J. Math. Soc. Japan
Vol. 60, No. 1 (2008) pp. 291–310
doi: 10.2969/jmsj/06010291

A new majorization between functions, polynomials, and operator inequalities II

By Mitsuru Uchiyama

(Received Feb. 13, 2007)
(Revised Jun. 14, 2007)

Abstract. Let $\boldsymbol{P}(I)$ be the set of all operator monotone functions defined on an interval I, and put $\boldsymbol{P}_+(I) = \{h \in \boldsymbol{P}(I) : h(t) \geqq 0, h \neq 0\}$ and $\boldsymbol{P}_+^{-1}(I) = \{h : h$ is increasing on $I, h^{-1} \in \boldsymbol{P}_+(0, \infty)\}$. We will introduce a new set $\boldsymbol{LP}_+(I) = \{h : h(t) > 0$ on $I, \log h \in \boldsymbol{P}(I)\}$ and show $\boldsymbol{LP}_+(I) \cdot \boldsymbol{P}_+^{-1}(I) \subset \boldsymbol{P}_+^{-1}(I)$ for every right open interval I. By making use of this result, we will establish an operator inequality that generalizes simultaneously two well known operator inequalities. We will also show that if $p(t)$ is a real polynomial with a positive leading coefficient such that $p(0) = 0$ and the other zeros of p are all in $\{z : Rz \leqq 0\}$ and if $q(t)$ is an arbitrary factor of $p(t)$, then $p(A)^2 \leqq p(B)^2$ for $A, B \geqq 0$ implies $A^2 \leqq B^2$ and $q(A)^2 \leqq q(B)^2$.

1. Introduction.

This paper is the sequel to [**11**]. A and B stand for bounded selfadjoint operators on a Hilbert space. Throughout the paper a function is assumed to be real and continuous, and "increasing" means "strictly increasing". A function $f(t)$ defined on an interval I in $\boldsymbol{R}$ is called an operator monotone function on I, provided $A \leqq B$ implies $f(A) \leqq f(B)$ for every pair A and B whose spectra $\sigma(A)$ and $\sigma(B)$ lie in I. $\boldsymbol{P}(I)$ denotes the set of all operator monotone functions on I, $\boldsymbol{P}_+(I)$ does $\{f \in \boldsymbol{P}(I) : f \neq 0,\ f \geqq 0\}$. When I is written in the concrete as $[a, b)$ we simply write $\boldsymbol{P}[a, b)$ instead of $\boldsymbol{P}([a, b))$. Suppose $I \subset J$. Then the restriction of $f \in \boldsymbol{P}(J)$ to I belongs to $\boldsymbol{P}(I)$. So we consider $\boldsymbol{P}(J)$ as the subset of $\boldsymbol{P}(I)$; we also consider as $\boldsymbol{P}_+(J) \subset \boldsymbol{P}_+(I)$; in particular, $\boldsymbol{P}_+[a, b) \subset \boldsymbol{P}_+(a, b)$. Since an operator monotone function is non-decreasing, $f \in \boldsymbol{P}_+(a, b)$ with $-\infty < a < \infty$ has the natural continuous extension to $[a, b)$, which belongs to $\boldsymbol{P}_+[a, b)$. This means $\boldsymbol{P}_+(a, b) \subset \boldsymbol{P}_+[a, b)$. Thus $\boldsymbol{P}_+(a, b) = \boldsymbol{P}_+[a, b)$. But we remark that $\boldsymbol{P}[a, b) \subsetneqq \boldsymbol{P}(a, b)$; in fact, $1/(a - t) \in \boldsymbol{P}(a, b)$ but $1/(a - t) \notin \boldsymbol{P}[a, b)$. Let us recall some notations introduced in [**11**]. For $-\infty < a < b \leqq \infty$

2000 *Mathematics Subject Classification.* Primary 47A63; Secondary 15A39.

Key Words and Phrases. matrix order, Löwner-Heinz inequality, operator inequality, operator monotone function, majorization.

$$\begin{aligned}\boldsymbol{P}_+^{-1}[a,b) = \{h |\ & h \text{ is increasing on } [a,b) \text{ and its range is } [0,\infty),\\ & h^{-1} \in \boldsymbol{P}[0,\infty)\},\end{aligned}$$

where h^{-1} stands for the inverse function of h. Now we put, for $-\infty \leqq a < b \leqq \infty$

$$\begin{aligned}\boldsymbol{P}_+^{-1}(a,b) = \{h |\ & h \text{ is increasing on} (a,b) \text{ and its range is } (0,\infty),\\ & h^{-1} \in \boldsymbol{P}(0,\infty)\}.\end{aligned}$$

If $-\infty < a$, by identifying $h \in \boldsymbol{P}_+^{-1}(a,b)$ as its natural extension to $[a,b)$ we have

$$\boldsymbol{P}_+^{-1}(a,b) = \boldsymbol{P}_+^{-1}[a,b).$$

Let h be a non-decreasing function on I and k an increasing function on J. Then h is said to be majorized by k, in symbols

$$h \preceq k$$

if $J \subset I$ and $h \circ k^{-1}$ is operator monotone on $k(J)$. This definition is equivalent with

$$\sigma(A), \sigma(B) \subset J,\ k(A) \leqq k(B) \Longrightarrow h(A) \leqq h(B).$$

If we need to make clear the domain J of k, we write as follows:

$$h \preceq k \quad (J).$$

The following theorem was shown in [**11**].

THEOREM A. *Suppose $-\infty < a < b \leqq \infty$. Then*

$$\boldsymbol{P}_+[a,b) \cdot \boldsymbol{P}_+^{-1}[a,b) \subset \boldsymbol{P}_+^{-1}[a,b), \quad \boldsymbol{P}_+^{-1}[a,b) \cdot \boldsymbol{P}_+^{-1}[a,b) \subset \boldsymbol{P}_+^{-1}[a,b).$$

Further, let $g_i(t)$ be a finite product of functions in $\boldsymbol{P}_+[a,b)$ for $1 \leqq i \leqq m$, and let $h_j(t) \in \boldsymbol{P}_+^{-1}[a,b)$ for $1 \leqq j \leqq n$. Then for every $\psi_i, \phi_j \in \boldsymbol{P}_+[0,\infty)$

$$\prod_{i=1}^m \psi_i(g_i) \prod_{j=1}^n \phi_j(h_j) \preceq \prod_{i=1}^m g_i \prod_{j=1}^n h_j \in \boldsymbol{P}_+^{-1}[a,b).$$

By making use of this theorem we obtained an operator inequality for non-negative operators (see Lemma 4.1). From $\boldsymbol{P}(-\infty,\infty) = \{\alpha t+\beta \colon \alpha \geqq 0, \beta \in \mathbf{R}\}$ it follows that $\boldsymbol{P}_+(-\infty,\infty) = \{c \colon c \geqq 0\}$. The first inclusion of the above theorem therefore holds for $(-\infty,\infty)$ as well. But it is meaningless. So we could not get an operator inequality for operators which are not necessarily non-negative. In this paper we will introduce a new class $\boldsymbol{LP}_+(I)$ including $\boldsymbol{P}_+(I)$ and extend Theorem A to all right open intervals I, namely $I=(a,b)$ or $I=[a,b)$, by using $\boldsymbol{LP}_+(I)$ instead of $\boldsymbol{P}_+(I)$. This will enable us to establish an operator inequality for (not necessarily non-negative) operators, which generalizes simultaneously two well known operator inequalities.

