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Linear approximation for equations of motion of vibrating membrane with one parameter

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Abstract. This article treats a one parameter family of equations of motion of vibrating membrane whose energy functionals converge to the Dirichlet integral as the parameter ε tends to zero. It is proved that both weak solutions satisfying energy inequality and generalized minimizing movements converge to a unique solution to the d'Alembert equation.

1. Introduction.

Let ε be a positive number. We treat a family of second order quasilinear hyperbolic equations

$$u_{tt} - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+\varepsilon^2|\nabla u|^2}}\right) = 0 \tag{1.1}$$

in a bounded domain $\Omega \subset \boldsymbol{R}^n$. We impose initial and boundary conditions

$$u(0,x) = u_0(x), \quad \frac{\partial u}{\partial t}(0,x) = v_0(x), \quad x \in \Omega, \tag{1.2}$$

$$u(t,x) = 0, \quad x \in \partial\Omega. \tag{1.3}$$

Consider the following equation

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$$u_{tt} - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \tag{1.4}$$

which is in [**7**], [**11**], [**12**] referred to as an equation of motion of vibrating membrane. In [**7**] approximate solutions to (1.4) are constructed by Ritz-Galerkin method and it is proved that the sequence of approximate solutions to (1.4) converges to a function u, and that, if u belongs to $W^{1,1}$ functions and satisfies energy conservation law, it is a weak solution to (1.4).[1] In [**11**], [**12**] the way of approximating is changed to Rothe's method and the condition of the limit u is simplified: the sequence of approximate solutions to (1.4) converges to a function u, and that, if u satisfies energy conservation law, it is a weak solution to (1.4) (in [**11**] the boundary condition is not essentially discussed and the observation is added in [**12**], compare to [**12**]). In all works mentioned above the limit should satisfy the energy conservation law, and existence theorem of a global weak solution has not been established yet without the energy conservation law.

Considering (1.4) with very small initial data, we find by putting $u_0 = \varepsilon\tilde{u}_0$, $v_0 = \varepsilon\tilde{v}_0$, and $u = \varepsilon\tilde{u}$, and by dividing the equation by ε that (1.4), (1.2), (1.3) are rewritten as (1.1), (1.2), (1.3) with u, u_0, v_0 being replaced with $\tilde{u}$, $\tilde{u}_0$, $\tilde{v}_0$, respectively.

In the end of [**12**, Section 1] a weak solution to (1.4) is defined as a weak solution to an evolution equation $u_{tt} + \partial\bar{J}(u,\overline{\Omega}) \ni 0$, where $\bar{J}$ denotes the area functional, namely,

$$\bar{J}(w,B) = \sqrt{1+|Dw|^2}(B)$$

for $w \in BV(\boldsymbol{R}^n)$ and for a Borel set $B \subset \boldsymbol{R}^n$ (in Appendix C we present a brief review about BV functions and some facts related to Equation (1.4)). Here we regard u as being extended to the whole space by null extention.[2] The area functional is a typical one that has linear growth energy with respect to the gradient and it is appropriate to handle it in the space of BV functions. Seemingly this definition of a weak solution is different from that of [**11**]. Moreover in [**12**] another seemingly different definition is presented ([**12**, Definition 2]). If $\partial\Omega$ is sufficiently smooth, they are equivalent except for the boundary condition. In

[1] This result is essentially obtained in [**7**], though the conditions stated there for the limit u are more complicated.

[2] Remark that $\bar{J}(u,\overline{\Omega}) = \bar{J}(u,\Omega) + \|\gamma u\|_{L^1(\partial\Omega)}$ (compare to Appendix C). In [**11**], [**12**] $\bar{J}(u,\Omega)$ and $\bar{J}(u,\overline{\Omega})$ are simply denoted by $J(u)$ and $I(u)$, respectively.

Appendix C we explain relations between these definitions. Since (1.1) is obtained by replacing u with εu in (1.4) and dividing the equation by ε, a weak solution to (1.1)–(1.3) could be defined as a weak solution to $u_{tt} + \partial \bar{J}_\varepsilon(u, \overline{\Omega}) \ni 0$, where

$$\bar{J}_\varepsilon(u, \overline{\Omega}) = \frac{1}{\varepsilon^2}\,(\bar{J}(\varepsilon u, \overline{\Omega}) - \mathscr{L}^n(\Omega)). \tag{1.5}$$

Remark that $\bar{J}_1(u, \overline{\Omega}) = J(u, \overline{\Omega}) - \mathscr{L}^n(\Omega)$. We subtract the area of Ω for the sake of convenience. Now we mention a definition of a weak solution to (1.1)–(1.3) exactly. Suppose that $\partial\Omega$ is Lipschitz continuous and that $u_0 \in L^2(\Omega) \cap BV(\Omega)$ and $v_0 \in L^2(\Omega)$. Putting

$$\mathscr{X} = \{\phi \in L^\infty((0,T); L^2(\Omega) \cap BV(\Omega)); \phi_t \in L^2((0,T) \times \Omega)\}, \tag{1.6}$$

we define

Definition 1.1. A function u is said to be a weak solution to (1.1)–(1.3) in $(0,T) \times \Omega$ if

i) $u \in L^\infty((0,T); BV(\Omega))$, $u_t \in L^2((0,T) \times \Omega)$
ii) $\text{s-}\lim_{t \searrow 0} u(t) = u_0$ in $L^2(\Omega)$
iii) for any $\phi \in C_0^0([0,T); L^2(\Omega)) \cap \mathscr{X}$,

$$\int_0^T \{\bar{J}_\varepsilon(u + \phi, \overline{\Omega}) - \bar{J}_\varepsilon(u, \overline{\Omega})\} dt \geq \int_0^T \int_\Omega u_t \phi_t(t,x) dx dt + \int_\Omega v_0(x)\phi(0,x) dx.$$

Formally, letting $\varepsilon \to 0$, we see (1.1) becomes the d'Alembert equation and it is natural to be interested in the behavior of weak solutions as $\varepsilon \to 0$. Although the existence of a global solution to (1.1) has not been established yet similarly to (1.4), we can show that, if for each ε there exists a weak solution to (1.1) that satisfies energy functional inequality, then it converges to a weak solution to the d'Alembert equation as $\varepsilon \to 0$. More precisely, as our first main result we have the following;

Theorem 1.1. *Suppose that $\partial\Omega$ is Lipschitz continuous and that $u_0 \in W_0^{1,2}(\Omega)$ and $v_0 \in L^2(\Omega)$. Let u^ε be a weak solution to* (1.1)–(1.3) *in* $(0,T) \times \Omega$ *that satisfies energy inequality*

$$\frac{1}{2}\int_\Omega |u_t^\varepsilon(t,x)|^2 dx + \bar{J}_\varepsilon(u^\varepsilon(t,\cdot), \overline{\Omega}) \leq \frac{1}{2}\int_\Omega |v_0|^2 dx + \bar{J}_\varepsilon(u_0, \overline{\Omega}). \tag{1.7}$$

Then there exists a function u *such that*

1) u^ε *converges to* u *as* $\varepsilon \to 0$ *weakly star in* $L^\infty((0,T);L^2(\Omega))$
2) u_t^ε *converges to* u_t *as* $\varepsilon \to 0$ *weakly star in* $L^\infty((0,T);L^2(\Omega))$
3) *for any* $T>0$, u^ε *converges to* u *as* $\varepsilon \to 0$ *strongly in* $L^p((0,T)\times\Omega)$ *for each* $1 \le p < n/(n-1)$
4) *for* $\mathscr{L}^1$*-a.e.* $t \in (0,T)$, $Du^\varepsilon(t,\cdot)$ *converges weakly star to* $Du(t,\cdot)$ *as* $\varepsilon \to 0$ *in the sense of Radon measures*
5) $u \in L^\infty((0,T);W_0^{1,2}(\Omega)) \cap W^{1,2}((0,T)\times\Omega)$
6) u *is a weak solution to*

$$\begin{cases} u_{tt} - \Delta u = 0, & (t,x) \in (0,T)\times\Omega \\ u(0,x) = u_0(x), & x \in \Omega \\ u_t(0,x) = v_0(x), & x \in \Omega \\ u(t,x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.8}$$

Rothe's approximation method employed in [**11**], [**12**] is a method of semidiscretization in time variable. Namely, letting u_0, v_0 be as in the initial condition and h be a positive number, we construct a sequence $\{u_l\}_{l=-1}^\infty$ by setting $u_{-1} = u_0 - hv_0$ for $l=-1$, letting u_0 be as in the initial condition for $l=0$, and for $l \ge 1$ letting u_l as a solution to the elliptic equation

$$\frac{u - 2u_{l-1} + u_{l-2}}{h^2} - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0. \tag{1.9}$$

In [**11**], [**12**] this equation is solved by finding a minimizer of

$$\mathscr{G}_l(u) = \frac{1}{2}\int_\Omega \frac{|u - 2u_{l-1} + u_{l-2}|^2}{h^2}\,dx + \bar{J}(u,\overline{\Omega}) \tag{1.10}$$

in $L^2(\Omega)\cap BV(\Omega)$ (details are reviewed in Appendix C). We define approximate solutions $u^h(t,x)$ and $\overline{u}^h(t,x)$ as, for $(l-1)h < t \le lh$,

$$\begin{cases} u^h(t,x) = \dfrac{t-(l-1)h}{h}u_l(x) + \dfrac{lh-t}{h}u_{l-1}(x) \\ \overline{u}^h(t,x) = u_l(x), \end{cases} \tag{1.11}$$

and then we have that, passing to a subsequence if necessary, u^h, $\overline{u}^h$ converge to the same function u in the space of BV functions. In the next section we show that the limit u can be regarded as a generalized minimizing movement associated with (1.4). Furthermore it is also the case for our problem (1.1)–(1.3), namely, we can construct a generalized minimizing movement u_ε associated with (1.1). Thus investigating its behavior as $\varepsilon \to 0$ is also a problem.

Minimizing movement is a new concept in mathematics introduced by De Giorgi ([**9**]) and it is closely related to Rothe's time semidiscretization method. Rothe has applied time semidiscretization method to solving a parabolic equation ([**18**]) and Rektorys has combined it with a method of minimizing functionals ([**17**]). In [**13**] N. Kikuchi has independently rediscovered Rektorys's idea and after [**13**] there are many works in applying this idea to constructing weak solutions to nonlinear partial differential equations (for example [**3**], [**16**], [**19**]). On the other hand, in [**1**] the method of combining time semidiscretization and minimizing functionals is also employed in constructing a flow of rectifiable currents by its mean curvature. Inspired by [**1**], E. De Giorgi has presented the theory of minimizing movements.

Following [**9**], we state the definition of a minimizing movement. Let S be a topological space.

Definition 1.2. Let $F : (1,\infty) \times \boldsymbol{Z} \times S \times S \to [-\infty,\infty]$ and $u : \boldsymbol{R} \to S$. We say that u is a minimizing movement associated with F and S, and we write $u \in MM(F,S)$, if there exists $w : (1,\infty) \times \boldsymbol{Z} \to S$ such that for any $t \in \boldsymbol{R}$

$$\lim_{\lambda\to\infty} w(\lambda,[\lambda t]) = u(t)$$

($[x] = \max\{z \in \boldsymbol{Z}; z \le x\}$) and for any $\lambda \in (1,\infty)$, $l \in \boldsymbol{Z}$

$$F(\lambda,l,w(\lambda,l+1),w(\lambda,l)) = \min_{s\in S} F(\lambda,l,s,w(\lambda,l)). \tag{1.12}$$

Several example of minimizing movements are presented in [**9**]. The simplest and the most typical one is as follows:

Example 1. ([**9**, Example 1.1]). Let $S = \boldsymbol{R}^n$, $f \in C^2(\boldsymbol{R}^n) \cap Lip(\boldsymbol{R}^n)$, and $\xi \in \boldsymbol{R}^n$. We set

$$F(\lambda,l,x,y) = \begin{cases} |x-\xi|^2 & \text{if } l \le 0 \\ \dfrac{\lambda}{2}|x-y|^2 + f(x) & \text{if } l > 0 \end{cases}$$

Then $u \in MM(F,S)$ if and only if $u \in Lip(\boldsymbol{R};\boldsymbol{R}^n)$ and

$$\begin{cases} \dfrac{du}{dt} + \nabla f(u) = 0 & (t>0), \\ u(t) = \xi & (t \le 0). \end{cases}$$

The following example is probably another typical example. In [**9**] the case that J is the Dirichlet integral is mentioned.

EXAMPLE 2. Let Ω be a bounded domain. Given a functional $J : L^1(\Omega) \to [c,\infty]$ ($c \in \boldsymbol{R}$), we suppose that (J1) J is convex, (J2) J is lower semicontinuous, (J3) the set $\{v; J(v) \le M\}$ is sequentially compact. Suppose that $u_0 \in L^2(\Omega)$ and $J(u_0) < \infty$. Let $S = L^1(\Omega)$ and we set

$$F(\lambda,l,u,v) = \begin{cases} \|u-u_0\|^2_{L^2(\Omega)} & \text{if } l \le 0 \\ \dfrac{\lambda}{2}\|u-v\|^2_{L^2(\Omega)} + J(u) & \text{if } l > 0 \end{cases}$$

Then $u \in MM(F,S)$ if and only if

$$\begin{cases} u_t + \partial J(u) \ni 0 & (t>0), \\ u(t) = u_0 & (t \le 0), \end{cases} \tag{1.13}$$

i.e., (S1) $u(t,\cdot) \in L^2(\Omega)$ for $\mathscr{L}^1$-a.e. $t \in (0,\infty)$, $u_t \in L^2_{\mathrm{loc}}([0,\infty);L^2(\Omega))$, (S2) s-$\lim_{t\searrow 0} u(t,x) = u_0(x)$ in $L^2(\Omega)$, and (S3) for each $v \in L^2(\Omega)$ and for $\mathscr{L}^1$-a.e. $t \in (0,\infty)$,

$$J(v) - J(u) \ge -\int_\Omega u_t(v-u)dx. \tag{1.14}$$

In [**9**] various examples in geometrical problems are also presented. Besides in [**9**] several examples and problems on minimizing movements related to some first order hyperbolic equations are presented. While, as pointed out in [**19**], Rothe's method is available for second order hyperbolic equations and, if the elliptic equation has a divergence form, then the elliptic equation is solved in a direct variational method. Thereby minimizing movements are also introduced for second order hyperbolic equations. For example, a function u is a minimizing movement associated with the d'Alembert equation if and only if u is a weak solution to it:

EXAMPLE 3. Let Ω be a bounded domain, $u_0 \in W_0^{1,2}(\Omega)$, and $v_0 \in L^2(\Omega)$. Let $S = L^1(\Omega) \times L^1(\Omega)$ and we set

$$F(\lambda, k, U, \tilde{U}) = \begin{cases} \|u - u_0\|^2_{L^2(\Omega)} + \|v - u_0 + \lambda^{-1} v_0\|^2_{L^2(\Omega)} & \text{if } k \leq 0 \\ \dfrac{\lambda^2}{2} \|u - 2\tilde{u} + \tilde{v}\|^2_{L^2(\Omega)} + \dfrac{1}{2} \displaystyle\int_\Omega |\nabla u|^2 dx + \|v - \tilde{u}\|^2_{L^2(\Omega)} & \text{if } k > 0 \end{cases}$$

($U = (u, v)$, $\tilde{U} = (\tilde{u}, \tilde{v})$). Then $U = (u, u) \in MM(F, S)$ if and only if $u(t) = u_0$ for $t < 0$ and u is a weak solution to (1.8) for $t \geq 0$.

