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On Gauss' formula for ψ and finite expressions for the L-series at 1

Dedicated to Professor Gilles Lachaud on his sixtieth birthday

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Abstract. In this paper, we shall prove in Theorem 1 that Gauss' famous closed formula for the values of the digamma function at rational arguments is equivalent to the well-known finite expression for the $L(1,\chi)$, which in turn gives rise to the finite expression for the class number of quadratic fields. We shall also prove several equivalent expressions for the arithmetic function $N(q)$ introduced by Lehmer and reveal the relationships among them.

1. Introduction and the main theorem.

Dirichlet's celebrated class number formula has two stages. To state them we introduce the notation. Let $k = \boldsymbol{Q}(\sqrt{d})$ be a quadratic field with discriminant d and let $h = h_k$ be its class number. Let $(\frac{d}{\cdot})$ be the Kronecker character associated to k, which is known to be a primitive Dirichlet character mod $|d|$. In general, let χ be a Dirichlet character to the modulus q and let $L(s,\chi)$ be the Dirichlet L-function (L-series) associated to χ:

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where the series on the right is absolutely convergent for $\sigma := \operatorname{Re} s > 1$ and is conditionally convergent for $\sigma > 0$ for non-principal χ. The value $L(1,\chi)$ for non-principal χ is therefore meaningful and a fortiori for the Kronecker character $(\frac{d}{\cdot})$. Let $\zeta_k(s)$ be the Dedekind zeta-function of k. Then, decomposing $\zeta_k(s)$ into h equivalent classes, we are led to considering the corresponding Epstein zeta-

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function, and the residue is known to be ([**10**, p.182])

$$\frac{2\pi h}{w\sqrt{|d|}} \quad \text{for} \quad d<0 \quad \text{and} \quad \frac{2h\log\varepsilon}{\sqrt{d}} \quad \text{for} \quad d>0,$$

where w and ε signify the number of roots of unity contained in the field and the funcdamental unit, usually so denoted. On the other hand, $\zeta_k(s)$ has the product decomposition

$$\zeta(s)L\left(s, \left(\frac{d}{\cdot}\right)\right),$$

which gives the residue $L(1, (\frac{d}{\cdot}))$, where $\zeta(s)$ is the Riemann zeta-function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1, \tag{1.1}$$

and $L\big(s, (\frac{d}{\cdot})\big)$ is the L-series. Equating these residues, we obtain the first stage of the Dirichlet class number formula.

The second stage consists in finite expressions for $L(1,\chi)$, where the underlying philosophy is that since h is finite, so is $L(1,\chi)$ to be (which is given in its inception as an infinite series).

The most well-known finite expressions for $L(1,\chi)$ are

$$L\left(1, \left(\frac{d}{\cdot}\right)\right) = -\frac{\pi i}{G\big((\frac{d}{\cdot})\big)} \sum_{a=1}^{|d|-1} \left(\frac{d}{a}\right) B_1\left(\frac{a}{|d|}\right), \quad d<0 \tag{1.2}$$

and

$$L\left(1, \left(\frac{d}{\cdot}\right)\right) = -\frac{1}{G\big((\frac{d}{\cdot})\big)} \sum_{a=1}^{d-1} \left(\frac{d}{a}\right) \log\left(2\sin\frac{a}{d}\pi\right), \quad d>0, \tag{1.3}$$

where $B_1(x) = x - \frac{1}{2}$ signifies the first (periodic) Bernoulli polynomial and $G(\chi)$ is the normalized Gauss sum defined by $\big(\chi(\cdot) = (\frac{d}{\cdot})\big)$

$$G(\chi) = \sum_{a \bmod |d|} \chi(a) e^{2\pi i \frac{a}{|d|}}. \tag{1.4}$$

(1.2) and (1.3) depend on the relation

$$L(s,\chi) = \frac{1}{G(\overline{\chi})} \sum_{a=1}^{q-1} \overline{\chi}(a) l_s\left(\frac{a}{q}\right), \tag{1.5}$$

where χ is a primitive Dirichlet character mod q and $l_s(x)$ is the polylogarithm (function) of the complex exponential argument

$$l_s(x) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s}, \quad \sigma > 1 \tag{1.6}$$

(also referred to as the Lerch zeta-function [**14**]). Its limiting case $s = 1$, $x \notin \boldsymbol{Z}$

$$l_1(x) = A_1(x) - \pi i B_1(x), \quad 0 < x < 1 \tag{1.7}$$

gives (1.2) and (1.3), where $A_1(x)$ signifies the first Clausen function $A_1(x) = -\log(2\sin \pi x)$ ([**17**], [**19**]).