We will also show that if $f(t) \in \boldsymbol{P}_+[a,\infty)$, then

$$\exp\biggl(\int \frac{1}{f(t)}\,dt\biggr) \in \boldsymbol{LP}_+[a,\infty);$$

moreover

$$\lim_{t\to a+0} \frac{t-a}{f(t)} \geqq 1 \Rightarrow \exp\biggl(\int \frac{1}{f(t)}\,dt\biggr) \in \boldsymbol{P}_+^{-1}[a,\infty) \cap \boldsymbol{LP}_+[a,\infty).$$

In [**11**] we have also shown

Theorem B. *For non-increasing sequences $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^m$, define the positive and increasing functions $u(t)$ and $v(t)$ by*

$$u(t) = \prod_{i=1}^n (t-a_i) \quad (t \geqq a_1), \quad v(t) = \prod_{i=1}^m (t-b_i) \quad (t \geqq b_1).$$

Then $u(t) \in \boldsymbol{P}_+^{-1}[a_1,\infty)$, and

$$m \leqq n, \quad \sum_{i=1}^k b_i \leqq \sum_{i=1}^k a_i \ (1 \leqq k \leqq m) \Longrightarrow v \preceq u \quad ([a_1,\infty)).$$

This theorem signifies that the 'majorization between functions' defined above is an extension of the classical submajorization between sequences, and it has an application to orthonormal polynomials (see Corollary 3.4 of [**11**]). It also follows from this theorem that for any factor $u_1(t)$ of $u(t)$

$$u_1(t) \preceq u(t) \quad ([a_1,\infty)).$$

In this paper we will deal with a real polynomial $p(t)$ with a positive leading coefficient such that $p(0) = 0$ and the zeros of p are all in $\{z\colon \Re z \leqq 0\}$, and show that for a factor $q(t)$ of $p(t)$

$$p(\sqrt{t})^2 \in \boldsymbol{P}_+^{-1}[0,\infty), \quad q(t)^2 \preceq p(t)^2,$$
i.e.,
$$p(A)^2 \leqq p(B)^2 \quad (0 \leqq A, B) \Rightarrow A^2 \leqq B^2, \quad q(A)^2 \leqq q(B)^2.$$

Further, if $p(0) = p'(0) = 0$, then

$$p(\sqrt{t}) \in \boldsymbol{P}_+^{-1}[0,\infty), \quad q(t) \preceq p(t) \quad ([0,\infty)).$$

2. Product Theorem.

We start by listing several properties of the majorization: (i) through (vi) are stated in [**11**] and (v) was proved there; (vii) is clear.

(i) $k^\alpha \preceq k^\beta$ for any increasing function $k \geqq 0$ and $0 < \alpha \leqq \beta$;
(ii) $g \preceq h,\ h \preceq k \Longrightarrow g \preceq k$;
(iii) if τ is an increasing function whose range is the domain of k, then
$h \preceq k \Longleftrightarrow h \circ \tau \preceq k \circ \tau$;
(iv) $k \in \boldsymbol{P}_+^{-1}[a,b) \Longleftrightarrow t \preceq k(t),\ k([a,b)) = [0,\infty)$;
(v) if the range of k is $[0,\infty)$ and $h \geqq 0$, then
$h \preceq k \Longrightarrow h^2 \preceq k^2$;
(vi) if h and k are increasing and unbounded to the above, then
$h \preceq k,\ k \preceq h \Longleftrightarrow h = ck + d$ for real numbers $c > 0,\ d$.
(vii) $h_1 \preceq k,\ h_2 \preceq k \Longrightarrow h_1 + h_2 \preceq k$.

REMARK 2.1. In (vi), the condition that h and k are unbounded to the above is necessary: indeed, $h(t) = t,\ \ k(t) = (t/(t+1))$ on $[0,\infty)$ satisfies $h \preceq k \preceq h$, but $h \neq ck + d$ for any real numbers $c > 0,\ d$. (vi) on page 225 of [**11**] misses this condition, though we have never used this property. So we give the proof of (vi) again. Since $h \circ k^{-1}$ is operator monotone on a right half line, it is operator concave and hence concave in the usual sense. $k \circ h^{-1}$ is concave as well. Since these are increasing, we get $h = ck + d$.

Before proceeding to the main part of the paper we recall that a given function is assumed to be continuous and "increasing" means "strictly increasing". The following Product Lemma was shown in the case where $I = [a, b)$ with

$-\infty < a < b \leqq \infty$ in Lemma 2.2 of [**11**]. We can prove it in the same way. So we give just an outline of the proof.

LEMMA 2.2 (PRODUCT LEMMA). *Let I be a right open interval with end points a, b such that $-\infty \leqq a < b \leqq \infty$ and h, g non-negative functions defined on I. Suppose the product hg is increasing, $(hg)(a+0) = 0$ and $(hg)(b-0) = \infty$. Then*

$$g \preceq hg \Longrightarrow h \preceq hg.$$

Moreover for every ψ_1, ψ_2 in $\boldsymbol{P}_+[0, \infty)$

$$g \preceq hg \Longrightarrow \psi_1(h)\psi_2(g) \preceq hg.$$

PROOF. Define the functions ϕ_i $(0 \leqq i \leqq 2)$ on $(0, \infty)$ by

$$\begin{aligned} &\phi_0(h(t)g(t)) = g(t), \quad \phi_1(h(t)g(t)) = h(t), \\ &\phi_2(h(t)g(t)) = \psi_1(h(t))\psi_2(g(t)) \quad (t \in I). \end{aligned}$$

We express ϕ_1 and ϕ_2, by putting $s = h(t)g(t)$, as

$$\phi_1(s) = \frac{s}{\phi_0(s)}, \quad \phi_2(s) = \psi_1\left(\frac{s}{\phi_0(s)}\right)\psi_2(\phi_0(s)).$$

Since $\phi_0 \in \boldsymbol{P}_+(0, \infty)$, by the Löwner theorem we obtain $\phi_1 \in \boldsymbol{P}_+[0, \infty)$ and $\phi_2 \in \boldsymbol{P}_+[0, \infty)$. $\square$

LEMMA 2.3. *Let $\{h_n\}$ be a sequence of increasing functions defined on a compact interval $[a, b]$. Suppose it converges pointwise to a continuous function h on $[a, b]$. Then it converges uniformly on $[a, b]$, and hence it is equicontinuous on $[a, b]$. Moreover, if h is increasing and $h_n(a) \leqq c < d \leqq h_n(b)$ for every n, then h_n^{-1} converges uniformly to h^{-1} on $[c, d]$.*

PROOF. Since h_n are assumed to be continuous, the first assertion follows from the monotonicity of h_n and the uniform continuity of h. To see the second one, it is sufficient to show $h_n^{-1}(s_0) \to h^{-1}(s_0)$ for each $s_0 \in [c, d]$. Put $t_0 = h^{-1}(s_0)$ and take an arbitrary $\epsilon > 0$. Since $h(t_0 - \epsilon) < h(t_0) < h(t_0 + \epsilon)$, there is N so that $h_n(t_0 - \epsilon) < h(t_0) < h_n(t_0 + \epsilon)$ for all $n \geqq N$, which implies $t_0 - \epsilon < h_n^{-1}(s_0) < t_0 + \epsilon$. This means h_n^{-1} converges pointwise (hence uniformly) to h^{-1} on $[c, d]$. $\square$

The following theorem is an extension of Theorem 2.9 of [**11**], in which I is confined to $[a,b)$.

THEOREM 2.4. *Let I be a right open interval with end points a, b such that $-\infty \leqq a < b \leqq \infty$. Suppose the sequence $\{h_n\}$ of functions in $\boldsymbol{P}_+^{-1}(I)$ converges pointwise to h on I such that $h(a+) = 0$, $h(t) > 0$ for $t > a$, and $h(b-0) = \infty$. Then*

$$h \in \boldsymbol{P}_+^{-1}(I).$$

Moreover, suppose the sequence $\{\tilde{h}_n\}$ of increasing functions converges pointwise to a continuous function $\tilde{h}$ on I. Then

$$\tilde{h}_n \preceq h_n \ (n = 1, 2, \cdots) \Longrightarrow \tilde{h} \preceq h.$$

PROOF. We first assume $I = (a,b)$. Since h_n^{-1} is operator monotone on $(0,\infty)$, it is operator concave and hence concave in the usual sense and increasing. Therefore h_n is increasing and convex. Thus h is non-decreasing and convex; it is therefore continuous. From $h(a+) = 0$ and $h(t) > 0$ for $t > a$ it follows that h is increasing and its range is $(0,\infty)$, because of $h(b-0) = \infty$. By Lemma 2.3 the sequence $\{h_n^{-1}\}$ of operator monotone functions converges uniformly to h^{-1} on every compact subinterval of $(0,\infty)$. This implies $h^{-1} \in \boldsymbol{P}(0,\infty)$, that is to say, $h \in \boldsymbol{P}_+^{-1}(I)$.