(This is precisely discussed in the next section, see Proposition 2.3.)

In [**11**], [**12**] convergence of the sequence of approximate solutions is not proved, but convergence of a subsequence is proved. Note that De Giorgi presents an example such that $MM(F, S) = \emptyset$ [**9**, Remark 1.2]. This motivates the definition of generalized minimizing movements.

DEFINITION 1.3. Let $F : (1, \infty) \times \boldsymbol{Z} \times S \times S \to [-\infty, \infty]$ and $u : \boldsymbol{R} \to S$. We say that u is a generalized minimizing movement associated with F, S, and we write $u \in GMM(F, S)$, if there exists a sequence $\{\lambda_i\}_{i=1}^\infty$ such that $\lim \lambda_i = \infty$ and a sequence $\{w_i\}$ of functions $w_i : \boldsymbol{Z} \to S$ such that for any $t \in \boldsymbol{R}$

$$\lim_{i\to\infty} w_i([\lambda_i t]) = u(t)$$

and for any $i \in \boldsymbol{N}$, $k \in \boldsymbol{Z}$

$$F(\lambda_i, k, w_i(k+1), w_i(k)) = \min_{s\in S} F(\lambda_i, k, s, w_i(k)).$$

In the next section we see that there exists a generalized minimizing movement associated with a second order hyperbolic equation having convex energy. In general it seemes to be a hard problem to establish existence theorem of a time global weak solution to a second order quasilinear hyperbolic equation. Thus, although a generalized minimizing movement is much a weaker concept than a weak solution, investigating its properties would probably supply some useful information for the study of it. Readers should remark that results of the author's previous works [**11**], [**12**] could be mentioned in terms of minimizing movement theory: if a generalized minimizing movement associated with (1.4) satisfies energy conservation law, then it is a weak solution.

For each $\varepsilon > 0$ there exists a generalized minimizing movement u_ε associated with (1.1)–(1.3) (compare to Proposition 2.4) and our second main purpose is to investigate the behavior of them as $\varepsilon \to 0$.

Theorem 1.2. *Suppose that* $\partial\Omega$ *is Lipschitz continuous and that* $u_0 \in W_0^{1,2}(\Omega)$ *and* $v_0 \in L^2(\Omega)$. *Let* u^ε *be a generalized minimizing movement associated with* (1.1) *with* (1.2), (1.3). *Then there exists a function* u *such that for any* $T > 0$ *the same assertions as in Theorem* 1.1 *hold.*

Up to the author's knowledge only a few facts are known in relations between a second order hyperbolic equation and a minimizing movement or a generalized minimizing movement associated with it. Problems in the most part of these relations seem to be open. This article presents an answer to one of such open problems.

In Section 2 we overview the theory of minimizing movements for second order equations and in Section 3 we investigate properties of a generalized minimizing movement associated with (1.1) with (1.2), (1.3) by the use of a technique of geometric measure theory. Theorems 1.1 and 1.2 are proved in Section 4. Proofs of them are carried out at the same time. Although each u^ε belongs to BV functions, the limit as $\varepsilon \to 0$ should belong to $W^{1,2}$ functions. The technique of geometric measure theory plays an important role to overcome this difficulty.

2. Minimizing movement theory for second order equations.

In this section we overview the theory of minimizing movements for second order equations. It is mentioned in a slightly abstract setting.

Let Ω be a bounded domain. Given a functional $J : L^1(\Omega) \to [c, \infty]$ ($c \in \boldsymbol{R}$), we suppose the same assumptions (J1) $\sim$ (J3) as in Example 2. Suppose that $u_0 \in L^2(\Omega)$ with $J(u_0) < \infty$ and $v_0 \in L^2(\Omega)$. We treat a second order equation

$$u_{tt} + \partial J(u) \ni 0 \tag{2.1}$$

in $L^2(\Omega)$ with initial conditions

$$u(0,x) = u_0, \quad \frac{\partial u}{\partial t}(0,x) = v_0. \tag{2.2}$$

Definition of a weak solution to (2.1), (2.2) is as follows:

Definition 2.1. A function $u : (0,T) \to L^2(\Omega)$ is said to be a weak solution to (2.1), (2.2) in $(0,T)$ if

i) $u \in W^{1,2}((0,T); L^2(\Omega))$, $J(u) \in L^1(0,T)$

ii) $\operatorname{s-lim}_{t \searrow 0} u(t) = u_0$ in $L^2(\Omega)$

iii) for any $\phi \in C_0^0([0,T);L^2(\Omega)) \cap W^{1,2}((0,T);L^2(\Omega))$,

$$\int_0^T \{J(u+\phi) - J(u)\}dt \geq \int_0^T \int_\Omega u_t(t,x)\phi_t(t,x)dxdt + \int_\Omega v_0(x)\phi(0,x)dx.$$

Now we set $S = L^1(\Omega) \times L^1(\Omega)$ and

$$F(\lambda, l, U, \tilde{U}) = \begin{cases} \|u-u_0\|^2_{L^2(\Omega)} + \|v - u_0 + \lambda^{-1}v_0\|^2_{L^2(\Omega)} & \text{if } l \leq 0 \\ \mathscr{F}(\lambda, u, \tilde{u}, \tilde{v}; J) + \|v - \tilde{u}\|^2_{L^2(\Omega)} & \text{if } l > 0 \end{cases} \tag{2.3}$$

($U = (u,v)$, $\tilde{U} = (\tilde{u}, \tilde{v})$), where

$$\mathscr{F}(\lambda, u, \tilde{u}, \tilde{v}; J) = \frac{\lambda^2}{2}\|u - 2\tilde{u} + \tilde{v}\|^2_{L^2(\Omega)} + J(u).$$

Proposition 2.1. *For these S and F, w as in* (1.12) *is given by*

$$w(\lambda, l) = \begin{cases} (u_0, u_0 - \lambda^{-1}v_0) & (l \leq 0) \\ (u_l, u_{l-1}) & (l > 0), \end{cases}$$

where u_l $(l > 0)$ satisfies

$$\mathscr{F}(\lambda, u_l, u_{l-1}, u_{l-2}; J) = \min_{u \in L^1(\Omega)} \mathscr{F}(\lambda, u, u_{l-1}, u_{l-2}; J)$$

(*here we suppose that* $u_{-1} = u_0 - \lambda^{-1}v_0$).

Proof. Clearly we have $w(\lambda, l) = (u_0, u_0 - \lambda^{-1}v_0)$ for $l \leq 0$. For $l > 0$ we first remark that

$$F(\lambda, l, (u,v), (\tilde{u}, \tilde{v})) \geq F(\lambda, l, (u, \tilde{u}), (\tilde{u}, \tilde{v}); J) = \mathscr{F}(\lambda, u, \tilde{u}, \tilde{v}; J)$$

for each $(u,v) \in L^1(\Omega) \times L^1(\Omega)$. Thus $(u, \tilde{u})$ minimizes $F(\lambda, l, (u,v), (\tilde{u}, \tilde{v}))$ if u minimizes $\mathscr{F}(\lambda, u, \tilde{u}, \tilde{v}; J)$. Then we have the conclusion by induction on l. □

Remark that $\mathscr{F}(h^{-1}, u, u_{l-1}, u_{l-2}; \bar{J}(\cdot, \overline{\Omega})) = \mathscr{G}_l(u)$, which is presented in

(1.10). This means that the sequence $\{u_l\}$ in the proposition coincides with the sequence constructed by Rothe's method.

A minimizing movement is defined as the limit

$$\lim_{\lambda\to\infty} w(\lambda, [\lambda t]) = \lim_{h\to 0} w\left(h^{-1}, \left[\frac{t}{h}\right]\right)$$

and a generalized minimizing movement is defined as a limit of a subsequence. Noting that $[t/h] = l$ implies $lh \le t < (l+1)h$, we put $\tilde{u}^h = u_l$ for $lh \le t < (l+1)h$ $(l = 0, 1, 2, \ldots)$ and $\tilde{u}^h = u_0 - hv_0$ for $-h \le t < 0$. Then we have

$$w\left(h^{-1}, \left[\frac{t}{h}\right]\right) = \begin{cases} (u_0, u_0 - hv_0) & (t < 0) \\ (\tilde{u}^h(t), \tilde{u}^h(t-h)) & (t \ge 0). \end{cases} \tag{2.4}$$

Clearly $\lim_{h\to 0} w(h^{-1}, [t/h]) = (u_0, u_0)$ if $t < 0$. Thus our purpose here is to investigate the limit of $\tilde{u}^h$ as $h \to 0$.

The convexity of the functional J and the minimality of each u_l implies

$$\frac{1}{2h^2}\|u_l - u_{l-1}\|^2 + J(u_l) \le \frac{1}{2}\|v_0\|^2 + J(u_0). \tag{2.5}$$

(compare to [**15**, Lemma 4.1]), and if we define $u^h(t)$ and $\overline{u}^h(t)$ as in (1.11), we have, for each $t \in \bigcup_{l=0}^{\infty}((l-1)h, lh)$,

$$\frac{1}{2}\int_\Omega |u_t^h(t)|^2 dx + J(\overline{u}^h(t)) \le \frac{1}{2}\int_\Omega |v_0|^2 dx + J(u_0). \tag{2.6}$$

Note that $\tilde{u}^h(t) = \overline{u}^h(t-h)$ for $t \in \bigcup_{l=0}^{\infty}((l-1)h, lh)$. This inequality supplies important information for u^h, $\overline{u}^h$, and also for $\tilde{u}^h$.

PROPOSITION 2.2. *Under the above notations it follows that*

1) $\{\|u_t^h\|_{L^\infty((0,\infty);L^2(\Omega))}\}$ *is uniformly bounded with respect to* h
2) $\{\|J(\overline{u}^h)\|_{L^\infty(-h,\infty)}\}$ *is uniformly bounded with respect to* h
3) $\{\|J(u^h)\|_{L^\infty(0,\infty)}\}$ *is uniformly bounded with respect to* h

and moreover there exist a sequence $\{h_j\}$ *with* $h_j \to 0$ *as* $j \to \infty$ *and a function* u *such that*

4) *for any* $T > 0$, u^{h_j} *converges to* u *as* $j \to \infty$ *weakly star in* $L^\infty((0,T); L^2(\Omega))$
5) $u_t^{h_j}$ *converges to* u_t *as* $j \to \infty$ *weakly star in* $L^\infty((0,\infty); L^2(\Omega))$

6) *for any* $T > 0$, u^{h_j} *converges to* u *as* $j \to \infty$ *strongly in* $L^\infty((0,T);L^1(\Omega))$
7) *for any* $T > 0$, $\overline{u}^{h_j}$ *converges to* u *as* $j \to \infty$ *strongly in* $L^\infty((0,T);L^1(\Omega))$
8) *for any* $T > 0$, $\tilde{u}^{h_j}$ *converges to* u *as* $j \to \infty$ *strongly in* $L^\infty((0,T);L^1(\Omega))$
9) *for any* $T > 0$, $\tilde{u}^{h_j}(t-h_j)$ *converges to* u *as* $j \to \infty$ *strongly in* $L^\infty((0,T);L^1(\Omega))$
10) $s\text{-}\lim_{t \searrow t_0} u(t) = u_0$ *in* $L^2(\Omega)$.

Proof. Assertions 1) and 2) immediately follow from (2.6). Since J is convex, we have

$$J(u^h(t,\cdot)) \le \frac{t-(l-1)h}{h} J(\overline{u}^h(t,\cdot)) + \frac{lh-t}{h} J(\overline{u}^h(t-h,\cdot))$$

and Assertion 3) also holds.

Assertion 5) is a direct consequence of Assertion 1). Since we have

$$u^h(t) - u^h(t') = \int_{t'}^{t} u_t^h(s)ds, \tag{2.7}$$

for each t, $t' \ge 0$, Assertion 1) implies that, for each $T > 0$, $\{\|u^h\|_{L^\infty((0,\infty);L^2(\Omega))}\}$ is uniformly bounded with respect to h, and hence Assersion 4) follows. Furthermore (2.7) and Assertion 1) imply that the function $t \mapsto u^h(t,\cdot) \in L^2(\Omega)$ is equicontinuous with respect to h. Moreover by 3) and Assumption (J3) on J we find, for any $T > 0$, $\{u^h(t,\cdot)\}$ is contained in a sequentially compact subset of $L^1(\Omega)$ which is independent of h and $t \in [0,T]$. Thus by Ascoli-Arzela theorem we obtain Assertion 6).

For $(l-1)h < t \le lh$,

$$\begin{aligned} \|\overline{u}^h(t) - u^h(t)\|_{L^2(\Omega)} &= \left\| u_l - \frac{t-(l-1)h}{h} u_l - \frac{lh-t}{h} u_{l-1} \right\|_{L^2(\Omega)} \\ &= \left\| \frac{lh-t}{h}(u_l - u_{l-1}) \right\|_{L^2(\Omega)} \le \|u_l - u_{l-1}\|_{L^2(\Omega)}. \end{aligned}$$

By (2.5) we have

$$\|u_l - u_{l-1}\|_{L^2(\Omega)} \le \sqrt{(\|v_0\|^2 + 2J(u_0))h}.$$

Hence

$$\sup_{t>0} \|\overline{u}^h(t) - u^h(t)\|_{L^2(\Omega)} \leq \sqrt{(\|v_0\|^2 + 2J(u_0))h}. \tag{2.8}$$

Now we have

$$\|\overline{u}^h - u\|_{L^\infty((0,T);L^1(\Omega))} \leq \|\overline{u}^h - u^h\|_{L^\infty((0,T);L^1(\Omega))} + \|u^h - u\|_{L^\infty((0,T);L^1(\Omega))},$$

the right hand side of which converges to 0 as $h \to 0$ by (2.8) and Assertion 6). Now we have Assertion 7).

Since $\|\overline{u}^h(t) - \tilde{u}^h(t)\|_{L^2(\Omega)} = \|u_l - u_{l-1}\|_{L^2(\Omega)}$ for $(l-1)h < t < lh$, we have

$$\|\overline{u}^h - \tilde{u}^h\|_{L^\infty((0,T);L^2(\Omega))} \leq \sqrt{(\|v_0\|^2 + 2J(u_0))h}$$

by (2.5). Thus we have Assertion 8) by Assertion 7). For $lh < t < (l+1)h$, we have $\|\tilde{u}^h(t) - \tilde{u}^h(t-h)\|_{L^2(\Omega)} = \|u_l - u_{l-1}\|_{L^2(\Omega)}$ and thus Assertion 9) follows from (2.5) and Assertion 8).

Assertion 10) is obtained in the same way as in the proof of [**15**, Theorem 4.1]. □

Theorem 2.1. *Let U be a function in $L^1(\Omega) \times L^1(\Omega)$.*

1) *$U \in GMM(F,S)$ if and only if*

$$U(t) = \begin{cases} (u_0, u_0) & (t \leq 0) \\ (u(t), u(t)) & (t > 0), \end{cases} \tag{2.9}$$

where u is as in Proposition 2.2.

