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Exponential growth of the numbers of particles for branching symmetric α-stable processes

Dedicated to Professor Shinzo Watanabe on the occasion of his seventieth birthday

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Abstract. We study the exponential growth of the numbers of particles for a branching symmetric α-stable process in terms of the principal eigenvalue of an associated Schrödinger operator. Here the branching rate and the branching mechanism can be state-dependent. In particular, the branching rate can be a measure belonging to a certain Kato class and is allowed to be singular with respect to the Lebesgue measure. We calculate the principal eigenvalues and give some examples.

1. Introduction.

In [**26**], we gave a criterion for extinction or local extinction of a branching symmetric α-stable process in terms of the principal eigenvalue for an associated time changed process of the symmetric α-stable process. We also proved in [**26**] that, if the branching process does not extinct, then the number of particles in the whole space at time t goes to infinity as $t \to \infty$ with positive probability. Our purpose in this paper is to study the exponential growth of the numbers of particles in the whole space and in every relatively compact open set by using the principal eigenvalue and the ground state of an associated Schrödinger operator. We also calculate the principal eigenvalues of the Schrödinger operators and apply our results to branching Brownian motions and branching symmetric α-stable processes.

Sevast'yanov [**25**] and S. Watanabe [**34**] considered the extinction problem for a branching Brownian motion on a bounded domain with state-independent branching rate and branching mechanism. They then gave a criterion for

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extinction by the principal eigenvalue of the Dirichlet Laplacian. Furthermore, S. Watanabe [**35**] established a limit theorem for a branching diffusion process by using the L^2-martingale theory (see also Ogura [**21**]). Engländer and Kyprianou [**15**] gave a criterion for local extinction of a branching diffusion process by the generalized principal eigenvalue of an associated Schrödinger operator (see [**22**] for the definition of generalized principal eigenvalues). They also studied the exponential growth of the number of particles in every relatively compact open set.

Here we consider more general branching processes than those studied in [**15**] and [**35**]. In particular, we discuss the exponential growth for a branching process whose motion component is a symmetric α-stable process and whose branching rate is a measure. Indeed, we allow the branching rate to be singular with respect to the Lebesgue measure. More precisely, let $\mathbf{M}^{\alpha} = (X_t, P_x)$ be the symmetric α-stable process on $\boldsymbol{R}^d$ and $\mathbf{M}^D$ the absorbing symmetric α-stable process on an open set D in $\boldsymbol{R}^d$. Let $\overline{\mathbf{M}^D} = (\mathbf{X}_t, \mathbf{P}_x)$ be a branching symmetric α-stable process such that each particle moves independently according to the law of $\mathbf{M}^D$. Denote by μ the branching rate measure, that is, the positive continuous additive functional A_t^{μ} in the Revuz correspondence to μ determines the distribution of the first splitting time of each particle. We assume that the branching rate μ is *Green-tight* (in notation, $\mu \in \mathscr{K}_{\infty}^{D}$). See Section 2 for the definition of $\mathscr{K}_{\infty}^{D}$. Let $\{p_n(x)\}_{n \geq 0}$ be the branching mechanism, that is, a particle splits into n particles at branching site $x \in D$ with probability $p_n(x)$. Further, let $Q(x) := \sum_{n=0}^{\infty} n p_n(x)$ be the expected number of particles which are born at branching site $x \in D$. We now define

$$\mathscr{L}^{(Q-1)\mu, D} := \mathscr{L}^{D} + (Q-1)\mu,$$

where $\mathscr{L}^D$ is the $L^2(D)$-infinitesimal generator of $\mathbf{M}^D$. Denote by λ the bottom of the spectrum of $\mathscr{L}^{(Q-1)\mu, D}$ and by h the corresponding ground state. Let

$$M_t := e^{\lambda t} \sum_{i=1}^{Z_t} h\big(\mathbf{X}_t^i\big), \quad t \geq 0,$$

where Z_t denotes the total number of particles at time t and $\mathbf{X}_t^i$, $1 \leq i \leq Z_t$, is the position of the ith particle at time t. Then, under the assumption that λ is negative, we prove that M_t is a square integrable martingale, that is, $\sup_{0<t<\infty} \mathbf{E}_x[M_t^2] < \infty$ (Lemma 3.4). As a result, the limit $M_{\infty} := \lim_{t \to \infty} M_t$ exists $\mathbf{P}_x$-a.s. and in $L^1(\mathbf{P}_x)$. Furthermore, we show that the limit M_{∞} is positive $\mathbf{P}_x$-a.s. on the event that the branching process survives (Theorem 3.7). This result says that Z_t grows exponentially at least with rate $-\lambda$. We also show that the number of particles in every relatively compact open set grows exponentially with rate $-\lambda$ (Theorem 3.12).

A crucial point is the square integrability of M_t. We now explain how to prove it. By the definition of the branching symmetric α-stable process, it follows that

$$\begin{aligned}\mathbf{E}_x\left[M_t^2\right] = e^{2\lambda t} E_x\left[\exp\left(A_t^{(Q-1)\mu}\right) h(X_t)^2; t < \tau_D\right] \\ + E_x\left[\int_0^{t\wedge\tau_D} \exp\left(2\lambda s + A_s^{(Q-1)\mu}\right) h(X_s)^2 \, dA_s^{R\mu}\right],\end{aligned} \tag{1.1}$$

where $A_t^{(Q-1)\mu} := A_t^{Q\mu} - A_t^{\mu}$ is the continuous additive functional, τ_D is the exit time of $\mathbf{M}^\alpha$ from D and $R(x) := \sum_{n=0}^{\infty} n(n-1)p_n(x)$. To show the uniform boundedness of the second term, we use a criterion for the gaugeability of measures (see Z.-Q. Chen [**6**], Takeda [**28**] and Takeda and Uemura [**32**]): for a signed measure $\mu = \mu^+ - \mu^- \in \mathscr{K}_\infty^D - \mathscr{K}_\infty^D$, it holds that

$$\sup_{x\in D} E_x\left[\exp\left(A_{\tau_D}^{\mu}\right)\right] < \infty$$

if and only if the principal eigenvalue for the time changed process of the $\exp\left(-A_t^{\mu^-}\right)$-subprocess with respect to μ^+ is greater than one (see also Theorem 2.2 below). Applying this result to the second term of (1.1), we establish the square integrability of M_t.

We note that Theorems 3.7 and 3.12 are applicable to more general branching symmetric Hunt processes under some assumptions (see Assumptions 3.15 and 3.16 below). For instance, they are applicable to branching Brownian motions on Riemannian manifolds and branching stable-like processes on $\boldsymbol{R}^d$ in the sense of Z.-Q. Chen and Kumagai [**8**] (see Remark 3.17).

2. Preliminaries.

2.1. Symmetric Hunt processes and two classes of measures.

Let X be a locally compact separable metric space and X_Δ its one point compactification. Let m be a positive Radon measure on X with full support. Let $\mathbf{M} = (\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, X_t, P_x, \zeta)$ be an m-symmetric Hunt process on X, where $\{\mathscr{F}_t\}_{t\geq 0}$ is the minimal admissible filtration, $\{\theta_t\}_{t\geq 0}$ is the time-shift operator satisfying $X_t \circ \theta_s = X_{t+s}$ identically for $s, t \geq 0$, and ζ is the lifetime, $\zeta = \inf\{t > 0 : X_t = \Delta\}$. We denote by p_t the Markovian transition semigroup of $\mathbf{M}$ given by

$$p_t f(x) = E_x[f(X_t)].$$

Let $\mathscr{S}$ be the set of smooth measures on X (see [**16**, p.80] for definition). It is then known in Theorem 5.1.4 of [**16**] that smooth measures and positive continuous additive functionals are in one to one correspondence under the so-called Revuz correspondence as follows: if we denote by A_t^μ the positive continuous additive functional corresponding to $\mu \in \mathscr{S}$, then for any γ-excessive function h ($\gamma \geq 0$) and any positive Borel measurable function f,

$$\lim_{t \downarrow 0} \frac{1}{t} \int_X E_x\left[\int_0^t f(X_s)\, dA_s^\mu\right] h(x)\, m(dx) = \int_X f(x) h(x)\, \mu(dx).$$

Let τ_t^μ be the right continuous inverse of A_t^μ,

$$\tau_t^\mu = \inf\Big\{s > 0 : A_{s\wedge\zeta}^\mu > t\Big\},$$

and let F^μ be the fine support of the measure μ defined by

$$F^\mu = \big\{x \in X : P_x\big(\tau_0^\mu = 0\big) = 1\big\}. \tag{2.1}$$

In the sequel, we assume that the transition density of $\mathbf{M}$ is absolutely continuous with respect to the measure m and denote by $p_t(x,y)$ the integral kernel of p_t,

$$p_t f(x) = \int_X p_t(x,y) f(y)\, m(dy).$$

Let $G_\alpha(x,y)$ be the α-resolvent density of $\mathbf{M}$,

$$G_\alpha(x,y) = \int_0^\infty e^{-\alpha t} p_t(x,y)\, dt, \quad \alpha > 0.$$

If $\mathbf{M}$ is transient, then the Green function

$$G_0(x,y) = \int_0^\infty p_t(x,y)\, dt$$

exists for $x \neq y$, and we put $G(x,y) = G_0(x,y)$.

We now introduce two classes of measures in $\mathscr{S}$.

DEFINITION 2.1.

(i) A positive smooth Radon measure on X is said to be in $\mathscr{K}_\infty(G_\alpha)$, if for any $\varepsilon > 0$, there exist a compact set $K \subset X$ and a positive constant $\delta > 0$ such that

$$\sup_{x\in X}\int_{X\setminus K} G_\alpha(x,y)\,\mu(dy) < \varepsilon,$$

and for all measurable sets $B \subset K$ with $\mu(B) < \delta$,

$$\sup_{x\in X}\int_{B} G_\alpha(x,y)\,\mu(dy) < \varepsilon.$$

Further, the class $\mathscr{K}_\infty$ is defined by

$$\mathscr{K}_\infty = \begin{cases} \mathscr{K}_\infty(G), & \mathbf{M} \text{ is transient} \\ \mathscr{K}_\infty(G_1), & \mathbf{M} \text{ is recurrent.} \end{cases}$$

(ii) A positive smooth Radon measure μ on X is said to be in $\mathscr{S}_\infty(G_\alpha)$, if for any $\varepsilon > 0$, there exist a compact set $K \subset X$ and a positive constant $\delta > 0$ such that

$$\sup_{(x,z)\in X\times X\setminus\triangle}\int_{X\setminus K} \frac{G_\alpha(x,y)G_\alpha(y,z)}{G_\alpha(x,z)}\,\mu(dy) < \varepsilon,$$

and for all measurable sets $B \subset K$ with $\mu(B) < \delta$,

$$\sup_{(x,z)\in X\times X\setminus\triangle}\int_{B} \frac{G_\alpha(x,y)G_\alpha(y,z)}{G_\alpha(x,z)}\,\mu(dy) < \varepsilon,$$

where $\triangle = \{(x,y) \in X \times X : x \neq y\}$. If $\mathbf{M}$ is transient, then $\mathscr{S}_\infty(G)$ is simply denoted by $\mathscr{S}_\infty$.

In the reminder of this subsection, we assume that $\mathbf{M}$ is transient. Then it holds that $\mathscr{S}_\infty \subset \mathscr{K}_\infty$ by Corollary 3.1 of [**11**]. It is also known in Proposition 2.2 of [**6**] that any measure μ in $\mathscr{K}_\infty$ is Green bounded, that is,

$$\sup_{x\in X} E_x\left[A^\mu_\zeta\right] = \sup_{x\in X}\int_X G(x,y)\,\mu(dy) < \infty. \tag{2.2}$$

Let μ be a signed measure on X which can be decomposed into $\mu = \mu^+ - \mu^-$ for some μ^+, $\mu^- \in \mathscr{K}_\infty$. Then the measure μ is said to be gaugeable if

$$\sup_{x\in X} E_x\Big[\exp\Big(A^{\mu}_{\zeta}\Big)\Big] < \infty,$$

where $A^{\mu}_t := A^{\mu^+}_t - A^{\mu^-}_t$. Let $(\mathscr{E},\mathscr{F})$ be the regular Dirichlet form on $L^2(X;m)$ generated by $\mathbf{M}$. It is then known in Theorem 2.1.3 of [**16**] that each $f\in\mathscr{F}$ admits a quasi continuous m-version (see p.67 of [**16**] for the definition of the quasi continuity). In the sequel, we always assume that each $f\in\mathscr{F}$ is quasi continuous. Define

$$\check{\lambda}(\mu) = \inf\bigg\{\mathscr{E}(f,f) + \int_X f^2\, d\mu^- : f\in\mathscr{F},\quad \int_X f^2\, d\mu^+ = 1\bigg\}.$$

Then the Dirichlet principle yields that $\check{\lambda}(\mu)$ is the bottom of the spectrum for the time changed process of the $\exp\big(-A^{\mu^-}_t\big)$-subprocess of $\mathbf{M}$ with respect to the positive continuous additive functional $A^{\mu^+}_t$. When we specify the positive and negative parts of the measure μ, we denote $\check{\lambda}(\mu)$ by $\check{\lambda}(\mu^+,\mu^-)$.