Polylogarithm functions are relatives of the Hurwitz zeta-function defined by

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad \sigma > 1, \ x > 0, \tag{1.8}$$

which for $x = 1$ reduces to the Riemann zeta-function defined by (1.1).

The polylogarithm function and the Hurwitz zeta-function are interrelated by the functional equation (sometimes referred to as Lerch's formula)

$$\zeta(1-s,x) = \frac{\Gamma(s)}{(2\pi)^s} \left(e^{-\frac{\pi i}{2}s} l_s(x) + e^{\frac{\pi i}{2}s} l_s(1-x)\right). \tag{1.9}$$

As suggested by (1.9), there is a counterpart of (1.5), which is a decomposition into residue classes mod q:

$$L(s,\chi) = \frac{1}{q^s} \sum_{a=1}^{q-1} \chi(a) \zeta\left(s, \frac{a}{q}\right) \tag{1.10}$$

being valid for any Dirichlet character mod q, not necessarily primitive.

Recall the Laurent expansion for $\zeta(s,x)$,

$$\zeta(s,x) = \frac{1}{s-1} - \psi(x) + O(s-1), \quad s \to 1, \tag{1.11}$$

where $\psi(x)$ signifies the Euler digamma function

$$\psi(x) = \frac{\Gamma'}{\Gamma}(x) = (\log \Gamma(x))'. \tag{1.12}$$

Also recall the orthogonality of characters

$$\sum_{a=1}^{q-1} \chi(a) = \begin{cases} 0, & \text{if } \chi \neq \chi_0, \\ \varphi(q), & \text{if } \chi = \chi_0, \end{cases} \tag{1.13}$$

where χ_0 and $\varphi(q)$ stand for the principal character mod q and the Euler function defined by $\varphi(q) = \sum_{1 \le a \le q, (a,q)=1} 1$, respectively.

From (1.10), (1.11) and (1.13) we obtain

$$L(s,\chi) = -\frac{1}{q} \sum_{a=1}^{q-1} \chi(a) \psi\left(\frac{a}{q}\right) + O(s-1), \quad s \to 1$$

and a fortiori

$$L(1,\chi) = -\frac{1}{q} \sum_{a=1}^{q-1} \chi(a) \psi\left(\frac{a}{q}\right). \tag{1.14}$$

For the values of $\psi\left(\frac{p}{q}\right)$, there is a remarkable formula of Gauss ($1 \le p < q$):

$$\begin{aligned} \psi\left(\frac{p}{q}\right) &= -\gamma - \log q - \frac{\pi}{2} \cot \frac{p}{q} \pi + \sum_{k=1}^{q-1} \cos \frac{2pk}{q} \pi \log\left(2 \sin \frac{k}{q} \pi\right) \\ &= -\gamma - \log q - \frac{\pi}{2} \cot \frac{p}{q} \pi + 2 \sum_{k \le \frac{q}{2}} \cos \frac{2pk}{q} \pi \log\left(2 \sin \frac{k}{q} \pi\right), \end{aligned} \tag{1.15}$$

where γ is the Euler constant ($\psi(1) = -\gamma$) and where the second equality is a consequence of Lemma 1 below ([**2**], [**4**], [**9**]).

It was D. H. Lehmer [**15**] who first used (1.15) in his study of the generalized Euler constant $\gamma(p,q)$ for an arithmetic progression p mod q. Using [**15**, (11)] and the relation [**15**, Theorem 7] between $\gamma(p,q)$ and $\psi\left(\frac{p}{q}\right)$, he deduced (1.15), and stated ([**15**, p.135]) "Our proof via finite Fourier series indicates that Gauss' remarkable result has a completely elementary basis."

