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Linear stability of projected canonical curves with applications to the slope of fibred surfaces

By Miguel Ángel Barja and Lidia Stoppino

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Abstract. Let $f\colon S \longrightarrow B$ be a non locally trivial relatively minimal fibred surface. We prove a lower bound for the slope of f depending increasingly from the relative irregularity of f and the Clifford index of the general fibres.

1. Introduction and preliminaries.

Let $f\colon S \longrightarrow B$ be a surjective holomorphic map with connected fibres from a complex smooth projective surface S onto a complex smooth curve B. We always assume that it is relatively minimal, i.e., that there is no (-1)–rational curve contained in a fibre of f. Let F be a general fibre. We call f a fibration of genus g whenever $g = g(F)$; we also set $b = g(B)$. The fibration is called smooth if all its fibres are smooth, isotrivial if all its smooth fibres are reciprocally isomorphic, and locally trivial if it is smooth and isotrivial (i.e. an holomorphic fibre bundle).

Let ω_S be the canonical line bundle of S and K_S any canonical divisor. Set $p_g = h^0(S, \omega_S)$, $q = h^1(S, \omega_S)$, $\chi\mathscr{O}_S = p_g - q + 1$ and let $e(X)$ be the topological Euler characteristic of X. We consider the following relative invariants:

$$\begin{aligned} K_f^2 &= (K_S - f^*K_B)^2 = K_S^2 - 8(b-1)(g-1) \\ \chi_f &= \deg f_*\omega_{S/B} = \chi\mathscr{O}_S - (b-1)(g-1) \\ e_f &= e(S) - e(B)e(F) = e(S) - 4(b-1)(g-1) \\ q_f &= q(S) - b. \end{aligned}$$

We have the following classical results, when $g \geq 2$:

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(i) (Noether) $12\chi_f = e_f + K_f^2$.
(ii) (Zeuthen-Segre) $e_f \geq 0$. Moreover, $e_f = 0$ if and only if f is smooth.
(iii) (Arakelov) $K_f^2 \geq 0$. Moreover, if $K_f^2 = 0$ then f is isotrivial.
(iv) $\chi_f \geq 0$. Moreover, $\chi_f = 0$ if and only if f is locally trivial.
(v) $0 \leq q_f \leq g$. When $b \geq 1$ $q_f = 0$ if and only if f is the Albanese map of S. On the other hand, $q_f = g$ if and only if $S = B \times F$ (cf. [**10**]).

We say that f is a non-Albanese fibration if $q_f > 0$.

When f is not locally trivial, Xiao (cf. [**30**]) defines the slope of f as

$$s(f) = \frac{K_f^2}{\chi_f}.$$

It follows immediately from Noether's equality that $0 \leq s(f) \leq 12$.

We are mostly concerned with a lower bound of the slope. The main known result in this direction is:

If $g \geq 2$ and f is not locally trivial, then $s(f) \geq 4 - \frac{4}{g}$,

which is known as the slope inequality. It was first proven by Horikawa and Persson for hyperelliptic fibrations. Xiao gives a proof for general fibrations (cf. [**30**]) and, independently, Cornalba and Harris prove it for semistable fibrations (i.e., for fibrations where all the fibres are semistable curves in the sense of Deligne and Mumford). Later on, in [**26**] it has been proved a generalization of their method which can be applied to any fibration.

The slope of a fibration turns out to be sensible to a lot of geometric properties, both of the fibres of f and of S (see [**2**] for a complete reference).

We like to pay attention to the influence of the relative irregularity of the the fibration, q_f. In view of our argument, also the Clifford index of f appears closely related to this problem. In [**16**], there is a very interesting attempt to exhibit the lower bound of the slope as an increasing function of the Clifford index. This seems very clear for hyperelliptic or trigonal fibrations (see also [**17**], [**24**]), and for general Clifford index, but in the intermediate cases, generality conditions are necessary ([**6**]).

In the case of the relative irregularity q_f, it seems again that the lower bound of the slope should be an increasing function of q_f. A crucial point where the relative irregularity q_f appears in a fibration is given by the so called Fujita decomposition:

$$f_*\omega_f = \mathscr{A} \oplus \mathscr{O}_B^{\oplus q_f}$$

which also produces a decomposition of the relative Jacobian fibration associated to f. In particular, notice that a general fibre of a non-Albanese fibration has non simple Jacobian.

The first result which manifests the influence of q_f on the slope is due to Xiao ([**30**]): $s(f) \geq 4$ whenever $q_f > 0$ and equality holds only if $q_f = 1$. Explicit lower bounds depending on q_f are given in [**19**] and in [**8**], but they are rather complicated and seem far to be sharp. However, from these results it seems clear that there should be a lower bound for the slope which is an increasing function of the relative irregularity.

We conjecture the following simple behavior for the bound.

CONJECTURE 1.1. *Let $f\colon S \longrightarrow B$ be a fibration of genus g, with relative irregularity $q_f < g-1$. Then*

$$s(f) \geq 4\,\frac{g-1}{g-q_f}\,.$$

This bound, if true, is sharp (Example 4.1). Apart from the aforementioned result of Xiao, some other evidences for this conjecture are the following.

- It is true when $\boldsymbol{P}(\mathscr{O}_B^{\oplus q_f})$ does not meet the general fibre and the projection from it induces a birational and linearly stable map (Remark 3.5).
- It is true when $\boldsymbol{P}(\mathscr{O}_B^{\oplus q_f})$ does not meet the general fibre and $\mathscr{A}$ is a semistable sheaf on B (Remark 3.3).
- There is an analogous canonical decomposition of $f_*\omega_f$ in case the fibration is a double cover fibration. In that situation, the corresponding conjectured bound holds (see Example 4.1 and [**12**]).
- In a semistable fibration with s singular fibres, Vojta proves the following inequality

  $$K_f^2 \leq (2g-2)(2b-2+s)$$

  which combined with slope inequality gives

  $$\chi_f \leq \frac{g}{2}\,(2b-2+s).$$

  However, a sharper bound of this type holds (cf. [**3**] and [**29**]), namely

  $$\chi_f \leq \frac{q-q_f}{2}\,(2b-2+s)$$

which is exactly the bound we would obtain using our conjectured bound instead of slope inequality in Vojta's formula.