2) *$U \in MM(F,S)$ if and only if U is as in (2.9) and we do not have to subtract a subsequence in Proposition 2.2.*

Proof. 1) It immediately follows from Proposition 2.2 8), 9) that $(\tilde{u}^{h_j}(t), \tilde{u}^{h_j}(t-h_j))$ converges to (u,u) in $L^1(\Omega) \times L^1(\Omega)$ for $t > 0$. Thus by (2.4) and the comment after (2.4) the function U as in (2.9) is a generalized minimizing movement.

Conversely, if $U \in L^1(\Omega) \times L^1(\Omega)$ is a generalized minimizing movement, then by the definition of a generalized minimizing movement and (2.4) a subsequence $\{\tilde{u}^{h_j}(t)\}$ converges to a function $u(t)$ in $L^1(\Omega)$ for each $t > 0$ and $U(t) = (u_0, u_0)$ for $t \leq 0$. By Proposition 2.2 1), 2), 3) we could show that, passing to a further subsequence if necessary, $\tilde{u}^{h_j}$ converges to u that satisfies all assertions of Proposition 2.2. In particular, $\tilde{u}^{h_j}(t-h_j)$ also converges to $u(t)$ and thus $U=(u(t),u(t))$.

2) This is clear by the definition of minimizing movement and Assertion 1) $\square$

A typical example of a minimizing movement associated with a second order hyperbolic equation is as follows:

PROPOSITION 2.3. *Let Ω be a bounded domain, $u_0 \in W_0^{1,2}(\Omega)$, and $v_0 \in L^2(\Omega)$. Let $S = L^1(\Omega)\times L^1(\Omega)$ and F as in* (2.3) *with*

$$J(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 dx & \text{if } u \in W_0^{1,2}(\Omega) \\ \infty & \text{if otherwise.} \end{cases}$$

Then $U=(u,u)\in MM(F,S)$ if and only if $u(t)=u_0$ for $t<0$ and u is a weak solution to (1.8) *for $t \geq 0$.*

PROOF. By Theorem 2.1 we should show that the function u as in Proposition 2.2 is a weak solution to (1.8) for $t \geq 0$ and that we do not have to subtract a subsequence in Proposition 2.2. Since u_l is a minimizer of $\mathscr{F}(h^{-1},u,u_{l-1},u_{l-2};J)$, we have

$$0 = \frac{d}{d\varepsilon}\mathscr{F}(h^{-1},u_l+\varepsilon\varphi,u_{l-1},u_{l-2};J)|_{\varepsilon=0} = \int_\Omega\left(\frac{u_l-2u_{l-1}+u_{l-2}}{h}\varphi(x)+\nabla u_l\cdot\nabla\varphi\right)dx$$

for any $\varphi \in C_0^1(\Omega)$. Thus, noting that, for $(l-1)h<t<lh$, $u_t^h(t,x) = ((u_l(x) - u_{l-1}(x))/h)$, we have for any $\phi \in C_0^1([0,\infty)\times\Omega)$,

$$\int_0^\infty\int_\Omega\left(\frac{u_t^h(t,x)-u_t^h(t-h,x)}{h}\phi(t,x)+\nabla\overline{u}^h\cdot\nabla\phi\right)dxdt = 0.$$

Proposition 2.2 implies that, passing to a subsequence if necessary, u_t^h and $\nabla\overline{u}^h$ converge weakly star to u_t and ∇u, respectively, in $L^\infty((0,\infty);L^2(\Omega))$. Hence we have

$$\int_0^\infty\int_\Omega\frac{u_t^h(t,x)-u_t^h(t-h,x)}{h}\phi(t,x)dxdt$$
$$\longrightarrow -\int_0^\infty\int_\Omega u_t\phi_t(t,x)dxdt - \int_\Omega v_0(x)\phi(0,x)dx$$

and

$$\int_0^\infty \int_\Omega \nabla \overline{u}^h \cdot \nabla \phi dx dt \longrightarrow \int_0^\infty \int_\Omega \nabla u \cdot \nabla \phi dx dt$$

(compare to, for example, [**19**]). Thus u is a weak solution to (1.8).

Since a solution to the linear wave equation is unique, the rest of the subsequence has another subsequence that converges in the same topology to the same function u. Thus we do not have to subtract a subsequence. □

For the sake of simplicity in the sequel we call u as in Proposition 2.2 a generalized minimizing movement associated with (2.1) with (2.2). If we do not have to subtract a subsequence, u should be called a minimizing movement. Remark that, since the functional $u \mapsto \mathscr{F}(h^{-1}, u, u_{l-1}, u_{l-2}; J)$ is strictly convex for each $l > 0$, a minimizing movement should be unique if it exists.

Our problem is the case $J(u) = \bar{J}_\varepsilon(u, \overline{\Omega})$. Recall that we are supposing $J(u_0) < \infty$ and note that $\bar{J}_\varepsilon(u_0, \overline{\Omega}) < \infty$ if and only if $u_0 \in BV(\Omega)$. Now, supposing $u_0 \in BV(\Omega)$ and $v_0 \in L^2(\Omega)$, we could obtain slightly stronger estimates than those in Proposition 2.2.

Proposition 2.4. *Under the above notations it follows that*

1) $\{\|u_t^h\|_{L^\infty((0,\infty);L^2(\Omega))}\}$ *is uniformly bounded with respect to h*
2) *for any* $T > 0$, $\{\|u^h\|_{L^\infty((0,T);L^2(\Omega)\cap BV(\Omega))}\}$ *is uniformly bounded with respect to h*
3) *for any* $T > 0$, $\{\|\overline{u}^h\|_{L^\infty((0,T);L^2(\Omega)\cap BV(\Omega))}\}$ *is uniformly bounded with respect to h*

and moreover there exist a sequence $\{h_j\}$ *with* $h_j \to 0$ *as* $j \to \infty$ *and a function u such that*

4) *for any* $T > 0$, $\overline{u}^{h_j}$ *converges to u as* $j \to \infty$ *weakly star in* $L^\infty((0,T); L^2(\Omega))$
5) $u_t^{h_j}$ *converges to* u_t *as* $j \to \infty$ *weakly star in* $L^\infty((0,\infty); L^2(\Omega))$
6) *for any* $T > 0$, u^{h_j} *converges to u as* $j \to \infty$ *strongly in* $L^p((0,T) \times \Omega)$ *for each* $1 \le p < n/(n-1)$
7) *for any* $T > 0$, $\overline{u}^{h_j}$ *converges to u as* $j \to \infty$ *strongly in* $L^p((0,T) \times \Omega)$ *for each* $1 \le p < n/(n-1)$
8) $u \in L^\infty((0,\infty); BV(\Omega))$
9) *for* $\mathscr{L}^1$*-a.e.* $t \in (0,\infty)$, $D\overline{u}^{h_j}(t,\cdot)$ *converges weakly star to* $Du(t,\cdot)$ *as* $j \to \infty$ *in the sense of Radon measures*
10) $s\text{-}\lim_{t \searrow 0} u(t) = u_0$ *in* $L^2(\Omega)$.

REMARK. The functional $\bar{J}_\varepsilon(u_0, \overline{\Omega})$ is depending on ε and the right hand side of (2.6) is not uniformly bounded with respect to ε. Hence the uniformity with respect to ε is not asserted in this proposition. In Theorems 1.1 and 1.2 u_0 is supposed to belong $W_0^{1,2}(\Omega)$ and in such a case $\bar{J}_\varepsilon(u_0, \overline{\Omega})$ is uniformly bounded with respect to ε (compare to the proof of Proposition 4.1).

PROOF. Assertions 1), 4), 5), 10) are direct consequences of Proposition 2.2 1), 4), 5), and 10), respectively.

By the use of Assertion 1) $\|u^h\|_{L^\infty((0,T);L^2(\Omega))}$ is uniformly bounded with respect to h (compare to the proof of [**11**, Theorem 3.3]). Noting that

$$\bar{J}(u, \overline{\Omega}) = \int_\Omega \sqrt{1 + |\nabla u(x)|^2}\, dx + |D^s u|(\Omega) + \int_{\partial\Omega} |\gamma u| \, d\mathscr{H}^{n-1},$$

where $D^s u$ denotes the singular part of Du with respect to the n dimensional Lebesgue measure $\mathscr{L}^n$ and ∇u denotes the Radon-Nikodym derivative of the absolutely continuous part with respect to $\mathscr{L}^n$, we have by (1.5)

$$\bar{J}_\varepsilon(u, \overline{\Omega}) = \int_\Omega \frac{1}{\varepsilon^2} \left(\sqrt{1 + \varepsilon^2 |\nabla u(x)|^2} - 1 \right) dx + \frac{1}{\varepsilon} |D^s u|(\Omega) + \frac{1}{\varepsilon} \int_{\partial\Omega} |\gamma u| \, d\mathscr{H}^{n-1}. \tag{2.10}$$

Thus we have

$$\bar{J}_\varepsilon(u, \overline{\Omega}) + \frac{1}{\varepsilon^2} \mathscr{L}^n(\Omega) \geq \frac{1}{\varepsilon} \int_\Omega |\nabla u(x)| \, dx + \frac{1}{\varepsilon} |D^s u|(\Omega) = \frac{1}{\varepsilon} |Du(\Omega)|. \tag{2.11}$$

This with Proposition 2.2 3) and 4) implies Assertion 3). Assertion 2) follows from (2.8), Assertion 3), (2.11), and Proposition 2.2 2).

By Sobolev's theorem $BV(\Omega) \subset L^p(\Omega)$ compactly for each $1 \leq p < n/(n-1)$. Then in the same way as in the proof of [**7**, Proposition 5.1] we obtain Assertions 6) and 7). The limits are the same because of (2.8). Assertion 8) immediately follows from 3), 4), and 7). Assertion 9) follows from 7). □

In our terminology u as in Proposition 2.4 should be called a generalized minimizing movement associated with (1.1) with (1.2), (1.3).

3. A generalized minimizing movement associated with (1.1).

In this section we investigate some properties of a generalized minimizing movement u associated with (1.1) with (1.2), (1.3). First we show energy inequality.

Proposition 3.1. *For $\mathscr{L}^1$-a.e. $t \in (0,T)$, (1.7) holds.*

Remark. Exactly u is depending on ε. However, in Proposition 2.4 we do not specify it and this omission is continued in this section.

Proof. For $\psi \in L^1(0,T)$ with $\psi \geq 0$,

$$\begin{aligned} 0 &\leq \int_0^T \psi \int_\Omega |u_t^h - u_t|^2 dx dt \\ &= \int_0^T \psi \int_\Omega |u_t^h|^2 dx dt - 2\int_0^T \psi \int_\Omega u_t^h u_t dx dt + \int_0^T \psi \int_\Omega |u_t|^2 dx dt \end{aligned}$$

By Proposition 2.4 5), the second term of the right hand side converges to $2\int_0^T \psi \int_\Omega |u_t|^2 dxdt$. Hence we have by dominated convergence theorem

$$\int_0^T \psi \int_\Omega |u_t|^2 dx dt \leq \limsup_{h \searrow 0} \int_0^T \psi \int_\Omega |u_t^h|^2 dx dt \leq \int_0^T \psi \limsup_{h \searrow 0} \int_\Omega |u_t^h|^2 dx dt.$$

Since ψ is arbitrary, we have, for $\mathscr{L}^1$-a.e. $t \in (0,T)$,

$$\limsup_{h \searrow 0} \int_\Omega |u_t^h(t,x)|^2 dx \geq \int_\Omega |u_t(t,x)|^2 dx.$$

By Proposition 2.4 7) we have, for $\mathscr{L}^1$-a.e. $t \in (0,T)$,

$$\liminf_{h \searrow 0} \bar{J}_\varepsilon(\overline{u}^h(t,\cdot), \overline{\Omega}) \geq \bar{J}_\varepsilon(u(t,\cdot), \overline{\Omega}).$$

Thus energy inequality (2.6) implies the conclusion. □

In [**11**] properties of a generalized minimizing movement associated with (1.4) is investigated and mentioned by the use of terms of oriented varifolds ([**11**, Section3]). Let G_0 denote the collection of all oriented n-dimensional vector subspaces of $\boldsymbol{R}^{n+1}$. Let us call a Radon measures in $\boldsymbol{R}^n \times \boldsymbol{R} \times G_0$ an oriented varifold in $\boldsymbol{R}^n \times \boldsymbol{R}$.[3]

In the sequel each BV function $v \in BV(\Omega)$ is extended to the whole space by

[3]Originally varifold is introduced for treating nonoriented surfaces in the context of geometric measure theory and a varifold in $\boldsymbol{R}^n \times \boldsymbol{R}$ is usually defined as a Radon measures in $\boldsymbol{R}^n \times \boldsymbol{R} \times G$, where G denotes the collection of all n-dimensional vector subspaces of $\boldsymbol{R}^{n+1}$. Our treatment of oriented varifolds is much closer to that of gradient Young measures (compare to [**14**]).

the null extension and it is still denoted by v. For $v \in BV(\boldsymbol{R}^n)$, we define $E_v \subset \boldsymbol{R}^{n+1}$ by

$$E_v = \{(x, y); x \in \boldsymbol{R}^n, y > v(x)\}.^4$$

It is a set of locally finite perimeter in $\boldsymbol{R}^{n+1}$, hence its reduced boundary $\partial^* E_v$ is a countably n-rectifiable set and the inward pointing approximate unit normal to $\partial^* E_v$ exists at $\mathscr{H}^n$-a.e. $z \in \partial^* E_v$. In this article it is denoted by $\nu_{E_v}(z)$.

Note that each element of G_0 is characterized by an n-vector ξ which is represented as $\xi = \tau_1 \wedge \cdots \wedge \tau_n$, where $\{\tau_1, \cdots, \tau_n\}$ is an orthonormal basis of this element. Thus G_0 is often identified with the set of all simple n-vectors having unit norm ([**6**, 1.6.2]). For each $\xi \in G_0$ there exists a unique vector ν such that $\xi \wedge \nu = e_1 \wedge \cdots \wedge e_{n+1}$. This map $\xi \mapsto \nu = \nu(\xi)$ is a homeomorphism from G_0 to the n-dimensional unit sphere S^n, and $\nu(\xi)$ is the unit normal to the vector subspace associated with ξ. Now for each BV function v we associate an oriented varifold

$$(\mathscr{H}^n \llcorner \partial^* E_v) \otimes \delta_{\nu^{-1}(\nu_{E_v}(z))}$$

and it is denoted by $\boldsymbol{v}_+(v)$ (compare to [**11**, Section 1]). For each oriented n-varifold V in $\boldsymbol{R}^{n+1}$ a Radon measure μ_V on $\boldsymbol{R}^{n+1}$ is defined by $\mu_V(A) = V(A \times G_0)$ for a Borel set $A \subset \boldsymbol{R}^{n+1}$. From [**4**, Theorem 10, p. 14] there exists a probability Radon measure $\eta_V^{(z)}$ on G_0 for μ_V-a.e. $z \in \boldsymbol{R}^{n+1}$ such that

$$\int_{\boldsymbol{R}^n \times \boldsymbol{R} \times G_0} \beta(z, \xi) dV = \int_{\boldsymbol{R}^n \times \boldsymbol{R}} \left(\int_{G_0} \beta(z, \xi) d\eta_V^{(z)} \right) d\mu_V \quad (\beta \in C_0^0(\boldsymbol{R}^n \times \boldsymbol{R} \times G_0)). \tag{3.1}$$

Now, for $\mathscr{L}^1$-a.e. t, we associate $\overline{u}^h(t, \cdot)$ with oriented varifolds and write $V_t^h = \boldsymbol{v}_+(\overline{u}^h(t, \cdot))$. By Proposition 2.4 3) we obtain the following proposition (compare to [**11**, Proposition 3.5] and [**7**, Proposition 4.3]).