Our main purpose is to elaborate on this statement of Lehmer and, on streamlining the argument, to show that (1.15) has a purely number-theoretic basis and that ψ is a number-theoretic function. As a converse to this, we shall also put into practice the statement of Deninger [**6**, p.180], to the effect that (1.15) can be used to evaluate $L(1,\chi)$. Indeed, Funakura was on these lines (cf. [**8**, (1)]) but he appealed to the integral representation of Legendre and applied Lehmer's argument of using $-\log(1-e^{2\pi ix})$, $0<x<1$.

We may now state our main theorem.

Theorem 1. *Gauss' formula* (1.15) *is equivalent to finite expressions for* $L(1,\chi)$:

$$L(1,\chi)=\frac{\pi}{2q}\sum_{a=1}^{q-1}\chi(a)\cot\frac{a}{q}\pi \tag{1.16}$$

for χ odd and

$$L(1,\chi)=-\frac{1}{\sqrt{q}}\sum_{a=1}^{q-1}\widehat{\chi}(a)\log\left(2\sin\frac{a}{q}\pi\right) \tag{1.17}$$

for χ even, where

$$\widehat{\chi}(a)=\frac{1}{\sqrt{q}}\sum_{k \bmod q}\chi(k)e^{-2\pi i\frac{k}{q}a} \tag{1.18}$$

is the finite Fourier transform of χ.

The finite Fourier transform of χ is intimately related to the generalized Gauss sum $G(a,\chi)$, see (2.3) below.

Corollary 1. *For primitive χ,* (1.16) *and* (1.17) *reduce, respectively, to*

$$L(1,\chi_{\mathrm{odd}})=-\frac{\pi i}{G(\overline{\chi})}\sum_{a=1}^{q-1}\overline{\chi}(a)B_1\left(\frac{a}{q}\right) \tag{1.16$'$}$$

and

$$L(1,\chi_{\mathrm{even}}) = -\frac{1}{G(\overline{\chi})}\sum_{a=1}^{q-1}\overline{\chi}(a)\log\left(2\sin\frac{a}{q}\pi\right), \tag{1.17$'$}$$

and a fortiori, finite expressions (1.2) *and* (1.3) *are consequences of Gauss' formula* (1.15).

REMARK 1. (i) On symmetry grounds, (1.16) may be stated as

$$L(1,\chi_{\mathrm{odd}}) = -\frac{\pi i}{\sqrt{q}}\sum_{a=1}^{q-1}\widehat{\chi}(a)B_1\left(\frac{a}{q}\right), \tag{1.16$''$}$$

which can be explicitly computed to be (1.16) (cf. e.g. [**8**]).

We note that both Funakura [**8**] and Ishibashi-Kanemitsu [**12**] treated the case of periodic functions $f(n)$ of period q, and obtained generalizations of the formulas (1.16)$''$ and (1.17), but they are already implicit in Yamamoto's work [**19**], depending on (1.5) and (1.7).

(ii) The last statement of Corollary 1 follows, on recalling that the Kronecker characters $\left(\frac{d}{\cdot}\right)$ are primitive odd or even characters mod $|d|$, according as $d<0$ or $d>0$, respectively.

In the course of proof of Theorem 1, we shall encounter an interesting number-theoretic function $N(q)=N_q$ defined by $\log N_q = -\sum_{d|q}(\mu(d)\log d)\frac{q}{d}$ which eventually cancels out in view of (1.20) below. We believe this function deserves wider attention and we state

THEOREM 2. *For* $q>1$, *the number-theoretic function* $\log N(q)=\log N_q$ *admits the following expressions.*

$$\log N_q = -q\sum_{d|q}\frac{\mu(d)}{d}\log d \tag{1.19}$$

$$= -\varphi(q)\sum_{a=1}^{q-1}\log\left(2\sin\left(\frac{a}{q}\pi\frac{\mu\left(\frac{q}{(a,q)}\right)}{\varphi\left(\frac{q}{(a,q)}\right)}\right)\right) \tag{1.20}$$

$$= \sum_{d|q}\Lambda(d)\varphi\left(\frac{q}{d}\right) \tag{1.21}$$

$$= \varphi(q)\sum_{p|q}\frac{\log p}{p-1}, \tag{1.22}$$

where the last sum is extended over all prime divisors p of q, and where μ and Λ signify the Möbius function and the von Mangoldt function, respectively.

