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The Turing degrees for some computation model with the real parameter

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Abstract. L. Blum, M. Shub and S. Smale defined a kind of computation model having parameters that take real values [**2**]. In this paper we will extend their computation theory on the basis of the conventional recursion theory which is constructed on the domain of integers. In particular, we will define the Turing degree for the set of reals similarly to the Turing degree for the set of integers, so as to derive some related results.

1. Definitions, Notations and Basic properties.

In order to define computation theory on real values, we will first introduce a kind of programming language and define a function that is computable with this programming language.

Definition (Program language $PL(\boldsymbol{R})$).

1. Variable symbols.

◦ $N_0, N_1, \ldots$	Integer type input output variables
◦ $TN_0, TN_1, \ldots$	Integer type auxiliary variables
◦ $A_0, A_1, \ldots$	Real type input output variables
◦ $TA_0, TA_1, \ldots$	Real type auxiliary variables
◦ $X_0, X_1, \ldots$	Finite real sequence type input output variables
◦ $TX_0, TX_1, \ldots$	Finite real sequence type auxiliary variables

Note:

1) Integer type variables take non-negative integer values.
2) If X is a finite real sequence type variable, then
   2.1) the value of X is a finite sequence of reals.
   2.2) Let $X[i]$ represent the i–th element of X.
   2.3) Let size(X) represent the length of the sequence X.
   2.4) Assume that $X[i] = 0$ if $i \geq$ size(X) holds.

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2. Statements.

Let N be an integer type variable, A, B, C and B_i $(i = 1, 2, \ldots)$ be real type variables, and X, Y be finite real sequence type variables.

S1)	$N := N + 1$	S2)	$N := N - 1$
S3)	IF $N \neq 0$ GOTO s	S4)	$A := B + C$
S5)	$A := B - C$	S6)	$A := B \cdot C$
S7)	$A := B/C$	S8)	$A := B$
S9)	$A := \alpha \quad (\alpha \in \boldsymbol{R})$	S10)	IF $A \leq B$ GOTO s
S11)	$A := X[N]$	S12)	$X[N] := A$
S13)	$N := \mathrm{size}(X)$	S14)	$X := \langle B_0, B_1, \ldots, B_{n-1} \rangle$
S15)	$X := \langle \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \rangle$ $(\alpha_i \in \boldsymbol{R})$	S16)	$X := Y$
S17)	END		

A program written in $PL(\boldsymbol{R})$ has line numbers assigned to every code line. Now initialize the variables as

$$N_0 = x_0, N_1 = x_1, \ldots, N_{n-1} = x_{n-1}$$
$$A_0 = \alpha_0, A_1 = \alpha_1, \ldots, A_{m-1} = \alpha_{m-1}$$
$$X_0 = Z_0, X_1 = Z_1, \ldots, X_{l-1} = Z_{l-1}$$

(0 for the rest of the variables) for input $\langle \vec{x}, \vec{\alpha}, \vec{Z} \rangle \in \omega^n \times \boldsymbol{R}^m \times (\boldsymbol{R}^{<\omega})^l$, run the program starting with the statement of line number 1, and halt at the line of 'END' statement. The output $(\in \omega^{n'} \times \boldsymbol{R}^{m'} \times (\boldsymbol{R}^{<\omega})^{l'})$ gives values of $N_0, N_1, \ldots, N_{n'-1}, A_0, A_1, \ldots, A_{m'-1}, X_0, X_1, \ldots, X_{l'-1}$.

DEFINITIONS (B.S.S. recursive map, B.S.S. recursive set, B.S.S. recursive predicate). Assume that $\mathfrak{X} = \omega^n \times \boldsymbol{R}^m \times (\boldsymbol{R}^{<\omega})^l$ and $\mathfrak{Y} = \omega^{n'} \times \boldsymbol{R}^{m'} \times (\boldsymbol{R}^{<\omega})^{l'}$.

- When a partial map $f : \mathfrak{X} \to \mathfrak{Y}$ is computable with a $PL(\boldsymbol{R})$ program p, then f is said to be a B.S.S. recursive map.
- When the characteristic function $\chi_A : \mathfrak{X} \to \{0, 1\}$ of the set A of $\mathfrak{X}$ is a total B.S.S. recursive function, then A is said to be a B.S.S. recursive set.
- When the set $\{\mathfrak{A} \in \mathfrak{X} | P(\mathfrak{A})\}$ of the predicate P on $\mathfrak{X}$ is a B.S.S. recursive set, then P is said to be a B.S.S. recursive predicate.

The above definition of B.S.S. recursive appears to be different from the one given in [**2**]. However it is easy to see that they are identical when we restrict the parameters to the real numbers.

DEFINITION (Computation with oracle). When $S \subseteq \mathfrak{X} = \omega^n \times \boldsymbol{R}^m \times (\boldsymbol{R}^{<\omega})^l$ is a set, a new programming language $PL(\boldsymbol{R})^S$ is defined as $PL(\boldsymbol{R})$ plus the following new type of statement:

SO) IF $(\langle M_0, \ldots, M_{n-1}, B_0, \ldots, B_{m-1}, Y_0, \ldots, Y_{l-1}\rangle \in S)$ GOTO s

M_i's are integer type variables, B_j's are real type variables, and Y_k's are finite real sequence type variables.

A partial map which is computable with $PL(\boldsymbol{R})^S$ is a B.S.S. recursive map in S. B.S.S. recursive sets in S and B.S.S. recursive predicates in S are defined similarly.

