(y)}dt \end{aligned}$$

holds for a.e. $r > 0$. In the last line we used the co-area formula.

Define for $\delta > 0$

$$E_\delta = \left\{ r \in (0,\infty) \middle| \int_{\partial B(r)} K(y) \frac{dA_r(y)}{\|\nabla u\|(y)} \right. \\ > \exp\left((1+\delta) \int_{u(x)}^r 2\mu(z)dz \right) \\ \left. \cdot \left(\int_0^r \gamma_r(t) \int_{\partial B(t)} K(y) \frac{dA_r(y)}{\|\nabla u\|(y)} dt \right)^{(1+\delta)^2} \right\}.$$

As usual argument on Nevanlinna theory (cf. [**18**], [**20**]), it is easy to see that E_δ is a disjoint union of intervals and the total length of E_δ is finite. Thus if $r \notin E_\delta$,

$$\int_{\partial B(r)} K(y) dA_r(y) \le \frac{C_3(x,r)}{C_2(x,r)^{(1+\delta)^2}} \left(\int_{B(r)} K(y) g_r(x,y) dv(y) \right)^{(1+\delta)^2},$$

where

$$C_3(x,r,\delta) = \left(\sup_{x \in B(r)} \|\nabla u\|(x) \right) \exp\left((1+\delta) \int_{u(x)}^r 2\mu(z)dz \right). \tag{2.4}$$

On the other hand, by Lemma 9

$$E_x[K(X_{\tau_r})] = \int_{\partial B(r)} K(y) d\pi_r^x(y) \\ \le C_1(x,r) \int_{\partial B(r)} K(y) dA_r(y).$$

Hence we can take

$$C(x,r,\delta) = \frac{C_1(x,r) C_3(x,r,\delta)}{C_2(x,r)^{(1+\delta)^2}}. \tag{2.5}$$

□

3. Second main theorem.

Let $R = (R_{ij})$ be the Ricci curvature of M. Define $R(x)$ by

$$R(x) = \inf_{\xi \in T_x M,\ \|\xi\|=1} R(\xi, \xi)$$

and

$$N(r, Ric) = -\int_{B(r)} g_r(o, z) R(x) dv(x).$$

We also define

$$N_1(r) = \lim_{\lambda \to \infty} \lambda P_o\Big(\sup_{0 \le t \le \tau_r} \log^- e(X_t) > \lambda \Big),$$

where $\log^- e(X)$ denotes the negative part of $\log e(X)$. Then we restate our second main theorem using these quantities.

Theorem 13. *Let f be a nonconstant meromorphic function on a complete Kähler manifold M and $a_1, \ldots, a_q$ distinct points in $\boldsymbol{C} \cup \{\infty\}$. Let $\delta > 0$. Then*

$$\sum_{j=1}^{q} m(r, a_j) + N_1(r) \le 2T(r) + 2N(r, Ric) + \log C(o, r, \delta) + O(\log T(r))$$

holds for all r except on a disjoint union of intervals of finite total length depending on δ, where $C(o, r, \delta)$ is given by (2.5) *in Proposition* 7.

Theorem 1 follows from Theorem 13 with

$$S(r) = \log C(o, r, \delta) + 2N(r, Ric). \tag{3.1}$$

In this section we give the proof of this theorem.

By the same calculation as in [**17**] (Theorem 4.1) or [**16**], we have

Lemma 14.

$$\frac{1}{2} \Delta_M \log e(x) \ge -2e(x) + 2R(x)$$

if $e(x) \neq 0$.

For the proof of Theorem 13, we need

LEMMA 15.

$$E_o[\log(k(f(X_{\tau_r}))e(X_{\tau_r}))]$$
$$\geq \sum_{i=1}^{q} m(r,a_i) + N_1(r) - 2T(r) - 2N(r,Ric) - const. \log T(r),$$

where $k(w) = \prod_{j=1}^{q}[w,a_j]^{-2}(\log[w,a_j])^{-4}$.