2. Proof of the theorems.

Let $f(n)$ be an arithmetic periodic function of period q:

$$f : \boldsymbol{Z} \to \boldsymbol{C}; \quad f(n+q) = f(n), \quad n \in \boldsymbol{Z}.$$

We define the parity of f as follows: f is called even if $f(-n) = f(n)$ and odd if $f(-n) = -f(n)$.

We prepare some lemmas, of which Lemma 1 is repeatedly used in what follows, without notice.

Lemma 1. *If f is odd, then*

$$\sum_{a=1}^{q-1} f(a) = 0$$

and if f is even, then

$$\sum_{a=1}^{q-1} f(a) = 2\sum_{a<\frac{q}{2}} f(a) + \frac{1+(-1)^q}{2} f\left(\frac{q}{2}\right).$$

In particular, if f and $\chi \bmod q$ are of opposite parity, then

$$\sum_{a=1}^{q-1} \chi(a) f(a) = 0$$

while if f and χ are of the same parity and $q > 2$, then

$$\begin{aligned}\sum_{a=1}^{q-1} \chi(a) f(a) &= 2\sum_{a\le\frac{q}{2}} \chi(a) f(a)\\ &= 2\sum_{a<\frac{q}{2}} \chi(a) f(a).\end{aligned}$$

Lemma 2. *The ψ function satisfies Gauss' multiplicative formula or the modified Kubert identity*

$$\psi(x) = \log q + \frac{1}{q}\sum_{a=0}^{q-1} \psi\left(\frac{x+a}{q}\right). \tag{2.1}$$

Lemma 3. *Let χ denote a Dirichlet character* $\bmod q$, $q \geq 3$. *Then*

$$\sum_{\chi \text{ even}} \chi(n) = \begin{cases} 0 & \textit{if } n \not\equiv \pm 1 \pmod q, \\ \dfrac{\varphi(q)}{2} & \textit{if } n \equiv \pm 1 \pmod q, \end{cases}$$

and

$$\sum_{\chi \text{ odd}} \chi(n) = \begin{cases} 0 & \textit{if } n \not\equiv \pm 1 \pmod q, \\ \dfrac{\varphi(q)}{2} & \textit{if } n \equiv 1 \pmod q, \\ -\dfrac{\varphi(q)}{2} & \textit{if } n \equiv -1 \pmod q, \end{cases}$$

where the sum is extended over all even and odd characters, respectively.

Proof. For $q \geq 3$, the set $\{\pm 1\}$ forms a subgroup of the reduced residue class group $G = (\boldsymbol{Z}/q\boldsymbol{Z})^{\times}$ of index 2. Hence the factor group $G/\{\pm 1\}$ has order $\frac{\varphi(q)}{2}$. Since the group of all even characters may be identified with the character group of $G/\{\pm 1\}$, it follows, from the orthogonality of characters, that

$$\begin{aligned} \sum_{\chi \text{ even}} \chi(n) &= \sum_{\chi \in \widehat{G/\{\pm 1\}}} \chi(n) \\ &= \begin{cases} 0 & \text{if } n \neq 1 \text{ in } G/\{\pm 1\}, \\ \dfrac{\varphi(q)}{2} & \text{if } n = 1 \text{ in } G/\{\pm 1\}, \end{cases} \end{aligned}$$

which proves the first assertion. The second assertion follows from the first and the orthogonality relation

$$\sum_{\chi \in \widehat{G}} \chi(n) = \begin{cases} 0 & \text{if } n \not\equiv 1 \pmod q \\ \varphi(q) & \text{if } n \equiv 1 \pmod q. \end{cases} \tag{2.2}$$

This completes the proof of Lemma 3. □

It is instructive to give a proof of Corollary 1 first. We introduce the generalized Gauss sum

$$\begin{aligned} G(k,\chi) &= \sum_{a=1}^{q-1} \chi(a) e^{2\pi i \frac{a}{q} k} \\ &= \sqrt{q}\chi(-1)\widehat{\chi}(k) \end{aligned} \tag{2.3}$$