Suppose $\tilde{h}_n \preceq h_n$ $(n = 1, 2, \cdots)$. Then $\tilde{h}_n \circ h_n^{-1} \in \boldsymbol{P}(0,\infty)$. By Lemma 2.3 $\{\tilde{h}_n\}$ is equicontinuous and converges uniformly to $\tilde{h}$ on every compact interval. Thus, by

$$\tilde{h}_n \circ h_n^{-1} - \tilde{h} \circ h^{-1} = \tilde{h}_n \circ h_n^{-1} - \tilde{h}_n \circ h^{-1} + \tilde{h}_n \circ h^{-1} - \tilde{h} \circ h^{-1}$$

$\tilde{h}_n \circ h_n^{-1}$ converges uniformly to $\tilde{h} \circ h^{-1}$ on every compact subinterval of $(0,\infty)$, which implies $\tilde{h} \circ h^{-1} \in \boldsymbol{P}(0,\infty)$. We consequently get $\tilde{h} \preceq h$.

We next assume $I = [a,b)$, where $-\infty < a$. By the above result we get $h \in \boldsymbol{P}_+^{-1}(a,b)$ and $\tilde{h} \circ h^{-1} \in \boldsymbol{P}(0,\infty)$. The assumption $h_n \in \boldsymbol{P}_+^{-1}(I)$ implies $h_n(a) = 0$ and hence $h(a) = 0$. Thus h is continuous on $[a,b)$ and so is h^{-1} on $[0,\infty)$. We therefore obtain $h \in \boldsymbol{P}_+^{-1}[a,b)$ and $\tilde{h} \circ h^{-1} \in \boldsymbol{P}[0,\infty)$, which implies $\tilde{h} \preceq h \quad ([a,b))$. □

REMARK 2.5. In the preceding theorem, we may assume more simply that h is continuous and increasing with the range $[0,\infty)$ or $(0,\infty)$ instead of the condition $h(a+) = 0,\ 0 < h(t)\ (a < t), h(b-0) = \infty$. We remark that the condition "$h(t) > 0$ for $t > a$" is necessary. In fact, there is a sequence $h_n(t) \in \boldsymbol{P}_+^{-1}[0,\infty)$ such that $\{h_n\}$ converges uniformly to $(t-1)_+ \notin \boldsymbol{P}_+^{-1}[0,\infty)$ (see [**6**], [**2**]).

Definition. Let $h(t)$ be an increasing and continuous function on an interval I. Then $h(t)$ is called a logarithmic operator monotone function on I and denoted by $h \in \boldsymbol{LP}_+(I)$ if $h(t) > 0$ and $\log h$ is operator monotone on the interior of I.

It is evident that for every $\alpha > 0$, $t^\alpha \in \boldsymbol{LP}_+[0,\infty)$ and $e^{\alpha t} \in \boldsymbol{LP}_+(I)$ for any I, and that $h \in \boldsymbol{LP}_+(I)$ if $h \in \boldsymbol{P}_+(I)$ and $h \neq 0$. It is also clear that $h,\ g \in \boldsymbol{LP}_+(I)$ implies $h^r g^s \in \boldsymbol{LP}_+(I)$ for $r, s > 0$; especially, $\boldsymbol{LP}_+(I) \cdot \boldsymbol{LP}_+(I) \subset \boldsymbol{LP}_+(I)$, and that $\boldsymbol{LP}_+[a,b) = \boldsymbol{LP}_+(a,b)$ if $-\infty < a < b \leqq \infty$. Note that a logarithmic operator monotone function is increasing unless it is constant, and that if $h_n(t) \in \boldsymbol{LP}_+(I)$, $h(t)$ is continuous on I, and $h_n(t) \to h(t) > 0$ in the interior of I as $n \to \infty$, then $h(t) \in \boldsymbol{LP}_+(I)$ too.

Proposition 2.6.

$$\boldsymbol{LP}_+(-\infty,\infty) = \{c_1 e^{c_2 t} \colon c_1 > 0,\ c_2 \geqq 0\}.$$

Proof. Suppose $g(t) \in \boldsymbol{LP}_+(-\infty,\infty)$. Then $\log g(t) \in \boldsymbol{P}(-\infty,\infty)$. Therefore $\log g(t) = ct + d$ $(c \geqq 0)$, which implies $g(t) = c_1 e^{c_2 t}$. □

If $I \neq (-\infty,\infty)$, then $\boldsymbol{LP}_+(I)$ includes a lot of functions in contrast to $\boldsymbol{LP}_+(-\infty,\infty)$. The following is a useful tool for the study of $\boldsymbol{LP}_+(I)$ with $I \neq (-\infty,\infty)$.

Lemma 2.7. *Suppose $-\infty < a < b \leqq \infty$. Let $g(t)$ be a function on $[a,b)$ such that $g(t) > 0$ for $a < t < b$. Then $g \in \boldsymbol{LP}_+[a,b)$ if and only if there is a sequence $\{g_n\}$ such that each g_n is a finite product of functions in $\boldsymbol{P}_+[a,b)$ and $\{g_n\}$ converges pointwise to g on $[a,b)$.*

Proof. Take $g \in \boldsymbol{LP}_+[a,b)$. We first assume $g(a) > 0$. Then $(1 + (1/n)\log g(t)) \in \boldsymbol{P}_+[a,b)$ for sufficiently large n. Hence $g_n := (1 + (1/n)\log g(t))^n$ satisfies the required condition. We next assume $g(a) = 0$. Since g is increasing, for sufficiently large n there is an $\{\epsilon_n\}$ such that $\log g(t) > -n$ for $t \geqq a + \epsilon_n$ and $\epsilon_n \downarrow 0$. Put $h_n(t) = g(t + \epsilon_n)$ if $b = \infty$, or put $h_n(t) = g((1 - (\epsilon_n/(b-a)))t + (b\epsilon_n/(b-a)))$ if $b < \infty$. Then $h_n \in \boldsymbol{LP}_+[a,b)$, $h_n(a) = g(a + \epsilon_n) > 0$ and $h_n(t) \downarrow g(t)$ on $[a,b)$. Thus we have $(1 + (1/n)\log h_n(t)) \in \boldsymbol{P}_+[a,b)$. Take an arbitrary $t_0 \in (a,b)$ and an m so that $a + \epsilon_m < t_0$. Then for $n > m$ we have $1 + (1/n)\log g(t_0) > 0$,

$$\left(1+\frac{1}{n}\log g(t_0)\right)^n \leqq \left(1+\frac{1}{n}\log h_n(t_0)\right)^n \leqq \left(1+\frac{1}{n}\log h_m(t_0)\right)^n \leqq h_m(t_0),$$

and hence

$$g(t_0) \leqq \liminf\left(1+\frac{1}{n}\log h_n(t_0)\right)^n \leqq \limsup\left(1+\frac{1}{n}\log h_n(t_0)\right)^n \leqq h_m(t_0).$$

Since $h_m(t_0) \downarrow g(t_0)$, $(1+(1/n)\log h_n(t_0))^n$ converges to $g(t_0)$. By

$$0 \leqq \left(1+\frac{1}{n}\log h_n(a)\right)^n \leqq h_m(a) \quad (m<n),$$

we also get $(1+(1/n)\log h_n(a))^n \to 0 = g(a)$. So $g_n(t) := (1+(1/n)\log h_n(t))^n$ satisfies the required condition.