Theorem 2.2 ([**6**, Corollary 2.9, Theorem 5.1]). *Suppose that a signed measure μ on X can be decomposed into $\mu=\mu^+-\mu^-$ for some $\mu^+,\mu^-\in\mathscr{K}_\infty$. Then the following conditions are equivalent*:
(i) *The measure μ is gaugeable*;
(ii) $\check{\lambda}(\mu^+,\mu^-) > 1$;
(iii) $\sup_{x\in X} E_x\Big[\int_0^\zeta \exp(A^{\mu}_t)\, dA^{\nu}_t\Big] < \infty$ *for any $\nu\in\mathscr{K}_\infty$.*

Proof. The implications (i) $\Leftrightarrow$ (ii) and (iii) $\Rightarrow$ (ii) are already proved in [**6**, Corollary 2.9, Theorem 5.1]. We now show the implication (ii) $\Rightarrow$ (iii) in a similar way to that yielding Proposition 3.2 of [**7**]. Let μ be a measure on X which can be decomposed into $\mu=\mu^+-\mu^-$ for some $\mu^+,\mu^-\in\mathscr{K}_\infty$ and assume that $\check{\lambda}(\mu^+,\mu^-)>1$. Since

$$\check{\lambda}(p\mu^+,p\mu^-) \ge \check{\lambda}(p\mu^+,\mu^-) = \frac{1}{p}\,\check{\lambda}(\mu^+,\mu^-)$$

for any $p>1$, we can take $p>1$ so that $\check{\lambda}(p\mu^+,p\mu^-)>1$ and the conjugate component of p is a positive integer. We fix such $p>1$ and denote its conjugate component by q, that is, $q\ge 2$ is the positive integer such that $1/p+1/q=1$. Then the Hölder inequality implies that for any measure $\nu\in\mathscr{K}_\infty$,

$$E_x\bigg[\int_0^\zeta \exp(A^{\mu}_t)\, dA^{\nu}_t\bigg] \le E_x\bigg[\sup_{0\le t\le\zeta}\big(\exp(A^{p\mu}_t)\big)\bigg]^{1/p} E_x\Big[\Big(A^{\nu}_{\zeta}\Big)^q\Big]^{1/q}. \tag{2.3}$$

As it holds that

$$\sup_{x\in X} E_x\left[\left(A^{\nu}_{\zeta}\right)^q\right] \leq q!\left(\sup_{x\in X} E_x\left[A^{\nu}_{\zeta}\right]\right)^q$$

(see Lemma 3.7 of [**12**]), we have $\sup_{x\in X} E_x\left[\left(A^{\nu}_{\zeta}\right)^q\right]^{1/q} < \infty$. A direct calculation yields that

$$\sup_{0\leq t\leq\zeta}(\exp(A^{p\mu}_t)) \leq \int_0^{\zeta} \exp(A^{p\mu}_t)dA^{p\tilde{\mu}^+}_t + 1,$$

where $\tilde{\mu}^+ - \tilde{\mu}^-$ is the Jordan decomposition of the measure μ. Since the measures $\tilde{\mu}^+$ and $\tilde{\mu}^-$ belong to the class $\mathscr{K}_\infty$ respectively, and the condition that $\check{\lambda}(p\mu^+, p\mu^-) > 1$ is equivalent to that $\check{\lambda}(p\tilde{\mu}^+, p\tilde{\mu}^-) > 1$ by [**32**, Lemma 3.1], we obtain

$$\sup_{x\in X} E_x\left[\int_0^{\zeta} \exp(A^{p\mu}_t)dA^{p\tilde{\mu}^+}_t\right] < \infty$$

by [**6**, Corollary 2.9, Theorem 5.1]. Therefore, the right hand side of (2.3) is bounded, which shows the implication (ii) ⇒ (iii). □

2.2. Branching symmetric Hunt processes.

Following [**18**], [**19**] and [**34**], we introduce the notion of branching symmetric Hunt processes. Let $\{p_n(x)\}_{n\geq 0}$, $x \in X$, be a sequence such that

$$0 \leq p_n(x) \leq 1 \quad \text{and} \quad \sum_{n=0}^{\infty} p_n(x) = 1.$$

For $\mu \in \mathscr{S}$, we denote by Z the random variable of the exponential distribution with rate A^{μ}_t:

$$P_x(t < Z \mid \mathscr{F}_\infty) = \exp(-A^{\mu}_t).$$

A particle of the branching symmetric Hunt process starts at $x \in X$ according to the law P_x. If $\zeta < Z$, then it dies at time ζ. On the other hand, if $Z < \zeta$, then it splits into n particles with probability $p_n(X_{Z-})$ at time Z. Then each of these particles starts at X_{Z-} independently according to the law $P_{X_{Z-}}$. Let $X^{(0)} = \{\Delta\}$ and $X^{(1)} = X$. Define the equivalent relation $\sim$ on $X^n = \underbrace{X\times\cdots\times X}_{n}$ as follows; let $\mathbf{x}^n = (x^1, x^2, x^3, \cdots, x^n)$, $\mathbf{y}^n = (y^1, y^2, y^3, \cdots, y^n) \in X^n$. If there exists a permutation σ on $\{1, 2, 3, \cdots, n\}$ such that $y^i = x^{\sigma(i)}$ for all i, then it is denoted by $\mathbf{x}^n \sim \mathbf{y}^n$. Let $X^{(n)} = X^n/\sim$ and $\mathbf{X} = \bigcup_{n=0}^{\infty} X^{(n)}$. When the branching process consists of n particles at time t, they determine a point in $X^{(n)}$. Hence it defines a branching symmetric Hunt process $\overline{\mathbf{M}} = (\mathbf{X}_t, \mathbf{P}_{\mathbf{x}}, \mathscr{G}_t)$ on $\mathbf{X}$ with motion component $\mathbf{M}$, branching rate μ and branching mechanism $\{p_n(x)\}_{n\geq 0}$.

Let T be the first splitting time of $\overline{\mathbf{M}}$:

$$\begin{aligned}\mathbf{P}_x(t < T \mid \sigma(X)) &= P_x(t < Z \mid \mathscr{F}_\infty) \\ &= \exp(-A_t^\mu).\end{aligned} \tag{2.4}$$

Denote by Z_t the total number of particles of $\overline{\mathbf{M}}$ at time t, that is,

$$Z_t = n \quad \text{if} \quad \mathbf{X}_t = (\mathbf{X}_t^1, \mathbf{X}_t^2, \mathbf{X}_t^3, \cdots, \mathbf{X}_t^n) \in X^{(n)}.$$

Let

$$e_0 = \inf\{t > 0 : Z_t = 0\}.$$

Then e_0 is called the extinction time of $\overline{\mathbf{M}}$. Let $u_e(x) = \mathbf{P}_x(e_0 < \infty) = \mathbf{P}_x(\lim_{t\to\infty} Z_t = 0)$. We then say that $\overline{\mathbf{M}}$ extincts if $u_e \equiv 1$ on X. Denote by $Z_t(A)$ the number of particles in a set $A \subset X$ at time t and let

$$L_A = \sup\{t > 0 : Z_t(A) > 0\}.$$

Let $u_A(x) = \mathbf{P}_x(L_A < \infty) = \mathbf{P}_x(\lim_{t\to\infty} Z_t(A) = 0)$. We then say that $\overline{\mathbf{M}}$ extincts locally if $u_A \equiv 1$ on X for every relatively compact open set A in X.

2.3. Symmetric $\boldsymbol{\alpha}$-stable processes.

Let $\mathbf{M}^\alpha = (\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, X_t, P_x)$, $0 < \alpha \leq 2$, be a symmetric α-stable process on $\boldsymbol{R}^d$ and denote by $(\mathscr{E}^\alpha, \mathscr{F}^\alpha)$ the Dirichlet form on $L^2(\boldsymbol{R}^d)$ generated by $\mathbf{M}^\alpha$. If $\alpha = 2$, then $\mathbf{M}^2$ is the Brownian motion on $\boldsymbol{R}^d$ and $(\mathscr{E}^2, \mathscr{F}^2) = (\mathbf{D}/2, H^1(\boldsymbol{R}^d))$, where $H^1(\boldsymbol{R}^d)$ is the Sobolev space of order one and $\mathbf{D}$ is the Dirichlet integral,

$$\mathbf{D}(f,f) = \int_{\boldsymbol{R}^d} |\nabla f|^2\, dx, \quad f \in H^1(\boldsymbol{R}^d).$$

On the other hand, if $0 < \alpha < 2$, then $\mathbf{M}^\alpha$ is a pure jump process and

$$\begin{aligned}\mathscr{E}^\alpha(f,f) &= \mathscr{A}(d,\alpha) \iint_{\boldsymbol{R}^d\times\boldsymbol{R}^d\setminus\triangle} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\, dxdy \\ \mathscr{F}^\alpha &= \left\{ f \in L^2(\boldsymbol{R}^d) : \iint_{\boldsymbol{R}^d\times\boldsymbol{R}^d\setminus\triangle} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\, dxdy < \infty \right\},\end{aligned}$$

where

$$\mathscr{A}(d,\alpha) = \frac{\alpha 2^{\alpha-3}\Gamma\left(\frac{d+\alpha}{2}\right)}{\pi^{d/2}\Gamma\left(1-\frac{\alpha}{2}\right)} \quad \text{and} \quad \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt.$$

If $d > \alpha$, then $\mathbf{M}^\alpha$ is transient and the Green function $G(x,y)$ is given by

$$G(x,y) = \frac{2^{1-\alpha}\Gamma\left(\frac{d-\alpha}{2}\right)}{\pi^{d/2}\Gamma\left(\frac{\alpha}{2}\right)}|x-y|^{\alpha-d}.$$

Let $\mathbf{M}^D = (X_t^D,\, P_x^D)$ be the absorbing symmetric α-stable process on an open set $D \subset \boldsymbol{R}^d$: set

$$X_t^D = \begin{cases} X_t, & 0 \le t < \tau_D \\ \Delta, & t \ge \tau_D, \end{cases}$$

where τ_D is the exit time of $\mathbf{M}^\alpha$ from D, that is, $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Then the Dirichlet form $(\mathscr{E}^D, \mathscr{F}^D)$ of $\mathbf{M}^D$ is the following:

$$\mathscr{F}^D = \{f \in \mathscr{F}^\alpha : f = 0 \quad \text{q.e. on } D^c\}$$

$$\mathscr{E}^D(f,f) = \begin{cases} \dfrac{1}{2}\displaystyle\int_D |\nabla f|^2\,dx, & \alpha = 2 \\ \dfrac{1}{2}\mathscr{A}(d,\alpha)\displaystyle\iint_{D\times D\setminus\triangle} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\,dxdy & \\ \quad +\mathscr{A}(d,\alpha)\displaystyle\int_D f(x)^2\left(\int_{D^c}\frac{1}{|x-y|^{d+\alpha}}\,dy\right)dx, & 0<\alpha<2 \end{cases}$$

([**16**, Theorem 4.4.2, Example 4.4.1]). Here q.e. is an abbreviation for quasi everywhere (see [**16**, p.66] for definition). Let p_t^D be the Markovian transition semigroup of $\mathbf{M}^D$ given by

$$p_t^D f(x) = E_x^D\left[f\left(X_t^D\right)\right].$$

Then by definition,

$$p_t^D f(x) = E_x[f(X_t) : t < \tau_D].$$

We denote by $p_t^D(x,y)$ the integral kernel of p_t^D,

$$p_t^D f(x) = \int_D p_t^D(x,y) f(y)\, dy.$$

Let $G_\beta^D(x,y)$ be the β-resolvent density of $\mathbf{M}^D$,

$$G_\beta^D(x,y) = \int_0^\infty e^{-\beta t} p_t^D(x,y)\, dt, \quad \beta > 0.$$

If $\mathbf{M}^D$ is transient, then the Green function

$$G_0^D(x,y) = \int_0^\infty p_t^D(x,y)\, dt$$

exists for $x \neq y$, and we put $G^D(x,y) = G_0^D(x,y)$.