PROOF. By Ito's formula or the first main theorem and Jensen's inequality it is easy to see that

$$E_o[\log(k(f(X_{\tau_r}))e(X_{\tau_r}))]$$
$$= \sum_{i=1}^{q}\{m(r,a_i) + E[\log(\log[f(X_{\tau_r}),a_i]^{-2})^{-4}\} + E[\log e(X_{\tau_r})]$$
$$\geq \sum_{i=1}^{q} m(r,a_i) - 4q\log T(r) + E[\log e(X_{\tau_r})] + O(1).$$

Note that $\log e(x)$ is upper bounded on $B(r)$. Thus the singularity of the function can be reduced to that of $\log^- e(x)$. Define $T_\lambda = \inf\{t > 0 : \log^- e(X_t) > \lambda\}$. By Ito's formula and Lemma 14 we have

$$E_o[\log e(X_{\tau_r\wedge T_\lambda})] = E_o\left[\frac{1}{2}\int_0^{\tau_r\wedge T_\lambda} \Delta_M \log e(X_t)dt\right]$$
$$\geq -2E_o\left[\int_0^{\tau_r\wedge T_\lambda} e(X_t)dt\right] + 2E_o\left[\int_0^{\tau_r\wedge T_\lambda} R(X_t)dt\right].$$

The left hand side equals

$$E_o[\log^+ e(X_{\tau_r})] - E_o[\log^- e(X_{\tau_r}) \ : \ \tau_r \leq T_\lambda] - \lambda P_o(\tau_r > T_\lambda).$$

Note $P_o(\tau_r > T_\lambda) = P_o(\sup_{0\leq t\leq \tau_r} \log^- e(X_t) > \lambda)$ and $P_o(\tau_r \leq T_\lambda) \to 1$ as $\lambda \to \infty$ since the set of the critical points of f is polar provided f is nonconstant. Letting $\lambda \to \infty$, by monotone convergence theorem we have

$$E_o[\log e(X_{\tau_r})] \geq N_1(r) - 2N(r,Ric) - 2T(r). \qquad \square$$

Proof of Theorem 13. From the above lemma we have to show that

$$E_o[\log(k(f(X_{\tau_r}))e(X_{\tau_r}))] \le \log C(o,r,\delta) + O(\log T(r))$$

holds except for r in a set of finite Lebesgue measure.

For $\delta > 0$ there exists $E_\delta \subset [0,\infty)$ with $|E_\delta| < \infty$ such that for $r \notin E_\delta$

$$\begin{aligned} E_o[\log(k(f(X_{\tau_r}))e(X_{\tau_r}))] &\le \log E_o[k(f(X_{\tau_r}))e(X_{\tau_r})] \quad \text{(by Jensen's inequality)} \\ &\le (1+\delta)^2 \log E_o\left[\int_0^{\tau_r} k(f(X_s))e(X_s)ds\right] + \log C(o,r,\delta) \\ &\qquad\qquad \text{(by Proposition 7)} \\ &\le (1+\delta)^2 \log T(r) + \log C(o,r,\delta) + O(1). \end{aligned}$$

In the last inequality we used Corollary 6. □

4. A generalization of the target manifolds.

We extend the above discussion to the cases where the target manifolds are general one dimensional compact Kähler manifolds.

Let N be a compact Riemann surface equipped with a metric ds_N^2 and its Kähler form ω_N. As before we can define Brownian motion W on N whose generator is $(1/2)\Delta_N$, where Δ_N is the Laplacian on N defined from ds_N^2. Let dv_N be the volume form defined from ω_N. W has a transition function $p(t,x,y)$ which is the fundamental solution of heat operator $\partial/\partial t - 1/2\Delta_N$ with respect to dv_N. We first introduce a new proximity function. Set

$$g(x,y) = 2\pi \int_0^\infty \left(p(t,x,y) - \frac{1}{Vol_N(N)}\right)dt + C,$$

where $Vol_N(N)$ is the volume of N with respect to dv_N and C is a positive constant such that $g(x,y) > 0$ for all $x, y \in N$.