(cf. (1.18)) and note that it decomposes into

$$G(k,\chi) = \overline{\chi}(k) G(\chi) \tag{2.4}$$

if and only if χ is primitive, where $G(\chi) = G(1,\chi)$ is the normalized Gauss sum (1.4) ([**1**], [**5**]). We derive (1.16)′ by appealing to Eisenstein's formula

$$\sum_{k=1}^{q-1} l_0\left(\frac{k}{q}\right) e^{-2\pi i \frac{p}{q} k} = B_1 - qB_1\left(\frac{p}{q}\right)$$

or rather its converse (cf. [**16**], [**11**], [**18**])

$$\begin{aligned} \sum_{k=1}^{q-1} B_1\left(\frac{k}{q}\right) e^{2\pi i \frac{p}{q} k} &= -l_0\left(\frac{p}{q}\right) - 1 - B_1 \\ &= -\frac{i}{2} \cot \frac{p}{q} \pi. \end{aligned} \tag{2.5}$$

Substituting (2.5) into (1.16), we find that

$$L(1,\chi) = \frac{\pi i}{q} \sum_{k=1}^{q-1} B_1\left(\frac{k}{q}\right) G(k,\chi).$$

Using (2.2) and other known facts

$$G(\overline{\chi}) = \chi(-1)\overline{G(\chi)}, \quad |G(\chi)|^2 = q,$$

we conclude (1.16)′.

We may deduce (1.17)′ from (1.17) in a similar way. Substituting (1.18) into (1.17), we obtain

$$L(1,\chi) = -\frac{1}{q}\sum_{k=1}^{q-1} \log\left(2\sin\frac{k}{q}\pi\right) \sum_{a=1}^{q-1} \chi(a)\cos\frac{2\pi k}{q}a \tag{2.6}$$

whose inner sum is again $G(k,\chi)$. Therefore for χ primitive, we have

$$L(1,\chi) = -\frac{G(\chi)}{q}\sum_{k=1}^{q-1} \overline{\chi}(k)\log\left(2\sin\frac{k}{q}\pi\right),$$

whence $(1.17)'$ follows in the same way. This completes the proof of Corollary 1.

We now turn to

Proof of Theorem 1. That (1.15) implies (1.16) and (1.17) is immediate. Indeed, substituting (1.15) in (1.14) and using Lemma 1, we obtain (1.16) for χ odd and (2.6) for χ even, which is the same as (1.17).

Now we are to prove the converse, i.e. we are to deduce (1.15) from (1.16) and (1.17).

With p, $(p,q)=1$, we multiply (1.14) by $\chi(p^{-1})$ and sum over χ mod q, $\chi \neq \chi_0$ to obtain

$$\begin{aligned} &\sum_{\chi_0\neq\chi \bmod q} \chi(p^{-1})L(1,\chi) \\ &= -\frac{1}{q}\sum_{a=1}^{q}\psi\left(\frac{a}{q}\right)\sum_{\chi_0\neq\chi \bmod q}\chi(ap^{-1}) \\ &= S_1+S_2, \end{aligned} \tag{2.7}$$

say, where

$$S_1 = -\frac{1}{q}\sum_{a=1}^{q}\psi\left(\frac{a}{q}\right)\sum_{\chi \bmod q}\chi(ap^{-1})$$

and

$$S_2 = \frac{1}{q}\sum_{a=1}^{q}\psi\left(\frac{a}{q}\right)\chi_0(ap^{-1}).$$

By the orthogonality (2.2) of characters,

$$S_1 = -\frac{\varphi(q)}{q}\psi\left(\frac{p}{q}\right). \tag{2.8}$$

The sum S_2 is

$$S_2 = \frac{1}{q}\sum_{a=1}^{q-1}{}^{*}\psi\left(\frac{a}{q}\right),$$

the star on the summation sign indicating that the sum runs over all a's, relatively prime to q, $(a,q)=1$. This last condition may be replaced by introducing the sum $\sum_{d|(a,q)}\mu(d)$. Expressing the condition $d|(a,q)$ as $d|q$, $a=a'd \le q-1$, we have

$$S_2 = \frac{1}{q}\sum_{d|q}\mu(d)\sum_{a'=1}^{\frac{q}{d}-1}\psi\left(\frac{a'}{\frac{q}{d}}\right)$$

whose inner sum is

$$\sum_{a=0}^{\frac{q}{d}-2}\psi\left(\frac{a+1}{\frac{q}{d}}\right) = -\frac{q}{d}\log\frac{q}{d} - \gamma\frac{q}{d} + \gamma$$

by Lemma 2. Hence

$$S_2 = -\frac{\log q}{q}\varphi(q) - \frac{1}{q}\log N_q - \frac{\varphi(q)}{q}\gamma, \tag{2.9}$$

where $\log N_q$ is defined by (1.19).