Definitions (Coding, Index). Programs written in $PL(\boldsymbol{R})$ or $PL(\boldsymbol{R})^S$ can be coded in the form of $\langle e, E\rangle$ ($e \in \omega$, $E \in \boldsymbol{R}^{<\omega}$) by employing a proper coding system. When a B.S.S. recursive map in S is computable with a program which has $\langle e, E\rangle$ as a code, $\langle e, E\rangle$ is called the index of f, and f is expressed as $f = \{\langle e, E\rangle\}^S$.

Definition (Computation path). Assume that f is a B.S.S. recursive map in S which has index $\langle e, E\rangle$, then:

- If the calculation of $\langle e, E\rangle$ for input $\mathfrak{A} = \langle \vec{x}, \vec{\alpha}, \vec{Z}\rangle$ halts at the T-th step, the sequence $p = \langle l_0, l_1, \ldots, l_{T-1}\rangle$ which contains the successive line numbers executed for the computation of $\{\langle e, E\rangle\}^S(\mathfrak{A})$ is called the computation path of $\{\langle e, E\rangle\}^S(\mathfrak{A})$.
- Let $V_p^{\langle e,E\rangle,S} = \{\mathfrak{A} \mid$ the computation path of $\{\langle e, E\rangle\}^S(\mathfrak{A})$ is $p\}$.

Definition (B.S.S. r.e. set).

- When f is a B.S.S. recursive map in S which has index $\langle e, E\rangle$, the event of the program $\langle e, E\rangle$ halting at input $\mathfrak{A}$ is denoted as $f(\mathfrak{A})\downarrow$, and the event of the program $\langle e, E\rangle$ not halting at input $\mathfrak{A}$ is denoted as $f(\mathfrak{A})\uparrow$. It is also assumed that $\mathrm{dom}(f) = \{\mathfrak{A} \mid f(\mathfrak{A})\downarrow\}$.
- If there exists a B.S.S. recursive map f in S for a set Ω so that $\Omega = \mathrm{dom}(f)$, then Ω is called a B.S.S. r.e. set in S. When $\{\mathfrak{A} \mid A(\mathfrak{A})\}$ is a B.S.S. r.e. set in S for predicate A, A is called a B.S.S. r.e. predicate in S.
- When $S = \emptyset$, we simply say B.S.S. r.e. set/predicate instead of B.S.S. set/predicate in $\emptyset$.

Definition (basic semi algebraic set).

- A subset X of $\boldsymbol{R}^m$ is called a basic semi algebraic set when there exist rational functions $h^1_1(x), \ldots, h^1_{n_1}(x)$, $h^2_1(x), \ldots, h^2_{n_2}(x)$ such that

$$x \in X \iff$$

$$h^1_1(x) > 0 \;\&\; \cdots \;\&\; h^1_{n_1}(x) > 0 \quad \& \quad h^2_1(x) \geq 0 \;\&\; \cdots \;\&\; h^2_{n_2}(x) \geq 0.$$

- A map $h : \boldsymbol{R}^n \to \boldsymbol{R}^m$, $h = (h_1, h_2, \dots, h_m)$ is said to be a rational map if every element h_i is a rational function.
- Let $S \subseteq \boldsymbol{R}^n$ be a set. A subset X of $\boldsymbol{R}^m$ is called a basic semi algebraic set in S when there exist rational functions $h^1_1(x), \dots, h^1_{n_1}(x)$, $h^2_1(x), \dots, h^2_{n_2}(x)$ and rational maps $h^3_1(x), \dots, h^3_{n_3}(x)$, $h^4_1(x), \dots, h^4_{n_4}(x)$ such that:

$$
\begin{aligned}
&x \in X \iff \\
&\qquad h^1_1(x) > 0 \ \& \ \cdots \ \& \ h^1_{n_1}(x) > 0 \quad \& \quad h^2_1(x) \geq 0 \ \& \ \cdots \ \& \ h^2_{n_2}(x) \geq 0 \\
&\quad \& \ h^3_1(x) \in S \ \& \ \cdots \ \& \ h^3_{n_3}(x) \in S \quad \& \quad h^4_1(x) \notin S \ \& \ \cdots \ \& \ h^4_{n_4}(x) \notin S.
\end{aligned}
$$

THEOREM. *Let f be a B.S.S. recursive map in S with the index $\langle e, E\rangle$.*

(1) $V_p^{\langle e,E\rangle,S}$ *is a basic semi algebraic set in S, and the coefficients of the rational maps of its definition are recursively determined by e, E and p.*
(2) *$dom(f) = \bigcup_{p\in\omega} V_p^{\langle e,E\rangle,S}$. So every B.S.S. r.e. set in S is a countable union of basic semi algebraic sets in S.*
(3) *The restriction of f to $V_p^{\langle e,E\rangle,S}$ is the rational map.*

In the theory of B.S.S. recursive functions, many theorems from the classical version of the theory ([**6**, Chapter 6 and 7]) remain valid. We present three of these theorems.