PROPOSITION 3.2. *There exists a subsequence of* $\{V_t^h\}$ *(still denoted by* $\{V_t^h\}$*) and a one parameter family of oriented varifolds* V_t *in* $\boldsymbol{R}^n \times \boldsymbol{R}$*,* $t \in (0, \infty)$*, such that, for each* $\psi(t) \in L^1(0, \infty)$ *and* $\beta \in C_0^0(\boldsymbol{R}^n \times \boldsymbol{R} \times G_0)$*,*

$$\lim_{h \to 0} \int_0^\infty \psi(t) \int_{\boldsymbol{R}^n \times \boldsymbol{R} \times G_0} \beta(z, \xi) dV_t^h(z, \xi) dt = \int_0^\infty \psi(t) \int_{\boldsymbol{R}^n \times \boldsymbol{R} \times G_0} \beta(z, \xi) dV_t(z, \xi) dt.$$

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[4] In [**11**] E_v is considered in $\Omega \times \boldsymbol{R}$. In this article null extension is introduced for the sake of controlling boundary condition.

For each $v \in BV(\boldsymbol{R}^n)$ we have by the definition of $\boldsymbol{v}_+(v)$ that, for any $\beta \in C_0^0(\boldsymbol{R}^n \times \boldsymbol{R} \times G_0)$,

$$\int_{\boldsymbol{R}^n \times \boldsymbol{R} \times G_0} \beta(z,\xi) d\boldsymbol{v}_+(v) = \int_{\overline{\Omega} \times \boldsymbol{R}} \beta(z, \nu^{-1}(\nu_{E_v}(z))) d(\mathscr{H}^n \llcorner \partial^* E_v). \tag{3.2}$$

The energy could be expressed as

$$\bar{J}_\varepsilon(\overline{u}^h, \overline{\Omega}) = \int_{\overline{\Omega} \times \boldsymbol{R}} \frac{1}{\varepsilon^2} \left(\sqrt{(\nu_{E_t^h}^{n+1})^2 + \varepsilon^2 |\nu'_{E_t^h}|^2} - \nu_{E_t^h}^{n+1} \right) d(\mathscr{H}^n \llcorner \partial^* E_t^h),$$

where $E_t^h = E_{\overline{u}^h(t,\cdot)}$ (compare to Theorems 1, 4, and 5 of [**8**, I Section 4.1.5]), moreover, since $\nu_{E_t^h}(z) = {}^t(0,\dots,0,1)$ for $z = (x,0)$, $x \in \boldsymbol{R}^n \setminus \overline{\Omega}$, we have

$$\begin{aligned} \bar{J}_\varepsilon(\overline{u}^h, \overline{\Omega}) &= \int_{\boldsymbol{R}^n \times \boldsymbol{R}} \frac{1}{\varepsilon^2} \left(\sqrt{(\nu_{E_t^h}^{n+1})^2 + \varepsilon^2 |\nu'_{E_t^h}|^2} - \nu_{E_t^h}^{n+1} \right) d(\mathscr{H}^n \llcorner \partial^* E_v) \\ &= \int_{\boldsymbol{R}^n \times \boldsymbol{R} \times G_0} \frac{1}{\varepsilon^2} \left(\sqrt{\nu^{n+1}(\xi)^2 + \varepsilon^2 |\nu'(\xi)|^2} - \nu^{n+1}(\xi) \right) dV_t^h. \end{aligned}$$

By the lower semicontinuity of Radon measures we have for each $\psi \in L^1(0,\infty)$ with $\psi \geq 0$

$$\begin{aligned} &\liminf_{h \to 0} \int_0^\infty \psi(t) \bar{J}_\varepsilon(\overline{u}^h, \overline{\Omega})\, dt \\ &\quad \geq \int_0^\infty \psi(t) \int_{\boldsymbol{R}^n \times \boldsymbol{R} \times G_0} \frac{1}{\varepsilon^2} \left(\sqrt{\nu^{n+1}(\xi)^2 + \varepsilon^2 |\nu'(\xi)|^2} - \nu^{n+1}(\xi) \right) dV_t dt. \end{aligned} \tag{3.3}$$

Let $\eta_{V_t}^{(z)}$ be the measure $\eta_V^{(z)}$ as in (3.1) for the oriented varifold V_t, let $E_t = E_{u(t,\cdot)}$ with u being a generalized minimizing movement associated with (1.1) with (1.2), (1.3), and let $\mu_{E_t} = \mathscr{H}^n \llcorner \partial^* E_t$.

Proposition 3.3. *For $\mathscr{L}^1$-a.e. $t \in (0,\infty)$,*

1) $\displaystyle\int_{\boldsymbol{R}^n \times \boldsymbol{R}} g(z) \nu_{E_t}(z) d\mu_{E_t} = \int_{\boldsymbol{R}^n \times \boldsymbol{R}} g(z) \left(\int_{G_0} \nu(\xi) d\eta_{V_t}^{(z)} \right) d\mu_{V_t}$ *for any* $g \in C_0^0(\boldsymbol{R}^n \times \boldsymbol{R}; \boldsymbol{R}^{n+1})$
2) $\mu_{V_t}(A) \geq \mu_{E_t}(A)$ *for each Borel set* $A \subset \boldsymbol{R}^n \times \boldsymbol{R}$
3) $\mu_{V_t}(A) = \displaystyle\int_A D_{\mu_{E_t}} \mu_{V_t}(z) d\mu_{E_t} + (\mu_{V_t} \llcorner Z)(A)$ *for* $A \subset \boldsymbol{R}^{n+1}$, *where* $D_{\mu_{E_t}} \mu_{V_t}$ *is the derivative of* μ_{V_t} *with respect to* μ_{E_t} *and* Z *is the* μ_{E_t}*-null set defined by* $Z = \{z; D_{\mu_{E_t}} \mu_{V_t}(z) = \infty\}$

4) $\displaystyle\int_{G_0} \nu(\xi) d\eta_{V_t}^{(z)} = 0$ *for* $\mu_{V_t} \llcorner Z$*-a.e.* z

5) $\operatorname{spt} \eta_{V_t}^{(z)} \subset \operatorname{irr}(G_0)$ *for* $\mu_{V_t} \llcorner Z$*-a.e.* z*, where* $\operatorname{irr}(G_0) = \{\xi; \nu^{n+1}(\xi) = 0\}$

6) $|D^s u|(\overline{\Omega}) \leq V_t(\overline{\Omega} \times \boldsymbol{R} \times \operatorname{irr}(G_0))$.

Proof. Assertions 1) $\sim$ 5) in the case of $\varepsilon = 1$ are proved in Lemma 3.6 and Theorem 3.7 of [**11**], and the proofs for $\varepsilon < 1$ are the same. Here we only have to show Assertion 6).

We fix a t at which Assertions 1) $\sim$ 5) hold. By 1) and 3) we have, for μ_{E_t}-a.e. $z \in \boldsymbol{R}^n \times \boldsymbol{R}$,

$$\nu_{E_t}^{n+1}(z) = D_{\mu_{E_t}} \mu_{V_t}(z) \int_{G_0} \nu^{n+1}(\xi) d\eta_{V_t}^{(z)}.$$

Then, since $D_{\mu_{E_t}} \mu_{V_t}(z) \geq 1$ for μ_{E_t}-a.e. $z \in \boldsymbol{R}^n \times \boldsymbol{R}$, $\nu_{E_t}^{n+1}(z) = 0$ implies

$$\int_{G_0} \nu^{n+1}(\xi) d\eta_{V_t}^{(z)} = 0,$$

which means $\operatorname{spt} \eta_{V_t}^{(z)} \subset \operatorname{irr}(G_0)$. Thus, for μ_{E_t}-a.e. $z \in \boldsymbol{R}^n \times \boldsymbol{R}$,

$$S_u := \{z \in \partial^* E_t; \nu_{E_t}^{n+1}(z) = 0\} \subset \{z \in \operatorname{spt} \mu_{V_t}; \operatorname{spt} \eta_{V_t}^{(z)} \subset \operatorname{irr}(G_0)\} =: S_{V_t},$$

more precisely, $\mu_{E_t}(S_u \setminus S_{V_t}) = 0$. Since $|D^s u|(\overline{\Omega}) = \mu_{E_t}(S_u)$ by Theorems 1 and 5 of [**8**, I Section 4.1.5], we have

$$\begin{aligned} |D^s u|(\overline{\Omega}) &= \mu_{E_t}(S_u) \leq \mu_{V_t}(S_{V_t}) = V_t(S_{V_t} \times G_0) = \int_{S_{V_t}} \eta_{V_t}^{(z)}(G_0) d\mu_{V_t} \\ &= \int_{S_{V_t}} \eta_{V_t}^{(z)}(\operatorname{irr}(G_0)) d\mu_{V_t} \leq \int_{\boldsymbol{R}^n \times \boldsymbol{R}} \eta_{V_t}^{(z)}(\operatorname{irr}(G_0)) d\mu_{V_t} = V_t(\overline{\Omega} \times \boldsymbol{R} \times \operatorname{irr}(G_0)). \end{aligned}$$

□

Remark that the left hand side of Assertion 1) is expressed as

$$\int_{\boldsymbol{R}^n \times \boldsymbol{R}} g(z) \nu_{E_t}(z) d\mu_{E_t} = \int_{\boldsymbol{R}^n} \{-g'(x, u(t,x)) \cdot dDu + g^{n+1}(x, u(t,x))\} dx.$$

Taking account of this fact we have

Proposition 3.4. *For* $\mathscr{L}^1$*-a.e.* t*,*

$$-\int_{\boldsymbol{R}^n} \phi(x) \cdot dDu = \int_{\boldsymbol{R}^n \times \boldsymbol{R} \times G_0} \phi(x) \cdot \nu'(\xi) dV_t$$

for any $\phi \in C_0^0(\boldsymbol{R}^n; \boldsymbol{R}^n)$.

PROOF. Set $g'(x,y) = \phi(x)\rho_R(y)$ with $\phi \in C_0^0(\boldsymbol{R}^n; \boldsymbol{R}^n)$ and $\rho_R \in C_0^0(\boldsymbol{R})$, $0 \le \rho_R \le 1$, $\rho_R(y) = 1$ for $|y| \le R$, $= 0$ for $|y| \ge R+1$. By Proposition 3.3 1) we have

$$\int_{\boldsymbol{R}^n \times \boldsymbol{R}} \phi(x)\rho_R(y) \cdot \nu'_{E_t}(z) d\mu_{E_t} = \int_{\boldsymbol{R}^n \times \boldsymbol{R} \times G_0} \phi(x)\rho_R(y) \cdot \nu'(\xi) dV_t, \tag{3.4}$$

Letting $R \to \infty$ in the left hand side of (3.4), we have by the above remark and dominated convergence theorem

$$\int_{\boldsymbol{R}^n \times \boldsymbol{R}} \phi(x)\nu'_{E_t}(z) d\mu_{E_t} = -\int_{\boldsymbol{R}^n} \phi(x) \cdot dDu,$$

while by the dominated convergence theorem again the right hand side of (3.4) converges to the right hand side of the assertion. □

In the same way as in the proof of Theorem 2.2 of [**11**] we have that, for any $\varphi \in C^1(\Omega \times \boldsymbol{R})$ having bounded first derivatives, $\bar{J}_\varepsilon(v + \sigma\varphi(x,v), \overline{\Omega})$ is differentiable at $\sigma = 0$ and it is expressed as

$$\frac{d}{d\sigma} \bar{J}_\varepsilon(v + \sigma\varphi(x,v), \overline{\Omega})|_{\sigma=0} = \int_{\partial^* E_v} \frac{-(\nabla_x \varphi \cdot \nu'_{E_v})\nu_{E_v}^{n+1} + |\nu'_{E_v}|^2 \varphi_y}{\sqrt{(\nu_{E_v}^{n+1})^2 + \varepsilon^2 |\nu'_{E_v}|^2}} d\mathscr{H}^n. \tag{3.5}$$

In particular we are able to insert $\phi(x) \in C_0^1(\Omega)$ or v itself as $\varphi(x,v)$.

Since u_l is the minimizer of $\mathscr{F}_l(v)$, we have, for any φ,

$$\begin{aligned} 0 &= \frac{d}{d\sigma} \mathscr{F}_l(u_l + \sigma\varphi(x,u_l))|_{\sigma=0} \\ &= \int_\Omega \frac{u_l(x) - 2u_{l-1}(x) + u_{l-2}(x)}{h^2} \varphi(x,u_l) dx + \frac{d}{d\sigma} \bar{J}_\varepsilon(u_l + \sigma\varphi(x,u_l), \overline{\Omega})|_{\sigma=0}. \end{aligned}$$

Then, noting that, for $(l-1)h < t < lh$, $u_t^h(t,x) = \dfrac{u_l(x) - u_{l-1}(x)}{h}$, we have for any $T > 0$ and for any $\varphi \in C^1([0,T] \times \Omega \times \boldsymbol{R})$ having bounded first derivatives

$$\int_0^T \Bigg[\int_\Omega \frac{u_t^h(t,x) - u_t^h(t-h,x)}{h} \varphi(t,x,\overline{u}^h(t,x)) dx \tag{3.6}$$

$$+ \int_{\partial^* E_t^h} \frac{-(\nabla_x \varphi \cdot \nu'_{E_t^h}) \nu_{E_t^h}^{n+1} + |\nu'_{E_t^h}|^2 \varphi_y}{\sqrt{(\nu_{E_t^h}^{n+1})^2 + \varepsilon^2 |\nu'_{E_t^h}|^2}} d\mathscr{H}^n \Bigg\} dt = 0.$$

Applying (3.2) to $\overline{u}^h$ and $\dfrac{-(\nabla_x \varphi \cdot \nu'(\xi))\nu^{n+1}(\xi) + |\nu'(\xi)|^2 \varphi_y}{\sqrt{\nu^{n+1}(\xi)^2 + \varepsilon^2 |\nu'(\xi)|^2}}$ for v and $\beta(z,\xi)$, respectively, we obtain

$$\int_{\partial^* E_t^h} \frac{-(\nabla_x \varphi \cdot \nu'_{E_t^h}) \nu_{E_t^h}^{n+1} + |\nu'_{E_t^h}|^2 \varphi_y}{\sqrt{(\nu_{E_t^h}^{n+1})^2 + \varepsilon^2 |\nu'_{E_t^h}|^2}} d\mathscr{H}^n \tag{3.7}$$

$$= \int_{\boldsymbol{R}^{n+1} \times G_0} \frac{-(\nabla_x \varphi \cdot \nu'(\xi))\nu^{n+1}(\xi) + |\nu'(\xi)|^2 \varphi_y}{\sqrt{\nu^{n+1}(\xi)^2 + \varepsilon^2 |\nu'(\xi)|^2}} dV_t^h(z,\xi).$$

PROPOSITION 3.5. *For each T at which u_t is left approximately continuous and for each $\phi \in C_0^1([0,T] \times \Omega)$*

$$\int_0^T \left\{ -\int_\Omega u_t \phi_t(t,x) dx + \int_{\Omega \times \boldsymbol{R} \times G_0} \frac{-(\nabla_x \phi \cdot \nu'(\xi))\nu^{n+1}(\xi)}{\sqrt{\nu^{n+1}(\xi)^2 + \varepsilon^2 |\nu'(\xi)|^2}} dV_t \right\} dt$$

$$= \int_\Omega v_0(x) \phi(0,x) dx - \int_\Omega u_t(T,x) \phi(T,x) dx$$

PROOF. It is easy to check that conditions for β in Proposition 3.2 could be weakened as $\beta(x,\xi) = \tilde{\beta}(x,\xi)\nu^{n+1}(\xi)$, $\tilde{\beta} \in C^0(\boldsymbol{R}^n \times \boldsymbol{R} \times G_0)$, $\tilde{\beta}(x,y,\xi) = 0$ if $|x|$ is large, and $|\tilde{\beta}| \le C$ (C is a constant). Hence, letting $h \to 0$ in (3.6) for $\varphi = \phi(t,x)$, we obtain the conclusion by (3.7). □

Readers should compare to the following proposition, which is immediately obtained by testing smooth functions and solution u itself and using (3.5) and (3.2). Compare also to Proposition A.1 in our Appendix.