Substituting (2.8) and (2.9) in (2.7), we conclude that

$$\sum_{\chi_0\neq\chi \bmod q}\chi(p^{-1})L(1,\chi) = \frac{\varphi(q)}{q}\left(-\psi\left(\frac{p}{q}\right) - \log q - \frac{1}{\varphi(q)}\log N_q - \gamma\right). \tag{2.10}$$

It remains to calculate the left-hand side of (2.10), which can be done by dividing the sum into two parts:

$$\sum_{\chi_0\neq\chi \text{ even}} \quad\text{and}\quad \sum_{\chi \text{ odd}}$$

substituting therewith (1.17) and (1.16), respectively.

First, by (1.17),

$$\begin{aligned}\sum_{\chi_0\neq\chi \bmod q}&\chi(p^{-1})L(1,\chi)\\&=-\frac{1}{\sqrt{q}}\sum_{a=1}^{q-1}\log\left(2\sin\frac{a}{q}\pi\right)\sum_{\chi_0\neq\chi\text{ even}}\overline{\chi}(p)\widehat{\chi}(a)\\&=-\frac{1}{\sqrt{q}}\sum_{a=1}^{q-1}\log\left(2\sin\frac{a}{q}\pi\right)\left(\sum_{\chi\text{ even}}\overline{\chi}(p)\widehat{\chi}(a)-\overline{\chi_0}(p)\widehat{\chi_0}(a)\right)\\&=T_1+T_2,\end{aligned}\tag{2.11}$$

say, where, by (1.18),

$$T_1=-\frac{1}{\sqrt{q}}\sum_{a=1}^{q-1}\log\left(2\sin\frac{a}{q}\pi\right)\sum_{\chi\text{ even}}\overline{\chi}(p)\frac{1}{\sqrt{q}}\sum_{k \bmod q}\chi(k)e^{-2\pi i\frac{k}{q}a}\tag{2.12}$$

and

$$T_2=\frac{1}{\sqrt{q}}\sum_{a=1}^{q-1}\log\left(2\sin\frac{a}{q}\pi\right)\frac{1}{\sqrt{q}}\sum_{k \bmod q}\chi_0(k)e^{-2\pi i\frac{k}{q}a}.\tag{2.13}$$

The inner double sum of T_1 is

$$\begin{aligned}\frac{1}{\sqrt{q}}\sum_{k \bmod q}&e^{-2\pi i\frac{k}{q}a}\sum_{\chi\text{ even}}\chi(kp^{-1})\\&=\frac{1}{\sqrt{q}}\frac{\varphi(q)}{2}\left(e^{-2\pi i\frac{p}{q}a}+e^{2\pi i\frac{p}{q}a}\right)\\&=\frac{\varphi(q)}{\sqrt{q}}\cos\left(2\pi\frac{p}{q}a\right)\end{aligned}$$

by Lemma 3, and so

$$T_1=-\frac{\varphi(q)}{q}\sum_{a=1}^{q-1}\cos\left(2\frac{p}{q}a\pi\right)\log\left(2\sin\frac{a}{q}\pi\right),\tag{2.14}$$

while the inner sum for T_2,

$$\sum_{k \bmod q}{}^{*} e^{-2\pi i \frac{k}{q} a},$$

is the Ramanujan sum, which is equal to

$$\varphi(q)\,\frac{\mu\left(\frac{q}{(a,q)}\right)}{\varphi\left(\frac{q}{(a,q)}\right)}$$

by Hölder's result (cf. e.g. [**15**, p.133]).