THEOREM (Enumeration Theorem). *Let $T^S_{n,m,l}(e, E, \mathfrak{A}, p) \equiv$ "$\mathfrak{A} \in V_p^{\langle e,E\rangle,S}$" ($\mathfrak{A} \in \mathfrak{X} = \omega^n \times \boldsymbol{R}^m \times (\boldsymbol{R}^{<\omega})^l$), then the predicate $T^S_{n,m,l}(e, E, \mathfrak{A}, p)$ is a B.S.S. recursive predicate in S. And if $A \subseteq \mathfrak{X}$ is a B.S.S. r.e. set in S, there exists $\langle e, E\rangle$ such that*

$$\mathfrak{A} \in A \iff \exists p \in \omega \ T^S_{n,m,l}(e, E, \mathfrak{A}, p).$$

THEOREM (Parameter Theorem). *Let $\mathfrak{X} = \omega^n \times \boldsymbol{R}^m \times (\boldsymbol{R}^{<\omega})^l$, $\mathfrak{Y} = \omega^{n'} \times \boldsymbol{R}^{m'} \times (\boldsymbol{R}^{<\omega})^{l'}$. There is a B.S.S. map $S^{n',m',l'}_{n,\,m,\,l}(e, E, \mathfrak{A}) : \omega \times \boldsymbol{R}^{<\omega} \times \mathfrak{X} \to \omega \times \boldsymbol{R}^{<\omega}$ such that*

$$\left\{S^{n',m',l'}_{n,\,m,\,l}(e, E, \mathfrak{A})\right\}^A(\mathfrak{B}) = \{\langle e, E\rangle\}^A(\mathfrak{A}, \mathfrak{B}) \ (\textit{where } \mathfrak{A} \in \mathfrak{X},\ \mathfrak{B} \in \mathfrak{Y}).$$

THEOREM. *Let $A(\mathfrak{A})$ be a predicate on $\mathfrak{X} = \omega^n \times \boldsymbol{R}^m \times (\boldsymbol{R}^{<\omega})^l$, then the following are equivalent*:

(1) *$A(\mathfrak{A})$ is B.S.S. r.e. in S,*
(2) *For some $\langle e, E\rangle$, $A(\mathfrak{A}) \iff \exists p \in \omega \ T^S_{n,m,l}(e, E, \mathfrak{A}, p)$,*

(3) *For some predicate R which is B.S.S. recursive in S,*
$A(\mathfrak{A}) \iff \exists n \in \omega\ R(n, \mathfrak{A})$.

We also note that Post's theorem remains valid in B.S.S. recursion theory.

In our theory, we deal with three types of existential quantifiers. The next theorem shows that "B.S.S. r.e. ness" is preserved under all three types of quantifications.

THEOREM 1.1.

(1) *If $R(n, \mathfrak{A})$ is a B.S.S. r.e. predicate, then $\exists n \in \omega\ R(n, \mathfrak{A})$ is also a B.S.S. r.e. predicate.*
(2) *If $R(\alpha, \mathfrak{A})$ is a B.S.S. r.e. predicate, then $\exists \alpha \in \boldsymbol{R}\ R(\alpha, \mathfrak{A})$ is also a B.S.S. r.e. predicate.*
(3) *If $R(X, \mathfrak{A})$ is a B.S.S. r.e. predicate, then $\exists X \in \boldsymbol{R}^{<\omega}\ R(X, \mathfrak{A})$ is also a B.S.S. r.e. predicate.*

PROOF.
(1) is Easy.
(2) Since $R(\alpha, \mathfrak{A})$ is a B.S.S. r.e. predicate, there exists $\langle e, E\rangle$ such that

$$R(\alpha, \mathfrak{A}) \equiv \exists p \in \omega\ T_{n,m+1,l}(e, E, \alpha, \mathfrak{A}, p).$$

And then

$$\begin{aligned}
\exists \alpha \in \boldsymbol{R}\ R(\alpha, \mathfrak{A}) &\equiv \exists \alpha \in \boldsymbol{R} \exists p \in \omega\ T_{n,m+1,l}(e, E, \alpha, \mathfrak{A}, p) \\
&\equiv \exists p \in \omega \exists \alpha \in \boldsymbol{R}\ T_{n,m+1,l}(e, E, \alpha, \mathfrak{A}, p) \\
&\equiv \exists p \in \omega \exists \alpha \in \boldsymbol{R}\ T_{0,1,0}(S(e, E, \mathfrak{A}), \alpha, p) \\
&\qquad \text{(where } S \text{ is the map } S^{0,\,1,\,0}_{n,\,m,\,l} \text{ in the parameter theorem.)} \\
&\equiv \exists p \in \omega\ \exists \alpha \in \boldsymbol{R}\ \alpha \in V_p^{S(e,E,\mathfrak{A})}.
\end{aligned}$$

Since $V_p^{S(e,E,\mathfrak{A})}$ is a basic semi algebraic set, it is defined by inequalities of rational functions. And $\exists \alpha \in \boldsymbol{R}\ \alpha \in V_p^{S(e,E,\mathfrak{A})}$ means that the inequalities have a solution. Now using the quantifier elimination theorem (effective version (see [**5**])) on real-closed fields, the existence is shown of inequalities of rational functions which are equivalent to $\exists \alpha \in \boldsymbol{R}\ \alpha \in V_p^{S(e,E,\mathfrak{A})}$. Furthermore, all coefficients thereof can be computed recursively from e, E, p, $\mathfrak{A}$. Therefore conjunction of these inequalities (let it be denoted as $R'(e, E, \mathfrak{A}, p)$) is a B.S.S. recursive predicate such that

$$\exists\alpha \in \boldsymbol{R}\ \alpha \in V_p^{S(e,E,\mathfrak{A})} \equiv R'(e,E,\mathfrak{A},p).$$