To our knowledge, the only known counterexamples to the bound above belong to the extremal case $q_f = g-1$ (cf. [**22**] and Remark 4.6).

Our approach is the following. Consider any vector subbundle $\mathscr{F} \subseteq f_*\omega_f$. The inclusion induces a linear system on F which is just the projection

$$\boldsymbol{P}(f_*\omega_f) \longrightarrow \boldsymbol{P}(\mathscr{F})$$

restricted to the canonical embedding of F (assume it is non hyperelliptic). Information about the degree and rank of this linear system is the main ingredient for applying Xiao's method. In some cases this information allows to conclude that the projection is linearly stable; roughly speaking, this means that any linear subsystem can only increase the ratio between the degree and the rank (see section 2 for a more precise definition). In the case of curves linear stability implies Hilbert stability and so we can also apply Cornalba-Harris method to study a lower bound of the slope $s(f)$.

With this purpose, we start in section 2 studying when a projection of a canonical curve is linearly stable. Our main result in this direction is

Theorem 1.2. *Let $C \subset \boldsymbol{P}^{g-1}$ be a canonical non-hyperelliptic curve. Let $\Sigma \subset \boldsymbol{P}^{g-1}$ be a proper $(s-1)$-space. Let k be any positive integer smaller or equal to $\min\{[s/2], [\mathrm{Cliff}(C)/2]\}$. Then there is a non-empty open set of $(k-1)$-spaces contained in Σ that induce linearly stable projections.*

In section 3 we use this information to study a lower bound of the slope of non-Albanese fibrations. We obtain

Theorem 1.3. *Let $f\colon X \to B$ be a fibred surface. Let $m := \min\{\mathrm{Cliff}(f), q_f\}$. Then the slope of f satisfies the inequality*

$$s(f) \geq 4\,\frac{g-1}{g-[m/2]}\,.$$

Although the main ingredient for the theorem is the result of linear stability of section 2, we give two different proofs of this result, one by applying Xiao's method and another one using the one of Cornalba-Harris. We present this fact as another instance that, at least in the case of surfaces, both methods, of different nature, produce the same results. We believe that this parallelism (which does not clearly hold for higher dimensions) merits further investigation.

The two invariants involved in our main result, the relative irregularity and the Clifford index, are of very different nature. Theorem 1.3 gives a strong inequality for big values of both these invariants. It is therefore important to verify that this two quantities are independent, and in particular that they can grow simultaneously. In section 4 we provide examples of fibred surfaces with both q_f and $\mathrm{Cliff}(f)$ large, but also of surfaces with large q_f and small $\mathrm{Cliff}(f)$, and vice versa.

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2. Linear stability of projections of a canonical curve.

In this section we prove, under suitable assumptions, the linear stability of general projections of canonical curves. This is the key property that allows us to apply both Xiao's and Cornalba-Harris method in the second part of the paper.

The notion of linear stability was first defined by Mumford in [**20**] for embeddings in projective spaces. The following is a natural generalization for curves with any map to projective spaces. For a more general treatment, see [**26**].

Let C be a smooth curve, together with a non-degenerate map in a projective space $\psi\colon C \to \boldsymbol{P}^r$. Consider the base point free linear series $\mathscr{A}$ associated to the morphism $\widetilde{\psi}$ obtained eliminating the base points of ψ. If d is the degree of $\mathscr{A}$ and r is its dimension (i.e. $\mathscr{A}$ is a g^r_d), we define the reduced degree of the pair (C,ψ) as

$$\mathrm{red.deg}(C,\psi) \colon = \frac{d}{r}$$

(we will also use the notation $\mathrm{red.deg}(C,\mathscr{A})$, or $\mathrm{red.deg}(C,V)$, where $V \subseteq H^0(C,\psi^*\mathscr{O}_{\boldsymbol{P}^r}(1))$ is the linear system such that $\mathscr{A} = \boldsymbol{P}(V)$).

DEFINITION 2.1. With the same notations as above, we say that $\psi\colon C \to \boldsymbol{P}^r$ is linearly semistable (resp. stable) if for any projection π on a positive dimensional projective space,

$$\mathrm{red.deg}(C,\pi\circ\psi) \geq \mathrm{red.deg}(C,\psi)$$

(resp. $\mathrm{red.deg}(C,\pi\circ\psi) > \mathrm{red.deg}(C,\psi)$).

In other words, we are asking that for any linear series $\mathscr{A}'$ (of degree d' and

dimension r') contained in the linear series associated to ψ the inequality $d'/r' \geq d/r$ has to be satisfied.

REMARK 2.2. It is easy to see that when ψ is induced from a line bundle, it is sufficient to check the inequality for any complete linear series contained in the one associated to ψ. The classical results on divisors on curves, such as Clifford's Theorem and its generalizations ([**10**], [**23**]) and the Riemann-Roch Theorem, imply quite easily the following results (cf. [**1**] and [**25**]).

(1) If C is a non-hyperelliptic curve, the canonical embedding of C in $\boldsymbol{P}^{g-1}$ is linearly stable.

(2) If C is hyperelliptic, the canonical morphism is linearly semistable, but not stable.

(3) If C is a smooth curve of genus $g \geq 1$ and L is a line bundle on C of degree $d > 2g$, the embedding induced by L is linearly stable.

We are interested in the linear stability of projections from the canonical image of a curve.

EXAMPLE 2.3. If C is a trigonal curve, the projection from a point outside the canonical image can be linearly unstable. Indeed, consider any effective divisor $P_1 + P_2 + P_3$ belonging to the g^1_3 on C (the P_i's need not be distinct). By the geometric Riemann-Roch Theorem these points span a line $\ell \subset \boldsymbol{P}^{g-1}$. Let P be a point of ℓ disjoint from C. It can be easily checked that the projection π from P is a birational morphism. The image of π, $\overline{C}$, has a triple point R. If we consider the projection ψ from R, we have, for $g \geq 5$

$$\operatorname{red.deg}(\overline{C}, \psi) = \frac{2g-5}{g-3} < \frac{2g-2}{g-2} = \operatorname{red.deg}(C, \pi).$$

From now on, C is a non-hyperelliptic curve embedded in $\boldsymbol{P}^{g-1}$ by its canonical system.

Let p_Λ be the projection from a $(k-1)$-plane Λ disjoint from C. We search for conditions for p_Λ to be linearly stable.