Hence

$$T_2 = \frac{\varphi(q)}{q}\sum_{a=1}^{q-1}\log\left(2\sin\frac{a}{q}\pi\right)\frac{\mu\left(\frac{q}{(a,q)}\right)}{\varphi\left(\frac{q}{(a,q)}\right)}$$

but this is $-\frac{1}{q}\log N_q$ by (1.20), i.e.

$$T_2 = -\frac{1}{q}\log N_q. \tag{2.15}$$

Substituting (2.14) and (2.15) in (2.11), we obtain

$$\sum_{\chi_0 \neq \chi \text{ even}} \chi(p^{-1})L(1,\chi) = -\frac{\varphi(q)}{q}\sum_{a=1}^{q-1}\cos 2\pi\frac{p}{q}a\pi\log\left(2\sin\frac{a}{q}\pi\right) - \frac{1}{q}\log N_q. \tag{2.16}$$

On the other hand, by (1.16) and Lemma 3,

$$\begin{aligned}\sum_{\chi \text{ odd}} \chi(p^{-1})L(1,\chi) &= \frac{\pi}{2q}\sum_{a=1}^{q-1}\cot\frac{a}{q}\pi\sum_{\chi \text{ odd}}\chi(ap^{-1})\\ &= \frac{\pi}{2q}\left(\frac{\varphi(q)}{2}\cot\frac{p}{q}\pi - \frac{\varphi(q)}{2}\cot\frac{-p}{q}\pi\right)\\ &= \frac{\pi\varphi(q)}{2q}\cot\frac{p}{q}\pi.\end{aligned} \tag{2.17}$$

Combining (2.16) and (2.17) implies

$$\begin{aligned}\sum_{\chi_0\neq\chi \bmod q} &\chi(p^{-1})L(1,\chi)\\ &= -\frac{\varphi(q)}{q}\sum_{a=1}^{q-1}\cos\left(2\frac{p}{q}a\pi\right)\log\left(2\sin\frac{a}{q}\pi\right) \\ &\quad -\frac{1}{q}\log N_q + \frac{\varphi(q)}{2q}\pi\cot\frac{p}{q}\pi.\end{aligned} \tag{2.18}$$

Equating (2.10) and (2.18) and eliminating the terms involving $\log N_q$, we conclude (1.15). This completes the proof. □

Before giving a proof of Theorem 2, we make the following

Remark 2. Lehmer [**15**] and Briggs [**3**] were the first who studied the generalized Euler constant $\gamma(r,q)$ for an arithmetic progression r mod q (cf. also [**7**], [**14**]), defined by

$$\gamma(r,q) = \lim_{x\to\infty}\left(\sum_{\substack{n\le x\\ n\equiv r \pmod q}} 1 - \frac{1}{q}\log x\right).$$

Lehmer considered the sum $\Phi(q) = \sum_{a=1}^{q}{}^{*}\gamma(a,q)$ and expressed it in the closed form in [**15**, Theorem 3]:

$$q\Phi(q) = \gamma\varphi(q) + \log N_q, \tag{2.19}$$

where N_q is some rational number. Then in [**15**, Theorem 4] he proves

$$N_q = \prod_{d|q} d^{-q\frac{\mu(d)}{d}},$$

which is (1.19) after exponentiation.

Then comparing (2.19) and [**15**, (16)], we have (1.22). Comparing then (1.22) with [**15**, Theorem 6] we obtain

$$\log N_q = \frac{\varphi(q)}{q}\log 2 - 2\frac{\varphi(q)}{q}\sum_{0<j<\frac{q}{2}}\frac{\mu\left(\frac{q}{(j,q)}\right)}{\varphi\left(\frac{q}{(j,q)}\right)},$$

to which we may apply Lemma 1 to deduce (1.20). Thus, Lehmer's argument, ingenious as it is, is rather rounding with respect to formulas (1.19) and (1.20) which are relevant to us.

Moreover, in Lehmer's consideration, formula (1.21) is missing, which is the key linking (1.19), (1.20) and (1.22) most naturally.

We now give a sketch of proof of Theorem 2, which consists of a set of instructive number-theoretic identities.