Hence

$$\exists\alpha \in \boldsymbol{R}\ R(\alpha,\mathfrak{A}) \equiv \exists p \in \omega\ R'(e,E,\mathfrak{A},p)$$

which is a B.S.S. r.e. predicate.
(3) For each $n \in \omega$, let

$$A_n(\mathfrak{A}) \equiv \exists\alpha_1 \cdots \exists\alpha_n \in \boldsymbol{R}\ R(\langle\alpha_1,\ldots,\alpha_n\rangle,\mathfrak{A})$$

then this is a B.S.S. r.e. predicate as shown in (2). Moreover, since it is shown that $A_n(\mathfrak{A})$ can be uniformly defined with respect to n by reviewing the proof of (2), $U(n,\mathfrak{A})$ expressed as $U(n,\mathfrak{A}) \equiv A_n(\mathfrak{A})$ is also a B.S.S. r.e. predicate. Thus

$$\exists X \in \boldsymbol{R}^{<\omega}\ R(X,\mathfrak{A}) \equiv \exists n \in \omega\ U(n,\mathfrak{A})$$

is also a B.S.S. r.e. predicate. □

COROLLARY 1.2. *If f is a B.S.S. recursive map then range(f) is a B.S.S. r.e. set.*

2. B.S.S. Turing Degrees.

In this section, we will define the degree of a set of reals using B.S.S. recursive maps, and show the existence of incomparable sets in two ways.

DEFINITION (B.S.S. Turing Degree). Suppose $\mathfrak{X} = \omega^n \times \boldsymbol{R}^m \times (\boldsymbol{R}^{<\omega})^l$, $\mathfrak{Y} = \omega^{n'} \times \boldsymbol{R}^{m'} \times (\boldsymbol{R}^{<\omega})^{l'}$.

- We say A is B.S.S. Turing reducible from B when there exists a function $f : \mathfrak{X} \to \{0,1\}$ for $A \subseteq \mathfrak{X}$ and $B \subseteq \mathfrak{Y}$ which is a total B.S.S. recursive function in B and satisfies

$$f(\mathfrak{A}) = 1 \iff \mathfrak{A} \in A$$

and express this statement as $A \leq_T^{BSS} B$.
- Define $A \leq_T^{BSS} B$ & $B \leq_T^{BSS} A$ as $A \equiv_T^{BSS} B$.
Also define $A \leq_T^{BSS} B$ & $B \not\leq_T^{BSS} A$ as $A <_T^{BSS} B$.
- That $A \not\leq_T^{BSS} B$ & $B \not\leq_T^{BSS} A$ for $A \subseteq \mathfrak{X}$ and $B \subseteq \mathfrak{Y}$ is expressed by the statement that A and B are incomparable in the sense of B.S.S. Turing

degree.

- Denote as $\mathscr{D}_T^{BSS} = (\wp(\mathfrak{X})/\equiv_T^{BSS})$ and call the element of $\mathscr{D}_T^{BSS}$ the B.S.S. Turing degree.

Method 1 (Topological method).

Here we will employ a topological method to show that a specific set, namely the set of all rational numbers, and Cantor's ternary set are incomparable in the sense of B.S.S. Turing degree.

THEOREM 2.1. *Let A, B be subsets of $\boldsymbol{R}^n$. B is a $\boldsymbol{\Delta}^0_\alpha$ set ($\alpha > 1$) and $A \leq_T^{BSS} B$ then A is also a $\boldsymbol{\Delta}^0_\alpha$ set.*

PROOF. Since $A \leq_T^{BSS} B$, assume that $f : \boldsymbol{R}^n \to \{0,1\}$, $f = \{\langle e, E\rangle\}^B$ is a total B.S.S. recursive function in B which satisfies $f(x) = 1 \iff x \in A$.

$\boldsymbol{R}^n = \bigcup_{p\in\omega} V_p^{\langle e,E\rangle,B}$ holds. $V_p^{\langle e,E\rangle,B}$ is basic semi algebraic in B for each p, and is therefore defined by the following inequalities using rational maps:

$$\begin{aligned} &h_1^1(x) > 0 \ \& \ \cdots \ \& \ h_{n_1}^1(x) > 0 \ \ \& \ \ h_1^2(x) \geq 0 \ \& \ \cdots \ \& \ h_{n_2}^2(x) \geq 0 \\ \& \ &h_1^3(x) \in B \ \& \ \cdots \ \& \ h_{n_3}^3(x) \in B \ \ \& \ \ h_1^4(x) \notin B \ \& \ \cdots \ \& \ h_{n_4}^4(x) \notin B. \end{aligned}$$

Thus since B is a $\boldsymbol{\Delta}^0_\alpha$ set, $V_p^{\langle e,E\rangle,B}$ is also a $\boldsymbol{\Delta}^0_\alpha$ set. Therefore

$$\begin{aligned} x \in A &\iff \exists p \in \omega\big[f(x) = 1 \ \& \ x \in V_p^{\langle e,E\rangle,B}\big] \\ x \notin A &\iff \exists p \in \omega\big[f(x) = 0 \ \& \ x \in V_p^{\langle e,E\rangle,B}\big]. \end{aligned}$$

Since f can be expressed as a rational function on $V_p^{\langle e,E\rangle,B}$, A and A^c are Σ^0_α sets, and consequently A is a $\boldsymbol{\Delta}^0_\alpha$ set (because it is continuous except for some closed set). $\square$

THEOREM 2.2. *If $Q \subseteq \boldsymbol{R}$ is an at most countable set and $C \subseteq \boldsymbol{R}$ is an uncountable measure 0 set [or meager set], then $C \not\leq_T^{BSS} Q$.*

The following lemmas are used to prove this theorem.