Call $V = \operatorname{Ann}(\Lambda) \subset H^0(\omega_C)$ the linear system associated to p_Λ, and $\mathscr{V} = |V|$ the associated linear series. Let $W \subseteq V$ be any proper subsystem. We call $\mathscr{W} = |W|$ the linear series, and $\overline{\mathscr{W}}$ the base point free linear series obtained from $\mathscr{W}$ by eliminating the base points.

If $\dim W = g - k - \alpha$, and $\deg \overline{\mathscr{W}} = 2g - 2 - d$, then W is not destabilising for V if and only if

$$\alpha \geq d\,\frac{g-k-1}{2g-2}\,.$$

Let D be the effective divisor of base points of $\mathscr{W}$. Roughly speaking, the inequality above implies that D should impose "enough" conditions on V itself. Indeed, a sufficient condition for W not to be destabilising is

$$\dim V(-D) \leq \dim V - d\frac{g-k-1}{2g-2}\,, \tag{2.1}$$

where, as usual, $V(-D) = V \cap H^0(\omega_C(-D))$. The geometric meaning of this condition is that the $(k-1)$-plane Λ intersect the $(d+h^0(D))$-plane spanned by D in a plane of dimension smaller or equal to $g - h^0(\omega_C(-D)) - d(g-k-1)/(2g-2) - 1$.

REMARK 2.4. Using the stronger versions of Clifford Theorem proved in [**10**] and [**23**], it can be shown that the projection of a canonical non-trigonal curve from any point not contained in it is linearly stable. Moreover, one can show that the projection of a trigonal canonical curve from a point not contained in a trisecant line is linearly stable (cf. [**25**]). In what follows, we generalize these results for projections from positive dimensional subspaces.

Given a line bundle L over a smooth curve C, its Clifford index is $\mathrm{Cliff}(L) = \deg L - 2(h^0(L)-1)$. If D is a divisor on C, $\mathrm{Cliff}(D) := \mathrm{Cliff}(\mathscr{O}_C(D))$.

DEFINITION 2.5. The Clifford index of a curve C of genus $g \geq 4$ is the integer:

$$\mathrm{Cliff}(C) = \min\{\mathrm{Cliff}(L) \mid L \in \mathrm{Pic}(C),\, h^0(L) \geq 2,\, h^1(L) \geq 2\}.$$

When $g = 2$ we set $\mathrm{Cliff}(C) = 0$; when $g = 3$ we set $\mathrm{Cliff}(C) = 0$ or 1 according to whether C is hyperelliptic or not.

A line bundle with h^0 and h^1 greater or equal to 2 is said to contribute to the Clifford index. Brill-Noether theory shows that $\mathrm{Cliff}(C) \leq [(g-1)/2]$, and that equality holds if C is general in moduli. Clifford's Theorem says that the curves with Clifford index 0 are exactly the hyperelliptic ones. It is easy to prove that the curves with Clifford index 1 are the trigonal ones and the smooth plane quintics. In general, the Clifford index and the gonality of a curve C are related by the following (cf. [**13**])

$$\operatorname{gon}(C) - 3 \leq \operatorname{Cliff}(C) \leq \operatorname{gon}(C) - 2.$$

Remark 2.6. As the Clifford index of a curve C measures how large is the ratio between the degree and the dimension of special linear series on C, it seems natural to guess that the canonical curves with higher Clifford index have linearly stable projections from positive-dimensional subspaces of $\boldsymbol{P}^{g-1}$. However, this guess is false. The problem is that the Clifford index does not control the divisors having H^1 of dimension 1. Indeed, consider a non-hyperelliptic curve C with arbitrary Clifford index, and let $D = P_0 + \dots P_k$ be an effective divisor consisting of $k+1$ points that impose independent conditions on $H^0(\omega_C)$. Consider a section φ of $H^0(\omega_C)$ not vanishing at anyone of the P_i's (a general section will do). The linear subsystem V of $H^0(\omega_C)$ spanned by $H^0(\omega_C(-D))$ and by φ has no base points by construction, and has dimension $g-k$. Hence, V induces the projection of the canonical image of C from a subspace of projective dimension $k-1$ disjoint from it. As soon as $k > 1$ this projection is linearly unstable, because

$$\operatorname{red.deg}(V) = \frac{2g-2}{g-k-1} > \operatorname{red.deg}(\omega_C(-D)) = \frac{2g-3-k}{g-k-2}.$$

Note that $h^1(\omega_C(-D)) = 1$, and hence $\omega_C(-D)$ is one of the divisors that does not contribute to the Clifford index of C.

Proposition 2.7. *Let $C \subset \boldsymbol{P}^{g-1}$ be a canonical curve, and k an integer such that $\operatorname{Cliff}(C) \geq 2k$. Let Λ be a $(k-1)$-plane in $\boldsymbol{P}^{g-1}$ disjoint from C such that*

$$\dim(\Lambda \cap \operatorname{span}(D)) < d\frac{g+k-1}{2g-2} - 1 \tag{2.2}$$

for any special effective divisor D on C with degree $d \leq 2k-1$ such that $\dim \operatorname{span}(D) = d-1$.

Then the projection with centre Λ is linearly stable.

Proof. Let $V \subset H^0(C, \omega_C)$ be the linear system associated to the projection with centre Λ. Let $W \subset V$ be any linear subsystem with $\dim W \geq 2$; we need to check that $\operatorname{red.deg}(C, W) \geq \operatorname{red.deg}(C, V)$. Let L_W be the line bundle generated by the sections of W. $L_W \cong \omega_C(-D)$, with $D = \operatorname{Ann}(W) \cap C$. Observe that

$$\Lambda \cap \operatorname{span}(D) = \boldsymbol{P}(\operatorname{Ann}(V + H^0(\omega_C(-D)))) \subseteq \boldsymbol{P}(H^0(\omega_C)^\vee) = \boldsymbol{P}^{g-1}.$$

Applying Grassman formula to $V + H^0(\omega_C(-D))$, condition (2.1) translates as:

$$\dim(\Lambda \cap \operatorname{span}(D)) \leq \frac{k}{2g-2}\, d + \frac{\operatorname{Cliff}(D)}{2} - 1. \tag{2.3}$$

Note that $\operatorname{span}(D)$ is a plane of dimension $(d+c)/2 - 1$.