We know

Lemma 16 (cf. [**4**]).

$$\frac{1}{2}\Delta_N g(x,y) = -2\pi\delta_x(y) + \frac{2\pi}{Vol_N(N)}.$$

Then the similar argument to the previous sections implies that

Proposition 17. *For $x,\ y \in N$ with $x \neq y$ and any stopping time T with $E_x[T] < \infty$,*

$$E_x[g(W_T, y)] - g(x, y) + N(T) = \frac{2\pi}{Vol_N(N)} E_x[T],$$

where

$$N(T) = \lim_{\lambda \to \infty} \lambda P_x\Big(\sup_{0 \leq t \leq T} g(W_t, y) > \lambda \Big).$$

We also note that for fixed $x \in N$

$$g(x, y) = \log \frac{1}{d(x, y)^2} + \text{a smooth function of } y$$

if y is in a neighborhood of x.

Let M a complete Kähler manifold satisfying the same assumptions as the previous sections and $f : M \to N$ be a nonconstant holomorphic map. We define the proximity function using $g(x, y)$ as before. Define

$$m(r, a) = \int_{\partial B(r)} g(f(x), a) d\pi_r^o(x)$$

for $a \in N$ with $f(o) \neq a$.

Define the characteristic function of f by

$$T(r) = c_m \int_{B(r)} g_r(o, y) f^* \omega_N \wedge \omega,$$

where $c_m = 2\pi^m/(m-1)!$ with $m = \dim M$.
It can be expressed as

$$\begin{aligned} T(r) &= \int_{B(r)} g_r(o, y) e(y) dv_M(y) \\ &= E_o\Big[\int_0^{\tau_r} e(X_s) ds \Big] \end{aligned}$$

where $e(x) = tr_M f^* ds_N^2$ as before.

Note that $g(f(x), a)$ is a δ-subharmonic function on M. It has a negative part of its Riesz charge, say $d\nu$, which is supported by $f^{-1}(a)$. Hence the counting function $N(r, a)$ can be defined by

$$N(r, a) = \frac{1}{2} \int_{B(r)} g_r(o, y) d\nu(y).$$

By Lemma 16

$$d\nu = 4\pi f^* \delta_a.$$

As the case of $\boldsymbol{P}^1(\boldsymbol{C})$, there exists a Brownian motion W on N such that

$$f(X_t) = W_{\rho_t} \qquad \text{with} \qquad \rho_t = \int_0^t e(X_s) ds.$$

Set $T = \int_0^{\tau_r} e(X_s) ds$ in Proposition 17. Then we have

Theorem 18 (First main theorem).

$$m(r, a) - g(f(o), a) + N(r, a) = \frac{2\pi}{Vol_N(N)} T(r).$$

As for the second main theorem we can carry out the same procedure as the case of $\boldsymbol{P}^1(\boldsymbol{C})$. Let $a_1, \ldots, a_q$ be distinct points in N. Define

$$h(x) = \prod_{j=1}^{q} e^{g(x, a_j)} (g(x, a))^{-4}.$$

Thanks to Lemma 4 and the asymptotics of $g(x, a)$ as $d(a, x) \to 0$, we can check the validity of

$$\sup_{y \in N} E_y \left[\int_0^1 h(W_t) dt \right] < \infty.$$

Then Proposition 5 is applicable to this case. Hence we have

$$E_o \left[\int_0^{\tau_r} h \circ f(X_t) e(X_t) dt \right] \leq c_1 T(r) + c_2$$

for some constants $c_1 > 0$, $c_2 > 0$.