PROPOSITION 3.6. *Let u be a weak solution to* (1.1)–(1.3) *and we define a one parameter family $\{V_t\}_{t>0}$ of oriented varifolds by $V_t = \boldsymbol{v}_+(u(t,\cdot))$. For each T at which u_t is left approximately continuous,*

1) *for each* $\phi \in C_0^1([0,T]\times\Omega)$

$$\int_0^T\left\{-\int_\Omega u_t\phi_t(t,x)dx+\int_{\Omega\times\boldsymbol{R}\times G_0}\frac{-(\nabla_x\phi\cdot\nu'(\xi))\nu^{n+1}(\xi)}{\sqrt{\nu^{n+1}(\xi)^2+\varepsilon^2|\nu'(\xi)|^2}}dV_t\right\}dt$$
$$=\int_\Omega v_0(x)\phi(0,x)dx-\int_\Omega u_t(T,x)\phi(T,x)dx$$

2)

$$\int_0^T\int_{\overline{\Omega}\times\boldsymbol{R}\times G_0}\frac{|\nu'(\xi)|^2}{\sqrt{\nu^{n+1}(\xi)^2+\varepsilon^2|\nu'(\xi)|^2}}dV_tdt$$
$$=\int_0^T\int_\Omega u_t^2dxdt+\int_\Omega v_0(x)u_0(x)dx-\int_\Omega u_t(T,x)u(T,x)dx.$$

4. Proof of Theorems 1.1 and 1.2.

In this section we prove Theorems 1.1 and 1.2 at the same time. Our strategy of the proof is as follows:

Step 1: Showing that u^ε converges to a function u by the use of Energy inequality (Proposition 4.1)

Step 2: Corresponding each u^ε to an oriented varifold V_t^ε that satisfies "equation" (compare to Propositions 3.5 and 3.6)

Step 3: Showing V_t^ε converges to an oriented varifold V_t and investigating properties of u and V_t (Lemmas 4.1–4.4 and Proposition 4.2)

Step 4: Letting $\varepsilon\to 0$ in the above "equation".

First we show the following proposition.

Proposition 4.1. *Suppose that* $\partial\Omega$ *is Lipschitz continuous and that* $u_0\in W_0^{1,2}(\Omega)$ *and* $v_0\in L^2(\Omega)$.

1) $\{\|u_t^\varepsilon\|_{L^\infty((0,T);L^2(\Omega))}\}$ *is uniformly bounded with respect to* ε
2) $\{\|u^\varepsilon\|_{L^\infty((0,T);L^2(\Omega)\cap BV(\Omega))}\}$ *is uniformly bounded with respect to* ε

Then there exist a sequence $\{\varepsilon_j\}$ *with* $\varepsilon_j\to 0$ *as* $j\to\infty$ *and a function* u *such that*

3) u^{ε_j} *converges to* u *as* $j\to\infty$ *weakly star in* $L^\infty((0,T);L^2(\Omega))$
4) $u_t^{\varepsilon_j}$ *converges to* u_t *as* $j\to\infty$ *weakly star in* $L^\infty((0,T);L^2(\Omega))$
5) *for any* $T>0$, u^{ε_j} *converges to* u *as* $j\to\infty$ *strongly in* $L^p((0,T)\times\Omega)$ *for each* $1\le p<n/(n-1)$

6) $u \in L^\infty((0,T); BV(\Omega) \cap L^2(\Omega))$
7) *for* $\mathscr{L}^1$*-a.e.* $t \in (0,T)$, $Du^{\varepsilon_j}(t,\cdot)$ *converges weakly star to* $Du(t,\cdot)$ *as* $j \to \infty$ *in the sense of Radon measures*
8) $s\text{-}\lim_{t \searrow 0} u(t) = u_0$ *in* $L^2(\Omega)$.

Proof. Since $u_0 \in W_0^{1,2}(\Omega)$, $|D^s u_0| = 0$ and $\gamma u_0 = 0$. Thus by (2.10) we have

$$\bar{J}_\varepsilon(u_0, \overline{\Omega}) \le \int_\Omega |\nabla u_0(x)|^2 dx, \tag{4.1}$$

This and energy inequality (1.7) immediately imply Assertion 1). Since the function $\varepsilon \mapsto \varepsilon^{-2}(\sqrt{1+\varepsilon^2|p|^2} - 1)$ is decreasing, we have $\|u^\varepsilon\|_{BV(\Omega)} \le J_1(u^\varepsilon, \Omega) + \mathscr{L}^n(\Omega) \le \bar{J}_\varepsilon(u_\varepsilon, \Omega) + \mathscr{L}^n(\Omega)$. Thus it also follows from (1.7) and (4.1) that $\operatorname{ess.sup}\{\|u^\varepsilon(t,\cdot)\|_{BV(\Omega)}; 0 < t < T\}$ is uniformly bounded with respect to ε. Other Assertions are proved in the same way as in the proof of Theorem 3.3 of [**11**]. □

Remark. In the sequel $\{u^{\varepsilon_j}\}$ is denoted by $\{u^\varepsilon\}$ for simplicity.

In the case of the proof of Theorem 1.1 we let $\{V_t^\varepsilon\}_{t>0}$ denote a one parameter family of oriented varifolds associated with the weak solutions $u^\varepsilon(t,\cdot)$ and in the case of that of Theorem 1.2 a one parameter family of oriented varifolds in $\boldsymbol{R}^n \times \boldsymbol{R}$ presented in Proposition 3.2 as V_t. In both cases V_t^ε satisfies

$$\begin{aligned} &\int_0^T \left\{ -\int_\Omega u_t^\varepsilon \phi_t(t,x)dx + \int_{\Omega\times\boldsymbol{R}\times G_0} \frac{-(\nabla_x\phi\cdot\nu'(\xi))\nu^{n+1}(\xi)}{\sqrt{\nu^{n+1}(\xi)^2 + \varepsilon^2|\nu'(\xi)|^2}} dV_t^\varepsilon \right\} dt \\ &\quad = \int_\Omega v_0(x)\phi(0,x)dx \end{aligned} \tag{4.2}$$

for each $\phi \in C_0^1([0,T) \times \Omega \times \boldsymbol{R})$ (Proposition 3.5 or Proposition 3.6).

Proposition 4.1 2) implies

$$\operatorname*{ess.sup}_{t>0} \left| \int_{K\times\boldsymbol{R}\times G_0} \beta(z,\xi) dV_t^\varepsilon(z,\xi) \right| \le C_K \sup|\beta|$$

for each compact set $K \subset \boldsymbol{R}^n$ and some constant C_K depending on K. Thus we obtain the following theorem in the same way as in the proof of [**7**, Proposition 4.3] (compare to Proposition 3.2).

Lemma 4.1. *There exists a subsequence of* $\{V_t^\varepsilon\}$ *(still denoted by* $\{V_t^\varepsilon\}$*) and a one parameter family of oriented varifolds* V_t *in* $\boldsymbol{R}^n \times \boldsymbol{R}$, $t \in (0,T)$, *such that, for each* $\psi(t) \in L^1(0,T)$ *and* $\beta \in C_0^0(\boldsymbol{R}^n \times \boldsymbol{R} \times G_0)$,

$$\lim_{\varepsilon \to 0} \int_0^T \psi(t) \int_{\boldsymbol{R}^n \times \boldsymbol{R} \times G_0} \beta(z,\xi) dV_t^\varepsilon(z,\xi) dt = \int_0^T \psi(t) \int_{\boldsymbol{R}^n \times \boldsymbol{R} \times G_0} \beta(z,\xi) dV_t(z,\xi) dt.$$

As a collorary of Lemma 4.1 we have

Lemma 4.2. *Put* $\overline{V}^\varepsilon = \int_0^T V_t^\varepsilon dt$ *and* $\overline{V} = \int_0^T V_t dt$. *Then* $\overline{V}^\varepsilon \overset{*}{\rightharpoonup} \overline{V}$ *in the sense of Radon measures in* $\boldsymbol{R}^n \times \boldsymbol{R} \times G_0$.

Proof. Letting $\psi = \chi_{[0,T]}$, the characteristic function of $[0,T]$, in Lemma 4.1, we immediately have the conclusion. □

Remark that each V_t^ε satisfies Proposition 3.3. Furthermore, repeating the process of the proof of Proposition 3.3, we have

Lemma 4.3. *For* $\mathscr{L}^1$*-a.e.* $t \in (0,T)$,

1) $\int_{\boldsymbol{R}^n \times \boldsymbol{R}} g(z) \nu_{E_t}(z) d\mu_{E_t} = \int_{\boldsymbol{R}^n \times \boldsymbol{R} \times G_0} g(z) \nu(\xi) dV_t$ *for any* $g \in C_0^0(\boldsymbol{R}^n \times \boldsymbol{R}; \boldsymbol{R}^{n+1})$
2) $\mu_{V_t}(A) \geq \mu_{E_t}(A)$ *for each Borel set* $A \subset \boldsymbol{R}^n \times \boldsymbol{R}$
3) $\mu_{V_t}(A) = \int_A D_{\mu_{E_t}} \mu_{V_t}(z) d\mu_{E_t} + (\mu_{V_t} \llcorner Z)(A)$ *for* $A \subset U$, *where* $D_{\mu_{E_t}} \mu_{V_t}$ *is the derivative of* μ_{V_t} *with respect to* μ_{E_t} *and* Z *is the* μ_{E_t}*-null set defined by* $Z = \{z; D_{\mu_{E_t}} \mu_{V_t}(z) = \infty\}$
4) $\int_{G_0} \nu(\xi) d\eta_{V_t}^{(z)} = 0$ *for* $\mu_{V_t} \llcorner Z$*-a.e.* z
5) $\operatorname{spt} \eta_{V_t}^{(z)} \subset \operatorname{irr}(G_0)$ *for* $\mu_{V_t} \llcorner Z$*-a.e.* z
6) $|D^s u|(\overline{\Omega}) \leq V_t(\overline{\Omega} \times \boldsymbol{R} \times \operatorname{irr}(G_0))$.

Lemma 4.4. $|D^s u(t,\cdot)|(\overline{\Omega}) = 0$ *for* $\mathscr{L}^1$*-a.e.* t.

Remark that Lemma 4.4 implies, in particular, $\gamma u = 0$.

Proof. If we have $V_t(\overline{\Omega} \times \boldsymbol{R} \times \operatorname{irr}(G_0)) = 0$ for $\mathscr{L}^1$-a.e. t, then the conclusion immediately follows from Lemma 4.3 6), and, since $V_t \geq 0$, this follows if we have

$$\overline{V}(\overline{\Omega} \times \boldsymbol{R} \times \operatorname{irr}(G_0)) = 0. \tag{4.3}$$

Hence we prove (4.3).

In the case of the proof of Theorem 1.1 we have by (1.7)

$$\begin{aligned}\bar{J}_\varepsilon(u^\varepsilon, \overline{\Omega}) &= \int_{\overline{\Omega}\times\boldsymbol{R}\times G_0} \frac{|\nu'(\xi)|^2}{\sqrt{\nu^{n+1}(\xi)^2 + \varepsilon^2|\nu'(\xi)|^2} + \nu^{n+1}(\xi)}\, dV_t^\varepsilon \\ &\leq \frac{1}{2}\int_\Omega |v_0|^2 dx + \bar{J}_\varepsilon(u_0, \overline{\Omega})\end{aligned} \tag{4.4}$$

In the case of the proof of Theorem 1.2 we have by (2.6) and (3.3) that for each $\psi \in L^1(0,T)$ with $\psi \geq 0$

$$\begin{aligned}&\int_0^T \psi(t) \int_{\overline{\Omega}\times\boldsymbol{R}\times G_0} \frac{|\nu'(\xi)|^2}{\sqrt{\nu^{n+1}(\xi)^2 + \varepsilon^2|\nu'(\xi)|^2} + \nu^{n+1}(\xi)}\, dV_t^\varepsilon dt \\ &\quad\leq \liminf_{h\to 0} \int_0^T \psi(t)\bar{J}_\varepsilon(\overline{u}^h, \overline{\Omega})dt \leq \left(\frac{1}{2}\int_\Omega |v_0|^2 dx + \bar{J}_\varepsilon(u_0, \overline{\Omega})\right)\int_0^T \psi(t)dt.\end{aligned} \tag{4.5}$$

Let σ be a positive number. Then

$$\begin{aligned}&\int_{\overline{\Omega}\times\boldsymbol{R}\times G_0\cap\{\nu^{n+1}<\sigma\}} \frac{|\nu'(\xi)|^2}{\sqrt{\nu^{n+1}(\xi)^2 + \varepsilon^2|\nu'(\xi)|^2} + \nu^{n+1}}\, dV_t^\varepsilon \\ &\qquad\geq \frac{1-\sigma^2}{\sqrt{\sigma^2+\varepsilon^2}+\sigma} V_t^\varepsilon(\overline{\Omega}\times\boldsymbol{R}\times G_0 \cap \{\nu^{n+1} < \sigma\}).\end{aligned}$$

Integrating from 0 to T, we have by (4.1) and (4.4) or (4.5)

$$\overline{V}^\varepsilon(\overline{\Omega}\times\boldsymbol{R}\times G_0\cap\{\nu^{n+1}<\sigma\}) \leq \frac{\sqrt{\sigma^2+\varepsilon^2}+\sigma}{1-\sigma^2}\left(\frac{1}{2}\int_\Omega |v_0|^2 dx + \int_\Omega |\nabla u_0(x)|^2 dx\right)T.$$

Thus, letting $\varepsilon \to 0$, we have by Lemma 4.2 and the lower semicontinuity of Radon measures

$$\overline{V}(\overline{\Omega}\times\boldsymbol{R}\times G_0\cap\{\nu^{n+1}<\sigma\}) \leq \frac{2\sigma}{1-\sigma^2}\left(\frac{1}{2}\int_\Omega |v_0|^2 dx + \int_\Omega |\nabla u_0(x)|^2 dx\right)T. \tag{4.6}$$

Letting $\sigma \to 0$, we have (4.3). □

Proposition 4.2. $u \in L^\infty((0,T); W_0^{1,2}(\Omega)) \cap W^{1,2}((0,T)\times\Omega)$