We show the converse statement. By the definition of $\boldsymbol{P}_+[a,b)$, $g_n(t)>0$ for $a<t<b$ and $\log g_n \in \boldsymbol{P}(a,b)$. Thus $\log g \in \boldsymbol{P}(a,b)$, and hence $g \in \boldsymbol{LP}_+[a,b)$. $\square$

REMARK 2.8. The above lemma does not hold for the case $(-\infty,\infty)$: in fact, by Proposition 2.6 $\boldsymbol{LP}_+(-\infty,\infty) = \{c_1 e^{c_2 t} \colon c_1>0,\ c_2 \geqq 0\}$, but $\boldsymbol{P}_+(-\infty,\infty) = \{c \colon c \geqq 0\}$.

From now on we will extend Theorem A to every right open interval $I=[a,b)$ or $I=(a,b)$, by considering $\boldsymbol{LP}_+(I)$ instead of $\boldsymbol{P}_+(I)$. We first deal with the case $-\infty<a<b\leqq\infty$. In this case we may suppose $I=[a,b)$, because $\boldsymbol{LP}_+[a,b) = \boldsymbol{LP}_+(a,b)$ and $\boldsymbol{P}_+^{-1}[a,b) = \boldsymbol{P}_+^{-1}(a,b)$.

LEMMA 2.9. *Suppose* $-\infty<a<b\leqq\infty$. *Then*

$$\boldsymbol{P}_+^{-1}[a,b)\cdot\boldsymbol{P}_+^{-1}[a,b) \subset \boldsymbol{P}_+^{-1}[a,b), \quad \boldsymbol{LP}_+[a,b)\cdot\boldsymbol{P}_+^{-1}[a,b) \subset \boldsymbol{P}_+^{-1}[a,b).$$

Further, let $g_i \in \boldsymbol{LP}_+[a,b)$ $(1\leqq i\leqq m)$ *and* $h_j \in \boldsymbol{P}_+^{-1}[a,b)$ $(1\leqq j\leqq n)$. *Then for every* $\psi_i, \phi_j \in \boldsymbol{P}_+[0,\infty)$

$$\prod_{i=1}^m \psi_i(g_i)\prod_{j=1}^n \phi_j(h_j) \preceq \prod_{i=1}^m g_i \prod_{j=1}^n h_j.$$

PROOF. The first inclusion is the same as Theorem A. Suppose $g\in\boldsymbol{LP}_+[a,b)$, $h\in\boldsymbol{P}_+^{-1}[a,b)$, and $\psi,\phi\in\boldsymbol{P}_+[0,\infty)$. Then by Lemma 2.7, there is a sequence $\{k_n\}$ such that each k_n is a finite product of functions in $\boldsymbol{P}_+[a,b)$ and $\{k_n\}$ converges pointwise to g on $[a,b)$. Theorem A says $k_n h \in \boldsymbol{P}_+^{-1}[a,b)$ and $h \preceq k_n h$. $k_n h$ converges pointwise to gh on $[a,b)$, and gh is increasing and the range is $[0,\infty)$. By Theorem 2.4, we get $gh \in \boldsymbol{P}_+^{-1}[a,b)$, namely the second inclusion, and $h \preceq gh$. Product Lemma deduces $\psi(g)\phi(h) \preceq gh$. Thus we have shown

$$gh \in \boldsymbol{P}_+^{-1}[a,b), \quad \psi(g)\phi(h) \preceq gh. \tag{1}$$

Suppose $g_i(t) \in \boldsymbol{LP}_+[a,b)$ and $h_j(t) \in \boldsymbol{P}_+^{-1}[a,b)$. By Theorem A, $\prod_{j=1}^n h_j \in \boldsymbol{P}_+^{-1}[a,b)$ and

$$\prod_{j=1}^n \phi_j(h_j) \preceq \prod_{j=1}^n h_j.$$

The function ϕ defined on $[0,\infty)$ by $\phi(\prod_{j=1}^n h_j(t)) = \prod_{j=1}^n \phi_j(h_j(t))$ is therefore in $\boldsymbol{P}_+[0,\infty)$. Thus by (1)

$$g_1 \prod_{j=1}^n h_j \in \boldsymbol{P}_+^{-1}[a,b), \quad \psi_1(g_1) \prod_{j=1}^n \phi_j(h_j) \preceq g_1 \prod_{j=1}^n h_j.$$

This means that the last statement of the lemma holds for $m=1$. One can see it by induction on m. □

We next deal with the case of $I = (-\infty, b)$.

Lemma 2.10. *Suppose $-\infty < b \leqq \infty$. Then*

$$\boldsymbol{P}_+^{-1}(-\infty,b) \cdot \boldsymbol{P}_+^{-1}(-\infty,b) \subset \boldsymbol{P}_+^{-1}(-\infty,b),$$
$$\boldsymbol{LP}_+(-\infty,b) \cdot \boldsymbol{P}_+^{-1}(-\infty,b) \subset \boldsymbol{P}_+^{-1}(-\infty,b).$$

Further, let $g_i \in \boldsymbol{LP}_+(-\infty,b)$ $(1 \leqq i \leqq m)$ and $h_j \in \boldsymbol{P}_+^{-1}(-\infty,b)$ $(1 \leqq j \leqq n)$. Then for $\psi_i, \phi_j \in \boldsymbol{P}_+(0,\infty)$

$$\prod_{i=1}^m \psi_i(g_i) \prod_{j=1}^n \phi_j(h_j) \preceq \prod_{i=1}^m g_i \prod_{j=1}^n h_j.$$

Proof. For $h_j \in \boldsymbol{P}_+^{-1}(-\infty,b)$ and for an arbitrary a with $-a < b$, define $h_{j,a}$ on $[-a,b)$ by $h_{j,a}(t) = h_j(t) - h_j(-a)$. Then $h_{j,a} \in \boldsymbol{P}_+^{-1}[-a,b)$. By Lemma 2.9,

$h_{1,a}h_{2,a} \in \boldsymbol{P}_+^{-1}[-a,b)$ and $h_{1,a} \preceq h_{1,a}h_{2,a}$. Note that an arbitrary compact subinterval of $(-\infty,b)$ is included in $[-a,b)$ for sufficiently large a. We can directly see that $h_{1,a}(t)h_{2,a}(t)$ converges uniformly to $h_1(t)h_2(t)$ on every compact subinterval of $(-\infty,b)$ as $a \to \infty$. It is not difficult to see, in a fashion similar to Lemma 2.3, that $(h_{1,a}h_{2,a})^{-1}$ and $h_{1,a} \circ (h_{1,a}h_{2,a})^{-1}$ converge uniformly to $(h_1h_2)^{-1}$ and $h_1 \circ (h_1h_2)^{-1}$, respectively, on every compact subinterval of $(0,\infty)$ as $a \to \infty$. These imply $(h_1h_2)^{-1} \in \boldsymbol{P}(0,\infty)$, $h_1 \circ (h_1h_2)^{-1} \in \boldsymbol{P}(0,\infty)$; hence

$$h_1h_2 \in \boldsymbol{P}_+^{-1}(-\infty,b), \quad h_1 \preceq h_1h_2.$$

Product Lemma says $\phi_1(h_1)\phi_2(h_2) \preceq h_1h_2$. By induction we can see $\Pi\phi_j(h_j) \preceq \Pi h_j$. The same argument leads from Lemma 2.9 to

$$\boldsymbol{LP}_+(-\infty,b)\cdot \boldsymbol{P}_+^{-1}(-\infty,b) \subset \boldsymbol{P}_+^{-1}(-\infty,b), \quad \Pi\psi_i(g_i)\Pi\phi_j(h_j) \preceq \Pi g_i \Pi h_j.$$