If D contributes to the Clifford index of C then inequality (2.3) is trivially satisfied as the right side term is bigger than $k-1$, which is the dimension of Λ.

If, on the other hand, D does not contribute to the Clifford index, necessarily we have $h^0(\mathscr{O}_C(D)) = 1$, because $h^1(\mathscr{O}_C(D)) = h^0(\omega_C(-D)) \geq \dim W \geq 2$. By the geometric version of the Riemann-Roch Theorem, the points of D are in general position (i.e. $\dim \operatorname{span}(D) = d-1$). Moreover, notice that in this case $d = \operatorname{Cliff}(D)$.

If $d > 2k(g-1)/(g+k-1)$, then condition (2.3) is satisfied with strict inequality, because the space $\operatorname{span}(D)$ has dimension strictly smaller than the number on the right hand side. Hence we can consider the case

$$d \leq \frac{2k(g-1)}{g+k-1}.$$

In particular, d has to be smaller or equal to $2k-1$, and under this assumption, inequality (2.3) is implied by (2.2). Hence, the proof is concluded. □

For the applications contained in the next section, we need to treat the following situation. Suppose that we are given a linear subspace Σ of $\boldsymbol{P}^{g-1}$ of dimension $\dim \Sigma = s-1$ (without any assumption on it). We want to find the biggest possible integer k such that there exists a linear subspace Λ of dimension $k-1$ contained in Σ such that the projection π_Λ with centre Λ is linearly stable. Of course k will depend on the dimension of Σ.

THEOREM 2.8. *Let $C \subset \boldsymbol{P}^{g-1}$ be a canonical non-hyperelliptic curve. Let $\Sigma \subset \boldsymbol{P}^{g-1}$ be a proper $(s-1)$-space. Let k be any positive integer smaller or equal to $\min\{[s/2], [\operatorname{Cliff}(C)/2]\}$. Then there is a non-empty open set of $(k-1)$-spaces contained in Σ that induce linearly stable projections of degree $2g-2$.*

PROOF. We try and find a linear space $\Lambda \subseteq \Sigma$ satisfying the assumptions of Proposition 2.7. We can replace conditions (2.2) with the following (more restrictive) ones:

$$\dim(\Lambda \cap \operatorname{span}(D)) \leq \frac{d}{2} - 1 \tag{2.4}$$

for any special divisor D on C with degree $d \leq 2k-1$ such that $\dim \operatorname{span}(D) = d-1$.

Observe that condition (2.4) for d even is implied by the same condition for $d+1$. Hence we can suppose d odd. We seek the existence of Λ in Σ that does not contain any $((d-1)/2)$-space contained in the span of d points in general position. Let us bound from the above the dimension of such "bad" spaces in the grassmanian $\operatorname{Gr}(k,g)$ of $(k-1)$-spaces in $\boldsymbol{P}^{g-1}$.

- The dimension of the spaces $\operatorname{span}(D)$ is d.
- The dimension of the $((d-1)/2)$-spaces contained in a fixed $(d-1)$-space $\operatorname{span}(D)$ is $\dim \operatorname{Gr}((d+1)/2, d) = d^2/4 - 1/4$.
- The dimension of the $(k-1)$-spaces contained in Σ that contain a fixed $(d-1)/2$-space is $\dim \operatorname{Gr}(k-(d-1)/2-1, s-(d-1)/2-1) = (k-(d+1)/2)(s-k)$.

Hence there exists a $(k-1)$-space in Σ satisfying conditions (2.4) as soon as the grassmanian of $(k-1)$-planes contained in Σ has dimension strictly greater than the dimension of the "bad" family, i.e.

$$k(s-k) = \dim \operatorname{Gr}(k,s) > d + \frac{d^2-1}{4} + \left(k - \frac{d+1}{2}\right)(s-k),$$

which becomes

$$s \geq k + 1 + \frac{d+1}{2}. \tag{2.5}$$

As d varies from 1 to $2k-1$, we see that the inequality obtained is $s \geq 2k+1$.

For $d = 2k-1$, we can slightly improve the bound arguing as follows. Inequality (2.4) for $d = 2k-1$ means that Λ in Σ is not entirely contained in any $(2k-2)$-space $\operatorname{span}(D)$. Let us make the following remark

If Σ is not the whole $\boldsymbol{P}^{g-1}$, then for any $r \leq s$, there is at most a finite number of r-secant $(r-1)$-spaces entirely contained in Σ.

Indeed, if there were a positive dimensional family of d secant $(r-1)$-spaces contained in Σ, then the whole curve C would be contained in Σ, contradicting the fact that the canonical morphism is non-degenerate.

Therefore, the $(k-1)$-spaces contained in Σ that are also contained in a $\operatorname{span}(D)$ are of dimension at most

$$2k-1+\dim \operatorname{Gr}(k,2k-2)=k^2-1,$$

and the same argument as above gives the bound $k(s-k)>k^2-1$, hence $s\geq 2k$.

Noting that for $d\leq 2k-3$ conditions (2.5) are satisfied for $s\geq 2k$, we can conclude the proof. □

REMARK 2.9. Note that the condition $\operatorname{Cliff}(C)\geq 2k$ implies necessarily that g has to be greater or equal to $4k+1$. Hence, if we consider for instance $\Sigma=\boldsymbol{P}^{g-1}$, the above result is empty for $4k\geq g$. However, it implies for instance that if C has general Clifford index (which is a general condition) then there exists a linear space of dimension $[(g-1)/4]-1$ such that the projection from it is a linearly stable map. It has to be remarked anyway that for k "big" with respect to g, the sufficient conditions made in the proof of Theorem 2.8 to simplify the original conditions for stability contained in Proposition 2.7, become consistently restrictive.

3. Application to the slope of fibred surfaces.

Let $f\colon S\longrightarrow B$ be a non-Albanese fibration. We are interested in giving a lower bound for the slope $s(f)$ as an increasing function of q_f. For this we will apply relative projections to the relative canonical map $S\dashrightarrow \boldsymbol{P}(f_*\omega_f)$ which induce, on the general fibre F, a linearly stable projection. The bigger the center of the projection is, the better is the bound we get. In the analysis of linear stability of projections of canonical curves in the previous section, appears as a fundamental ingredient the Clifford index. As we will see, the bound we get involves naturally this two invariants: the Clifford index of the general fibre and the relative irregularity q_f.