Proof. Lemma 4.4 implies that the distributional derivative Du coincides with ∇u and hence $u(t,\cdot) \in W_0^{1,1}(\Omega)$ for $\mathscr{L}^1$-a.e. t. Thus it is sufficient to show

$$u \in L^\infty((0,T); W^{1,2}(\Omega)) \tag{4.7}$$

for each $T > 0$. Since

$$\begin{aligned}
&\int_{\overline{\Omega}\times\boldsymbol{R}\times G_0\cap\{\nu^{n+1}>\varepsilon\}} \frac{|\nu'(\xi)|^2}{\sqrt{\nu^{n+1}(\xi)^2+\varepsilon^2|\nu'(\xi)|^2}+\nu^{n+1}}\, dV_t^\varepsilon \\
&\quad\geq \int_{\overline{\Omega}\times\boldsymbol{R}\times G_0\cap\{\nu^{n+1}>\varepsilon\}} \frac{|\nu'(\xi)|^2}{\nu^{n+1}(\xi)(\sqrt{1+|\nu'(\xi)|^2}+1)}\, dV_t^\varepsilon \\
&\quad\geq \frac{1}{\sqrt{2}+1}\int_{\overline{\Omega}\times\boldsymbol{R}\times G_0\cap\{\nu^{n+1}>\varepsilon\}} \frac{|\nu'(\xi)|^2}{\nu^{n+1}(\xi)}\, dV_t^\varepsilon,
\end{aligned} \tag{4.8}$$

we have by (4.1) and (4.4) or (4.5), for each $\psi \in L^1(0,T)$ with $\psi \geq 0$,

$$\begin{aligned}
&\int_0^T \psi(t)\int_{\overline{\Omega}\times\boldsymbol{R}\times G_0\cap\{\nu^{n+1}>\varepsilon\}} \frac{|\nu'(\xi)|^2}{\nu^{n+1}(\xi)}\, dV_t^\varepsilon dt \\
&\quad\leq (\sqrt{2}+1)\left(\frac{1}{2}\int_\Omega |v_0|^2 dx + \int_\Omega |\nabla u_0(x)|^2 dx\right)\int_0^T \psi(t)dt.
\end{aligned} \tag{4.9}$$

For each $\sigma > \varepsilon$, since $\{\nu^{n+1} > \sigma\} \subset \{\nu^{n+1} > \varepsilon\}$, we have by (4.9)

$$\begin{aligned}
&\int_0^T \psi(t)\int_{\overline{\Omega}\times\boldsymbol{R}\times G_0\cap\{\nu^{n+1}>\sigma\}} \frac{|\nu'|^2}{\nu^{n+1}}\, dV_t^\varepsilon dt \\
&\quad\leq (\sqrt{2}+1)\left(\frac{1}{2}\int_\Omega |v_0|^2 dx + \int_\Omega |\nabla u_0(x)|^2 dx\right)\int_0^T \psi(t)dt.
\end{aligned}$$

On $\{\nu^{n+1} > \sigma\}$ the integrand of the left hand side of above is positive and continuous. Thus, letting $\varepsilon \to 0$, we have by Lemma 4.1 and the lower semi-continuity of Radon measures

$$\int_0^T \psi(t) \int_{\overline{\Omega}\times\boldsymbol{R}\times G_0\cap\{\nu^{n+1}>\sigma\}} \frac{|\nu'|^2}{\nu^{n+1}}\, dV_t dt \\ \le (\sqrt{2}+1)\Big(\frac{1}{2}\int_\Omega |v_0|^2 dx + \int_\Omega |\nabla u_0(x)|^2 dx\Big)\int_0^T \psi(t)dt$$

and thus

$$\operatorname*{ess.\,sup}_{t\ge 0} \int_{\overline{\Omega}\times\boldsymbol{R}\times G_0\cap\{\nu^{n+1}>\sigma\}} \frac{|\nu'|^2}{\nu^{n+1}}\, dV_t \le (\sqrt{2}+1)\Big(\frac{1}{2}\int_\Omega |v_0|^2 dx + \int_\Omega |\nabla u_0(x)|^2 dx\Big).$$

Hence, for $\mathscr{L}^1$-a.e. t, we have by letting $\sigma \to 0$

$$\int_{\overline{\Omega}\times\boldsymbol{R}\times(G_0\setminus\operatorname{irr}(G_0))} \frac{|\nu'|^2}{\nu^{n+1}}\, dV_t \le (\sqrt{2}+1)\Big(\frac{1}{2}\int_\Omega |v_0|^2 dx + \int_\Omega |\nabla u_0(x)|^2 dx\Big). \tag{4.10}$$

By a mapping $\iota : G_0 \setminus \operatorname{irr}(G_0) \ni \xi \mapsto \nu'(\xi)/\nu^{n+1}(\xi) \in \boldsymbol{R}^n$ the measure $\eta^{(z)}_{V_t} \llcorner \nu^{n+1}$ in $G_0 \setminus \operatorname{irr}(G_0)$ induces a measure in $\boldsymbol{R}^n$. Namely we have

$$\int_{\overline{\Omega}\times\boldsymbol{R}\times(G_0\setminus\operatorname{irr}(G_0))} \frac{|\nu'|^2}{\nu^{n+1}}\, dV_t = \int_{\overline{\Omega}\times\boldsymbol{R}} \int_{G_0\setminus\operatorname{irr}(G_0)} \frac{|\nu'|^2}{\nu^{n+1}}\, d\eta^{(z)}_{V_t} d\mu_{V_t} \\ = \int_{\overline{\Omega}\times\boldsymbol{R}} \int_{\boldsymbol{R}^n} |p|^2 d\iota_\#(\eta^{(z)}_{V_t} \llcorner \nu^{n+1}) d\mu_{V_t}.$$

By (4.3) we have $\displaystyle\int_{G_0\setminus\operatorname{irr}(G_0)} \nu^{n+1} d\eta^{(z)}_{V_t} \ne 0$ for $\mathscr{L}^1$-a.e. t and μ_{V_t}-a.e. z. Then we define a measure Φ on $\boldsymbol{R}^n$ by

$$\Phi = \Big(\int_{G_0\setminus\operatorname{irr}(G_0)} \nu^{n+1} d\eta^{(z)}_{V_t}\Big)^{-1} \iota_\#(\eta^{(z)}_{V_t} \llcorner \nu^{n+1}).$$

Then, noting

$$\Phi(\boldsymbol{R}^n) = \Big(\int_{G_0\setminus\operatorname{irr}(G_0)} \nu^{n+1} d\eta^{(z)}_{V_t}\Big)^{-1} \iota_\#(\eta^{(z)}_{V_t} \llcorner \nu^{n+1})(\boldsymbol{R}^n) \\ = \Big(\int_{G_0\setminus\operatorname{irr}(G_0)} \nu^{n+1} d\eta^{(z)}_{V_t}\Big)^{-1} \int_{G_0\setminus\operatorname{irr}(G_0)} \nu^{n+1} d\eta^{(z)}_{V_t} = 1,$$

we have by Jensen's inequality

$$\int_{\boldsymbol{R}^n} |p|^2 d\Phi \geq \left| \int_{\boldsymbol{R}^n} p d\Phi \right|^2.$$

Here, note that

$$\begin{aligned}\int_{\boldsymbol{R}^n} p d\Phi &= \left(\int_{G_0 \setminus \mathrm{irr}(G_0)} \nu^{n+1} d\eta_{V_t}^{(z)} \right)^{-1} \int_{\boldsymbol{R}^n} p d\iota_{\#}(\eta_{V_t}^{(z)} \llcorner \nu^{n+1}) \\ &= \left(\int_{G_0 \setminus \mathrm{irr}(G_0)} \nu^{n+1} d\eta_{V_t}^{(z)} \right)^{-1} \int_{G_0 \setminus \mathrm{irr}(G_0)} \nu' d\eta_{V_t}^{(z)}.\end{aligned}$$

On the other hand Lemma 4.3 1), 3) imply

$$\int_{G_0 \setminus \mathrm{irr}(G_0)} \nu d\eta_{V_t}^{(z)} = D_{\mu_{E_t}} \mu_{V_t}(z)^{-1} \nu_{E_t}(z).$$

Finally we have

$$\begin{aligned}\int_{\overline{\Omega} \times \boldsymbol{R} \times (G_0 \setminus \mathrm{irr}(G_0))} \frac{|\nu'|^2}{\nu^{n+1}} dV_t &= \int_{\overline{\Omega} \times \boldsymbol{R}} \int_{G_0 \setminus \mathrm{irr}(G_0)} \nu^{n+1} d\eta_{V_t}^{(z)} \int_{\boldsymbol{R}^n} |p|^2 d\Phi d\mu_{V_t}. \\ \geq \int_{\overline{\Omega} \times \boldsymbol{R}} &\left(\int_{G_0 \setminus \mathrm{irr}(G_0)} \nu^{n+1} d\eta_{V_t}^{(z)} \right)^{-1} \left| \int_{G_0 \setminus \mathrm{irr}(G_0)} \nu' d\eta_{V_t}^{(z)} \right|^2 d\mu_{V_t} \\ = \int_{\overline{\Omega} \times \boldsymbol{R}} &(D_{\mu_{E_t}} \mu_{V_t}(z)^{-1} \nu_{E_t}^{n+1}(z))^{-1} |D_{\mu_{E_t}} \mu_{V_t}(z)^{-1} \nu'_{E_t}(z)|^2 D_{\mu_{E_t}} \mu_{V_t}(z) d\mu_{E_t} \\ = \int_{\overline{\Omega} \times \boldsymbol{R}} &\frac{|\nu'_{E_t}(z)|^2}{\nu_{E_t}^{n+1}(z)} d\mu_{E_t} = \int_{\Omega} |\nabla u(t,x)|^2 dx.\end{aligned}$$

This implies (4.7) by (4.10). □

Lemma 4.5. *For $\mathscr{L}^1$-a.e. t, $V_t^{\varepsilon}(\overline{\Omega} \times \boldsymbol{R} \times G_0 \cap \{\nu^{n+1} \leq \varepsilon\}) \to 0$ as $\varepsilon \to 0$.*

Proof. Since

$$\int_0^T \int_{\overline{\Omega}\times \boldsymbol{R}\times G_0\cap\{\nu^{n+1}\le\varepsilon\}} \frac{|\nu'(\xi)|^2}{\sqrt{\nu^{n+1}(\xi)^2+\varepsilon^2|\nu'(\xi)|^2}+\nu^{n+1}}\, dV_t^\varepsilon dt$$
$$\ge \frac{1-\varepsilon^2}{\varepsilon\sqrt{2-\varepsilon^2}+\varepsilon}\int_0^T V_t^\varepsilon(\overline{\Omega}\times \boldsymbol{R}\times G_0\cap\{\nu^{n+1}\le\varepsilon\})dt,$$

we have by (4.1) and (4.4) or (4.5)

$$\int_0^T V_t^\varepsilon(\overline{\Omega}\times \boldsymbol{R}\times G_0\cap\{\nu^{n+1}\le\varepsilon\})dt\to 0$$

as $\varepsilon\to 0$. Then by Fatou's lemma

$$0\le \int_0^T \liminf_{\varepsilon\to 0} V_t^\varepsilon(\overline{\Omega}\times \boldsymbol{R}\times G_0\cap\{\nu^{n+1}\le\varepsilon\})dt$$
$$\le \liminf_{\varepsilon\to 0}\int_0^T V_t^\varepsilon(\overline{\Omega}\times \boldsymbol{R}\times G_0\cap\{\nu^{n+1}\le\varepsilon\})dt=0,$$

which implies the conclusion. □

End of the proof of Theorems 1.1 and 1.2. Up to a subsequence Assertions 1)–4) are obtained in Proposition 4.1. Assertion 5) is obtained in Proposition 4.2. Now we prove 6).

For $\mathscr{L}^1$-a.e. $t\in[0,T]$, Proposition 4.1 7) and Lemma 4.5 hold. Now we fix one of such t. For each $\phi\in C_0^1([0,T)\times\Omega)$

$$\int_{\overline{\Omega}\times \boldsymbol{R}\times G_0} \frac{\nabla\phi\cdot\nu'(\xi)\nu^{n+1}(\xi)}{\sqrt{\nu^{n+1}(\xi)^2+\varepsilon^2|\nu'(\xi)|^2}}\, dV_t^\varepsilon \tag{4.11}$$
$$=\int_{\overline{\Omega}\times \boldsymbol{R}\times G_0}\nabla\phi\cdot\nu'(\xi)dV_t^\varepsilon+\int_{\overline{\Omega}\times \boldsymbol{R}\times G_0}\nabla\phi\cdot\nu'(\xi)\left(\frac{\nu^{n+1}(\xi)}{\sqrt{\nu^{n+1}(\xi)^2+\varepsilon^2|\nu'(\xi)|^2}}-1\right)dV_t^\varepsilon$$
$$=\int_{\overline{\Omega}\times \boldsymbol{R}\times G_0}\nabla\phi\cdot\nu'(\xi)dV_t^\varepsilon+\int_{\overline{\Omega}\times \boldsymbol{R}\times \{\nu^{n+1}\le\varepsilon\}}\nabla\phi\cdot\nu'(\xi)\left(\frac{\nu^{n+1}(\xi)}{\sqrt{\nu^{n+1}(\xi)^2+\varepsilon^2|\nu'(\xi)|^2}}-1\right)dV_t^\varepsilon$$

$$+\int_{\overline{\Omega}\times\boldsymbol{R}\times\{\nu^{n+1}>\varepsilon\}}\nabla\phi\cdot\nu'(\xi)\left(\frac{\nu^{n+1}(\xi)}{\sqrt{\nu^{n+1}(\xi)^2+\varepsilon^2|\nu'(\xi)|^2}}-1\right)dV_t^\varepsilon$$

$$=: I+II+III.$$

By Proposition 3.4 and Proposition 4.1 7) we have

$$I=-\int_{\overline{\Omega}}\nabla\phi\cdot dDu^\varepsilon\to-\int_{\overline{\Omega}}\nabla\phi\cdot dDu\left(=-\int_\Omega\nabla\phi\cdot\nabla u dx\right)$$

and by Lemma 4.5

$$|II|\le 2\sup|\nabla\phi|V_t^\varepsilon(\overline{\Omega}\times\boldsymbol{R}\times G_0\cap\{\nu^{n+1}\le\varepsilon\})\to 0.$$

For each ξ that satisfies $\nu^{n+1}(\xi)>\varepsilon$, we have

$$\left|\frac{\nu^{n+1}(\xi)}{\sqrt{\nu^{n+1}(\xi)^2+\varepsilon^2|\nu'(\xi)|^2}}-1\right| \tag{4.12}$$

$$=\left|-\frac{\varepsilon^2|\nu'(\xi)|^2}{\nu^{n+1}(\xi)\sqrt{\nu^{n+1}(\xi)^2+\varepsilon^2|\nu'(\xi)|^2}+\varepsilon^2|\nu'(\xi)|^2+\nu^{n+1}(\xi)^2}\right|$$

$$\le\frac{\varepsilon^2|\nu'(\xi)|^2}{\nu^{n+1}(\xi)\varepsilon(\sqrt{1+|\nu'(\xi)|^2}+\varepsilon|\nu'(\xi)|^2+\varepsilon)}$$

$$\le\varepsilon\frac{|\nu'(\xi)|^2}{\nu^{n+1}(\xi)}.$$

Thus we have by (4.9)

$$\left|\int_0^T III\,dt\right|\le\varepsilon\sup|\nabla\phi|\int_0^T\int_{\overline{\Omega}\times\boldsymbol{R}\times\{\nu^{n+1}>\varepsilon\}}\frac{|\nu'(\xi)|^2}{\nu^{n+1}(\xi)}dV_t^\varepsilon dt$$

$$\le\varepsilon\sup|\nabla\phi|(\sqrt{2}+1)\left(\frac{1}{2}\int_\Omega|v_0|^2dx+\int_\Omega|\nabla u_0(x)|^2dx\right)T\to 0$$

as $\varepsilon \to 0$. Hence, letting $\varepsilon \to 0$ in (4.2), we have

$$\int_0^T \left\{ -\int_\Omega u_t \phi_t(t,x)dx + \int_\Omega \nabla\phi \cdot \nabla u dx \right\} dt = \int_\Omega v_0(x)\phi(0,x)dx,$$

which with Proposition 4.1 8) and Proposition 4.2 means u satisfies (1.8) in $(0,T) \times \Omega$ in a weak sense.