Let μ be a signed measure on D which can be decomposed into $\mu = \mu^+ - \mu^-$ for some μ^+, $\mu^- \in \mathscr{K}_\infty^D$, where $\mathscr{K}_\infty^D$ denotes the class $\mathscr{K}_\infty$ associated with $\mathbf{M}^D$. Define

$$\mathscr{E}^{\mu,D}(f,f) = \mathscr{E}^D(f,f) - \int_D f^2\, d\mu, \quad f \in \mathscr{F}^D.$$

Since any measure in $\mathscr{K}_\infty^D$ charges no set of zero capacity by [**1**, Theorem 3.3], the form $(\mathscr{E}^{\mu,D}, \mathscr{F}^D)$ is well-defined. Moreover, it follows from [**1**, Theorem 4.1] that $(\mathscr{E}^{\mu,D}, \mathscr{F}^D)$ is a lower semibounded and closed form. Denote by $p_t^{\mu,D}$ the $L^2(D)$-semigroup generated by $(\mathscr{E}^{\mu,D}, \mathscr{F}^D)$. Then the Feynman-Kac formula shows that

$$p_t^{\mu,D} f(x) = E_x[\exp(A_t^\mu) f(X_t); t < \tau_D].$$

We now note that $p_t^D f$ is a bounded and continuous function for any $f \in \mathscr{B}_b(D)$ and $\|p_t^D\|_{1,\infty} < \infty$ for any $t > 0$, where $\mathscr{B}_b(D)$ stands for the set of bounded Borel measurable functions on D and $\|\cdot\|_{p,q}$ denotes the operator norm from $L^p(D)$ to $L^q(D)$. We then obtain the following from [**1**]:

THEOREM 2.3. *Suppose that a signed measure μ on D can be decomposed into $\mu = \mu^+ - \mu^- \in \mathscr{K}_\infty^D - \mathscr{K}_\infty^D$.*
(i) *For any $f \in \mathscr{B}_b(D)$, $p_t^{\mu,D} f$ is a bounded and continuous function on D.*
(ii) *For any $t > 0$, it holds that $\left\|p_t^{\mu,D}\right\|_{p,q} < \infty$ for any $1 \le p \le q \le \infty$.*

Theorem 2.3 (i) assures the existence of the integral kernel $p_t^{\mu,D}(x,y)$ of $p_t^{\mu,D}$,

$$p_t^{\mu,D} f(x) = \int_D p_t^{\mu,D}(x,y) f(y)\, dy. \tag{2.5}$$

Let $\sigma(\mathscr{E}^{\mu,D})$ be the totality of the spectrum of the self-adjoint operator associated with $(\mathscr{E}^{\mu,D}, \mathscr{F}^D)$. Set $\mathscr{E}_1^D(f,f) = \mathscr{E}^D(f,f) + \int_D f^2\, dx$. We can then show that the embedding from $(\mathscr{F}^D, \mathscr{E}_1^D)$ to $L^2(D;\nu)$ is compact for any $\nu \in \mathscr{K}_\infty^D$ by the same way as in Theorem 2.8 of [**29**]. Hence, if we put

$$\lambda(D) = \inf\left\{\mathscr{E}^D(f,f) : f \in C_0^\infty(D),\ \int_D f^2\, dx = 1\right\},$$

then, by the Friedrichs theorem [**20**, 2.5.4, Lemma 1], the spectrum in $\sigma(\mathscr{E}^{\mu,D})$ less than $\lambda(D)$ consists of isolated eigenvalues with finite multiplicities.

In the remainder of this section, we fix a signed measure μ on D which can be decomposed into $\mu = \mu^+ - \mu^- \in \mathscr{K}_\infty^D - \mathscr{K}_\infty^D$. Denote by $\lambda(\mu; D)$ the bottom of $\sigma(\mathscr{E}^{\mu,D})$:

$$\lambda(\mu; D) = \inf\left\{\mathscr{E}^{\mu,D}(f,f) : f \in C_0^\infty(D),\ \int_D f^2\, dx = 1\right\}.$$

Assume that $\lambda := \lambda(\mu; D) < 0$, that is, λ is the principal eigenvalue. Then, since the ground state h satisfies $h = e^{\lambda t} p_t^{\mu,D} h$ on D, we see that h is bounded and continuous by Theorem 2.3, and strictly positive by combining the irreducibility of $\mathbf{M}^D$ with the strict positivity of $\exp(A_t^\mu)$. Let $G_\beta^{\mu^-,D}(x,y)$ be the β-resolvent density of the $\exp(-A_t^{\mu^-})$-subprocess of $\mathbf{M}^D$,

$$\int_D G_\beta^{\mu^-,D}(x,y) f(y)\, dy = E_x\left[\int_0^{\tau_D} \exp\left(-\beta t - A_t^{\mu^-}\right) f(X_t)\, dt\right].$$

We can then see in a similar way to Section 4 of [**31**] that

$$h(x) = \int_D G_{-\lambda}^{\mu^-,D}(x,y) h(y)\, \mu^+(dy). \tag{2.6}$$

Remark 2.4. Assume that $\mathbf{M}^D$ is transient. We now show that, if the support of a measure $\nu \in \mathscr{K}_\infty^D$ is compact, then ν belongs to $\mathscr{S}_\infty\left(G_\beta^D\right)$ for any $\beta > 0$. Let $\nu \in \mathscr{K}_\infty^D$ be a measure with compact support and put $F = \mathrm{supp}[\nu]$. Let

O be a finite union of bounded $C^{1,1}$ domains in D such that $F \subset O$. Here we say that a set A is a $C^{1,1}$ domain, if for any $x \in \partial A$, there exists a positive constant $r > 0$ such that $B_x(r) \cap \partial A$ is the graph of a function whose first derivatives are Lipschitz continuous, where $B_x(r) = \{y \in \boldsymbol{R}^d : |x - y| \leq r\}$. Since $G^D(x, y) \leq G(x, y)$, Corollary 1.3 of [**10**] implies that

$$G^O(x, y) \leq G^D(x, y) \leq CG^O(x, y)$$

for any $x, y \in F$, where $C \geq 1$ is a positive constant depending on F. Furthermore, since

$$\sup_{(x,z)\in O\times O\setminus\triangle} \int_O \frac{G^O(x, y)G^O(y, z)}{G^O(x, z)}\, dy < \infty$$

by Theorem 1.8 of [**10**], it follows from Theorem 5.3 of [**6**] and Lemma 3.3 of [**28**] that

$$G^O_\beta(x, y) \leq G^O(x, y) \leq CG^O_\beta(x, y)$$

for any $x, y \in O$, which leads us to that

$$G^D_\beta(x, y) \leq G^D(x, y) \leq CG^D_\beta(x, y)$$

for any $x, y \in F$. Here the constants C above are different and depend on β, respectively. Therefore, for any nonnegative Borel function f on D,

$$\begin{aligned}
&\sup_{(x,z)\in D\times D\setminus\triangle} \int_D \frac{G^D_\beta(x, y)G^D_\beta(y, z)}{G^D_\beta(x, z)} f(y)\, \nu(dy) \\
&= \sup_{(x,z)\in F\times F\setminus\triangle} \int_F \frac{G^D_\beta(x, y)G^D_\beta(y, z)}{G^D_\beta(x, z)} f(y)\, \nu(dy) \qquad (2.7)\\
&\leq C \sup_{(x,z)\in F\times F\setminus\triangle} \int_F \frac{G^D(x, y)G^D(y, z)}{G^D(x, z)} f(y)\, \nu(dy).
\end{aligned}$$

Here we note that the following $3G$-inequality holds locally for $G^D(x, y)$:

$$\frac{G^D(x, y)G^D(y, z)}{G^D(x, z)} \leq C(G^D(x, y) + G^D(y, z)), \quad (x, z) \in F \times F \setminus \triangle,$$

where C is a constant depending on F. Thereby the right hand side of (2.7) is not greater than

$$2C \sup_{x \in F} \int_F G^D(x,y) f(y)\, \nu(dy) = 2C \sup_{x \in D} \int_D G^D(x,y) f(y)\, \nu(dy),$$

which shows that ν belongs to $\mathscr{S}_\infty\left(G^D_\beta\right)$.

Let μ be a signed measure on D which can be decomposed into $\mu = \mu^+ - \mu^- \in \mathscr{K}^D_\infty - \mathscr{K}^D_\infty$ such that the supports of μ^+ and μ^- are compact. Assume that $\lambda := \lambda(\mu; D) < 0$ and denote by h the corresponding ground state. Since μ^+ and μ^- belong to $\mathscr{S}_\infty\left(G^D_{-\lambda}\right)$ as discussed above, we can show that, by the same way as in Section 4 of [**31**],

$$C^{-1} G^D_{-\lambda}(o,x) \le h(x) \le C G^D_{-\lambda}(o,x), \quad x \in D \setminus K \tag{2.8}$$

for a compact set $K \subset D$ and a fixed point $o \in K$, where $C \ge 1$ is a positive constant depending on K.

REMARK 2.5. Let μ be a measure belonging to $\mathscr{K}^D_\infty$ and let O be a finite union of bounded $C^{1,1}$ domains in D. Then there exists a positive constant $C = C(O,\alpha) > 1$ such that

$$\frac{G^O(x,y) G^O(y,z)}{G^O(x,z)} \le C\left(\frac{1}{|x-y|^{d-\alpha}} + \frac{1}{|y-z|^{d-\alpha}}\right), \quad x,y,z \in O$$

by [**10**, Theorem 1.6] and there exists a positive constant $C = C(D,O,\alpha) > 1$ such that

$$\frac{C^{-1}}{|x-y|^{d-\alpha}} \le G^D(x,y) \le \frac{C}{|x-y|^{d-\alpha}}, \quad x,y \in O.$$

Hence it follows that $\mu|_O \in \mathscr{S}^O_\infty$, where $\mathscr{S}^O_\infty$ denotes the class $\mathscr{S}_\infty$ associated with $\mathbf{M}^O$.

3. Exponential growth of the numbers of particles.

Let $\overline{\mathbf{M}^D} = (\mathbf{X}_t, \mathbf{P}_x, \mathscr{G}_t)$ be the branching symmetric α-stable process with motion component $\mathbf{M}^D$, branching rate $\mu \in \mathscr{K}^D_\infty$ and branching mechanism $\{p_n(x)\}_{n \ge 0}$. Let

$$Q(x) = \sum_{n=0}^{\infty} np_n(x)$$

and suppose that $\sup_{x \in D} Q(x) < \infty$. Define

$$\check{\lambda}(\mu, Q; D) = \inf\left\{\mathscr{E}^D(f,f) + \int_D f^2\, d\mu : f \in C_0^\infty(D),\ \int_D f^2 Q\, d\mu = 1\right\}.$$

Then $\check{\lambda}(\mu, Q; D)$ is the principal eigenvalue for the time changed process of the $\exp(-A_t^\mu)$-subprocess of $\mathbf{M}^D$ with respect to $A_t^{Q\mu}$. Define $\lambda(\mu, Q; D) = \lambda((Q-1)\mu; D)$, that is,

$$\lambda(\mu, Q; D) = \inf\left\{\mathscr{E}^D(f,f) - \int_D f^2(Q-1)\, d\mu : f \in C_0^\infty(D),\ \int_D f^2\, dx = 1\right\}. \quad (3.1)$$

We then see that $\lambda(\mu, Q; D) \geq 0$ if and only if $\check{\lambda}(\mu, Q; D) \geq 1$ by the same way as in Lemma 2.2 of [**31**]. We can thus rephrase Theorem 3.1 of [**26**] as follows:

Theorem 3.1. *Assume that $P_x(\tau_D < \infty) = 1$ for any $x \in D$. If the branching rate μ belongs to $\mathscr{S}_\infty^D$, then $\overline{\mathbf{M}^D}$ extincts if and only if $\lambda(\mu, Q; D) \geq 0$.*

We also proved in Lemma 3.8 of [**26**] the following:

Lemma 3.2. *Assume that $P_x(\tau_D < \infty) = 1$ for any $x \in D$. Then*

$$\{e_0 = \infty\} = \left\{\lim_{t \to \infty} Z_t = \infty\right\} \quad \mathbf{P}_x\text{-a.s.}$$

for any $x \in D$.

Lemma 3.2 says that, if the branching process $\overline{\mathbf{M}^D}$ does not extinct, then

$$\mathbf{P}_x\left(\lim_{t \to \infty} Z_t = \infty \,\middle|\, e_0 = \infty\right) = 1.$$

We first study the exponential growth of Z_t in terms of the principal eigenvalue $\lambda(\mu, Q; D)$. Let

$$R(x) = \sum_{n=1}^{\infty} n(n-1)p_n(x).$$

We now prove the following:

LEMMA 3.3. *If* $\sup_{x \in D} Q(x) < \infty$, *then*

$$\mathbf{E}_x\left[\sum_{i=1}^{Z_t} f(\mathbf{X}_t^i)\right] = E_x\left[\exp\left(A_t^{(Q-1)\mu}\right) f(X_t); t < \tau_D\right] \tag{3.2}$$

for any $f \in \mathscr{B}_b(D)$. *If* $\sup_{x \in D} R(x) < \infty$, *then*

$$\begin{aligned} \mathbf{E}_x\left[\left(\sum_{i=1}^{Z_t} f(\mathbf{X}_t^i)\right)\left(\sum_{i=1}^{Z_t} g(\mathbf{X}_t^i)\right)\right] &= E_x\left[\exp\left(A_t^{(Q-1)\mu}\right) f(X_t) g(X_t); t < \tau_D\right] \\ + E_x\left[\int_0^{t \wedge \tau_D} \exp\left(A_s^{(Q-1)\mu}\right) \mathbf{E}_{X_s}\right.&\left.\left[\sum_{i=1}^{Z_{t-s}} f(\mathbf{X}_{t-s}^i)\right] \mathbf{E}_{X_s}\left[\sum_{i=1}^{Z_{t-s}} g(\mathbf{X}_{t-s}^i)\right] dA_s^{R\mu}\right] \end{aligned} \tag{3.3}$$

for any $f, g \in \mathscr{B}_b(D)$.