□

Now we can get, by putting Lemma 2.9 and Lemma 2.10 together, the main theorem:

THEOREM 2.11 (PRODUCT THEOREM). *For every right open interval I,*

$$\boldsymbol{P}_+^{-1}(I)\cdot \boldsymbol{P}_+^{-1}(I) \subset \boldsymbol{P}_+^{-1}(I), \quad \boldsymbol{LP}_+(I)\cdot \boldsymbol{P}_+^{-1}(I) \subset \boldsymbol{P}_+^{-1}(I).$$

Further, let $g_i(t) \in \boldsymbol{LP}_+(I)$ for $1 \leqq i \leqq m$ and $h_j(t) \in \boldsymbol{P}_+^{-1}(I)$ for $1 \leqq j \leqq n$. Then for every $\psi_i, \phi_j \in \boldsymbol{P}_+[0,\infty)$

$$\prod_{i=1}^{m}\psi_i(g_i)\prod_{j=1}^{n}\phi_j(h_j) \preceq \prod_{i=1}^{m} g_i \prod_{j=1}^{n} h_j \in \boldsymbol{P}_+^{-1}(I).$$

3. Applications to concrete functions.

In this section we apply Product Theorem to concrete functions. We first deal with $t^\alpha e^{-t^{-\beta}}$ defined on $(0,\infty)$: the author was asked by S. Pereverzev and U. Tautenhahn if it belongs to $\boldsymbol{P}_+^{-1}(0,\infty)$. Let $\alpha,\beta > 0$. Then $t^\alpha e^{-t^{-\beta}}$ has the continuous extension to $[0,\infty)$ and its range is $[0,\infty)$.

PROPOSITION 3.1. *For $0 < \beta \leqq \alpha$*

$$t^\alpha \preceq t^\alpha e^{-t^{-\beta}} \quad ([0,\infty)).$$

Moreover, if $1 \leqq \alpha$,

$$t^{\alpha}e^{-t^{-\beta}} \in \boldsymbol{P}_+^{-1}[0,\infty).$$

PROOF. Put $c = (\beta/\alpha)$. Since $t \in \boldsymbol{P}_+^{-1}[0,\infty)$ and $e^{-t^{-c}} \in \boldsymbol{LP}_+(0,\infty)$ $(= \boldsymbol{LP}_+[0,\infty))$, Product Theorem says $te^{-t^{-c}} \in \boldsymbol{P}_+^{-1}[0,\infty)$, which implies that $t \preceq te^{-t^{-c}}$. The substitution of t^{α} for t, by the property (iii) in the previous section, yields $t^{\alpha} \preceq t^{\alpha}e^{-t^{-\beta}}$. If $\alpha \geqq 1$, we have $t \preceq t^{\alpha}$. Thus, the transitive property (ii) gives $t \preceq t^{\alpha}e^{-t^{-\beta}}$, which implies $t^{\alpha}e^{-t^{-\beta}} \in \boldsymbol{P}_+^{-1}[0,\infty)$. □

We next deal with a real polynomial with imaginary zeros. Before we do so, recall that a function f defined on $[0,\infty)$ is said to be semi-operator monotone if $f(\sqrt{t})^2$ is operator monotone. It is evident that $h(\sqrt{t})^2 \in \boldsymbol{P}_+^{-1}[0,\infty)$ if and only if the range of h is $[0,\infty)$ and h^{-1} is semi-operator monotone there.

THEOREM 3.2. *Let* $p(t)$ *be a real polynomial with a positive leading coefficient such that* $p(0) = 0$ *and zeros of* p *are all in* $\{z\colon \Re z \leqq 0\}$. *Let* $q(t)$ *be a factor of* $p(t)$. *Then*

$$p(\sqrt{t})^2 \in \boldsymbol{P}_+^{-1}[0,\infty), \quad q(t)^2 \preceq p(t)^2,$$

that is

$$p(A)^2 \leqq p(B)^2 \quad (0 \leqq A, B) \Rightarrow A^2 \leqq B^2, \quad q(A)^2 \leqq q(B)^2.$$

Furthermore, if $p(0) = p'(0) = 0$, *then*

$$p(\sqrt{t}) \in \boldsymbol{P}_+^{-1}[0,\infty), \quad q(t) \preceq p(t),$$

that is

$$p(A) \leqq p(B) \quad (0 \leqq A, B) \Rightarrow A^2 \leqq B^2, \quad q(A) \leqq q(B).$$

PROOF. There is no loss of generality in assuming that the leading coefficient of $p(t)$ is 1. That p^{-1} is semi-operator monotone on $[0,\infty)$, i.e. $p(\sqrt{t})^2 \in \boldsymbol{P}_+^{-1}[0,\infty)$, has been shown in Theorem 4.1 of [**10**]. However we now give a simple and direct proof of it by making use of Product Theorem. Note that $p(t)$ can be represented as

$$p(t) = t^{\lambda}\prod_k \{(t+a_k)^2 + b_k^2\}^{\lambda_k} \quad (a_k, b_k \geqq 0),$$

where $\lambda_k \geqq 1$ if $b_k > 0$, or $\lambda_k \geqq 1/2$ if $b_k = 0$, and $\lambda \geqq 1$. Since $t^\lambda \in \boldsymbol{P}_+^{-1}[0,\infty)$ and $\{(\sqrt{t}+a_k)^2 + b_k^2\}^{\lambda_k} \in \boldsymbol{LP}_+[0,\infty)$, by Product Theorem

$$p(\sqrt{t})^2 = t^\lambda \prod \{(\sqrt{t}+a_k)^2 + b_k^2\}^{2\lambda_k} \in \boldsymbol{P}_+^{-1}[0,\infty),$$
$$\phi_0(t^\lambda) \prod \phi_k(\{(\sqrt{t}+a_k)^2 + b_k^2\}^{2\lambda_k}) \preceq t^\lambda \prod \{(\sqrt{t}+a_k)^2 + b_k^2\}^{2\lambda_k},$$

where $\phi_k \in \boldsymbol{P}_+[0,\infty)$. Since $q(\sqrt{t})^2$ is represented as the left hand side of the above relation with $\phi_k(t) = t^\alpha$, where $0 \leqq \alpha \leqq 1$, we get $q(\sqrt{t})^2 \preceq p(\sqrt{t})^2$. By considering the mapping $\tau(t) = t^2$ on $[0,\infty)$, in virtue of the property (iii), we get $q(t)^2 \preceq p(t)^2$.

Suppose $p(0) = p'(0) = 0$, that is $\lambda \geqq 2$. Since $t^{\frac{\lambda}{2}} \in \boldsymbol{P}_+^{-1}$, by a similar argument as the above, we have

$$p(\sqrt{t}) = t^{\frac{\lambda}{2}} \prod_k \{(\sqrt{t}+a_k)^2 + b_k^2\}^{\lambda_k} \in \boldsymbol{P}_+^{-1}, \quad q(\sqrt{t}) \preceq p(\sqrt{t}),$$

which implies $q(t) \preceq p(t)$. The operator inequalities are obvious. □

It is evident that $t \in \boldsymbol{P}_+[0,\infty)$, $\int \frac{1}{t}\,dt = \log t \in \boldsymbol{P}(0,\infty)$, and $\exp(\log t) = t \in \boldsymbol{P}_+^{-1}[0,\infty) \cap \boldsymbol{P}_+[0,\infty)$. This fact deserves some notice. In Theorem 3.5 we will generalize it. To do so, we need the following lemma, whose first assertion was stated in Remark 1 of [**9**] and the second one was essentially proved in the proof of Theorem 3.2 of [**10**]. But for the sake of completeness we give a proof.