Taking any linear subspace of the canonical embedding of a concrete fibre F we are not sure we can extend it to a relative linear subspace over B (in order to make a relative projection), except it is contained in the trivial part $\boldsymbol{P}(\mathscr{O}_B^{\oplus q_f})$ of the Fujita decomposition

$$f_*\omega_f=\mathscr{A}\oplus\mathscr{O}_B^{\oplus q_f}.$$

Moreover, such an election allows us to control the degree of the sheaves involved, since $\deg\mathscr{A}=\deg f_*\omega_f$.

We present here two different proofs. To the, yet classical, method of Xiao to

study the slope of fibrations, has recently joined the generalized Cornalba-Harris method. Although they are of different nature, the application of both seem to give very similar results in several situations (cf. [**25**], [**26**]), at least in the case of fibred surfaces ([**7**] gives a higher dimensional example). It is an intriguing question whether both methods are in fact equivalent or not. Our aim is to show how either method provides, in this case, exactly the same bound, and to present this fact as an instance of this parallelism.

As we will see, Xiao and Cornalba-Harris start from a subsheaf of the pushforward of a line bundle on the total space (in our, and most cases, $f_*\omega_f$); from this, they give as an output an inequality involving divisor classes on the base. However, while Xiao's method needs almost no hypothesis, the one of Cornalba-Harris requires a GIT stability condition on the maps induced by the subsheaf on the general fibres. Nevertheless, as hopefully the computations made here will show, the linear stability of the maps induced on the general fibres, although not required by Xiao, is a fundamental ingredient for both the approaches.

Applying our results on linear stability of projections, we are able to find a direct factor $\mathscr{E}$ of $f_*\omega_f$ which induces linearly stable projections on the general fibres, and such that $\deg \mathscr{E} = \chi_f$.

Given a fibred surface $f\colon S \longrightarrow B$, we define its Clifford index $\mathrm{Cliff}(f)$ as the maximum of the Clifford indices of the fibres (cf. [**16**]). As Cliff is a lower semicontinuous locally constant function, $\mathrm{Cliff}(f)$ is the Clifford index of the general fibres.

Proposition 3.1. *Let $f\colon S \longrightarrow B$ be a fibred surface. If $k = \min\{[\mathrm{Cliff}(f)/2], [q_f/2]\}$, there exists a decomposition*

$$f_*\omega_f = \mathscr{E} \oplus \mathscr{O}_B^{\oplus k}$$

such that the fibre of $\mathscr{E}$ on general $t \in B$ is a linear system inducing a linearly stable degree $2g-2$ morphism of the fibre $f^{-1}(t) = F_t$.

Proof. If f is an Albanese fibration, or if $\mathrm{Cliff}(f) \leq 1$, $\mathscr{E}$ is the whole sheaf $f_*\omega_f$, and the statement is satisfied, because for a general fibre F, $H^0(F, \omega_F)$ is base-point-free, and it induces a linearly stable embedding (Remark 2.2).

Otherwise, let us consider the Fujita decomposition

$$f_*\omega_f = \mathscr{A} \oplus \mathscr{O}_B^{\oplus q_f}.$$

The sheaf $\mathscr{A}$ induces on a fibre F_t a projection of the canonical image from the (q_f-1)-plane $\Sigma_t = \boldsymbol{P}(\mathrm{Ann}(\mathscr{A}\otimes \boldsymbol{k}(t))$ (of course Σ_t is canonically identified with $\boldsymbol{P}(\mathscr{O}_B^{\oplus q_f}\otimes \boldsymbol{k}(t))$).

A general fibre F is smooth and $\mathrm{Cliff}(F) = \mathrm{Cliff} f$. Let us fix such a general fibre, and drop the small t from the notations.

Let Λ be a $k-1$-plane contained in Σ and let $A' \subseteq H^0(\omega_F)$ be the linear system associated with the projection from it. Note that, as $\mathscr{O}_B^{\oplus q_f}$ is trivial, we can extend Λ to a trivial direct factor of $f_*\omega_f$ and get a decomposition

$$f_*\omega_f = \mathscr{E}\oplus \mathscr{O}_B^{\oplus k}.$$

By Theorem 2.8, as conditions $2k \le q_f$ and $\mathrm{Cliff}(F)\ge 2k$ are satisfied, on any general fibre there exists a dense open set of $(k-1)$-plane contained in Σ inducing linearly stable, base-point-free projections of degree $2g-2$. So we can choose one Λ in our fixed fibre F such that the fibre of the corresponding $\mathscr{E}$ on general t enjoys the same properties. □

We now come to the two proofs of Theorem 1.3.

Via Xiao's method.

Xiao's method is a well established way of studying slopes of fibrations (cf. [**30**], [**2**], [**5**], [**21**], [**17**], [**18**]). We just sketch it and refer to [**2**] and to [**30**] for details. Consider the Harder-Narashimann filtration of any subsheaf $\mathscr{F}$ of $f_*\omega_f$:

$$0 =: \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_n = \mathscr{F}$$

and let $\mu_1 > \cdots > \mu_n$ ($\mu_i := \mu(\mathscr{E}_i/\mathscr{E}_{i-1})$) be the associated slopes. Set $r_i = \mathrm{rk}\mathscr{E}_i$. We have

$$\deg \mathscr{F} = \sum_{i=1}^{n} r_i(\mu_i - \mu_{i+1}), \qquad (\text{where } \mu_{n+1}=0)$$

For technical reasons, it is necessary that all the sequence of slopes is decreasing (including $\mu_{n+1}=0$), so we need $\mu_n \ge 0$. This is always achieved if $\mathscr{F}$ is not only a subsheaf but also a direct summand of $f_*\omega_f$ (which is a nef vector bundle on B).