Finally the uniqueness of a solution to the linear wave equation implies the rest of the subsequence has another subsequence that converges in the same topology to the same function u. Thus we do not have to subtract a subsequence. □

Appendix

A. Remark on the proof of Theorem 1.1.

In Section 4 the proof of Theorem 1.1 is carried out at the same time as that of Theorem 1.2 by putting $V_t^\varepsilon = \boldsymbol{v}_+(u^\varepsilon)$. However most of the readers are probably not familiar to varifold theory, and thus in this appendix we try to mention the proof of Theorem 1.1 without using the varifold theory except for that of Proposition 4.2. This proposition does require varifold theory.

Before proof we remark the following fact. It is obtained by testing smooth functions and solution u itself and using expression formula (2.10) of $\bar{J}_\varepsilon(u, \overline{\Omega})$. Compare to Proposition 3.6.

Proposition A.1. Suppose that a function u is a weak solution to (1.1)–(1.3) in $(0,T) \times \Omega$. Then

iii)$_1{}'$ for any $\phi \in C_0^1([0,T) \times \Omega)$,

$$\int_0^T \left\{ -\int_\Omega u_t \phi_t(t,x)dx + \int_\Omega \frac{\nabla u}{\sqrt{1+\varepsilon^2|\nabla u|^2}} \nabla\phi(t,x)dx \right\} dt = \int_\Omega v_0(x)\phi(0,x)dx$$

iii)$_2{}'$ for any $\psi \in C_0^1([0,T))$,

$$\int_0^T \Big\{ -\int_\Omega u_t(\psi'(t)u + \psi(t)u_t)dx$$
$$+ \psi(t)\int_\Omega \frac{|\nabla u|^2}{\sqrt{1+\varepsilon^2|\nabla u|^2}}\,dx + \frac{1}{\varepsilon}\psi(t)|D^s u|(\Omega)$$
$$+ \frac{1}{\varepsilon}\psi(t)\int_{\partial\Omega}|\gamma u|d\mathscr{H}^{n-1}\Big\}dt = \psi(0)\int_\Omega v_0(x)u_0(x)dx.$$

Remark. If $\partial\Omega$ is of C^2 class, then the converse of Proposition A.1 holds. Namely, a function u is a weak solution to (1.1)–(1.3) in $(0,T)\times\Omega$ if and only if u satisfies i), ii) of Definition 1.1 and $\text{iii})_1{}'$, $\text{iii})_2{}'$ hold (compare to [**12**, Theorem A.1]).

Proposition 4.1 is proved without using varifold theory. Proof of Proposition 4.2 requires varifold theory and it should be the same as that of Section 4. Thus we do not repeat them. The following lemma corresponds to Lemma 4.5 in Section 4.

Lemma A.1. $\displaystyle\int_{\{|\nabla u^\varepsilon|\geq 1/\varepsilon\}} |\nabla u^\varepsilon(x)|dx + |D^s u^\varepsilon|(\overline{\Omega}) \to 0$ *as* $\varepsilon\to 0$.

Proof. By (1.7) and (4.1) we have

$$\bar{J}_\varepsilon(u^\varepsilon,\overline{\Omega}) = \int_\Omega \frac{|\nabla u^\varepsilon|^2}{\sqrt{1+\varepsilon^2|\nabla u^\varepsilon|^2}+1}\,dx + \frac{1}{\varepsilon}|D^s u^\varepsilon|(\overline{\Omega}) \tag{A.1}$$
$$\leq \frac{1}{2}\int_\Omega |v_0|^2dx + \int_\Omega |\nabla u_0(x)|^2dx.$$

Since

$$\int_{\{|\nabla u^\varepsilon|\geq 1/\varepsilon\}} \frac{|\nabla u^\varepsilon|^2}{\sqrt{1+\varepsilon^2|\nabla u^\varepsilon|^2}+1}\,dx + \frac{1}{\varepsilon}|D^s u^\varepsilon|(\overline{\Omega})$$
$$\geq \frac{1}{\varepsilon(\sqrt{2}+1)}\int_{\{|\nabla u^\varepsilon|\geq 1/\varepsilon\}}|\nabla u^\varepsilon|dx + \frac{1}{\varepsilon}|D^s u^\varepsilon|(\overline{\Omega}),$$

we have by (A.1)

$$\int_{\{|\nabla u^\varepsilon|\geq 1/\varepsilon\}} |\nabla u^\varepsilon(x)|dx + |D^s u^\varepsilon|(\overline{\Omega}) \leq \text{Const. } \varepsilon$$

as $\varepsilon \to 0$. □

PROOF OF THEOREM 1.1. Up to a subsequence Assertions 1)–4) are obtained in Proposition 4.1. Assertion 5) is obtained in Proposition 4.2. Now we prove 6).

Noting

$$\int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}} \frac{|\nabla u^\varepsilon|^2}{\sqrt{1+\varepsilon^2|\nabla u^\varepsilon|^2}+1}\,dx \geq \frac{1}{\sqrt{2}+1}\int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}} |\nabla u^\varepsilon|^2 dx,$$

we have by (A.1)

$$\int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}} |\nabla u^\varepsilon|^2 dx \leq (\sqrt{2}+1)\left(\frac{1}{2}\int_\Omega |v_0|^2 dx + \int_\Omega |\nabla u_0(x)|^2 dx\right). \qquad \text{(A.2)}$$

Now, for each $\phi \in C_0^1([0,T)\times\Omega)$

$$\begin{aligned}\int_\Omega \frac{\nabla\phi\cdot\nabla u^\varepsilon}{\sqrt{1+\varepsilon^2|\nabla u^\varepsilon|^2}}\,dx &= \int_{\{|\nabla u^\varepsilon|\geq 1/\varepsilon\}} \frac{\nabla\phi\cdot\nabla u^\varepsilon}{\sqrt{1+\varepsilon^2|\nabla u^\varepsilon|^2}}\,dx + \int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}} \nabla\phi\cdot\nabla u^\varepsilon dx \\ &\quad + \int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}} \nabla\phi\cdot\nabla u^\varepsilon\left(\frac{1}{\sqrt{1+\varepsilon^2|\nabla u^\varepsilon|^2}}-1\right)dx \\ &=: I + II + III.\end{aligned}$$

First we have by Lemma A.1

$$|I| \leq \sup|\nabla\phi| \int_{\{|\nabla u^\varepsilon|\geq 1/\varepsilon\}} |\nabla u^\varepsilon| dx \to 0.$$

Again by Lemma A.1

$$
\begin{aligned}
II &= \int_{\Omega} \nabla\phi \cdot dDu^{\varepsilon} - \left(\int_{\{|\nabla u^{\varepsilon}|\geq 1/\varepsilon\}} \nabla\phi \cdot \nabla u^{\varepsilon} dx + \int_{\Omega} \nabla\phi \cdot dD^{s}u^{\varepsilon} \right) \\
&\to \int_{\Omega} \nabla\phi \cdot dDu \left(= \int_{\Omega} \nabla\phi \cdot \nabla u dx \right).
\end{aligned}
$$

Since

$$
\frac{1}{\sqrt{1+\varepsilon^2|\nabla u^{\varepsilon}|^2}} - 1 = -\frac{\varepsilon^2|\nabla u^{\varepsilon}|^2}{1+\varepsilon^2|\nabla u^{\varepsilon}|^2 + \sqrt{1+\varepsilon^2|\nabla u^{\varepsilon}|^2}},
$$

we have by (A.2)

$$
\begin{aligned}
|III| &\leq \varepsilon \sup|\nabla\phi| \int_{\{|\nabla u^{\varepsilon}|<1/\varepsilon\}} |\nabla u^{\varepsilon}|^2 dx \\
&\leq (\sqrt{2}+1)\varepsilon \sup|\nabla\phi| \left(\frac{1}{2}\int_{\Omega} |v_0|^2 dx + \int_{\Omega} |\nabla u_0(x)|^2 dx \right) \to 0.
\end{aligned}
$$

Hence, letting $\varepsilon \to 0$, we have by Proposition A.1 iii)$_1{}'$

$$
\int_0^T \left\{ -\int_{\Omega} u_t \phi_t(t,x) dx + \int_{\Omega} \nabla\phi \cdot \nabla u dx \right\} dt = \int_{\Omega} v_0(x)\phi(0,x) dx,
$$

which with Proposition 4.1 8) and Proposition 4.2 means u satisfies (1.8) in a weak sense.

Finally the uniqueness of a solution to the linear wave equation implies the rest of the subsequence has another subsequence that converges in the same topology to the same function u. Thus we do not have to subtract a subsequence. □

B. The case $u_0 \in W_0^{1,p}(\Omega)$, $1 < p < 2$.

In Theorems 1.1 and 1.2 we suppose that the initial data u_0 belongs to $W_0^{1,2}(\Omega)$. However a weak solution to (1.1)–(1.3) is defined just for $u_0 \in BV(\Omega)$. In order to fill this gap we observe the case that $u_0 \in W_0^{1,p}(\Omega)$, $1 < p < 2$. In such a case energy inequality (1.7) does not imply the uniform estimate with respect to ε. We only have the linear approximation holds provided that (a subsequence of) u^ε converges to a distribution u. In the case that u_0 belongs only to $BV(\Omega)$ we still have no answer to our problems.

THEOREM B.1. *Suppose that u^ε is a weak solution to* (1.1)–(1.3) *in* $(0,T)\times\Omega$ *that satisfies energy inequality* (1.7). *Further suppose that* (*a subsequence of*) u^ε *converges to a distribution u in* $(0,T)\times\Omega$. *Then u satisfies the d'Alembert equation in the sense of distributions in* $(0,T)\times\Omega$.

THEOREM B.2. *Suppose that u^ε is a generalized minimizing movement associated with* (1.1)–(1.3). *Further suppose that* (*a subsequence of*) u^ε *converges to a distribution u in* $(0,T)\times\Omega$. *Then u satisfies the d'Alembert equation in the sense of distributions in* $(0,T)\times\Omega$.

Similar to the case of our main theorems we prove these two theorems at the same time by letting $\{V_t^\varepsilon\}_{t>0}$ denote a one parameter family of oriented varifolds corresponding to the weak solutions $u^\varepsilon(t,\cdot)$ in the case of the proof of Theorem B.1 and a one parameter family of oriented varifolds in $\boldsymbol{R}^n\times\boldsymbol{R}$ presented in Proposition 3.2 as V_t in the case of that of Theorem B.2. In both cases V_t^ε satisfies (4.2).

LEMMA B.1. *For $\mathscr{L}^1$-a.e. t, $V_t^\varepsilon(\overline{\Omega}\times\boldsymbol{R}\times G_0\cap\{\nu^{n+1}\le\varepsilon\})\to 0$ as $\varepsilon\to 0$.*

PROOF. Since u_0 belongs to $W^{1,p}(\Omega)$, we have

$$\bar{J}_\varepsilon(u_0,\overline{\Omega})=\int_\Omega\frac{|\nabla u_0(x)|^2}{\sqrt{1+\varepsilon^2|\nabla u_0(x)|^2}+1}\,dx\le\varepsilon^{-(2-p)}\int_\Omega|\nabla u_0(x)|^p dx. \tag{B.1}$$

Note that (4.4) or (4.5) still holds in the case of the proof of Theorem B.1 or B.2, respectively. Thus, since $p>1$, we have the conclusion by (4.4) or (4.5) and (B.1) in the same way as in the proof of Lemma 4.5. □

PROOF OF THEOREMS B.1 AND B.2. Since (4.8) still holds under assumptions here, (4.4) or (4.5) and (B.1) imply

$$\begin{aligned}&\int_0^T\int_{\overline{\Omega}\times\boldsymbol{R}\times G_0\cap\{\nu^{n+1}>\varepsilon\}}\frac{|\nu'(\xi)|^2}{\nu^{n+1}}\,dV_t^\varepsilon dt\\&\quad\le(\sqrt{2}+1)\Big(\frac{1}{2}\int_\Omega|v_0|^2dx+\varepsilon^{-(2-p)}\int_\Omega|\nabla u_0(x)|^p dx\Big)T.\end{aligned}\tag{B.2}$$

Let us fix $t\in(0,T)$ at which Lemma B.1 holds. Now, for each $\phi\in$

$C_0^\infty((0,T)\times\Omega)$, (4.11) holds. By Proposition 3.4 we have

$$I = -\int_\Omega \nabla\phi \cdot dDu^\varepsilon = \int_\Omega u^\varepsilon \Delta\phi dx$$

and thus

$$\int_0^T I dt = -\langle u^\varepsilon, \Delta\phi\rangle \to -\langle u, \Delta\phi\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the coupling of distributions in $(0,T)\times\Omega$, while by Lemma B.1

$$|II| \le 2\sup|\nabla\phi| V_t^\varepsilon(\overline{\Omega}\times \boldsymbol{R}\times G_0 \cap \{\nu^{n+1}\le\varepsilon\}) \to 0.$$

Noting that (4.12) still holds under assumptions here, we have by (B.2)

$$\begin{aligned}\left|\int_0^T III\, dt\right| &\le \varepsilon \sup|\nabla\phi| \int_0^T \int_{\overline{\Omega}\times\boldsymbol{R}\times\{\nu^{n+1}>\varepsilon\}} \frac{|\nu'(\xi)|^2}{\nu^{n+1}(\xi)} dV_t^\varepsilon dt \\ &\le (\sqrt{2}+1)\sup|\nabla\phi|\left(\varepsilon\frac{1}{2}\int_\Omega |v_0|^2 dx + \varepsilon^{p-1}\int_\Omega |\nabla u_0(x)|^p dx\right)T \to 0\end{aligned}$$

as $\varepsilon\to 0$. Hence, letting $\varepsilon\to 0$ in (4.2), we have

$$\langle u, \phi_{tt} - \Delta\phi\rangle = 0,$$

which means u satisfies the d'Alembert equation in the sense of distributions. □

C. Review.

Finally we review some properties of BV functions. For details about BV functions, consult to, for example, [**2**], [**5**], [**10**]. We also review several facts which are discussed in [**11**], [**12**] and give some additional comments.

C.1. BV functions.