LEMMA 3.3. *Let $f(t)$ be an increasing and differentiable function defined on (a,∞). Then*

$$-f'(t) \in \boldsymbol{P}(a,\infty) \Rightarrow f(\infty) = \infty,\ f(t) \in \boldsymbol{P}(a,\infty).$$

Furthermore,

$$\gamma := \lim_{t \to a+0} (t-a) f'(t) > 0 \Rightarrow f(a+) = -\infty,\ 0 > -f'(t) + \frac{\gamma}{t-a} \in \boldsymbol{P}(a,\infty).$$

PROOF. By considering $f(t+a)$, we may assume $a = 0$. Since $-f'(t)$ is non-positive and operator monotone on $(0,\infty)$, by Lemma 2 of [**9**] it is represented as

$$-f'(t) = -f'(\infty) - \int_0^\infty \frac{1}{t+s}\,d\nu(s), \tag{2}$$

where

$$\int_0^\infty \frac{s}{s^2+1}\,d\nu(s) < \infty.$$

From this formula it follows that for an arbitrary $\epsilon > 0$ and for $x \geqq \epsilon$

$$f(x) - f(\epsilon) = f'(\infty)(x-\epsilon) + \int_\epsilon^x \left(\int_0^\infty \frac{1}{t+s}\,d\nu(s)\right)dt.$$

Since $0 \leqq 1/(t+s)$ is continuous on $0 \leqq s < \infty, \epsilon \leqq t \leqq x$, by Fubini's theorem

$$f(x) - f(\epsilon) = f'(\infty)(x-\epsilon) + \int_0^\infty \log\frac{x+s}{\epsilon+s}\,d\nu(s).$$

Since f is not constant, $f'(\infty) > 0$ or $\nu([0,\infty)) > 0$ because of (2). Thus, by the above equality, we obtain $f(\infty) = \infty$. Notice that $\log((x+s)/(\epsilon+s))$ is integrable w.r.t. $d\nu(s)$ and that it is operator monotone, as a function of x, on $\epsilon \leqq x < \infty$ for each s. Thus $f(x)$ is operator monotone on $\epsilon \leqq x < \infty$, that is, $f \in \boldsymbol{P}(0,\infty)$.

Assume $\lim_{t\to 0+} tf'(t) = \gamma > 0$. Since $f'(t) \geqq 0$ is decreasing on $(0,\infty)$, $f'(0+)$ and $f'(\infty)$ both exist. Since $\gamma > 0$, we have $f'(0+) = \infty$ and $f(0+) = -\infty$; indeed, the former fact is evident and the latter follows from $tf'(t) \leqq f(t) - f(0+)$, for $f(t)$ is concave. Since $0 \leqq f'(\infty) < \infty$, by (2) we get

$$\gamma = \lim_{t\to+0} tf'(t) = \lim_{t\to+0} \int_0^\infty \frac{t}{t+s}\,d\nu(s).$$

Since ν is finite on each bounded Borel set and $s/(s^2+1)$ is integrable with respect to $\nu(s)$,

$$k(s) := \begin{cases} 1 & \text{if } 0 \leqq s \leqq 1, \\ \frac{1}{s} & \text{if } 1 \leqq s \end{cases}$$

is integrable with respect to ν. Since $t/(s+t) \leqq k(s)$ for $0 < t < 1$, we have

$$\gamma = \int_0^\infty \lim_{t\to+0} \frac{t}{s+t}\,d\nu(s) = \nu(\{0\}).$$

Denote the Dirac measure by δ and put $\mu = \nu - \gamma\delta$. Then μ is a positive Borel measure on $[0,\infty)$ and $s/(1+s^2)$ is integrable with respect to μ. Hence $1/(s+t)$ is integrable with respect to μ for each $t > 0$. Thus

$$-f'(\infty) - \int_0^\infty \frac{1}{s+t}\, d\mu(s) = -f'(t) + \frac{\gamma}{t} \quad (t > 0).$$

Since the left hand side is negative and operator monotone on $0 < t < \infty$, so is the right hand side. □

REMARK 3.4. We will not use $f(\infty) = \infty$ nor $f(a+) = -\infty$ in this paper. But this says the range of $\exp f(t)$ is $(0, \infty)$, so it might be helpful when we check if $\exp f(t) \in \boldsymbol{P}_+^{-1}[a, \infty)$.

THEOREM 3.5. *Suppose $f(t) \in \boldsymbol{P}_+[a, \infty)$. Then*

$$\int \frac{1}{f(t)}\, dt \in \boldsymbol{P}(a, \infty), \quad \exp\left(\int \frac{1}{f(t)}\, dt\right) \in \boldsymbol{LP}_+[a, \infty).$$

Furthermore,

$$\lim_{t \to a+0} \frac{t-a}{f(t)} \geqq 1 \Rightarrow \exp\left(\int \frac{1}{f(t)}\, dt\right) \in \boldsymbol{P}_+^{-1}[a, \infty) \cap \boldsymbol{LP}_+[a, \infty).$$

PROOF. Since $f(t)$ is positive and operator monotone, $-(1/f(t)) \in \boldsymbol{P}(a, \infty)$. By the first assertion of Lemma 3.3, $\int (1/f(t)) dt \in \boldsymbol{P}(a, \infty)$. This implies

$$y := \exp\left(\int \frac{1}{f(t)}\, dt\right) \in \boldsymbol{LP}_+[a, \infty).$$

We remark that y is defined on (a, ∞), but it has the natural extension to $[a, \infty)$.

Assume

$$\lim_{t \to a+0} \frac{t-a}{f(t)} = \gamma \geqq 1.$$

Then by the second assertion of Lemma 3.3 we gain

$$0 > -\left(\frac{1}{f(t)} - \frac{\gamma}{t-a}\right) \in \boldsymbol{P}(a, \infty).$$

From the fact shown above, it follows that

$$\exp\left(\int\left(\frac{1}{f(t)}-\frac{\gamma}{t-a}\right)dt\right)\in \boldsymbol{LP}_+[a,\infty),$$

which yields

$$\frac{y}{(t-a)^\gamma}\in \boldsymbol{LP}_+[a,\infty).$$

Since $(t-a)^\gamma\in \boldsymbol{P}_+^{-1}[a,\infty)$ and $y=(t-a)^\gamma(y/(t-a)^\gamma)$, Product Theorem says $y\in \boldsymbol{P}_+^{-1}[a,\infty)$. □

COROLLARY 3.6 (cf. THEOREM 3.2 of [**9**]). *Let $u(t)$ be an increasing function defined on $[a,\infty)$ with the range $[0,\infty)$ and differentiable on (a,∞). Then*

$$\frac{u(t)}{u'(t)}\in \boldsymbol{P}_+(a,\infty)\Rightarrow u\in \boldsymbol{LP}_+[a,\infty).$$

Furthermore,

$$\lim_{t\to a+0}(t-a)\,\frac{u'(t)}{u(t)}=\gamma\geqq 1\Rightarrow u\in \boldsymbol{P}_+^{-1}[a,\infty)\cap \boldsymbol{LP}_+[a,\infty).$$

PROOF. Since

$$u(t)=\exp\left(\int\frac{u'(t)}{u(t)}\,dt\right),$$

we deduce the required result from Theorem 3.5. □

EXAMPLE 3.7. Let us consider $\frac{t}{t+1}\in \boldsymbol{P}_+[0,\infty)$. Since $\lim_{t\to 0+}t((t+1)/t)=1$, by Theorem 3.5 we get

$$t\exp t\in \boldsymbol{P}_+^{-1}[0,\infty)\cap \boldsymbol{LP}_+[0,\infty).$$

We note that this directly follows from Product Theorem too. We give another example: $\log(t+1)\in \boldsymbol{P}_+[0,\infty)$ and $\lim_{t\to 0+}(t/(\log(t+1))=1$, so Theorem 3.5 says

$$\exp\left(\int \frac{1}{\log(t+1)}\,dt\right) \in \boldsymbol{P}_+^{-1}[0,\infty) \cap \boldsymbol{LP}_+[0,\infty).$$

4. Simultaneous generalization of two operator inequalities.

This section is devoted to applying Product Theorem to operator inequalities. The next lemma for non-negative operators was shown in Theorem 4.4 of [**11**] by using Theorem A.