For each i, the composite of the natural sheaf homomorphisms

$$f^*\mathscr{E}_i \to f^*f_*\omega_f \to \omega_f$$

induces a rational map $S \to \boldsymbol{P}_B(\mathscr{E}_i)$. Up to a suitable sequence of blowing-ups $\epsilon : \widehat{S} \longrightarrow S$ (which does not modify the general fibre F), the above map becomes a morphism for every i. Let M_i be the moving part of the pull-back of the tautological line bundle H_i on $\boldsymbol{P}_B(\mathscr{E}_i)$. $M_i|_F$ is a base point free linear system on F which induces a map into $\boldsymbol{P}^{r_i-1}$ (a fibre of $\boldsymbol{P}_B(\mathscr{E}_i) \to B$), of degree d_i.

Proposition 3.2. (cf. [**30**]) *For any sequence of indices with* $1 \leq i_1 < \cdots < i_m \leq n$ *we have*

$$M_n^2 \geq \sum_{p=1}^{m}(d_{i_p} + d_{i_{p+1}})(\mu_{i_p} - \mu_{i_{p+1}})$$

where $i_{m+1} = n + 1$.

Then, we can proceed to give a proof of 1.3:

Proof of 1.3.

Following the notations of Proposition 3.1, we put $\mathscr{F} = \mathscr{E} = \mathscr{A} \oplus \mathscr{O}^{\oplus(q_f-k)}$ and apply 3.2 for the whole set of indexes $\{1, 2, \ldots, n\}$:

$$M_n^2 \geq \sum_{i=1}^{n}(d_i + d_{i+1})(\mu_i - \mu_{i+1}).$$

Since by construction the linear system $M_{n|F}$ is linearly stable and of degree $d_n = 2g - 2$, then for all $i = 1, \ldots, n$ we have

$$\frac{d_i}{r_i - 1} \geq \frac{d_n}{r_n - 1} = \frac{2g-2}{q_f - k - 1} =: \alpha.$$

Using that $r_{i+1} \geq r_i + 1$ and that $\deg \mathscr{E} = \deg f_*\omega_f = \chi_f$ we conclude

$$M_n^2 \geq 2\alpha\chi_f - \alpha\mu_1.$$

On the other hand, $M_n \leq \epsilon^*K_f$ and both are nef, so we have

$$K_f^2 = (\epsilon^*K_f)^2 \geq M_n^2 \geq 2\alpha\chi_f - \alpha\mu_1.$$

Finally, defining now $\mathscr{F} = f_*\omega_f$ and taking the set of indexes $\{1, n\}$ we obtain

$$K_f^2 \geq (2g-2)\mu_1$$

which combined with the previous inequality produces the desired result

$$K_f^2 \geq 4\,\frac{g-1}{g-k}\,\chi_f.$$

□

REMARK 3.3. In the particular case when $\mathscr{A}$ is semistable, we can take in the previous proof $\mathscr{F} = \mathscr{A}$. Then the same argument produce

$$K_f^2 \geq 2\,\frac{d}{g-q_f}\,\chi_f.$$

If, moreover, we know that $\boldsymbol{P}(\mathscr{O}_B^{\oplus q_f})$ does not meet the general fibre F, then $d = 2g-2$ and so

$$K_f^2 \geq 4\,\frac{g-1}{g-q_f}\,\chi_f.$$

Via a Theorem of Cornalba and Harris.

The method of Cornalba-Harris is introduced in [**11**]. Let us summarize the version for fibred surfaces[1] following the generalization presented in [**26**].

THEOREM 3.4 (Cornalba-Harris). *Let $f\colon S \to B$ be a fibred surface. Let L be a line bundle on S and $\mathscr{F}$ a coherent subsheaf of f_*L of rank r such that for general $t \in B$ the linear system*

$$\mathscr{F} \otimes \boldsymbol{k}(t) \subseteq H^0(F_t, L_{|F_t})$$

induces a linearly stable map. Let $\mathscr{G}_h$ be a coherent subsheaf of $f_*L^{\otimes h}$ that contains the image of the morphism

$$\mathrm{Sym}^h \mathscr{F} \longrightarrow f_*L^{\otimes h},$$

[1]The original Cornalba-Harris Theorem requires the assumption of Hilbert instead of linear stability. For curves, linear stability implies Hilbert stability as proved in [**1**] or in [**25**]. It is not known whether the converse implication holds.

and coincides with it at general t. If $N = \mathrm{rank}\mathscr{G}_h$ is of the form $Ah + O(1)$ and $\deg \mathscr{G}_h$ of the form $Bh^2 + O(h)$, the following inequality holds:

$$rB - A \deg \mathscr{F} \geq 0. \tag{3.6}$$

Let us consider the particular case in which $\mathscr{F} = f_*L$ and $\mathscr{G}_h = f_*L^{\otimes h}$. By the Riemann-Roch Theorem

$$\deg f_*L^{\otimes h} = \deg f_!L^{\otimes h} + \deg R^1 f_* L^{\otimes h} = \frac{h^2}{2} L^2 - \frac{h}{2} LK_f + \deg f_*\omega_f + \deg R^1 f_* L^{\otimes h}.$$

Let d be the relative degree of L. For large enough h, By Riemann-Roch on the general fibre, $N = dh - g + 1$, where g is the genus of the fibration. Suppose that $\deg R^1 f_* L^{\otimes h} = Ch^2 + O(h)$; in this case the computation of the leading coefficient of $\deg \mathscr{G}_h$ gives:

$$rL^2 + r\,C - 2d \deg f_*L \geq 0. \tag{3.7}$$

Proof of Theorem 1.3.

Let us use Proposition 3.1. If $k = 0$ ($\mathrm{Cliff} f \leq 1$, or f is an Albanese morphism), the statement of Theorem 1.3 is just the slope inequality.

Otherwise, observe that the sheaf $\mathscr{E}$ of Proposition 3.1 satisfies the assumptions of Theorem 3.4. Consider the morphism of sheaves

$$\mathrm{Sym}^h \mathscr{E} \longrightarrow f_*\omega_f^{\otimes h},$$

and call $\mathscr{G}_h$ its image.