PROOF. Let us denote by $Z_t(m)$ the total number of particles at time t such that each of their trajectories over time interval $[0, t]$ has m branching points, and by

$$\mathbf{X}_t(m) = \left(\mathbf{X}_t^1(m), \mathbf{X}_t^2(m), \cdots, \mathbf{X}_t^{Z_t(m)}(m)\right)$$

the positions of all such particles at time t. Define

$$Z_t(f) = \sum_{i=1}^{Z_t} f(\mathbf{X}_t^i) \quad \text{and} \quad Z_t(m; f) = \sum_{i=1}^{Z_t(m)} f(\mathbf{X}_t^i(m)),$$

respectively for $f \in \mathscr{B}_b(D)$. Then

$$Z_t(f) = \sum_{m=0}^{\infty} Z_t(m; f).$$

We first show (3.2). It follows from (2.4) that

$$\mathbf{E}_x[Z_t(0; f)] = E_x[\exp(-A_t^{\mu}) f(X_t); t < \tau_D].$$

Since each particle moves independently, the strong Markov property yields that

$$
\begin{aligned}
\mathbf{E}_x[Z_t(m;f)] &= \mathbf{E}_x[\mathbf{E}_{\mathbf{X}_T}[Z_{t-T}(m-1;f)];T\le t] \\
&= \mathbf{E}_x\left[\sum_{i=1}^{Z_T}\mathbf{E}_{\mathbf{X}_T^i}[Z_{t-T}(m-1;f)];T\le t\right] \\
&= E_x\left[\int_0^{t\wedge\tau_D}\exp\bigl(-A_s^{\mu}\bigr)\mathbf{E}_{X_s}[Z_{t-s}(m-1;f)]\,dA_s^{Q\mu}\right].
\end{aligned}
$$

Hence

$$
\mathbf{E}_x[Z_t(m;f)] = E_x\left[\exp(-A_t^{\mu})\,\frac{\left(A_t^{Q\mu}\right)^m}{m!}\,f(X_t);t<\tau_D\right]
$$

by iterations, which implies (3.2).

We next show (3.3). Denote by $Z_t^j(m)$ the total number of children of x^j at time t such that each of their trajectories over time interval $[0,t]$ has m branching points under the law $\mathbf{P}_{\mathbf{x}^{\mathbf{n}}}$, $\mathbf{x}^n=(x^1,x^2,x^3,\cdots,x^n)\in X^{(n)}$, and by

$$
\mathbf{X}_t^j(m) = \left(\mathbf{X}_t^{j,1}(m),\mathbf{X}_t^{j,2}(m),\mathbf{X}_t^{j,3}(m),\cdots,\mathbf{X}_t^{j,Z_t^j(m)}(m)\right)
$$

the positions of all such particles at time t. Let us define

$$
Z_t^j(m;f) = \sum_{i=1}^{Z_t^j(m)} f(\mathbf{X}_t^{j,i}(m)).
$$

Then the strong Markov property shows that

$$
\begin{aligned}
&\mathbf{E}_x[Z_t(m;f)Z_t(n;g)] = \mathbf{E}_x[\mathbf{E}_{\mathbf{X}_T}[Z_t(m-1;f)Z_t(n-1;g)];T\le t] \\
&= \mathbf{E}_x\Biggl[\mathbf{E}_{\mathbf{X}_T}\Biggl[\sum_{j=1}^{Z_T} Z_{t-T}^j(m-1;f)Z_{t-T}^j(n-1;g) \\
&\quad + \sum_{1\le j,k\le Z_T,j\ne k} Z_{t-T}^j(m-1;f)Z_{t-T}^k(n-1;g)\Biggr];T\le t\Biggr]
\end{aligned}
$$

for $m,n\ge 1$. Moreover, since each particle moves independently, (2.4) yields that the last term above is equal to

$$\begin{aligned}
&\mathbf{E}_x\left[\sum_{i=1}^{Z_T}\mathbf{E}_{\mathbf{X}_T^i}[Z_{t-T}(m-1;f)Z_{t-T}(n-1;g)];T\le t\right]\\
&\quad+\mathbf{E}_x\left[\sum_{1\le j,k\le Z_T,j\ne k}\mathbf{E}_{\mathbf{X}_T^j}[Z_{t-T}(m-1;f)]\mathbf{E}_{\mathbf{X}_T^k}[Z_{t-T}(n-1;f)];T\le t\right]\\
&\quad=E_x\left[\int_0^{t\wedge\tau_D}\exp\bigl(-A_s^\mu\bigr)\mathbf{E}_{X_s}[Z_{t-s}(m-1;f)Z_{t-s}(n-1;g)]\,dA_s^{Q\mu}\right]\\
&\qquad+E_x\left[\int_0^{t\wedge\tau_D}\exp\bigl(-A_s^\mu\bigr)\mathbf{E}_{X_s}[Z_{t-s}(m-1;f)]\mathbf{E}_{X_s}[Z_{t-s}(n-1;g)]\,dA_s^{R\mu}\right].
\end{aligned}$$

Here we note that

$$\mathbf{E}_x[Z_t(0;f)Z_t(0;g)]=E_x[\exp(-A_t^\mu)f(X_t)g(X_t);t<\tau_D]$$

and $Z_t(0;f)Z_t(m;g)=0$ for any $m\ge1$ by definition. Then, by iterations and Fubini's theorem,

$$\begin{aligned}
&\mathbf{E}_x[Z_t(m;f)Z_t(m;g)]=E_x\left[\exp(-A_t^\mu)\,\frac{\left(A_t^{Q\mu}\right)^m}{m!}\,f(X_t)g(X_t);t<\tau_D\right]\\
&+E_x\left[\int_0^{t\wedge\tau_D}\exp\bigl(-A_s^\mu\bigr)\sum_{k=1}^m\mathbf{E}_{X_s}[Z_{t-s}(m-k;f)]\mathbf{E}_{X_s}[Z_{t-s}(m-k;g)]\,\frac{\bigl(A_s^{Q\mu}\bigr)^{k-1}}{(k-1)!}\,dA_s^{R\mu}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{E}_x[Z_t(m;f)Z_t(n;g)]\\
&\quad=E_x\left[\int_0^{t\wedge\tau_D}\exp\bigl(-A_s^\mu\bigr)\sum_{k=1}^n\mathbf{E}_{X_s}[Z_{t-s}(m-k;f)]\mathbf{E}_{X_s}[Z_{t-s}(n-k;g)]\,\frac{\bigl(A_s^{Q\mu}\bigr)^{k-1}}{(k-1)!}\,dA_s^{R\mu}\right]
\end{aligned}$$

for $m>n\ge1$. Since

$$Z_t(f)Z_t(g)=\sum_{m=0}^\infty Z_t(m;f)Z_t(m;g)+\sum_{n=1}^\infty\sum_{m=n+1}^\infty\bigl(Z_t(m;f)Z_t(n;g)+Z_t(m;g)Z_t(n;f)\bigr),$$

we obtain (3.3) by Fubini's theorem. □

In the sequel, we assume that $\lambda:=\lambda(\mu,Q;D)<0$. We denote by h the ground state corresponding to λ. Then

$$h(x)=e^{\lambda t}E_x\Big[\exp\Big(A_t^{(Q-1)\mu}\Big)h(X_t);t<\tau_D\Big]. \tag{3.4}$$

Define

$$M_t=e^{\lambda t}\sum_{i=1}^{Z_t}h(\mathbf{X}_t^i),\quad t\geq 0. \tag{3.5}$$

Then M_t is a $\mathbf{P}_x$-martingale by (3.2) and (3.4). Furthermore, it follows from (3.3) and (3.4) that

$$\begin{aligned}\mathbf{E}_x\big[M_t^2\big]&=e^{2\lambda t}E_x\Big[\exp\Big(A_t^{(Q-1)\mu}\Big)h(X_t)^2;t<\tau_D\Big]\\&\quad+E_x\bigg[\int_0^{t\wedge\tau_D}\exp\Big(2\lambda s+A_s^{(Q-1)\mu}\Big)h(X_s)^2\,dA_s^{R\mu}\bigg].\end{aligned} \tag{3.6}$$

The following lemma is crucial in this paper.

LEMMA 3.4. *Assume that* $\sup_{x\in D}R(x)<\infty$. *Then* M_t *is square integrable.*

PROOF. Since

$$e^{2\lambda t}E_x\Big[\exp\Big(A_t^{(Q-1)\mu}\Big)h(X_t)^2;t<\tau_D\Big]\leq e^{\lambda t}\|h\|_\infty h(x)$$

by (3.4) and the right hand side converges to 0 as $t\to\infty$, it follows from (3.6) that

$$\begin{aligned}\lim_{t\to\infty}\mathbf{E}_x\big[M_t^2\big]&=E_x\bigg[\int_0^{\tau_D}\exp\Big(2\lambda s+A_s^{(Q-1)\mu}\Big)h(X_s)^2\,dA_s^{R\mu}\bigg]\\&\leq\|h\|_\infty^2\|R\|_\infty\sup_{x\in D}E_x\bigg[\int_0^{\tau_D}\exp\Big(2\lambda s+A_s^{(Q-1)\mu}\Big)\,dA_s^{\mu}\bigg].\end{aligned} \tag{3.7}$$

Since

$$\begin{aligned}&\inf\bigg\{\mathscr{E}^D(f,f)-\int_D f^2\,(Q-1)d\mu-2\lambda\int_D f^2\,dx:f\in C_0^\infty(D),\ \int_D f^2\,dx=1\bigg\}\\&=-\lambda>0\end{aligned}$$

by the definition of λ, Lemma 3.5 of [**28**] shows that

$$\inf\left\{\mathscr{E}^D(f,f)+\int_D f^2\,d\mu-2\lambda\int_D f^2\,dx : f\in C_0^\infty(D),\ \int_D f^2\,Qd\mu=1\right\}>1.$$

Hence the last term of (3.7) is finite by Theorem 2.2, which implies the square integrability of M_t. $\square$

Lemma 3.4 tells us that there exists the limit $M_\infty=\lim_{t\to\infty}M_t\in[0,\infty)$ $\mathbf{P}_x$-a.s. and in $L^1(\mathbf{P}_x)$, say, $\mathbf{E}_x[M_\infty]=h(x)>0$, which yields that $\mathbf{P}_x(M_\infty\in(0,\infty))>0$ for any $x\in D$. It also holds that

$$\mathbf{E}_x\left[M_\infty^2\right]=E_x\left[\int_0^{\tau_D}\exp\Big(2\lambda s+A_s^{(Q-1)\mu}\Big)h(X_s)^2\,dA_s^{R\mu}\right].$$

We now consider the following equation:

$$\begin{aligned}u(x)=E_x\left[\exp\Big(-A^\mu_{\tau_D}\Big);\tau_D<\infty\right]+E_x\left[\int_0^{\tau_D}\exp(-A_t^\mu)F(u)(X_t)\,dA_t^\mu\right],&\\ 0\le u(x)\le 1,\quad x\in D,&\end{aligned}\tag{3.8}$$

where

$$F(u)(\cdot)=\sum_{n=0}^\infty p_n(\cdot)u(\cdot)^n.$$

Then the function $u\equiv 1$ is a solution to (3.8). Moreover, as proved in Proposition 3.1 of [**26**], the extinction probability u_e (see Section 2.2 for definition) is a minimal solution to (3.8). Here we give a sufficient condition for the solutions of (3.8) to be just u_e and $u\equiv 1$ in terms of the branching rate and the Green function. To be precise, let $G^{\mu,D}(x,y)$ be the Green function of the $\exp(-A_t^\mu)$-subprocess of $\mathbf{M}^D$,

$$E_x\left[\int_0^{\tau_D}\exp(-A_t^\mu)f(X_t)\,dt\right]=\int_D G^{\mu,D}(x,y)f(y)\,dy.$$

We then have

LEMMA 3.5. *Assume that* $P_x(\tau_D<\infty)=1$ *for any* $x\in D$. *If* $\iint_{D\times D}G^{\mu,D}(x,y)\,\mu(dx)\mu(dy)<\infty$, *then the equation* (3.8) *has just two solutions,* $u\equiv 1$ *and* u_e.

PROOF. Let u be a solution to (3.8) such that $u(x_0) < 1$ for some $x_0 \in D$. Since u is finely continuous by Lemma 3.2 of [**26**], it follows from (3.8) that $P_{x_0}(\sigma_{O \cap F^\mu} < \infty) > 0$, where $O = \{x \in D : u(x) < 1\}$ and F^μ is the fine support of the measure μ defined in (2.1). Moreover, by the irreducibility of the process $\mathbf{M}^D$, it holds that $P_x(\sigma_{O \cap F^\mu} < \infty) > 0$ for any $x \in D$, which implies that $u < 1$ on D.