Lemma 4.1. *Let $h(t) \in \boldsymbol{P}_+^{-1}[0,\infty)$, and let $\tilde{h}(t)$ be a non-decreasing function on $[0,\infty)$ such that*

$$0 \leqq \tilde{h} \preceq h.$$

Let $g_n(t)$ be a finite product of functions in $\boldsymbol{P}_+[0,\infty)$ for each n, and let the sequence $\{g_n(t)\}$ converge pointwise to $g(t)$. Suppose $g(0+) = g(0)$ and $g(t) > 0$ for $t > 0$. Then the function φ on $[0,\infty)$ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$ belongs to $\boldsymbol{P}_+[0,\infty)$ and satisfies

$$0 \leqq A \leqq B \Rightarrow \begin{cases} \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geqq g(A)^{\frac{1}{2}}\tilde{h}(B)g(A)^{\frac{1}{2}}, \\ \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \leqq g(B)^{\frac{1}{2}}\tilde{h}(A)g(B)^{\frac{1}{2}}. \end{cases}$$

Furthermore, if $\tilde{h} \in \boldsymbol{P}_+[0,\infty)$, then

$$0 \leqq A \leqq B \Rightarrow \begin{cases} \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geqq \varphi(g(A)^{\frac{1}{2}}h(A)g(A)^{\frac{1}{2}}), \\ \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \leqq \varphi(g(B)^{\frac{1}{2}}h(B)g(B)^{\frac{1}{2}}). \end{cases} \tag{3}$$

We note that (3) is a generalization of the Furuta inequality [**5**]: for $p \geqq 1, r > 0$

$$0 \leqq A \leqq B \Rightarrow \begin{cases} (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geqq (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}, \\ (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1+r}{p+r}} \leqq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1+r}{p+r}}. \end{cases} \tag{4}$$

In fact, put $h(t) = t^p \in \boldsymbol{P}_+^{-1}[0,\infty)$, $g_n(t) = g(t) = t^r$ and $\tilde{h}(t) = t$. Then $0 \leqq \tilde{h} \preceq h$ and φ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$ is $t^{(1+r)/(p+r)}$. Thus (3) yields (4).

The following inequality relevant to (4) was shown by Ando [**1**], M.Fujii-

T.Furuta-E.Kamei [**4**] (also see [**7**]):
for $p, r > 0$

$$A \leqq B \Rightarrow \begin{cases} (e^{\frac{r}{2}A} e^{pB} e^{\frac{r}{2}A})^{\frac{r}{p+r}} \geqq (e^{\frac{r}{2}A} e^{pA} e^{\frac{r}{2}A})^{\frac{r}{p+r}}, \\ (e^{\frac{r}{2}B} e^{pA} e^{\frac{r}{2}B})^{\frac{r}{p+r}} \leqq (e^{\frac{r}{2}B} e^{pB} e^{\frac{r}{2}B})^{\frac{r}{p+r}}. \end{cases} \tag{5}$$

Here A, B are not assumed to be non-negative. So we could not get a generalization of (5), for Theorem A is useless for the case $I = (-\infty, \infty)$. The aim of this section is to extend Lemma 4.1 to every I by using $\boldsymbol{LP}_+(I)$ instead of $\boldsymbol{P}_+(I)$ so that the extended operator inequality is a simultaneous generalization of (4) and (5). Precisely, we will show

Theorem 4.2. *Let I be a right open interval. Suppose $h(t) \in \boldsymbol{P}_+^{-1}(I)$, $g(t) \in \boldsymbol{LP}_+(I)$ and $\tilde{h}(t)$ is a non-decreasing function on I such that*

$$0 \leqq \tilde{h} \preceq h.$$

Then the function φ on $[0, \infty)$ defined by

$$\varphi(g(t)h(t)) = g(t)\tilde{h}(t) \quad (t \in I)$$

belongs to $\boldsymbol{P}_+[0, \infty)$, and for A, B with $\sigma(A), \sigma(B) \subset I$

$$A \leqq B \Rightarrow \begin{cases} \varphi(g(A)^{\frac{1}{2}} h(B) g(A)^{\frac{1}{2}}) \geqq g(A)^{\frac{1}{2}} \tilde{h}(B) g(A)^{\frac{1}{2}}, \\ \varphi(g(B)^{\frac{1}{2}} h(A) g(B)^{\frac{1}{2}}) \leqq g(B)^{\frac{1}{2}} \tilde{h}(A) g(B)^{\frac{1}{2}}. \end{cases} \tag{6}$$

Furthermore, if $\tilde{h} \in \boldsymbol{P}_+(I)$, then

$$A \leqq B \Rightarrow \begin{cases} \varphi(g(A)^{\frac{1}{2}} h(B) g(A)^{\frac{1}{2}}) \geqq \varphi(g(A)^{\frac{1}{2}} h(A) g(A)^{\frac{1}{2}}), \\ \varphi(g(B)^{\frac{1}{2}} h(A) g(B)^{\frac{1}{2}}) \leqq \varphi(g(B)^{\frac{1}{2}} h(B) g(B)^{\frac{1}{2}}). \end{cases} \tag{7}$$

Proof. That φ belongs to $\boldsymbol{P}_+[0, \infty)$ follows from Product Theorem. We will only show the first inequality of (6), for the second one can be shown in the same way, and (7) follows from (6).

Assume first $I = [0, \infty)$. Lemma 4.1 and Lemma 2.7 say Theorem 4.2 holds for $I = [0, \infty)$.

Assume secondly $I = [a, b)$, where $-\infty < a < b < \infty$. Consider the bijective mapping τ from $[a,b)$ to $[0,\infty)$ defined by $\tau(t)=(b-a)/(b-t)-1$. For a function f on $[a,b)$ define the function F on $[0,\infty)$ by $F(\tau(t))=f(t)$ for $t\in[a,b)$. The operator monotonicity of $\tau(t)$ and that of $\tau^{-1}(t)=b-(b-a)/(1+t)$ yield

$$f\in \boldsymbol{P}_+[a,b) \text{ (or } \boldsymbol{LP}_+[a,b)) \Leftrightarrow F\in \boldsymbol{P}_+[0,\infty) \text{ (or } \boldsymbol{LP}_+[0,\infty)),$$
$$f\in \boldsymbol{P}_+^{-1}[a,b) \Leftrightarrow F\in \boldsymbol{P}_+^{-1}[0,\infty).$$

Suppose $h\in \boldsymbol{P}_+^{-1}[a,b)$, $g\in \boldsymbol{LP}_+[a,b)$, $0\leqq \tilde{h}\preceq h \quad ([a,b))$, and

$$\varphi(h(t)g(t))=\tilde{h}(t)g(t) \quad (a\leqq t<b).$$

Put $H(\tau(t))=h(t)$, $G(\tau(t))=g(t)$, $\tilde{H}(\tau(t))=\tilde{h}(t)$ for $t\in[a,b)$. Then $H\in \boldsymbol{P}_+^{-1}[0,\infty)$, $G\in \boldsymbol{L}P_+^{-1}[0,\infty)$, $0\leqq \tilde{H}\preceq H$ $([0,\infty))$, and

$$\varphi(H(t)G(t))=\tilde{H}(t)G(t) \quad (0\leqq t<\infty).$$

Suppose $a\leqq A\leqq B<b$. Since $0\leqq \tau(A)\leqq \tau(B)<\infty$, by the fact shown above we have

$$\varphi(G(\tau(A))^{\frac{1}{2}}H(\tau(B))G(\tau(A))^{\frac{1}{2}})\geqq G(\tau(A))^{\frac{1}{2}})\tilde{H}(\tau(B))G(\tau(A))^{\frac{1}{2}}.$$

This implies

$$\varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}})\geqq g(A)^{\frac{1}{2}}\tilde{h}(B)g(A)^{\frac{1}{2}}.$$

Thus we get the first inequality of (6).