On general t, the morphism induced by $\mathscr{E} \otimes \boldsymbol{k}(t)$ has degree $2g - 2$. Moreover, we now prove that it is birational. Indeed, as a consequence of Castelnuovo's bound (cf. [**1**] Exercise B-7), either the map induced by $\mathscr{E} \otimes \boldsymbol{k}(t)$ is birational or it factors through a double cover over a curve of genus at most k. This last case is impossible, because it would imply that

$$\mathrm{Cliff}(f) = \mathrm{Cliff}(F_t) \leq \mathrm{gon} F_t - 2 \leq \gamma \leq k,$$

contrary to the assumption. Hence,

$$\mathrm{rank}\mathscr{G}_h = h^0(\overline{F}, (j^*\mathscr{O}_{\boldsymbol{P}^{g-k-1}}(1))^{\otimes h}) = (2g-2)h + O(1),$$

where $\overline{F}_t$ is the image of F_t. Moreover,

$$\deg \mathscr{G}_h \leq \deg f_* \omega_f^{\otimes h}$$

because $f_*\omega_f^{\otimes h}$ is nef (cf. [**28**]). Hence, the coefficient of h^2 in $\deg \mathscr{G}_h$ is smaller than $K_f^2/2$, and inequality (3.6) implies

$$(g-k)\frac{K_f^2}{2} - (2g-2)\chi_f \geq 0,$$

as claimed. □

REMARK 3.5. Suppose that, under suitable assumptions, the fibre of $\mathscr{A}$ itself on general $t \in B$ was a base point free linear system of degree d which induced a linear semistable morphism. Both the Cornalba-Harris Theorem and the method of Xiao would give as a result the inequality

$$s(f) \geq 2\frac{d}{g-q_f},$$

which coincides with the bound of Conjecture 1.1 if $d = 2g-2$, that's to say if $\boldsymbol{P}(\mathscr{O}_B^{\oplus q_f})$ is disjoint from the general fibre.

4. Examples.

EXAMPLE 4.1. This example is constructed in [**4**], sec 4.5 (see also [**12**, Example 4.1]). Let Γ and B be smooth curves. Let $\gamma > 0$ be the genus of Γ. We consider $B \times \Gamma$. Let p_1 and p_2 be the two projections, and H_1, H_2 their general fibres. Consider a smooth divisor $R \in |2nH_1 + 2mH_2|$ (by Bertini's Theorem such a divisor exists, at least for sufficiently large n and m). Let $\rho\colon X \to B \times \Gamma$ be the double cover ramified over R. Call $\mathscr{L}$ the associated line bundle such that $\mathscr{L}^{\otimes 2} = \mathscr{O}_{B\times\Gamma}(R)$.

Consider the fibration $f := p_1 \circ \rho\colon X \to B$; its general fibre is a double cover of Γ, and its genus is $g = 2\gamma + m - 1$. A computation shows that its slope is

$$s(f) = 4\frac{2\gamma+m-2}{\gamma+m-1} = 4\frac{g-1}{g-\gamma}.$$

The relative irregularity is exactly $q_f = \gamma$. Indeed,

$$q = h^1(X, \mathscr{O}_X) = h^1(B \times \Gamma, \mathscr{O}_{B\times\Gamma}) + h^1(B \times \Gamma, \mathscr{L}^{-1}) =$$
$$= b + \gamma + h^1(B, K_B(nP_1)) + h^1(\Gamma, K_\Gamma(mP_2)) = b + \gamma.$$

Hence, notice that this fibrations have slope reaching the expected bound of Conjecture 1.1, (regardless to the Clifford index). Quite interestingly, these fibrations are also examples of slope minimal with respect to the bound for double cover fibrations established in [**12**].

What about the Clifford index? In general, the gonality of the general fibre of these fibrations is at most twice the gonality of the quotient Γ, and so it is smaller or equal to $\gamma + 3$ if γ is odd, and $\gamma + 2$ if γ is even. Hence, the Clifford index of the general fibre X_t is smaller or equal to $\gamma + 1$ for odd γ and to γ for even γ. Under suitable assumptions, the Clifford index is "almost" γ, as shown by the following standard argument.

Lemma 4.2. *Suppose that* Γ *has general gonality* $\mathrm{gon}(\Gamma) = \alpha = [(\gamma + 3)/2]$, *and suppose that* α *is a prime number. If we choose* $m \geq \gamma^2 + \gamma + 4$ *in the above construction, then* $\mathrm{Cliff}(X_t) \geq \gamma$ *for odd* γ, *and* $\mathrm{Cliff}(X_t) \geq \gamma - 1$ *for even* γ.

Proof. Let β be the gonality of X_t. We want to prove that $\beta = 2\alpha$. By definition of gonality, $\beta \leq 2\alpha$. Let us suppose then that β is strictly smaller than 2α. Consider the following diagram

$$\begin{array}{ccccc} & & X_t & & \\ & \overset{\sigma_1}{\swarrow} & \downarrow \sigma & \overset{\sigma_2}{\searrow} & \\ \boldsymbol{P}^1 & \xleftarrow{\beta_1} & \overline{X_t} & \xrightarrow{\beta_2} & \boldsymbol{P}^1 \\ & \underset{\pi_1}{\nwarrow} & \cap & \underset{\pi_2}{\nearrow} & \\ & & \boldsymbol{P}^1 \times \boldsymbol{P}^1 & & \end{array}$$

where σ_1 is a degree β morphism, and σ_2 the composition of the quotient morphism $\psi\colon X_t \to \Gamma$ with a morphism $\Gamma \to \boldsymbol{P}^1$ of degree α; the π_i are the projections, $\sigma = \sigma_1 \times \sigma_2$ and $\overline{X}_t = \sigma(X_t)$. Let d be the degree of σ; clearly $d \mid 2\alpha$ and $d \mid \beta$, hence $d = 1, 2, \alpha$ are the only possibilities. By the adjunction formula,

$$2p_a(\overline{X}_t) - 2 = (K_{\boldsymbol{P}^1\times\boldsymbol{P}^1}\overline{X}_t + \overline{X}_t^2) = 2\,\frac{(\alpha\beta)}{d^2} - \frac{\beta}{d} - \frac{2\alpha}{d} + 1.$$

If $d = 1$, then $g \leq p_a(\overline{X}_t) \leq (2\alpha - 1)(\beta - 1) \leq (\gamma + 2)(\gamma + 1)$. Remembering that $g = 2\gamma + m - 1$, we deduce that m has to be smaller or equal to $\gamma^2 + \gamma + 3$, contrary to the assumption.

If $d = 2$, then observe that by assumption $g > \gamma(\gamma+1)+1 \geq 4(p_a(\overline{X}_t))+1$, then it follows from Lemma 1.7 of [**12**], that $\overline{X}_t$ is isomorphic to Γ. But then β_1 would be a morphism from Γ to $\boldsymbol{P}^1$ of degree strictly smaller than α, which is a contradiction.