A function $u\in L^1(\Omega)$ is said to have bounded variation in Ω if

$$\sup\left\{\int_\Omega u\,\mathrm{div} g\, dx\,;\, g=(g_1,\dots,g_n)\in C_0^1(\Omega;\boldsymbol{R}^n),\ |g|\le 1\right\} < \infty.$$

This means that the distributional derivative Du is an $\boldsymbol{R}^n$-valued finite Radon measure in Ω. The vector space of all BV functions in Ω is denoted by $BV(\Omega)$. It is a Banach space equipped with the norm $\|u\|_{BV} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$.[5]

Since the distributional derivative Du is an $\boldsymbol{R}^n$-valued finite Radon measure in Ω, we have a decomposition $Du = D^a u + D^s u$, where $D^a u$ is the absolutely continuous part and $D^s u$ is the singular part with respect to the n dimensional Lebesgue measure $\mathscr{L}^n$.

Suppose that $\partial\Omega$ is Lipschitz continuous. Then each BV function u has its trace γu. γ is a bounded operator from $BV(\Omega)$ to $L^1(\partial\Omega)$ such that, for each $g \in C^1(\overline{\Omega}; \boldsymbol{R}^n)$,

$$\int_\Omega u \operatorname{div} g \, dx = -\int_\Omega g \cdot Du + \int_{\partial\Omega} \gamma u \, g \cdot \vec{n} \, d\mathscr{H}^{n-1}, \tag{C.1}$$

where $\vec{n}$ is the outer unit normal to $\partial\Omega$.

Remark that $W^{1,1}(\Omega)$ is not dense in $BV(\Omega)$. We only have

THEOREM C.1. *For each $u \in BV(\Omega)$ there exists $\{u_j\}_{j=1}^\infty \subset C^\infty(\Omega)$ such that*

i) *$u_j \to u$ strongly in $L^1(\Omega)$*
ii) $|Du_j|(\Omega) \to |Du|$

as $j \to \infty$.

([**2**, Theorem 3.9], [**5**, Theorem 2, Section 5.2.2], [**10**, Theorem 1.17])

C.2. Constructing approximate solutions to (1.4) with (1.2), (1.3).

When we construct approximate solutions to (1.4) with (1.2), (1.3) in Rothe's method, we should solve elliptic equation (1.9). Note that a C^2 function u is a solution to (1.9) if it is a minimizer of the functional

$$\frac{1}{2}\int_\Omega \frac{|u - 2u_{l-1} + u_{l-2}|^2}{h^2} dx + J(u, \Omega), \tag{C.2}$$

where J denotes the area of the graph, namely,

$$J(u, \Omega) = \int_\Omega \sqrt{1 + |\nabla u|^2} dx. \tag{C.3}$$

This functional has linear growth order with respect to ∇u and thus it is finite for

[5]Given a vector valued Radon measure μ, we write its total variation as $|\mu|$.

$u \in W^{1,1}(\Omega)$. If we extend it to $L^1(\Omega)$ by $J(u,\Omega) = \infty$ for $u \in L^1(\Omega) \setminus W^{1,1}(\Omega)$, we obtain a convex functional on $L^1(\Omega)$. However it is not lower semicontinuous and thus the existence of a minimizer is not assured. This brings us to the idea of relaxation (lower semicontinuous envelope), and the relaxed functional of J in the $L^1(\Omega)$ norm is defined as

$$\bar{J}(u,\Omega) := \inf\left\{\liminf_{j\to\infty} J(u_j,\Omega); \{u_j\} \subset W_0^{1,1}(\Omega),\ \text{s-lim}_{j\to\infty} u_j = u \text{ in } L^1(\Omega)\right\}.$$

In the same way as in the proof of Theorem C.1 we could obtain that for each $u \in BV(\Omega)$ there exists $\{u_j\} \subset C^\infty(\Omega)$ such that $\text{s-lim}_{j\to\infty} u_j = u$ in $L^1(\Omega)$ and $\lim_{j\to\infty} J(u_j,\Omega) = \sqrt{1+|Du|^2}(\Omega)$, where $\sqrt{1+|Du|^2}$ denotes the total variation of $\boldsymbol{R}^{n+1}$-valued Radon measure ${}^t(Du, \mathscr{L}^n)$. Using this fact we easily obtain

$$\bar{J}(u,\Omega) = \sqrt{1+|Du|^2}(\Omega). \tag{C.4}$$

Note that

$$\sqrt{1+|Du|^2}(\Omega) = \sup\left\{\int_\Omega (u \operatorname{div} g' + g^0)dx; g = (g', g^0) \in C_0^1(\Omega; \boldsymbol{R}^{n+1}),\ |g| \le 1\right\}$$

and that

$$\sqrt{1+|Du|^2}(\Omega) = \int_\Omega \sqrt{1+|\nabla u(x)|^2}dx + |D^s u|(\Omega),$$

where ∇u denotes the Radon-Nikodym derivative of $D^a u$ with respect to $\mathscr{L}^n$.

Suppose that $\partial\Omega$ is sufficiently smooth. Considering the boundary condition, we should solve (1.9) with (1.3). Hence we should find a minimizer of (C.2) in a class of functions satisfying (1.3). In order to carry out it, we replace J in (C.2) with another functional

$$I(u,\Omega) = \begin{cases} J(u,\Omega) & \text{if } u \in W_0^{1,1}(\Omega) \\ \infty & \text{if otherwise.} \end{cases}$$

This is not lower semicontinuous, either. Similar to the case of (C.4), the relaxed functional $\bar{I}$ is obtained as

$$\bar{I}(u,\Omega)=\sup\left\{\int_\Omega (u\,\mathrm{div} g'+g^0)dx; g=(g',g^0)\in C^1(\overline{\Omega};\boldsymbol{R}^{n+1}),\ |g|\le 1\right\}.$$

Let u still denote the null extension of u to the whole space. Then $u\in BV(\boldsymbol{R}^n)$ and thus Du is a $\boldsymbol{R}^n$-valued Radon measure in the whole space $\boldsymbol{R}^n$. Moreover we have $\bar{I}(u,\Omega)=\sqrt{1+|Du|^2}(\overline{\Omega})$. Taking account of this fact, we write $\bar{I}(u,\Omega)$ as $\bar{J}(u,\overline{\Omega})$. Noting that $|D^s u|(\partial\Omega)=\int_{\partial\Omega}|\gamma u|d\mathscr{H}^{n-1}$, we have

$$\bar{J}(u,\overline{\Omega})=\int_\Omega \sqrt{1+|\nabla u(x)|^2}dx+|D^s u|(\Omega)+\int_{\partial\Omega}|\gamma u|d\mathscr{H}^{n-1}.$$

Thereby a weak solution to (1.9) with (1.3) is obtained as a minimizer of $\mathscr{G}_l$ in $L^2(\Omega)\cap BV(\Omega)$, where $\mathscr{G}_l$ is as in (1.10). Indeed in [**11**], [**12**] (1.9) is solved by finding a minimizer of $\mathscr{G}_l$.

C.3. Definition of a weak solution to (1.4) with (1.2), (1.3).

Finally we review definitions of a weak solution to (1.4) with (1.2), (1.3) that are discussed in [**11**], [**12**].

Originally Equation (1.4) is derived as the Euler-Lagrange equation of the action integral

$$\int_0^T\left(\frac{1}{2}\int_\Omega |u_t(t,x)|^2dx-\bar{J}(u,\Omega)\right)dt. \tag{C.5}$$

However $\bar{J}$ is not always Gâteaux differentiable on $BV(\Omega)$ and thus we cannot calculate $(d/d\sigma)\bar{J}(u+\sigma\varphi,\Omega)|_{\sigma=0}$ directly. The area functional $\bar{J}(u,\Omega)$ coincides with the n-dimensional Hausdorff measure of $\partial^*E_u\cap\Omega\times\boldsymbol{R}$ (E_u is as in Section 3) and we should only calculate a variation of $\mathscr{H}^n(\partial^*E_u\cap\Omega\times\boldsymbol{R})$. Noticing that the equation describes the longitudinal vibration, we could calculate the variation by the use of a one parameter family of diffeomorphisms of $\Omega\times\boldsymbol{R}$, each of which is written as $\Omega\times\boldsymbol{R}\ni(x,y)\mapsto(x,y+\sigma\varphi(x,y))\in\Omega\times\boldsymbol{R}$, where σ is the parameter and φ is a given function on $\Omega\times\boldsymbol{R}$. If $\varphi\in C_0^1(\Omega\times\boldsymbol{R})$, the function $\sigma\mapsto\bar{J}(u+\sigma\varphi(x,u),\Omega)$ is differentiable and its derivative at $\sigma=0$ is expressed by the use of ν_{E_u} (which denotes the inward pointing approximate unit normal to ∂^*E_u, see Section 3):

$$\begin{aligned}&\frac{d}{d\sigma}\bar{J}(u+\sigma\varphi(x,u),\Omega)|_{\sigma=0}\\&\quad=\int_{\partial^*E_u\cap\Omega\times\boldsymbol{R}}[-(\nabla_x\varphi\cdot\nu'_{E_u})\nu^{n+1}_{E_u}+|\nu'_{E_u}|^2\varphi_y]d\mathscr{H}^n\ (\nu_{E_u}=(\nu'_{E_u},\nu^{n+1}_{E_u}))\end{aligned}$$

(compare to [**11**, Theorem 2.2]).

In [**11**], taking account of these facts, a weak solution to (1.4) with (1.2), (1.3) is given as follows:

Definition C.1. A function u is said to be a weak solution to (1.4) with (1.2), (1.3) in $(0,T)\times\Omega$ if

i) $u\in L^\infty((0,T);BV(\Omega))$, $u_t\in L^2((0,T)\times\Omega)$
ii) $\text{s-}\lim_{t\searrow 0} u(t)=u_0$ in $L^2(\Omega)$
iii) $\gamma u=0$ for $\mathscr{L}^1$-a.e. $t\in(0,T)$
iv) for any $\varphi\in C_0^1([0,T)\times U)$,

$$\int_0^T\left\{-\int_\Omega u_t(\varphi_t(t,x,u)+\varphi_y(t,x,u)u_t)dx+\int_{\partial^*E_{u(t,\cdot)}\cap\Omega\times\boldsymbol{R}}[-(\nabla_x\varphi\cdot\nu'_{E_{u(t,\cdot)}})\nu^{n+1}_{E_{u(t,\cdot)}}\right.$$
$$\left.+|\nu'_{E_{u(t,\cdot)}}|^2\varphi_y]d\mathscr{H}^n\right\}dt=\int_\Omega v_0(x)\varphi(0,x,u_0(x))dx.$$

In [**11**, Theorem A.1] it is proved that, if $\partial\Omega$ is of C^2 class, Definition C.1 is equivalent to the definition of a weak solution to $u_{tt}+\partial\bar{J}(u,\Omega)\ni 0$: putting

$$\mathscr{X}_0=\{\phi\in\mathscr{X};\gamma\phi=0\text{ for }\mathscr{L}^1\text{-a.e. }t\in(0,T)\},$$

where $\mathscr{X}$ is as in (1.6), we define

Definition C.2. A function u is said to be a weak solution to (1.4) with (1.2), (1.3) in $(0,T)\times\Omega$ if i), ii), iii), and

iv)′ for any $\phi\in C_0^0([0,T);L^2(\Omega))\cap\mathscr{X}_0$,

$$\int_0^T\{\bar{J}(u+\phi,\Omega)-\bar{J}(u,\Omega)\}dt\ge\int_0^T\int_\Omega u_t\phi_t(t,x)dxdt+\int_\Omega v_0(x)\phi(0,x)dx.$$

Seemingly the main theorem of [**11**] asserts that the function u satisfies condition iii); however this condition is in fact implicitly assumed in the assumption of the energy conservation law (compare to [**12**, Section 1]). In [**12**] Dirichlet condition (1.3) is weakened by replacing $\bar{J}(u,\Omega)$ with $\bar{J}(u,\overline{\Omega})$. Namely, in [**12**] a solution is defined as in the following and in this article we employ this definition. Remark that this weaker formulation of (1.3) makes the condition of energy conservation law weaker. In [**12**] it is proved that the same result as in [**11**] still holds even if we only suppose this weaker condition.

Definition C.3. A function u is said to be a weak solution to (1.4) with (1.2), (1.3) in $(0,T)\times\Omega$ if and only if i), ii), and

v) for any $\phi \in C_0^0([0,T);L^2(\Omega))\cap\mathscr{X}$,

$$\int_0^T \{\bar{J}(u+\phi,\overline{\Omega})-\bar{J}(u,\overline{\Omega})\}dt \geq \int_0^T\int_\Omega u_t\phi_t(t,x)dxdt+\int_\Omega v_0(x)\phi(0,x)dx.$$

Further in [**12**] it is proved that, if $\partial\Omega$ is of C^2 class, Definition C.3 is equivalent to

Definition C.4. A function u is said to be a weak solution to (1.4) with (1.2), (1.3) in $(0,T)\times\Omega$ if and only if i), ii), v)$_1{}'$ (= iv) of Definition C.1), and

v)$_2{}'$ for any $\psi\in C_0^1([0,T))$,

$$\begin{aligned}\int_0^T \Bigg\{ &-\int_\Omega u_t(\psi'(t)u+\psi(t)u_t)dx+\psi(t)\int_{\partial^* E_{u(t,\cdot)}\cap\Omega\times\boldsymbol{R}} |\nu'_{E_{u(t,\cdot)}}|^2 d\mathscr{H}^n \\ &+\psi(t)\int_{\partial\Omega}|\gamma u|d\mathscr{H}^{n-1}\Bigg\}dt=\psi(0)\int_\Omega v_0(x)u_0(x)dx.\end{aligned}$$

Finally we remark that, looking at the proof of the equivalence between Definitions C.3 and C.4 carefully, we find that it is obtained by testing only smooth functions and u itself. Namely, if $\partial\Omega$ is of C^2 class, Definitions C.3 and C.4 are also equivalent to

Definition C.5. A function u is said to be a weak solution to (1.4) with (1.2), (1.3) in $(0,T)\times\Omega$ if and only if i), ii),

v)$_1{}''$ for any $\phi\in C_0^1([0,T)\times\Omega)$,

$$\int_0^T\left\{-\int_\Omega u_t\phi_t(t,x)dx+\int_\Omega\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\nabla\phi(t,x)dx\right\}dt=\int_\Omega v_0(x)\phi(0,x)dx$$

v)$_2{}''$ for any $\psi\in C_0^1([0,T))$,

$$\int_0^T \left\{ - \int_\Omega u_t(\psi'(t)u + \psi(t)u_t)dx + \psi(t) \int_\Omega \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}}dx + \psi(t)|D^s u|(\Omega) \right.$$
$$\left. + \psi(t) \int_{\partial\Omega} |\gamma u| d\mathscr{H}^{n-1} \right\} dt = \psi(0) \int_\Omega v_0(x)u_0(x)dx.$$

Implication relations among these definitions are as follows:

$$\text{Definition C.2} \begin{array}{c} \Longrightarrow \\ \Longrightarrow \end{array} \begin{array}{c} \text{Definition C.1} \\ \text{Definition C.3} \end{array} \begin{array}{c} \Longrightarrow \\ \Longrightarrow \end{array} \text{Definition C.4} \Longrightarrow \text{Definition C.5}\ .$$

If $\partial\Omega$ is of C^2 class, the converses except for C.3 $\Rightarrow$ C.2 and C.4 $\Rightarrow$ C.1 also hold.

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