PROOF OF THEOREM 2. First we prove that (1.19) and (1.21) are equivalent. Recall the well-known generating functions ($\sigma > 1$)

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}$$

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

$$\zeta'(s) = \sum_{n=1}^{\infty} \frac{-\log n}{n^s},$$

etc. Then (1.19) amounts to

$$\left(\frac{1}{\zeta(s)}\right)' \zeta(s-1) = \sum_{n=1}^{\infty} \frac{\log N_n}{n^s},$$

whose left-hand side is

$$-\frac{\zeta'}{\zeta}(s) \frac{\zeta(s-1)}{\zeta(s)} = \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^s} \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}$$

$$= \sum_{n=1}^{\infty} \frac{\sum_{d|n} \Lambda(d)\varphi\left(\frac{n}{d}\right)}{n^s},$$

which is (1.21).

To prove (1.20), it is the easiest to transform [**15**, (11)] rather than to follow the lines of proof of Lehmer mentioned in Remark 2.

[**15**, (11)] reads

$$q\gamma(r,q) = \gamma + \log 2 + \frac{\pi}{2}\cot\frac{r}{q}\pi \\ - 2\sum_{j<\frac{q}{2}} \cos\left(\frac{2rj}{q}\pi\right)\log\left(\sin\frac{j}{q}\pi\right),$$

which, by Lemma 1, amounts to

$$q\gamma(r,q) = \gamma + \log 2 + \frac{\pi}{2}\cot\frac{r}{q}\pi \\ - \sum_{j=1}^{q-1} \cos\frac{2rj}{q}\pi\log\left(\sin\frac{j}{q}\pi\right).$$

Now noting that

$$(\log 2)\sum_{j=1}^{q-1}\cos\frac{2rj}{q}\pi = -\log 2,$$

we may absorb the term $\log 2$ in the sum:

$$q\gamma(r,q) = \gamma + \frac{\pi}{2}\cot\frac{r}{q}\pi \\ - \sum_{j=1}^{q-1} \cos\frac{2rj}{q}\pi\log\left(2\sin\frac{j}{q}\pi\right).$$

The rest is the same as in Lehmer: summing over r, $(r,q)=1$ and applying Hölder's result (cf. the statement immediately after (2.14)) leads to (1.20) on appealing to (2.19).

The most interesting part of the proof is the deduction of (1.22) from (1.21).

We recall that $\varphi(p^\alpha) = p^\alpha - p^{\alpha-1} = p^\alpha\left(1-\frac{1}{p}\right)$ for p a prime and that φ is multiplicative. Denote by $\alpha_p(n)$ the highest exponent of p that divides n: $p^{\alpha_p(n)}||n$.

Then by (1.21),

$$\log N_n = \sum_{p|n}\sum_{\alpha=1}^{\alpha_p(n)} \Lambda(p^\alpha)\varphi\left(\frac{n}{p^\alpha}\right) \\ = \sum_{p|n}(\log p)\sum_{\alpha=1}^{\alpha_p(n)}\varphi\left(p^{\alpha_p(n)-\alpha}\frac{n}{p^{\alpha_p(n)}}\right)$$

$$= \sum_{p|n} (\log p) \sum_{\alpha=1}^{\alpha_p(n)} \varphi\left(p^{\alpha_p(n)-\alpha}\right) \varphi\left(\frac{n}{p^{\alpha_p(n)}}\right)$$

$$= \sum_{p|n} (\log p) \left(\left(\sum_{\alpha=1}^{\alpha_p(n)-1} \varphi(p^{\alpha_p(n)-\alpha}) \right) \frac{\varphi(n)}{\varphi(p^{\alpha_p(n)})} + \frac{\varphi(n)}{\varphi(p^{\alpha_p(n)})} \right),$$

where we used the definition of $\Lambda(n)$ to be $\log p$ if $n = p^m$ and 0 otherwise. Factoring $\varphi(n)$ out and substituting the explicit expression for $\varphi(p^m)$, we get

$$\log N_n = \varphi(n) \sum_{p|n} (\log p) \left(\sum_{\alpha=1}^{\alpha_p(n)-1} \left(\frac{1}{p^\alpha} \right) + \frac{1}{p^{\alpha_p(n)-1}(p-1)} \right),$$

the geometric series summing to $p^{\alpha_p(n)-1}/(p^{\alpha_p(n)}(p-1))$, and (1.22) follows.

Since (1.22) implies (1.20) as is explained in Remark 2, we complete the proof of Theorem 2. □

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