It remains to deal with the case $d = \alpha$. In this case, It has to be $\overline{X}_t = \boldsymbol{P}^1$, and $\sigma = \sigma_1$. Then, one can consider the composite morphism $X_t \xrightarrow{\sigma\times\psi} \boldsymbol{P}^1 \times \Gamma$, which has degree either 1 or α (remember that α is prime). The case of degree 1 would imply (again by adjunction) that $g \leq (\gamma+3)(3\gamma-1)/4+1$, so it can be excluded. The case of degree α would imply that $\Gamma \cong \boldsymbol{P}^1$, a contradiction, because we assumed that $\gamma = g(\Gamma) > 0$. □

EXAMPLE 4.3. The following construction leads in particular to examples of fibrations with $q_f = 2$ and Clifford index big. Let S be an abelian surface, and let C be a smooth curve of genus $g \geq 3$ contained in it. By the adjunction formula, $C^2 = (CK_S + C^2) = 2g-2$. By Riemann-Roch

$$h^0(S, \mathscr{O}_S(C)) = \frac{(CK_S + C^2)}{2} + \chi_S = g - 1 \geq 2.$$

Hence we can consider an algebraic pencil (i.e. a linear series of dimension 1) in $|C|$. Let $\widehat{S}$ be the blow up of S in the $2g-2$ base points. The pencil induces a fibration $f\colon \widehat{S} \longrightarrow \boldsymbol{P}^1$, which clearly has relative irregularity $q_f = q(S) = 2$, and whose Clifford index is the Clifford index of C.

In the following we shall prove that there exist abelian surfaces containing curves of arbitrary genus and Clifford index big. We will use an argument suggested to us by A. Knutsen.

Let us first recall the following definitions and results.

A line bundle L on a variety X is said to be k-very ample if for any 0-dimensional scheme Z of length $k+1$, the restriction map

$$H^0(X, L) \longrightarrow H^0(Z, \mathscr{O}_Z(L))$$

is surjective; hence, in particular, a line bundle is 0-very ample iff it is globally generated, 1-very ample iff it is very ample, 2-very ample iff it separates tangent vectors, and so on. If C is a smooth curve then the gonality of C is $k+1$ if and only if ω_C is $(k-1)$-very ample but not k-ample (this is a straightforward consequence of Riemann-Roch).

If C is a smooth curve contained in a smooth projective surface S, by adjunction

$$\omega_C \simeq (\omega_S \otimes \mathscr{O}_S(C)) \otimes \mathscr{O}_C.$$

Hence, we derive immediately that if $\operatorname{gon}(C) \leq k+1$, then $\omega_S \otimes \mathscr{O}_S(C)$ is not k-very ample. We will use the following result.

THEOREM 4.4 (Bauer-Szemberg, [**9**]). *Let S be an abelian surface with Picard number 1, and L a line bundle on S of type $(1,d)$, $d \geq 1$. Then L is k-very ample if and only if $d \geq 2k+3$.*

We are now ready to prove the following

LEMMA 4.5. *Let S be an abelian surface with Picard number 1 and let L be an ample line bundle of type $(1,d)$, with $d \geq 1$. Then if C is a smooth curve of genus g contained in the linear system associated to L, $\operatorname{gon}(C) \geq (d/2) = ((g+1)/2)$.*

PROOF. Let $k+1$ be the gonality of C. Remember that $g(C) = d-1$. Suppose by contradiction that $k+1 < d/2$. This implies that $d \geq 2k+3$, and by Theorem 4.4, $\mathscr{O}_S(C) \simeq \omega_S \otimes \mathscr{O}_S(C)$ is k-very ample. From the above remarks it follows that $\operatorname{gon}(C) > k+1$, which is the desired contradiction. □

It is worth noticing that the construction above can be made in much more generality using the results of [**27**].

Hence, we can construct fibrations from an Abelian surface to $\boldsymbol{P}^1$ with "almost general" Clifford index.

REMARK 4.6. Note that these fibrations all have slope 6. Indeed, given any such fibration $f\colon \widehat{S} \longrightarrow \boldsymbol{P}^1$,

$$K_f = K_{\widehat{S}} - f^* K_{\boldsymbol{P}^1} \sim \sum_{i=1}^{2g-2} E_i + 2C,$$

where E_i are the exceptional divisors of the blow up $\widehat{S} \to S$. Hence,

$$\begin{aligned} K_f^2 &= \left(\sum_{i=1}^{2g-2} E_i + 2C \cdot \sum_{i=1}^{2g-2} E_i + 2C \right) \\ &= 4\left(\sum_{i=1}^{2g-2} E_1 \cdot C \right) + \left(\sum_{i=1}^{2g-2} E_i \cdot \sum_{i=1}^{2g-2} E_i \right) = 6(g-1), \end{aligned}$$

and

$$\deg f_*\omega_f = \chi_{\widehat{S}} - \chi_{\boldsymbol{P}^1}\chi_F = g-1.$$

This slope is coherent, and indeed bigger than, the bound given by Theorem 1.3, which is 4.

It is also coherent with the bound of Conjecture 1.1, for any genus except for $g=3$, when it gives a counterexample for the case $q_f = g-1$.

REMARK 4.7. One could make an analogous construction starting from a $K3$ surface. By a result of Knutsen ([**15**]) there are $K3$ surfaces containing curves of any possible gonality. Hence this construction leads to fibrations with $q_f = 0$ and Cliff(f) arbitrary. In this case the slope is $6(g-1)/(g+1)$. Note that this slope reaches exactly the bound for fibrations with general Clifford index and odd genus found by Konno (cf. [**16**], [**2**]) and by Eisenbud-Harris for semistable fibrations([**14**]).

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Miguel Ángel Barja

Departament de Matemàtica Aplicada I
Universitat Politècnica de Catalunya
ETSEIB Avda. Diagonal
08028 Barcelona, Spain
E-mail: miguel.angel.barja@upc.edu

Lidia Stoppino

Dipartimento di Matematica
Università di Roma TRE
Largo S. L. Murialdo
1 I-00146, Roma, Italy
E-mail: stoppino@mat.uniroma3.it