LEMMA 2.3. *Let $h : \boldsymbol{R} \to \boldsymbol{R}$ be a non-constant rational function and Q be a countable set, then $\{x \in \boldsymbol{R} \mid h(x) \in Q\}$ is an at most countable set.*

LEMMA 2.4. *Let $f : \boldsymbol{R} \to \boldsymbol{R}$, $f = \{\langle e, E\rangle\}^B$ be a total function whose range is a finite set. Then if $V_p^{\langle e,E\rangle,B}$ is an infinite set, f is constant over $V_p^{\langle e,E\rangle,B}$.*

PROOF OF THEOREM 2.2. Suppose that $C \leq_T^{BSS} Q$, and that $f : \boldsymbol{R} \to \{0,1\}$, $f = \{\langle e, E\rangle\}^Q$, that is a total function which satisfies $f(x) = 1 \iff x \in C$. Since $\boldsymbol{R} = \bigcup_{p\in\omega} V_p^{\langle e,E\rangle,Q}$, $V_p^{\langle e,E\rangle,Q} \cap C$ is an uncountable set for some $p \in \omega$. In the definition inequalities of $V_p^{\langle e,E\rangle,Q}$

$$\begin{gathered} h_1^1(x) > 0 \;\&\; \cdots \;\&\; h_{n_1}^1(x) > 0 \;\;\&\;\; h_1^2(x) \geq 0 \;\&\; \cdots \;\&\; h_{n_2}^2(x) \geq 0 \\ \&\; h_1^3(x) \in Q \;\&\; \cdots \;\&\; h_{n_3}^3(x) \in Q \;\;\&\;\; h_1^4(x) \notin Q \;\&\; \cdots \;\&\; h_{n_4}^4(x) \notin Q \end{gathered}$$

it may be assumed that each rational function is not constant. Then since such a set of x as $h_i^3(x) \in Q$ is at most countable, expressions having the form of $h_i^3(x) \in Q$ are not included. Further, define a set V' by

$$\begin{aligned} h_1^1(x) > 0 \;\&\; \cdots \;\&\; h_{n_1}^1(x) > 0 \;\;\&\;\; h_1^2(x) > 0 \;\&\; \cdots \;\&\; h_{n_2}^2(x) > 0 \\ \&\;\; h_1^4(x) \notin Q \;\&\; \cdots \;\&\; h_{n_4}^4(x) \notin Q \end{aligned}$$

which is derived by excluding the symbol of equality from expressions having the form $h_j^2(x) \geq 0$ among the preceding inequalities. Then, since $V' \subseteq V_p^{\langle e,E\rangle,Q}$ and the difference thereof is at most finite, V', too, is an uncountable set. Since V' is a non-empty open set minus at most countable points, $V' - C \neq \emptyset$ because C is a measure 0 set [or meager set]. This leads to $V_p^{\langle e,E\rangle,Q} \cap C \neq \emptyset$, $V_p^{\langle e,E\rangle,Q} - C \neq \emptyset$ which contradicts the proposition that f is a constant function on $V_p^{\langle e,E\rangle,Q}$. $\square$

COROLLARY 2.5. *Let $\boldsymbol{Q}$ be the set of all rational numbers, and C be Cantor's ternary set, then $\boldsymbol{Q}$ and C are incomparable in the sense of B.S.S. Turing degree.*

PROOF. C is a closed set, and is therefore a $\boldsymbol{\Delta}_2^0$ set. However, since Baire's category theorem dictates that $\boldsymbol{Q}$ is not $\boldsymbol{\Delta}_2^0$, $\boldsymbol{Q} \not\leq_T^{BSS} C$ according to Theorem 2.1. $C \not\leq_T^{BSS} \boldsymbol{Q}$ is derived from Theorem 2.2. $\square$

Method 2 (Algebraic method).

Now we show the existence of incomparable sets using algebraic methods.

THEOREM 2.6. *Let K, $L \subseteq \boldsymbol{R}$ be finitely generated fields of $\boldsymbol{Q}$. Then $K \subseteq L \iff K \leq_T^{BSS} L$.*

PROOF (of $K \subseteq L \Rightarrow K \leq_T^{BSS} L$). We show this for the case where L is a simple extension field of K. Because K and L are finitely generated fields, we can find reals $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that $L = K(\alpha_1, \alpha_2, \ldots, \alpha_m)$. Then

$$K \subseteq K(\alpha_1) \subseteq K(\alpha_1, \alpha_2) \subseteq \cdots \subseteq K(\alpha_1, \alpha_2, \ldots, \alpha_m),$$

every extension is simple, and the relation $\leq_T^{BSS}$ is transitive.

Build a $PL(\boldsymbol{R})^L$ program which accepts K where $L = K(\alpha)$. Let $n = [L : K]$ $(1 \leq n \leq \infty)$.