We now define

$$G_\mu^{\mu,D} f(x) = E_x\left[\int_0^{\tau_D} \exp(-A_t^\mu) f(X_t)\, dA_t^\mu\right].$$

Then the right hand side above is equal to

$$\int_D G^{\mu,D}(x,y) f(y)\, \mu(dy).$$

As a direct calculation yields that

$$E_x\left[\exp\left(-A_{\tau_D}^\mu\right)\right] = 1 - E_x\left[\int_0^{\tau_D} \exp(-A_t^\mu)\, dA_t^\mu\right],$$

the equation (3.8) is equivalent to that

$$v = G_\mu^{\mu,D}(F(1) - F(1-v))$$

on D, where $v = 1 - u > 0$. Since the function $v_e = 1 - u_e > 0$ is a solution to the equation above, we see that

$$\begin{aligned}
\int_D v(F(1) - F(1-v_e))\, d\mu &= \int_D G_\mu^{\mu,D}(F(1) - F(1-v))(F(1) - F(1-v_e))\, d\mu \\
&= \int_D G_\mu^{\mu,D}(F(1) - F(1-v_e))(F(1) - F(1-v))\, d\mu \\
&= \int_D v_e(F(1) - F(1-v))\, d\mu.
\end{aligned}$$

Here the integrability of the terms above follows by the assumption on μ and the second equality holds by the symmetry of the operator $G_\mu^{\mu,D}$ with respect to μ (see Theorem 3.2 (iv) of [**2**]). Since $F(\cdot)$ is strictly convex and $v_e \geq v > 0$, it holds that

$$\frac{F(1)-F(1-v)}{1-(1-v)}=\frac{F(1)-F(1-v_e)}{1-(1-v_e)}\quad \mu\text{-a.e.},$$

which shows that $u=u_e$ μ-a.e. Using (3.8), we have $u=u_e$ on D. $\square$

PROPOSITION 3.6. *Assume that $P_x(\tau_D<\infty)=1$ for any $x\in D$. If $\sup_{x\in D}R(x)<\infty$ and $\iint_{D\times D}G^{\mu,D}(x,y)\,\mu(dx)\mu(dy)<\infty$, then*

$$\{e_0=\infty\}=\{M_\infty>0\}\quad \mathbf{P}_x\text{-}a.s.$$

for any $x\in D$.

PROOF. Since $\lambda<0$ and

$$M_t\le e^{\lambda t}Z_t\|h\|_\infty, \tag{3.9}$$

it holds that

$$\{M_\infty>0\}\subset\{e_0=\infty\}.$$

It also holds that, by the assumption on the exit time τ_D,

$$\mathbf{P}_x(T=\infty,e_0=\infty)=E_x\Big[\exp\Big(-A^\mu_{\tau_D}\Big);\tau_D=\infty\Big]=0.$$

Hence, by noting that

$$\{T=\infty\}\subset\{e_0<\infty\}\subset\{M_\infty=0\},$$

we see that

$$\begin{aligned}\mathbf{P}_x(M_\infty=0)&=\mathbf{P}_x(M_\infty=0,T=\infty)+\mathbf{P}_x(M_\infty=0,T<\infty)\\&=\mathbf{P}_x(T=\infty)+\mathbf{P}_x(M_\infty=0,T<\infty)\\&=E_x\Big[\exp\Big(-A^\mu_{\tau_D}\Big);\tau_D<\infty\Big]\\&\quad+E_x\bigg[\int_0^{\tau_D}\exp(-A^\mu_t)F(\mathbf{P}_\cdot(M_\infty=0))(X_t)\,dA^\mu_t\bigg],\end{aligned}$$

that is, the function $\mathbf{P}_x(M_\infty=0)$ is a solution to (3.8). Since $\mathbf{P}_x(M_\infty=0)<1$, it follows from Lemma 3.5 that $\mathbf{P}_x(M_\infty = 0) = u_e(x)$ for any $x \in D$. Namely, $\mathbf{P}_x(M_\infty > 0) = \mathbf{P}_x(e_0 = \infty)$ for any $x \in D$, which completes the proof. $\square$

THEOREM 3.7. *Suppose that $P_x(\tau_D < \infty) = 1$ for any $x \in D$.*
(i) *If $\sup_{x\in D} R(x) < \infty$ and $\iint_{D\times D} G^{\mu,D}(x,y)\,\mu(dx)\mu(dy) < \infty$, then*

$$\mathbf{P}_x(M_\infty \in (0,\infty) \mid e_0 = \infty) = 1, \quad x \in D. \tag{3.10}$$

As a consequence,

$$\mathbf{P}_x\left(\liminf_{t\to\infty} e^{\lambda t} Z_t > 0 \,\middle|\, e_0 = \infty\right) = 1, \quad x \in D. \tag{3.11}$$

(ii) *If $\sup_{x\in D} R(x) < \infty$ and $\iint_{D\times D} G^{\mu,D}(x,y)\,\mu(dx)\mu(dy) < \infty$, then for any $\kappa > \lambda$,*

$$\mathbf{P}_x\left(\lim_{t\to\infty} e^{\kappa t} Z_t = \infty \,\middle|\, e_0 = \infty\right) = 1, \quad x \in D. \tag{3.12}$$

(iii) *For any $\kappa < \lambda$,*

$$\mathbf{P}_x\left(\lim_{t\to\infty} e^{\kappa t} \sum_{i=1}^{Z_t} h(\mathbf{X}_t^i) = 0\right) = 1, \quad x \in D \tag{3.13}$$

and

$$\mathbf{P}_x\left(\liminf_{t\to\infty} e^{\kappa t} Z_t = 0\right) = 1, \quad x \in D. \tag{3.14}$$

Furthermore, if the open set D is Green bounded, that is, $\sup_{x\in D} E_x[\tau_D] < \infty$, then for any $\kappa < \lambda$,

$$\mathbf{P}_x\left(\lim_{t\to\infty} e^{\kappa t} Z_t = 0\right) = 1, \quad x \in D. \tag{3.15}$$

PROOF. The equation (3.10) follows from Proposition 3.6. Since

$$\{M_\infty > 0\} \subset \left\{\liminf_{t\to\infty} e^{\lambda t} Z_t > 0\right\} \subset \left\{\lim_{t\to\infty} e^{\kappa t} Z_t = \infty\right\}$$

for $\kappa > \lambda$ by (3.9), we have (3.11) and (3.12).

Suppose that $\kappa < \lambda$. Then the equation (3.13) holds by Lemma 3.4. By (3.2),

$$\begin{aligned}e^{\kappa t}\mathbf{E}_x[Z_t] &= E_x\left[\exp\left(\kappa t + A_t^{(Q-1)\mu}\right); t < \tau_D\right] \\ &= e^{\kappa t}E_x\left[\exp(-A_t^\mu)\int_0^t \exp\left(A_s^{Q\mu}\right) dA_s^{Q\mu}; t < \tau_D\right] + e^{\kappa t}E_x[\exp(-A_t^\mu); t < \tau_D].\end{aligned}$$

Choose a positive constant ε such that $0 < \varepsilon < \lambda - \kappa$. Then the last term above is not greater than

$$e^{(\kappa-\lambda+\varepsilon)t}E_x\left[\int_0^{\tau_D} \exp\left((\lambda-\varepsilon)s + A_s^{(Q-1)\mu}\right) dA_s^{Q\mu}\right] + e^{\kappa t}E_x[\exp(-A_t^\mu); t < \tau_D]. \quad (3.16)$$

Further, by the same argument as in Lemma 3.4, it follows that

$$\sup_{x\in D} E_x\left[\int_0^{\tau_D} \exp\left((\lambda-\varepsilon)s + A_s^{(Q-1)\mu}\right) dA_s^{Q\mu}\right] < \infty,$$

and thus the term (3.16) converges to 0 as $t \to \infty$. Hence by Fatou's lemma,

$$\mathbf{E}_x\left[\liminf_{t\to\infty} e^{\kappa t}Z_t\right] \le \lim_{t\to\infty} e^{\kappa t}\mathbf{E}_x[Z_t] = 0,$$

which implies (3.14).

In the sequel, we further assume that the open set D is Green bounded. Let

$$u_\kappa(x) = E_x\left[\exp\left(\kappa\tau_D + A_{\tau_D}^{(Q-1)\mu}\right)\right].$$

Then $\sup_{x\in D} u_\kappa(x) < \infty$ by Theorem 2.2. Moreover, Jensen's inequality yields that

$$\inf_{x\in D} u_\kappa(x) \ge \exp\left(\kappa \sup_{x\in D} E_x[\tau_D] - \sup_{x\in D} E_x\left[A_{\tau_D}^\mu\right]\right) > 0,$$

where we note that $\sup_{x\in D} E_x[A_{\tau_D}^\mu] < \infty$ by (2.2). By the definition of u_κ and (3.2),

$$e^{\kappa t}\mathbf{E}_x\left[\sum_{i=1}^{Z_t} u_\kappa(\mathbf{X}_t^i)\right] = e^{\kappa t}E_x\left[\exp\left(A_t^{(Q-1)\mu}\right)u_\kappa(X_t); t < \tau_D\right]$$

$$= e^{\kappa t} E_x\Big[\exp\Big(A_t^{(Q-1)\mu}\Big) E_{X_t}\Big[\exp\Big(\kappa\tau_D + A_{\tau_D}^{(Q-1)\mu}\Big)\Big]; t < \tau_D\Big].$$

Then the last term above is equal to

$$E_x\Big[\exp\Big(\kappa\tau_D + A_{\tau_D}^{(Q-1)\mu}\Big); t < \tau_D\Big] \leq u_\kappa(x)$$

by the Markov property. Since $e^{\kappa t}\sum_{i=1}^{Z_t} u_\kappa(\mathbf{X}_t^i)$ is a nonnegative $\mathbf{P}_x$-supermartingale such that

$$\sup_{(x,t)\in D\times[0,\infty)} e^{\kappa t}\mathbf{E}_x\left[\sum_{i=1}^{Z_t} u_\kappa(\mathbf{X}_t^i)\right] \leq \sup_{x\in D} u_\kappa(x) < \infty,$$

there exists a limit $\lim_{t\to\infty} e^{\kappa t}\sum_{i=1}^{Z_t} u_\kappa(\mathbf{X}_t^i) < \infty$ $\mathbf{P}_x$-a.s. for any $x \in D$. Furthermore, we see that $\limsup_{t\to\infty} e^{\kappa t} Z_t < \infty$ $\mathbf{P}_x$-a.s. because $\inf_{x\in D} u_\kappa(x) > 0$ and

$$\Big(\inf_{x\in D} u_\kappa(x)\Big) e^{\kappa t} Z_t \leq e^{\kappa t}\sum_{i=1}^{Z_t} u_\kappa(\mathbf{X}_t^i).$$

Noting that $\kappa < \lambda$ is arbitrary, we have (3.15). □

We next study the exponential growth of the number of particles in every relatively compact open set. In Theorem 3.2 of [**26**], we gave a criterion for local extinction of a branching symmetric α-stable process in terms of the principal eigenvalue for an associated time changed process of the symmetric α-stable process. By using $\lambda(\mu, Q; D)$ defined in (3.1), we can rephrase Theorem 3.2 of [**26**] as follows:

Theorem 3.8. *Suppose that, for any relatively compact open set A in D, $P_x^D(\rho_A < \infty) = 1$ for any $x \in D$, where $\rho_A = \sup\{t > 0 : X_t^D \in A\}$. If the branching rate μ belongs to $\mathscr{S}_\infty^D$, then $\overline{\mathbf{M}}^D$ extincts locally if and only if $\lambda(\mu, Q; D) \geq 0$.*

In the sequel, we assume that $\lambda := \lambda(\mu, Q; D) < 0$. We then have

Lemma 3.9. *For any non-empty open set A in D,*

$$\mathbf{P}_x\Big(\limsup_{t\to\infty} Z_t(A) = \infty\Big) > 0, \quad x \in D. \tag{3.17}$$

Moreover, if $P_x^D(\rho_A < \infty) = 1$ for any $x \in D$ and $\iint_{D\times D} G^{\mu,D}(x,y)\,\mu(dx)\mu(dy) < \infty$, then

$$\mathbf{P}_x\left(\limsup_{t\to\infty} Z_t(A) = 0 \text{ or } \infty\right) = 1, \quad x \in D. \tag{3.18}$$

Namely,

$$\{L_A = \infty\} = \left\{\limsup_{t\to\infty} Z_t(A) = \infty\right\} \quad \mathbf{P}_x\text{-a.s.}, \quad x \in D.$$

To prove Lemma 3.9, we consider the following equation:

$$\begin{aligned} u(x) = E_x\Big[\exp\Big(-A^\mu_{\tau_D}\Big)\Big] + E_x\left[\int_0^{\tau_D} \exp(-A^\mu_t)F(u)(X_t)\,dA^\mu_t\right], \\ 0 \le u(x) \le 1, \quad x \in D. \end{aligned} \tag{3.19}$$

We can then prove the following by the same way as in Lemma 3.5.