Assume thirdly $I=[a,\infty)$, where $-\infty<a<\infty$. Then by considering the bijective mapping $\tau(t)=t-a$ from I to $[0,\infty)$ and by using the inequality for the case of $[0,\infty)$ we can easily gain the first inequality of (6).

Assume last $I=(-\infty,b)$, where $-\infty<b\leqq\infty$. Let $h\in \boldsymbol{P}_+^{-1}(-\infty,b), g\in \boldsymbol{LP}_+(-\infty,b)$ and $0\leqq \tilde{h}\preceq h$. Define the function $h_n(t)$ on $[-n,b)$ by

$$h_n(t)=h(t)-h(-n) \quad (-n\leqq t<b)$$

for each n. Then $h_n\in \boldsymbol{P}_+^{-1}[-n,b)$ and $0\leqq \tilde{h}\preceq h_n \quad ([-n,b))$. Since $g\in \boldsymbol{LP}_+[-n,b)$, where we consider g as a function on $[-n,b)$, the function φ_n defined by

$$\varphi_n(h_n(t)g(t)) = \tilde{h}(t)g(t) \quad (-n \leqq t < b)$$

belongs to $\boldsymbol{P}_+[0,\infty)$. Since the function $h_n(t)g(t)$ is well defined on every compact subinterval of $(-\infty, b)$ for sufficiently large n and $h(-n) \to 0$ as $n \to \infty$, $h_n(t)g(t)$ converges uniformly to $h(t)g(t)$ on every compact subinterval of $(-\infty, b)$, from which it follows that the sequence of the inverses of $h_n(t)g(t)$ converges uniformly to the inverse of $h(t)g(t)$ on every compact subinterval of $(0,\infty)$. We can consequently see that φ_n converges uniformly to φ on every compact subinterval of $(0,\infty)$; therefore $\{\varphi_n\}$ is equicontinuous there. Suppose $A \leqq B < b$. Take n so that $n \geqq \|A\|, \|B\|$. Then $h_n(B)$ is well defined, and by (6) for the case of $[-n, b)$

$$\varphi_n(g(A)^{\frac{1}{2}} h_n(B) g(A)^{\frac{1}{2}}) \geqq g(A)^{\frac{1}{2}} \tilde{h}(B) g(A)^{\frac{1}{2}}. \tag{8}$$

$\|h_n(B) - h(B)\| = h(-n) \to 0$ implies

$$\|g(A)^{\frac{1}{2}} h_n(B) g(A)^{\frac{1}{2}} - g(A)^{\frac{1}{2}} h(B) g(A)^{\frac{1}{2}}\| \to 0 \quad (n \to \infty).$$

Since $\{\varphi_n\}$ converges uniformly to φ on every compact subinterval of $(0,\infty)$ as we mentioned above,

$$\begin{aligned}
&\|\varphi_n(g(A)^{\frac{1}{2}} h_n(B) g(A)^{\frac{1}{2}}) - \varphi(g(A)^{\frac{1}{2}} h(B) g(A)^{\frac{1}{2}})\| \\
&\leqq \|(\varphi_n - \varphi)(g(A)^{\frac{1}{2}} h_n(B) g(A)^{\frac{1}{2}})\| \\
&+ \|\varphi(g(A)^{\frac{1}{2}} h_n(B) g(A)^{\frac{1}{2}}) - \varphi(g(A)^{\frac{1}{2}} h(B) g(A)^{\frac{1}{2}})\| \to 0 \quad (n \to \infty).
\end{aligned}$$

Thus, by letting $n \to \infty$ in (8), we arrive at the first inequality of (6). □

Needless to say, (7) is a generalization of (4). To see that it is also a generalization of (5), consider $I = (-\infty, \infty)$, $e^{pt} \in \boldsymbol{P}_+^{-1}(-\infty,\infty)$, $e^{rt} \in \boldsymbol{LP}_+(-\infty,\infty)$. Since $1 \preceq e^{pt}$ and the function φ on $[0,\infty)$ defined by

$$\varphi(e^{pt}e^{rt}) = 1e^{rt}$$

is $t^{r/(p+r)}$, (7) yields (5).

Corollary 4.3. *Suppose $h \in \boldsymbol{P}_+^{-1}(I) \cap \boldsymbol{LP}_+(I)$, $p \geqq 1, r > 0$ and $0 < \alpha \leqq r/(p+r)$. Then for A, B with $\sigma(A), \sigma(B) \subset I$*

$$A \leqq B \Rightarrow \begin{cases} (h(A)^{\frac{r}{2}} h(B)^p h(A)^{\frac{r}{2}})^\alpha \geqq (h(A)^{\frac{r}{2}} h(A)^p h(A)^{\frac{r}{2}})^\alpha, \\ (h(B)^{\frac{r}{2}} h(A)^p h(B)^{\frac{r}{2}})^\alpha \leqq (h(B)^{\frac{r}{2}} h(B)^p h(B)^{\frac{r}{2}})^\alpha. \end{cases}$$

PROOF. One can see that $h(t)^p \in \boldsymbol{P}_+^{-1}(I)$ for $p \geqq 1$, $h(t)^r \in \boldsymbol{LP}_+(I)$ for $r > 0$, $1 \preceq h(t)^p$ and that the function φ on $[0, \infty)$ defined by $\varphi(h(t)^p h(t)^r) = h(t)^r$ $(t \in I)$ is $t^{r/(p+r)}$. So the above inequalities follow from (7) in the case of $\alpha = r/(p+r)$, and then from the Löwner-Heinz inequality in the case of $0 < \alpha < r/(p+r)$. $\square$

ACKNOWLEDGEMENTS. The author would like to thank the referee for a lot of valuable comments.

References

[1] T. Ando, On some operator inequalities, Math. Ann., **279** (1987), 157–159.
[2] J. C. Bourin, M. Uchiyama, A matrix subadditivity inequality for $f(A+B)$ and $f(A)+f(B)$, Linear Algebra Appl., **423** (2007), 512–518.
[3] R. Bhatia, Matrix Analysis, Springer, 1996.
[4] M. Fujii, T. Furuta, E. Kamei, Furuta's inequality and its application to Ando's theorem, Linear Algebra Appl., **179** (1993), 161–169.
[5] T. Furuta, $A \geqq B \geqq 0$ assures $(B^r A^p B^r)^{1/q} \geqq B^{(p+2r)/q}$ for $r \geqq 0, p \geqq 0, q \geqq 1$ with $(1+2r)q \geqq p+2r$, Proc. Amer. Math. Soc., **101** (1987), 85–88.
[6] T. Kosem, Inequalities between $\|f(A+B)\|$ and $\|f(A)+f(B)\|$, Linear Algebra Appl., **418** (2006) 153–160.
[7] M. Uchiyama, Some exponential operator inequalities, Math Inequal. Appl., **2** (1999), 469–472.
[8] M. Uchiyama, Operator monotone functions which are defined implicitly and operator inequalities, J. Funct. Anal., **175** (2000), 330–347.
[9] M. Uchiyama, M. Hasumi, On some operator monotone functions, Integral Equations Operator Theory, **42** (2002), 243–251.
[10] M. Uchiyama, Inverse functions of polynomials and orthogonal polynomials as operator monotone functions, Transaction of Amer. Math. Soc., **355** (2003) 4111–4123.
[11] M. Uchiyama, A new majorization between functions, polynomials, and operator inequalities, J. Funct. Anal., **231** (2006) 231–244.

Mitsuru UCHIYAMA
Department of Mathematics
Interdisciplinary Faculty
of Science and Engineering
Shimane University
Matsue city, Shimane 690-8504, Japan
E-mail: uchiyama@riko.shimane-u.ac.jp