1. **if** $x \notin L$ **then output** 0 **end**.
2. Since $x \in L = K(\alpha)$, find a K-coefficient irreducible rational function $f(t)/g(t)$ ($f(t)$, $g(t)$ are polynomials) which satisfies $x = f(\alpha)/g(\alpha)$ and the degree of f, g are less than n. (In the case $n < \infty$, we take $g(t) \equiv 1$.) (Since K is expressed as $K = \boldsymbol{Q}(\beta_1, \ldots, \beta_l)$, elements of K and the entire K-coefficient polynomial can be enumerated by a B.S.S. recursive function. Furthermore such polynomials f and g are determined uniquely for each x.)
3. **if** $\deg f = \deg g = 0$ **then output** 1 **else output** 0.
4. **end**. □

To prove the converse, we show the following lemmas:

LEMMA 2.7. *Suppose that $K \subseteq \boldsymbol{R}$ is a subfiled of $\boldsymbol{R}$, and $h(x)$ is a real coefficient rational function, such that $h(r) \in K$ for infinitely many $r \in K$. Then $h(x)$ has elements of K as its coefficients.*

PROOF. Assume $h(x) = f(x)/g(x)$, ($f(x)$, $g(x)$ are polynomials). We show this by induction with respect to $n = \deg(f) + \deg(g)$.
When $n = 0$, this is clear.
When $n > 0$, we can assume $\deg(f) \geq \deg(g)$ (by considering the inverse $1/h(x)$).
Let $h(r) = q$, then

$$h(x) = (f(x) - qg(x))/g(x) + q = (x - r)f_1(x)/g(x) + q$$

and then the induction hypothesis may be applied to $f_1(x)/g(x)$. □

LEMMA 2.8. *Suppose that $K \subseteq \boldsymbol{R}$ is a subfiled of $\boldsymbol{R}$, and $h(x)$ is a real coefficient rational function. If $\alpha \notin K$ and $q_1, q_2 \in K$ $(q_1 \neq q_2)$ satisfy*

$$\exists^\infty r \in K[h(r + q_i\alpha) \in K] \qquad (i = 1, 2)$$

then $h(x)$ is a constant function.

PROOF. Since the rational function $h(x + q_i\alpha)$ takes values belonging to K for an infinite number of $r \in K$, Lemma 2.7 asserts that this is a K-coefficient rational function. Thus $h(x + q_i\alpha) = p_i(x)$ where $p_i(x)$ is a K-coefficient rational function for $i = 1, 2$. Since $p_1(x) = h(x + q_1\alpha)$ and $p_2(x - q_2\alpha) = h(x)$, we conclude $p_1(x) = h(x + q_1\alpha) = p_2((x + q_1\alpha) - q_2\alpha) = p_2(x + q\alpha)$ $(q = q_1 - q_2 \neq 0)$.

Comparison of coefficients on both sides of this equation shows that $p_1(x)$ and hence $h(x)$ is constant. $\square$

PROOF (of $K \leq_T^{BSS} L \Rightarrow K \subseteq L$). Assume $K \leq_T^{BSS} L$ and $K \not\subseteq L$. We will derive a contradiction. Suppose that $f : \boldsymbol{R} \to \{0,1\}$ is $f = \{\langle e, E\rangle\}^L$, total, and satisfies

$$f(x) = 1 \iff x \in K$$

Since $\boldsymbol{R} = \bigcup_{p\in\omega} V_p^{\langle e,E\rangle,L}$, $V_p^{\langle e,E\rangle,L}$ is an uncountable set for some $p \in \omega$, then clearly $V_p^{\langle e,E\rangle,L} - K \neq \emptyset$.

It can be assumed that each rational map $h_j^i(x)$ is not constant in the definition inequalities of $V_p^{\langle e,E\rangle,L}$

$$\begin{gathered} h_1^1(x) > 0 \ \& \ \cdots \ \& \ h_{n_1}^1(x) > 0 \ \ \& \ \ h_1^2(x) \geq 0 \ \& \ \cdots \ \& \ h_{n_2}^2(x) \geq 0 \\ \& \ h_1^3(x) \in L \ \& \ \cdots \ \& \ h_{n_3}^3(x) \in L \ \ \& \ \ h_1^4(x) \notin L \ \& \ \cdots \ \& \ h_{n_4}^4(x) \notin L. \end{gathered}$$

Then, since such a set of x as $h_i^3(x) \in L$ is at most countable, expressions having the form of $h_i^3(x) \in L$ are not included. Let U be an open set defined by

$$h_1^1(x) > 0 \ \& \ \cdots \ \& \ h_{n_1}^1(x) > 0 \ \ \& \ \ h_1^2(x) > 0 \ \& \ \cdots \ \& \ h_{n_2}^2(x) > 0$$

and W be a set defined by

$$h_1^4(x) \notin L \ \& \ \cdots \ \& \ h_{n_4}^4(x) \notin L.$$

Then $U \cap W$ is not empty since $V_p^{\langle e,E\rangle,L}$ is an uncountable set.

CLAIM. $U \cap W \cap K \neq \emptyset$.

If $U \cap W \cap K$ is empty, then

$$x \in U \cap K \Rightarrow h_1^4(x) \in L \ \vee \ \cdots \ \vee \ h_{n_4}^4(x) \in L.$$

Since the set $K - L$ is a dense subset of $\boldsymbol{R}$, we take an $\alpha \in U \cap (K - L)$. Now consider $x = r + q\alpha \in U \cap K (r, q \in \boldsymbol{Q})$, then there exist j and q_1, $q_2 (q_1 \neq q_2)$ such that there are infinitely many $r \in \boldsymbol{Q}$ which satisfy $h_j^4(r + q_i\alpha) \in L$ for $i = 1, 2$. Therefore, from Lemma 2.8, $h_j^4(x)$ is a constant function, which contradicts the proposition.