LEMMA 3.10. *Suppose that $\iint_{D\times D} G^{\mu,D}(x,y)\,\mu(dx)\mu(dy) < \infty$. If the functions u_1 and u_2 are solutions to* (3.19) *respectively, and $u_1 \le u_2 < 1$ on D, then $u_1 = u_2$ on D.*

PROOF OF LEMMA 3.9. Let O be a finite union of bounded $C^{1,1}$ domains in D such that $\lambda < \lambda(\mu, Q; O) < 0$. Since the measure $\mu|_O$ belongs to $\mathscr{S}^O_\infty$ by Remark 2.5, we see from Theorem 3.8 that $\overline{\mathbf{M}^O} = (\mathbf{P}^O_{\mathbf{x}})$ does not extinct, and thus

$$\mathbf{P}_x\left(\lim_{t\to\infty} Z_t(O) = \infty\right) \ge \mathbf{P}^O_x\left(\lim_{t\to\infty} Z_t = \infty\right) > 0, \quad x \in O.$$

Furthermore, the left hand side above is positive for any $x \in D$ by the irreducibility of $\mathbf{M}^D$.

Let us denote by $p_t^{(Q-1)\mu,D}(x,y)$ the integral kernel of the Feynman-Kac semigroup $p_t^{(Q-1)\mu,D}$ as defined in (2.5). Then $p_t^{(Q-1)\mu,D}(x,A) := \int_A p_t^{(Q-1)\mu,D}(x,y)\,dy$ is bounded and continuous on D by Theorem 2.3 (i) and

$$p := \inf_{x\in O} p_1^{(Q-1)\mu,D}(x,A) > 0$$

by the irreducibility of $\mathbf{M}^D$. Since

$$\mathbf{E}_x[Z_t(A)] = E_x\Big[\exp\Big(A_t^{(Q-1)\mu}\Big);\, t<\tau_D, X_t\in A\Big]$$

by (3.2), it holds that

$$\inf_{x\in O}\mathbf{E}_x[Z_1(A)] = p > 0,$$

and thus

$$\inf_{x\in O}\mathbf{P}_x(Z_1(A)\geq 1) > 0. \tag{3.20}$$

Let q be a nonnegative constant such that

$$e^{-q} = \sup_{x\in O}\mathbf{E}_x[\exp(-Z_1(A))].$$

Then it holds that $0<q\leq p$ because the right hand side above is less than one by (3.20) and

$$\sup_{x\in O}\mathbf{E}_x[\exp(-Z_1(A))] \geq \exp\Big(-\inf_{x\in O}\mathbf{E}_x[Z_1(A)]\Big)$$

by Jensen's inequality. Choose a positive constant $\overline{q}$ such that $0<\overline{q}<q$. Then for any $\mathbf{x}^n=(x^1,x^2,x^3,\cdots,x^n)\in O^{(n)}$,

$$\begin{aligned}\mathbf{P}_{\mathbf{x}^n}(Z_1(A)<\overline{q}Z_0(O)) &= \mathbf{P}_{\mathbf{x}^n}(\exp(-Z_1(A))>\exp(-\overline{q}Z_0(O)))\\ &\leq e^{n\overline{q}}\prod_{i=1}^n\mathbf{E}_{x^i}[\exp(-Z_1(A))]\end{aligned}$$

by Chebyshev's inequality. Since the last term above is not greater than $e^{\overline{q}-q}<1$ for any $n\geq 1$ by the definition of q, it holds that

$$\sup_{n\geq 1,\,\mathbf{x}^n\in O^{(n)}}\mathbf{P}_{\mathbf{x}^n}(Z_1(A)<\overline{q}Z_0(O))<1.$$

Namely,

$$\inf_{n\geq 1,\,\mathbf{x}^n\in O^{(n)}}\mathbf{P}_{\mathbf{x}^n}(Z_1(A)\geq\overline{q}Z_0(O))>0.$$

Let us define

$$A_m = \{Z_m(A) \geq \overline{q} Z_{m-1}(O)\}$$

for any positive integer $m \geq 1$ and

$$\Omega_0 = \left\{ \lim_{t\to\infty} Z_t(O) = \infty \right\}. \tag{3.21}$$

Then, by the Markov property,

$$\begin{aligned}\mathbf{P}_x(A_{m+1} \mid \mathscr{G}_m)(\omega) &= \mathbf{P}_{\mathbf{X}_m(\omega)}(Z_1(A) \geq \overline{q} Z_0(O)) \\ &\geq \inf_{n\geq 1,\ \mathbf{x}^n \in O^{(n)}} \mathbf{P}_{\mathbf{x}^n}(Z_1(A) \geq \overline{q} Z_0(O)) > 0\end{aligned}$$

for any $x \in D$ and $\omega \in \Omega_0$, and hence

$$\sum_{m=0}^{\infty} \mathbf{P}_x(A_{m+1} \mid \mathscr{G}_m)(\omega) = \infty.$$

Noting that

$$\left\{ \sum_{m=0}^{\infty} \mathbf{P}_x(A_{m+1} \mid \mathscr{G}_m) = \infty \right\} = \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} A_m$$

by [**14**, p.237, Corollary 3.2], we obtain (3.17).

In the sequel, let A be an open set in D such that $P_x^D(\rho_A < \infty) = 1$ for any $x \in D$ and assume that $\iint_{D\times D} G^{\mu,D}(x,y)\,\mu(dx)\mu(dy) < \infty$. Set

$$u_1(x) = \mathbf{P}_x\left(\lim_{t\to\infty} Z_t(A) = 0 \right)$$

and

$$u_2(x) = \mathbf{P}_x\left(\limsup_{t\to\infty} Z_t(A) < \infty \right).$$

We then see in a similar way to Proposition 3.6 that the functions u_1 and u_2 are solutions to (3.19) respectively, by the assumption on A. Since it holds that

$u_1 \le u_2 < 1$ by definition, Lemma 3.10 implies that $u_1 = u_2$ on D, which leads us to (3.18). $\square$

PROPOSITION 3.11. *For any non-empty open set A in D and $\kappa > \lambda$,*

$$\mathbf{P}_x\left(\limsup_{t\to\infty} e^{\kappa t} Z_t(A) = \infty\right) > 0, \quad x \in D. \tag{3.22}$$

Moreover, if $P_x^D(\rho_A < \infty) = 1$ for any $x \in D$ and $\iint_{D\times D} G^{\mu,D}(x,y)\,\mu(dx)\mu(dy) < \infty$, then

$$\{L_A = \infty\} = \left\{\limsup_{t\to\infty} e^{\kappa t} Z_t(A) = \infty\right\} \quad \mathbf{P}_x\text{-}a.s., \quad x \in D,$$

and

$$\{L_A < \infty\} = \left\{\lim_{t\to\infty} e^{\kappa t} Z_t(A) = 0\right\} \quad \mathbf{P}_x\text{-}a.s., \quad x \in D.$$

PROOF. For any $\kappa > \lambda$, there exists a finite union of bounded $C^{1,1}$ domains O in D such that $\lambda < \lambda(\mu, Q; O) < \kappa$. Then, by Theorem 3.7 (ii),

$$\mathbf{P}_x\left(\lim_{t\to\infty} e^{\kappa t} Z_t(O) = \infty\right) \ge \mathbf{P}_x^O\left(\lim_{t\to\infty} e^{\kappa t} Z_t = \infty\right) > 0, \quad x \in O.$$

Moreover, the left hand side above is positive for any $x \in D$ by the irreducibility of $\mathbf{M}^D$. Hence if we replace Ω_0 defined in (3.21) with

$$\left\{\lim_{t\to\infty} e^{\kappa t} Z_t(O) = \infty\right\},$$

then (3.22) follows by the same way as in Lemma 3.9.

In the sequel, let A be an open set in D such that $P_x^D(\rho_A < \infty) = 1$ for any $x \in D$ and assume that $\iint_{D\times D} G^{\mu,D}(x,y)\,\mu(dx)\mu(dy) < \infty$. Set

$$u_1(x) = \mathbf{P}_x\left(\lim_{t\to\infty} e^{\kappa t} Z_t(A) = 0\right)$$

and

$$u_2(x) = \mathbf{P}_x\left(\limsup_{t\to\infty} e^{\kappa t} Z_t(A) < \infty\right).$$

Then it follows from (3.22) that $u_A \le u_1 \le u_2 < 1$ on D, where $u_A(x) = \mathbf{P}_x(L_A < \infty)$. Furthermore, by noting that u_A, u_1 and u_2 are solutions to (3.19) respectively, Lemma 3.10 implies that $u_A = u_1 = u_2$ on D, which completes the proof. □

THEOREM 3.12.
(i) *For any relatively compact open set A in D,*

$$\mathbf{P}_x\left(\limsup_{t\to\infty} e^{\lambda t} Z_t(A) < \infty\right) = 1, \quad x \in D. \tag{3.23}$$

As a consequence, for any $\kappa < \lambda$,

$$\mathbf{P}_x\left(\lim_{t\to\infty} e^{\kappa t} Z_t(A) = 0\right) = 1, \quad x \in D.$$

(ii) *Assume that $\iint_{D\times D} G^{\mu,D}(x,y)\,\mu(dx)\mu(dy) < \infty$. Then, for any non-empty open set A in D such that $P_x^D(\rho_A < \infty) = 1$ for any $x \in D$ and $\kappa > \lambda$,*

$$\mathbf{P}_x\left(\limsup_{t\to\infty} e^{\kappa t} Z_t(A) = \infty \;\middle|\; L_A = \infty\right) = 1, \quad x \in D. \tag{3.24}$$

PROOF. Let A be a relatively compact open set in D. Then

$$e^{\lambda t} Z_t(A) \le \frac{1}{\inf_{x\in A} h(x)} M_t.$$

Since

$$\limsup_{t\to\infty} e^{\lambda t} Z_t(A) \le \frac{1}{\inf_{x\in A} h(x)} M_\infty < \infty \quad \mathbf{P}_x\text{-a.s.},$$

(3.23) holds. The equation (3.24) follows from Proposition 3.11. □

REMARK 3.13. Engländer and Kyprianou [**15**] studied the exponential growth of the number of particles in every relatively compact open set for a branching diffusion process such that the branching rate is a bounded, nonnegative and continuous function. On the other hand, we can take unbounded functions as branching rate in (3.22) of Proposition 3.11 and Theorem 3.12 (i). For instance, suppose that $\alpha = 2$ and $D = \boldsymbol{R}^3$. Since the measure $\mu(dx) = 1/|x|\chi_{|x|\le 1}\,dx$ belongs to $\mathscr{K}_\infty^{\boldsymbol{R}^3}$, we can take the measure μ as branching rate. Moreover, the ground state of $\lambda(\mu;\boldsymbol{R}^3)$ satisfies (2.8) because the support of μ is compact.

REMARK 3.14. Assume that $d = 1$ and $1 < \alpha \le 2$ or $d = \alpha = 2$, that is, the symmetric α-stable process $\mathbf{M}^\alpha$ on $\boldsymbol{R}^d$ is Harris recurrent. Let us consider the branching symmetric α-stable process $\overline{\mathbf{M}^\alpha} = (\mathbf{P}_x)$ on $\boldsymbol{R}^d$ with branching rate $\mu \in \mathscr{K}_\infty^{\boldsymbol{R}^d}$. We denote by T the first splitting time of $\overline{\mathbf{M}^\alpha}$. Then, since $P_x(A_\infty^\mu = \infty) = 1$ for any $x \in \boldsymbol{R}^d$ (see [**24**, p.426, Proposition 3.11]), it follows that

$$\mathbf{P}_x(T = \infty) = E_x\big[\exp\big(-A_\infty^\mu\big)\big] = 0$$

for any $x \in \boldsymbol{R}^d$. Using this fact, we can show Theorems 3.7 and 3.12 by the same argument. Here the condition $\iint_{D\times D} G^{\mu,D}(x,y)\,\mu(dx)\mu(dy) < \infty$ is replaced with $\mu(\boldsymbol{R}^d) < \infty$ and the condition on the exit time τ_D or the last exit times is not imposed.

Let us recall that $\mathbf{M} = (X_t, P_x)$ is an m-symmetric Hunt process on X, where X is a locally compact separable metric space and m is a positive Radon measure on X with full support. Let $(\mathscr{E},\mathscr{F})$ be the regular Dirichlet form on $L^2(X;m)$ generated by $\mathbf{M}$. We make the following assumption on $\mathbf{M}$:

ASSUMPTION 3.15.

(i) (Irreducibility) If a Borel set A is p_t-invariant, that is, if $p_t(\chi_A f) = \chi_A p_t f(x)$ for any $f \in L^2(X;m) \cap \mathscr{B}_b(X)$ and $t > 0$, then $m(A) = 0$ or $m(X \setminus A) = 0$.

(ii) (Strong Feller property) For any $f \in \mathscr{B}_b(X)$, $p_t f$ is a bounded and continuous function on X.

(iii) (Ultracontractivity) For any $t > 0$, it holds that $\|p_t\|_{1,\infty} < \infty$, where $\|\cdot\|_{p,q}$ denotes the operator norm from $L^p(X;m)$ to $L^q(X;m)$.