Note that Lemma 14 holds for general cases by replacing "2" in the term before $-e(x)$ by $K(x)$: the Gaussian curvature of N. Namely,

LEMMA 19 (cf. [**16**], [**17**]).

$$\frac{1}{2}\Delta_M \log e(x) \geq -K(x)e(x) + 2R(x)$$

for $x \in M$ *such that* $e(x) \neq 0$.

Hence we have

$$\begin{aligned}
&E_o[\log(h(f(X_{\tau_r}))e(X_{\tau_r}))] \\
&\quad \geq \sum_{i=1}^{q} m(r, a_i) + N_1(r) - E_o\left[\int_0^{\tau_r} K \circ f(X_t)e(X_t)dt\right] \\
&\quad\quad - 2N(r, Ric) - \text{const.} \log T(r).
\end{aligned}$$

Set $u(x) = \int_N g(x,y)K(y)dv_N(y)$. Note that u is bounded on N. Let W be a Brownian motion on N starting from w and T a stopping time with $E_w[T] < \infty$. By Proposition 16 and Ito's formula, we have

$$E_w[u(W_T)] - u(w) = -2\pi E_w\left[\int_0^T K(W_t)dt\right] + \frac{2\pi C(N)}{Vol_N(N)}E_w[T],$$

where $C(N)$ is the total curvature of N defined by

$$C(N) = \int_N K(x)dv_N(x).$$

Recall the expression that $f(X_t) = W_{\rho_t}$, which is described before Theorem 18. By the time-change argument as we used before,

$$\begin{aligned}
&E_o[u(f(X_{\tau_r}))] - u(f(o)) \\
&\quad = -2\pi E_o\left[\int_0^{\tau_r} K(f(X_t))e(X_t)dt\right] + \frac{2\pi C(N)}{Vol_N(N)}E_o\left[\int_0^{\tau_r} e(X_t)dt\right].
\end{aligned}$$

Recall $E_o[\int_0^{\tau_r} e(X_t)dt] = T(r)$. Hence we have

$$E_o\left[\int_0^{\tau_r} K(f(X_t))e(X_t)dt\right] = \frac{C(N)}{Vol_N(N)}T(r) + O(1).$$

Therefore we have a second main theorem in this case.

Theorem 20. *Let f be a nonconstant holomorphic map from a complete Kähler manifold M to a compact Riemann surface N without boundary and $a_1, \ldots, a_q$ $(\neq f(o))$ distinct points in N. Let $\delta > 0$. Then*

$$\sum_{j=1}^{q} m(r, a_j) + N_1(r) \leq \frac{C(N)}{Vol_N(N)}T(r) + 2N(r, Ric) + \log C(o, r, \delta) + O(\log T(r))$$

holds for all r except on a disjoint union of intervals of finite total length depending on δ, where $C(o, r, \delta)$ is given by (2.5).

Taking account of Theorem 18 we define the defect $\delta(a, f)$ by

$$\delta(a, f) = \frac{Vol_N(N)}{2\pi} \liminf_{r\to\infty} \frac{m(r, a)}{T(r)}.$$

Then we have

Theorem 21 (Defect relation). *If $T(r) \to \infty$ as $r \to \infty$, then*

$$\sum_{i=1}^{q} \delta(a_i, f) \leq \frac{C(N)}{2\pi} + \frac{Vol_N(N)}{2\pi} \limsup_{r\to\infty} \frac{2N(r, Ric) + \log C(o, r, \delta)}{T(r)}.$$

By Gauss-Bonnet theorem, we have

Corollary 22.

$$\sum_{i=1}^{q} \delta(a_i, f) \leq \chi(N) + \frac{Vol_N(N)}{2\pi} \limsup_{r\to\infty} \frac{2N(r, Ric) + \log C(o, r, \delta)}{T(r)},$$

where $\chi(N)$ is the Euler characteristic of N.

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Atsushi Atsuji
Keio University
4-1-1, Hiyoshi
Yokohama, 223-8521, Japan
E-mail: atsuji@econ.keio.ac.jp