$V_p^{\langle e,E\rangle,L} \cap K \neq \emptyset$ is derived from $U \cap W \cap K \neq \emptyset$ and $U \cap W \subseteq V_p^{\langle e,E\rangle,L}$ which, together with $V_p^{\langle e,E\rangle,L} - K \neq \emptyset$, contradict the proposition that f is a constant function on $V_p^{\langle e,E\rangle,L}$. □

3. The embedding of the real line to B.S.S. Turing degrees.

Deduction from Theorem 2.6 shows the existence of a countable ascending sequence, a countable descending sequence and countable incomparable elements within B.S.S. Turing degrees. In this section we prove that there exist continuum many of these.

In this section, we suppose that $I = \{\alpha_k \mid k \in \omega\}$ satisfies the conditions

[1] $\forall k \in \omega\ \alpha_k \notin \overline{\boldsymbol{Q}(\alpha_0, \alpha_1, \ldots, \alpha_{k-1})}$ (Overline means algebraic closure),
[2] $I \subseteq \boldsymbol{R}$ is dense.

Then it is readily shown that I satisfies the condition:

[1'] $\forall k \in \omega\ \alpha_k \notin \overline{\boldsymbol{Q}(I - \{\alpha_k\})}$.

Theorem 3.1. *Let A, $B \subseteq \boldsymbol{R}$ be B.S.S. recursive sets, then the following statements hold*:

(1) *If $A \subseteq B$ then $A \cap I \leq_T^{BSS} B \cap I$.*
(2) *If $(B - A)^\circ \neq \emptyset$ then $B \cap I \not\leq_T^{BSS} A \cap I$.*

Proof.
(1) clear.
(2) Assume that $B \cap I \leq_T^{BSS} A \cap I$. Let $f : \boldsymbol{R} \to \{0, 1\}$ be a total B.S.S. recursive function in $A \cap I$ that satisfies

$$f(x) = 1 \iff x \in B \cap I$$

and set $f = \{\langle e, E\rangle\}^{A \cap I}$. Since f is a total function and

$$\boldsymbol{R} = \bigcup_{p \in \omega} V_p^{\langle e,E\rangle, A\cap I}$$

then

$$V_p^{\langle e,E\rangle, A\cap I} \cap (B - A)^\circ \text{ is uncountable for some } p \in \omega.$$

Claim. $V_p^{\langle e,E\rangle, A\cap I} \cap (B - A)^\circ \cap I = \emptyset.$

Since $V_p^{\langle e,E\rangle,A\cap I}$ is an infinite set, f is constant on this set. If $V_p^{\langle e,E\rangle,A\cap I} \cap (B-A)^\circ \cap I \neq \emptyset$, then $V_p^{\langle e,E\rangle,A\cap I} \cap B \cap I$ is not empty, and therefore $f|V_p^{\langle e,E\rangle,A\cap I} = 1$. This means $V_p^{\langle e,E\rangle,A\cap I} \subseteq B \cap I$, which contradicts that I is a countable set.

Let the definition inequalities of $V_p^{\langle e,E\rangle,A\cap I}$ be

$$\begin{aligned} &h_1^1(x) > 0 \quad \& \cdots \& \; h_{n_1}^1(x) > 0 \quad \& \;\; h_1^2(x) \geq 0 \quad \& \cdots \& \; h_{n_2}^2(x) \geq 0 \\ &\& \; h_1^3(x) \in A \cap I \; \& \cdots \& \; h_{n_3}^3(x) \in A \cap I \;\; \& \;\; h_1^4(x) \notin A \cap I \; \& \cdots \& \; h_{n_4}^4(x) \notin A \cap I \end{aligned}$$

(it is assumed that every rational function is not constant.) Then the set of x such that $h_j^3(x) \in A \cap I$ is an at most countable set, so that expressions having the form $h_j^3(x) \in \ A \cap I$ are not included.

Let U be an open set defined by

$$h_1^1(x) > 0 \; \& \; \cdots \; \& \; h_{n_1}^1(x) > 0 \;\; \& \;\; h_1^2(x) > 0 \; \& \; \cdots \; \& \; h_{n_2}^2(x) > 0$$

and W be a set defined by

$$h_1^4(x) \notin A \cap I \; \& \; \cdots \; \& \; h_{n_4}^4(x) \notin A \cap I.$$

Then since $V_p^{\langle e,E\rangle,A\cap I}$ is uncountable, $U \cap W$ is not empty. From the claim that $U \cap W \cap (B-A)^\circ \cap I = \emptyset$ it can be derived that

$$x \in U \cap (B-A)^\circ \cap I \Rightarrow h_1^4(x) \in A \cap I \;\vee\; \cdots \;\vee\; h_{n_4}^4(x) \in A \cap I.$$

Since $U \cap (B-A)^\circ$ is a non-empty open set, and I is dense in $\boldsymbol{R}$, $U \cap (B-A)^\circ \cap I$ is an infinite set. Thus there is a j which satisfies

$$\exists^\infty x \in U \cap (B-A)^\circ \cap I[h_j^4(x) \in A \cap I] \tag{$*$}$$

Denote $K = \boldsymbol{Q}(I)$, and

$$\exists^\infty x \in K[h_j^4(x) \in K]$$

is derived from the above equation, and therefore, from Lemma 2.7, $h_j^4(x)$ is a K-coefficient rational function. Thus when $h_j^4(x)$ is expressed as $h(x, \alpha_{k_1}, \ldots, \alpha_{k_m})$ (coefficients of h are rational numbers) and set $C = \{\alpha_{k_1}, \ldots, \alpha_{k_m}\}$, then $(*)$ leads to

$$\exists^\infty x \in U \cap (B - A)^\circ \cap I - C[h(x, \alpha_{k_1}, \ldots, \alpha_{k_m}) \in A \cap I] \qquad (**)$$