Let $\overline{\mathbf{M}}$ be the branching symmetric Hunt process with motion component $\mathbf{M}$, branching rate $\mu \in \mathscr{K}_\infty$ and branching mechanism $\{p_n(x)\}_{n\ge 0}$. Put $Q(x) = \sum_{n=0}^\infty n p_n(x)$ and assume that $\sup_{x\in X} Q(x) < \infty$. We now define

$$\lambda(\mu,Q) = \inf\left\{\mathscr{E}(f,f) - \int_X f^2\,(Q-1)d\mu : f \in \mathscr{F},\ \int_X f^2\,dm = 1\right\}. \tag{3.25}$$

We also make the following assumption on $\mathbf{M}$:

ASSUMPTION 3.16. (Compact embedding) The embedding from $(\mathscr{F}, \mathscr{E}_1)$ to $L^2(X;\mu)$ is compact, where $\mathscr{E}_1(f,f) = \mathscr{E}(f,f) + \int_X f^2\,dm$.

Let

$$\lambda_0 = \inf\left\{\mathscr{E}(f,f) : f \in \mathscr{F},\ \int_X f^2\,dm = 1\right\}.$$

Then, as discussed in Section 2.3 for symmetric α-stable processes, Assumption 3.16 implies that, if $\lambda(\mu,Q) < \lambda_0$, then $\lambda(\mu,Q)$ is the principal eigenvalue and Assumption 3.15 yields that the corresponding ground state h is bounded, continuous and strictly positive on X. Hence, if the motion component $\mathbf{M}$ is transient or Harris recurrent, then Theorem 3.7 holds. In addition, if the support of μ is compact, then Theorem 3.12 holds.

REMARK 3.17.

(i) Let M be a simply connected, complete and non-compact Riemannian manifold and consider the Brownian motion on M. Denote by $(\mathscr{E}, \mathscr{F})$ the associated regular Dirichlet form on $L^2(M;V)$:

$$\mathscr{E}(f,f) = \frac{1}{2}\int_M |\nabla f|^2\,dV$$
$$\mathscr{F} = \text{the closure of } C_0^\infty(M) \text{ with respect to } \mathscr{E}(\cdot,\cdot) + \|\cdot\|^2_{L^2(M;V)},$$

where V is the Riemannian volume of M. We then see in a similar way to [**30**, Section 3] that $(\mathscr{E}, \mathscr{F})$ satisfies Assumption 3.16. Hence, if the Brownian motion on M fulfills Assumption 3.15, then Theorems 3.7 and 3.12 are applicable to branching Brownian motions on M. For example, we can find in [**13**, Section 5] some sufficient conditions for the Brownian motion on M to satisfy Assumption 3.15.

(ii) Let $(\mathscr{E}, \mathscr{F})$ be a regular Dirichlet form on $L^2(\boldsymbol{R}^d)$ and $\mathbf{M}$ the associated symmetric Hunt process. If $(\mathscr{E}_1, \mathscr{F})$ is comparable to that of the symmetric α-stable process, then, by applying the same argument as in [**29**], we can show that the embedding from $(\mathscr{F}, \mathscr{E}_1)$ to $L^2(\boldsymbol{R}^d;\mu)$ is compact for any $\mu \in \mathscr{K}_\infty$.

For instance, we consider stable-like processes on $\boldsymbol{R}^d$ in the sense of Z.-Q. Chen and Kumagai [**8**]: let $c(x,y)$ be a symmetric function on $\boldsymbol{R}^d \times \boldsymbol{R}^d$ which is bounded between two positive constants $c_2 > c_1 > 0$, that is,

$$c_1 \le c(x,y) \le c_2, \quad \text{a.e. } (x,y) \in \boldsymbol{R}^d \times \boldsymbol{R}^d.$$

Fix $0 < \alpha < 2$ and define

$$\mathscr{E}(f,f) = \iint_{\boldsymbol{R}^d \times \boldsymbol{R}^d \setminus \triangle} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} c(x,y)\, dxdy$$
$$\mathscr{F} = \left\{ f \in L^2(\boldsymbol{R}^d) : \iint_{\boldsymbol{R}^d \times \boldsymbol{R}^d \setminus \triangle} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\, dxdy < \infty \right\}.$$

Since $(\mathscr{E},\mathscr{F})$ is a regular Dirichlet form on $L^2(\boldsymbol{R}^d)$, there exists the associated symmetric Hunt process on $\boldsymbol{R}^d$, which is called the α-stable-like process. Clearly the Dirichlet form $(\mathscr{E},\mathscr{F})$ is comparable to that of the symmetric α-stable process. Moreover, it is proved in [**8**, Theorem 4.14] that the α-stable-like process on $\boldsymbol{R}^d$ admits a Hölder continuous transition density which is comparable to that of the symmetric α-stable process. These facts imply that stable-like processes on $\boldsymbol{R}^d$ fulfill Assumptions 3.15 and 3.16, and thus Theorems 3.7 and 3.12 are applicable to branching stable-like processes. Note that the class $\mathscr{K}_\infty$ of the α-stable-like process on $\boldsymbol{R}^d$ is identified with that of the symmetric α-stable process on $\boldsymbol{R}^d$.

We announce that, if the motion component $\mathbf{M}$ satisfies Assumptions 3.15 and 3.16, then the following limit theorem is established for the branching process $\overline{\mathbf{M}}$ in [**9**]: for any $x \in X$, there exists a subspace Ω_0 of the sample path space of $\overline{\mathbf{M}}$ with $\mathbf{P}_x(\Omega_0) = 1$ such that, for any $\omega \in \Omega_0$,

$$\lim_{t\to\infty} e^{\lambda(\mu,Q)t} Z_t(A)(\omega) = M_\infty(\omega) \int_A h\, dm$$

for any relatively compact Borel set A in X such that $m(\partial A) = 0$. Hence Theorem 3.7 (i) gives a sufficient condition for the right hand side above to be positive under the condition that $\overline{\mathbf{M}}$ survives.

4. Examples.

We calculate the principal eigenvalues and the ground states of the Schrödinger operators and apply Theorems 3.7 and 3.12 to branching Brownian motions and branching symmetric α-stable processes.

4.1. In case of $\boldsymbol{\alpha = 2}$.

In this subsection, we suppose that $\alpha = 2$, that is, we consider the Brownian case.

EXAMPLE 4.1. Suppose that $d = 1$. Let us take first $D = (-R, R)$ for $R > 0$ and $\mu = \sum_{i=1}^{n} \alpha_i \delta_{a_i}$, $\alpha_i > 0$, $-R < a_1 < a_2 < \cdots < a_n < R$, where δ_{a_i} is the Dirac measures at $a_i \in (-R, R)$. Denote by h the ground state corresponding to $\lambda := \lambda(\mu; (-R, R))$. We then see from (2.6) that

$$h(x) = \sum_{i=1}^{n} \alpha_i G_{-\lambda}^R(x, a_i) h(a_i),$$

where $G_\beta^R(x, y)$, $\beta > 0$, is the β-resolvent density of the absorbing Brownian motion on $(-R, R)$. Let G_β^R be the $n \times n$-matrix defined by $(\alpha_j G_\beta^R(a_i, a_j))_{1 \le i,j \le n}$. Then the relation above implies that

$$\lambda = \min\{\kappa : |G_{-\kappa}^R - I| = 0\},$$

where I is the $n \times n$-unit matrix.

First suppose that $n = 1$, $a = a_1$ and $\alpha_1 = 1$. Since

$$G_{-\lambda}^R(x, y) = \frac{2}{\sqrt{-2\lambda}\sinh(\sqrt{-2\lambda}R)} \sinh\{\sqrt{-2\lambda}(R - x)\} \sinh\{\sqrt{-2\lambda}(R + y)\} \tag{4.1}$$

for $-R < y \le x < R$ ([**4**, p.105]) and $G_{-\lambda}^R(a, a) = 1$, it holds that

$$\frac{\sqrt{-2\lambda}(e^{2\sqrt{-2\lambda}R} - e^{-2\sqrt{-2\lambda}R})}{e^{2\sqrt{-2\lambda}R} + e^{-2\sqrt{-2\lambda}R} - e^{2\sqrt{-2\lambda}a} - e^{-2\sqrt{-2\lambda}a}} = 1. \tag{4.2}$$

If we take $h(a) = \sqrt{-2\lambda}\sinh(2\sqrt{-2\lambda}R)/2$, then

$$h(x) = \begin{cases} \sinh\{2\sqrt{-2\lambda}(R - a)\}\sinh\{2\sqrt{-2\lambda}(R + x)\}, & -R < x \le a \\ \sinh\{2\sqrt{-2\lambda}(R + a)\}\sinh\{2\sqrt{-2\lambda}(R - x)\}, & a < x < R. \end{cases}$$

For instance, suppose that $a = 0$. Since the equation (4.2) becomes

$$\frac{\sqrt{-2\lambda}(e^{2\sqrt{-2\lambda}R} + 1)}{e^{2\sqrt{-2\lambda}R} - 1} = 1,$$

we can find that if $R > 1$, then λ is a unique solution to the equation above and $-1/2 < \lambda < 0$. Otherwise, $\lambda = 0$.

Consider the binary branching absorbing Brownian motion on $(-R, R)$ with branching rate δ_0. Then this process does not extinct if and only if $R > 1$. Note that $(-R, R)$ is Green bounded because

$$E_x[\tau_R] = 2(R^2 - x^2),$$

where τ_R is the exit time of the one-dimensional Brownian motion from $(-R, R)$. Hence if $R > 1$, then (3.10), (3.12) and (3.15) hold.

Next suppose that $\mu = \delta_a + \delta_{-a}$ for $a \in (0, R)$. Then it follows from (4.1) that

$$\frac{\sqrt{-2\lambda}\sinh(2\sqrt{-2\lambda}R)}{2\sinh\{\sqrt{-2\lambda}(R-a)\}(\sinh\{\sqrt{-2\lambda}(R-a)\} + \sinh\{\sqrt{-2\lambda}(R+a)\})} = 1. \tag{4.3}$$

If we take $h(a) = \sqrt{-2\lambda}\sinh(2\sqrt{-2\lambda}R)/2$, then

$$h(x) = \begin{cases} \sinh\{2\sqrt{-2\lambda}(R-a)\}\sinh\{2\sqrt{-2\lambda}(R+a)\}\sinh\{2\sqrt{-2\lambda}(R+x)\}, \\ \hfill -R < x \le -a \\ \sinh\{2\sqrt{-2\lambda}(R-a)\}\sinh\{2\sqrt{-2\lambda}(R-x)\}\sinh\{2\sqrt{-2\lambda}(R+x)\}, \\ \hfill -a < x \le a \\ \sinh\{2\sqrt{-2\lambda}(R-a)\}\sinh\{2\sqrt{-2\lambda}(R+a)\}\sinh\{2\sqrt{-2\lambda}(R-x)\}, \\ \hfill a < x < R. \end{cases}$$

Assume that $a = 1$. If $R > 3/2$, then the principal eigenvalue λ is a negative unique solution to (4.3). Otherwise, $\lambda = 0$.

Let us consider the binary branching absorbing Brownian motion on $(-R, R)$ with branching rate $\delta_1 + \delta_{-1}$. Then this process does not extinct if and only if $R > 3/2$. Furthermore, (3.10), (3.12) and (3.15) hold if $R > 3/2$.

EXAMPLE 4.2. Suppose that $d = 1$. Let us take first $D = (0, \infty)$ and $a \in (0, \infty)$. Denote by $G^0_\beta(x, y)$, $\beta > 0$, the β-resolvent density of the absorbing Brownian motion on $(0, \infty)$:

$$G^0_\beta(x, y) = \frac{2}{\sqrt{2\beta}} e^{-\sqrt{2\beta}x} \sinh\left(\sqrt{2\beta}y\right)$$

for $0 < y < x$ ([**4**, p.107]). By the same way as in Example 4.1, it follows that $\lambda := \lambda(\delta_a; (0, \infty))$ satisfies

$$\frac{\sqrt{-2\lambda}e^{2\sqrt{-2\lambda}a}}{e^{2\sqrt{-2\lambda}a} - 1} = 1.$$

Moreover, a direct calculation implies that this equation has a negative unique solution $-1/2 < \lambda < 0$ if $a > 1/2$. We denote by h the ground state corresponding to λ. If we take $h(a) = \sqrt{-2\lambda}/2$, then

$$h(x) = \begin{cases} e^{-\sqrt{-2\lambda}a} \sinh(\sqrt{-2\lambda}x), & 0 < x \le a \\ e^{-\sqrt{-2\lambda}x} \sinh(\sqrt{-2\lambda}a), & a < x. \end{cases}$$

Consider the binary branching absorbing Brownian motion on $(0, \infty)$ with branching rate δ_a. Then this process does not extinct if $a > 1/2$. Since $(0, \infty)$ is not Green bounded, (3.10), (3.12) and (3.14) hold if $a > 1/2$.