However, since h is not constant, it can be seen that

$$\exists x \in U \cap (B - A)^\circ \cap I - C[h(x, \alpha_{k_1}, \ldots, \alpha_{k_m}) \in A \cap I - C].$$

Let x be α_t and $h(\alpha_t, \alpha_{k_1}, \ldots, \alpha_{k_m})$ be α_s, then $\alpha_s \neq \alpha_t$ is deduced from $\alpha_s \in A$ and $\alpha_t \in (B - A)^\circ$. Also because $\alpha_s \notin C$,

$$\alpha_s \in \boldsymbol{Q}(\alpha_t, \alpha_{k_1}, \ldots, \alpha_{k_m}) \subseteq \boldsymbol{Q}(I - \{\alpha_s\})$$

which contradicts condition [1']. □

COROLLARY 3.2. *Suppose $\varphi(a) = [(-\infty, a) \cap I]$ ([ ] is the equivalence class of B.S.S. Turing degrees), then φ is an ordered embedding from $\boldsymbol{R}$ to B.S.S. Turing Degrees.*

COROLLARY 3.3. *Let $X_\sigma = \bigcup_{n\in\sigma}(n, n+1)$ for $\sigma \subseteq \omega$, then X_σ is a B.S.S. recursive set. Therefore $\mathscr{F}$ is an almost disjoint family of $[\omega]^\omega (= \{X \subseteq \omega \mid |X| = \omega\})$ which has continuum cardinality, so $\{[X_\sigma \cap I] \mid \sigma \in \mathscr{F}\}$ is an incompatible set of B.S.S. Turing degrees which have the continuum cardinality.*

4. Open problems.

Now we state some open problems.

In classical recursion theory, we can unite two consecutive quantifiers of type 1. However since $\boldsymbol{R}$ and $\boldsymbol{R}^2$ are not isomorphic, it is not clear that we can do this in B.S.S. recursion theory.

QUESTION 1. When $R(\alpha, \beta, n, \mathfrak{A})$ is a B.S.S. recursive predicate, can $\forall\alpha \in \boldsymbol{R}\ \forall\beta \in \boldsymbol{R}\ \exists n \in \omega\ R(\alpha, \beta, n, \mathfrak{A})$ be expressed in the form of $\forall\alpha \in \boldsymbol{R} \exists n \in \omega\ R'(\alpha, n, \mathfrak{A})$ ($R'(\alpha, n, \mathfrak{A})$: B.S.S. recursive predicate)?

The use of elimination of quantifiers for the theory of real-closed fields was essential in the proof of Theorem 1.1 (2). Therefore the same proof cannot be used when we try to relativize this theorem to S.

QUESTION 2. When $R(\alpha, \mathfrak{A})$ is B.S.S. r.e. in S, is $\exists\alpha \in \boldsymbol{R}\ R(\alpha, \mathfrak{A})$ also B.S.S. r.e. in S?

In the theory of Turing degrees, the jump operator plays an important role. In the theory of B.S.S. Turing degrees, we can define the jump operator by $S' = \{\langle e, E\rangle \mid \{\langle e, E\rangle\}^S(e, E)\downarrow\}$. However in this case, for $S \subseteq \boldsymbol{R}^n$, S' will be a subset of $\omega \times \boldsymbol{R}^{<\omega}$. So we ask the following question:

QUESTION 3. Assume $S \subseteq \boldsymbol{R}^n$. We set $j(S) = \{\langle e, \alpha_1, \dots, \alpha_n\rangle \mid \{\langle e, \alpha_1, \dots, \alpha_n\rangle\}^S(e, \alpha_1, \dots, \alpha_n)\downarrow\}$. Does $S <_T^{BSS} j(S)$ hold?

In B.S.S. recursion theory, topology and algebraic structure play important roles. Thus in the sense of B.S.S. recursion theory, $\boldsymbol{R}$ and $\boldsymbol{R}^2$ are not isomorphic. It is not clear that the degree for subsets of $\boldsymbol{R}^2$ corresponds to the degree for subsets of $\boldsymbol{R}$.

QUESTION 4. Does $\forall A \subseteq \boldsymbol{R}^2 \exists B \subseteq \boldsymbol{R}(A \equiv_T^{BSS} B)$ hold?

References

[1] B. M. Trager, Algebraic factoring and rational function integration, Proceedings of 1976 ACM Symposium on Symbolic and Algebraic Computation, New York, (ed. Richard D. Jenks), ACM, August 1976, pp. 219–226.

[2] L. Blum, M. Shub and S. Smale, On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions and universal machines, Bull. Amer. Math. Soc. (N.S.), **21** (1989), 1–46.

[3] L. Blum and S. Smale, The Gödel Incompleteness Theorem and Decidability over a Ring, From Topology to Computation, Proceedings of the Smalefest, 1993, pp. 321–339.

[4] L. Blum, F. Cucker, M. Shub and S. Smale, Complexity and Real Computation, Springer, 1998.

[5] A. Seidenberg, A new decision method for elementary algebra, Ann. of Math., **60** (1954), 365–374.

[6] J. R. Shoenfield, Mathematical Logic, Addison–Wesley Publishing Company, 1967.

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