Next take $D = (0, \infty)$ and $\mu = \delta_a + \delta_b$ for $0 < a < b$ and $\lambda := \lambda(\delta_a + \delta_b; (0, \infty))$. We then see in a similar way to Example 4.1 that λ satisfies

$$G^0_{-\lambda}(a, b)^2 = (1 - G^0_{-\lambda}(a, a))(1 - G^0_{-\lambda}(b, b)).$$

Denote by h the ground state corresponding to λ. If we take $h(a) = \sqrt{-2\lambda}\{1 + e^{2\sqrt{-2\lambda}b}(\sqrt{-2\lambda} - 1)\}/2$, then

$$h(x) = \begin{cases} e^{-\sqrt{-2\lambda}a}\{e^{2\sqrt{-2\lambda}a} + e^{2\sqrt{-2\lambda}b}(\sqrt{-2\lambda} - 1)\} \sinh(\sqrt{-2\lambda}x), & 0 < x \le a \\ e^{-\sqrt{-2\lambda}x}\{e^{2\sqrt{-2\lambda}x} + e^{2\sqrt{-2\lambda}b}(\sqrt{-2\lambda} - 1)\} \sinh(\sqrt{-2\lambda}a), & a < x \le b \\ \sqrt{-2\lambda}e^{\sqrt{-2\lambda}(2b-x)} \sinh(\sqrt{-2\lambda}a), & b < x. \end{cases}$$

If we assume that $a = 1/4$, then $-2 < \lambda < 0$ for $b > 1/4$.

We now consider the binary branching absorbing Brownian motion on $(0, \infty)$ with branching rate $\delta_{1/4} + \delta_b$. If $b > 1/4$, then this process does not extinct. Furthermore, (3.10), (3.12) and (3.14) hold.

Example 4.3. Suppose that $d = 3$. Let us take $D = \boldsymbol{R}^3$ and $\mu = \delta_R$, the surface measure on $\{x \in \boldsymbol{R}^3 : |x| = R\}$. It is then known in [**3**] that if $R > 1$, then $\lambda := \lambda(\delta_R; \boldsymbol{R}^3)$ is a unique solution to

$$\frac{2\sqrt{-2\lambda}e^{2\sqrt{-2\lambda}R}}{e^{2\sqrt{-2\lambda}R} - 1} = 1$$

and $\lambda \in (-1/8, 0)$. On the other hand, if $R \le 1$, then $\lambda = 0$. Thus the binary branching Brownian motion on $\boldsymbol{R}^3$ with branching rate δ_R does not extinct locally if and only if $R > 1$. Moreover, if $R > 1$, then (3.23) and (3.24) hold.

4.2. In case of $0 < \alpha \leq 2$.

In this subsection, we assume that $0 < \alpha \leq 2$.

EXAMPLE 4.4. Suppose that $d = 1$ and $1 < \alpha \leq 2$. Let $D = \boldsymbol{R}$ and $\mu = \sum_{i=1}^{n} \alpha_i \delta_{a_i}$, $\alpha_i > 0$, $-\infty < a_1 < a_2 < \cdots < a_n < \infty$. Denote by h the ground state corresponding to $\lambda(\alpha) := \lambda(\mu; \boldsymbol{R})$. Let $G_\beta(x, y)$ be the β-resolvent density of $\mathbf{M}^\alpha$,

$$G_\beta(x,y) = \begin{cases} \dfrac{2^{1/\alpha}}{\pi} \displaystyle\int_0^\infty \frac{\cos\{2^{1/\alpha}(x-y)z\}}{\beta + z^\alpha}\, dz, & 1 < \alpha < 2 \\ \dfrac{1}{\sqrt{2\beta}} e^{-\sqrt{2\beta}|x-y|}, & \alpha = 2. \end{cases}$$

We then see in a similar way to Example 4.1 that

$$h(x) = \sum_{i=1}^{n} \alpha_i G_{-\lambda(\alpha)}(x, a_i) h(a_i)$$

and

$$\lambda(\alpha) = \min\{\kappa : |G_{-\kappa} - I| = 0\},$$

where G_β is the $n \times n$-matrix defined by $(\alpha_j G_\beta(a_i, a_j))_{1 \leq i,j \leq n}$. We now assume that $n = 1$ and $a_1 = 0$. Let $\alpha_1 = Q - 1 > 0$. For $1 < \alpha < 2$, since $(Q-1)G_{-\lambda(\alpha)}(0,0) = 1$ and

$$\begin{aligned} G_{-\lambda(\alpha)}(0,0) &= \frac{2^{1/\alpha}}{\pi(-\lambda(\alpha))^{(\alpha-1)/\alpha}} \int_0^\infty \frac{1}{1+z^\alpha}\, dz \\ &= \frac{2^{1/\alpha}}{\alpha \sin(\pi/\alpha)(-\lambda(\alpha))^{(\alpha-1)/\alpha}}, \end{aligned}$$

it follows that

$$\lambda(\alpha) = -\left\{ \frac{(Q-1)2^{1/\alpha}}{\alpha \sin(\pi/\alpha)} \right\}^{\alpha/(\alpha-1)}.$$

This value is also true for $\alpha = 2$. It also follows that

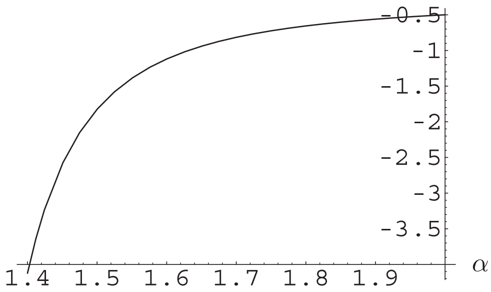

Figure 1. $\lambda(\alpha)$, $1.4 < \alpha \le 2$.

$$h(x) = \begin{cases} \dfrac{(Q-1)2^{1/\alpha}}{\pi} \displaystyle\int_0^\infty \dfrac{\cos(2^{1/\alpha}xz)}{-\lambda(\alpha)+z^\alpha}\, dz h(0), & 1 < \alpha < 2 \\ e^{-(Q-1)|x|}h(0), & \alpha = 2. \end{cases}$$

Here we note that there exists a positive constant $C > 1$ such that

$$\frac{C^{-1}}{|x|^{1+\alpha}} \le h(x) \le \frac{C}{|x|^{1+\alpha}}, \quad |x| \ge 1$$

for $1 < \alpha < 2$ by (II.18) of [**5**]. Figure 1 is the graph of $\lambda(\alpha)$ for $1.4 < \alpha \le 2$ when $Q = 2$. We note that $\lim_{\alpha\downarrow 1} \lambda(\alpha) = -\infty$.

Let $\overline{\mathbf{M}} = (\mathbf{X}_t, \mathbf{P}_x)$ be a branching symmetric α-stable process on $\boldsymbol{R}$ with branching rate δ_0. Assume that the branching mechanism satisfies $p_0(0) + p_2(0) = 1$. Then $Q(0) = 2p_2(0)$. Since the extinction probability is a minimal solution to (3.8) as can be proved in a similar way to that yielding Proposition 3.1 of [**26**], we obtain

$$\mathbf{P}_x(e_0 < \infty) = \begin{cases} 1, & 0 \le p_2(0) \le 1/2 \\ (1-p_2(0))/p_2(0), & 1/2 < p_2(0) \le 1. \end{cases}$$

Hence if $1/2 < p_2(0) \le 1$, then it holds that

$$\mathbf{P}_x\left(\lim_{t\to\infty} e^{\lambda(\alpha)t} \sum_{i=1}^{Z_t} h(\mathbf{X}_t^i) \in (0,\infty) \,\middle|\, e_0 = \infty\right) = 1$$

and

$$\mathbf{P}_x\left(\liminf_{t\to\infty} e^{\lambda(\alpha)t} Z_t > 0 \;\middle|\; e_0 = \infty\right) = 1$$

by Remark 3.14, where each Q in $\lambda(\alpha)$ and h is replaced by $Q(0)$. It also holds that, for any relatively compact open set A,

$$\mathbf{P}_x\left(\limsup_{t\to\infty} e^{\lambda(\alpha)t} Z_t(A) < \infty\right) = 1,$$

and for any non-empty open set A and any $\kappa > \lambda(\alpha)$,

$$\mathbf{P}_x\left(\limsup_{t\to\infty} e^{\kappa t} Z_t(A) = \infty \;\middle|\; L_A = \infty\right) = 1.$$

EXAMPLE 4.5. Suppose that $d = 1$ and $1 < \alpha \leq 2$. Let us take $D = (-R, R)$ and $\mu = \delta_a$, $a \in (-R, R)$. Then, combining Example 3.11 of [**27**] with Lemma 3.5 of [**28**], we see that $\lambda(\delta_a; (-R, R)) < 0$ if and only if

$$R > \frac{A + \sqrt{A^2 + 4a^2}}{2}, \tag{4.4}$$

where

$$A = \left\{(\alpha - 1)2^{\alpha-2}\Gamma\left(\frac{\alpha}{2}\right)^2\right\}^{1/(\alpha-1)}.$$

Note that $\lambda(\delta_a; (-R, R)) \downarrow \lambda(\alpha)$ as $R \to \infty$ for each $\alpha \in (1, 2]$. Denote by h the ground state corresponding to $\lambda := \lambda(\delta_a; (-R, R))$. Then

$$h(x) = G^R_{-\lambda}(x, a)h(a),$$

where $G^R_{-\lambda}(x, y)$ is the $-\lambda$-resolvent density of the absorbing symmetric α-stable process on $(-R, R)$. It also follows from (3) of [**23**] and (2.8) that

$$h(x) = \begin{cases} O((x - R)^{\alpha/2}), & x \to R \\ O((x + R)^{\alpha/2}), & x \to -R. \end{cases}$$

Let us consider the binary branching absorbing symmetric α-stable process on $(-R, R)$ with branching rate δ_a. Then this process does not extinct if and only if a and R satisfies (4.4). Here we note that $(-R, R)$ is Green bounded because

$$E_x[\tau_R] = \frac{2}{\Gamma(\alpha+1)} \left(R^2 - x^2\right)^{\alpha/2}$$

as proved by Getoor [**17**, Section 5] or S. Watanabe [**33**, Theorem 2.1], where τ_R is the exit time of the one-dimensional symmetric α-stable process from $(-R, R)$. Therefore, if a and R satisfies (4.4), then (3.10), (3.12) and (3.15) hold.

Example 4.6. Suppose that $d = 1$ and $1 < \alpha \leq 2$. Let us take $D = (0, \infty)$ and $\mu = \delta_a$, $a \in (0, \infty)$. Denote by $G^0(x, y)$ the Green function of the absorbing symmetric α-stable process on $(0, \infty)$. It is then shown in [**23**] that

$$G^0(x, y) = \frac{2}{\Gamma\left(\dfrac{\alpha}{2}\right)^2} \int_0^{x \wedge y} z^{(\alpha-2)/2} (z + |y - x|)^{(\alpha-2)/2} \, dz.$$

Hence, by the same way as in Example 3.11 of [**27**], it holds that

$$\begin{aligned} \inf\left\{\mathscr{E}^{(0,\infty)}(f, f) : f \in C_0^\infty((0, \infty)), f(a)^2 = 1\right\} &= \frac{1}{G^0(a, a)} \\ &= \frac{(\alpha-1)\Gamma\left(\frac{\alpha}{2}\right)^2}{2a^{\alpha-1}}. \end{aligned}$$

We then see that the left hand side above is less than 1 if and only if

$$a > \left\{\frac{(\alpha-1)\Gamma\left(\frac{\alpha}{2}\right)^2}{2}\right\}^{1/(\alpha-1)}. \tag{4.5}$$

Moreover Lemma 3.5 of [**28**] implies that (4.5) is also equivalent to that $\lambda(\delta_a; (0, \infty)) < 0$. Denote by h the ground state corresponding to $\lambda(\delta_a; (0, \infty))$. Then it follows from (3) of [**23**] and (2.8) that

$$h(x) = \begin{cases} O\left(x^{\alpha/2}\right), & x \to 0 \\ O\left(x^{-(1+\alpha)}\right), & x \to \infty. \end{cases}$$

Consider the binary branching absorbing symmetric α-stable process on $(0,\infty)$ with branching rate δ_a. Then this process does not extinct if and only if a satisfies (4.5). Since $(0,\infty)$ is not Green bounded, (3.10), (3.12) and (3.14) hold if a satisfies (4.5).

Example 4.7. Suppose that $1<\alpha\le 2$ and $d>\alpha$. Let us take $D=\boldsymbol{R}^d$ and $\mu=\delta_R$, the surface measure on $\{x\in\boldsymbol{R}^d : |x|=R\}$. Then it follows from Example 4.1 of [**32**] and Lemma 3.5 of [**28**] that

$$\lambda(\delta_R;\boldsymbol{R}^d)<0 \iff R>\left\{\frac{\sqrt{\pi}\Gamma\left(\frac{d+\alpha}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha-1}{2}\right)\Gamma\left(\frac{d-\alpha}{2}\right)}\right\}^{1/(\alpha-1)}. \tag{4.6}$$

Hence, the binary branching symmetric α-stable process on $\boldsymbol{R}^d$ with branching rate δ_R does not extinct locally if and only if $R>0$ satisfies the right hand side of (4.6). Furthermore, under this condition, (3.23) and (3.24) hold.

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