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Fourier-Borel transformation on the hypersurface of any reduced polynomial

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Abstract. For a polynomial p on $\boldsymbol{C}^n$, the variety $V_p = \{z \in \boldsymbol{C}^n; p(z) = 0\}$ will be considered. Let $\mathrm{Exp}(V_p)$ be the space of entire functions of exponential type on V_p, and $\mathrm{Exp}'(V_p)$ its dual space. We denote by ∂p the differential operator obtained by replacing each variable z_j with $\partial/\partial z_j$ in p, and by $\mathscr{O}_{\partial p}(\boldsymbol{C}^n)$ the space of holomorphic solutions with respect to ∂p. When p is a reduced polynomial, we shall prove that the Fourier-Borel transformation yields a topological linear isomorphism: $\mathrm{Exp}'(V_p) \to \mathscr{O}_{\partial p}(\boldsymbol{C}^n)$. The result has been shown by Morimoto, Wada and Fujita only for the case $p(z) = z_1^2 + \cdots + z_n^2 + \lambda\,(n \geq 2)$.

1. Introduction and Preliminaries.

Let $\mathscr{O}(\boldsymbol{C}^n)$ be the space of entire functions on $\boldsymbol{C}^n$ equipped with the topology of uniform convergence on compact subsets. $\mathscr{O}(\boldsymbol{C}^n)$ is an FS (Fréchet-Schwartz) space. We put

$$\|f\|_A = \sup\{|f(z)| \exp(-A|z|) \ ; \ z \in \boldsymbol{C}^n\} \quad \text{and}$$
$$E_A = \{f \in \mathscr{O}(\boldsymbol{C}^n) \ ; \ \|f\|_A < \infty\}$$

for $A > 0$, where $|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ for $z = (z_1, \ldots, z_n) \in \boldsymbol{C}^n$. The space E_A is a Banach space with respect to the norm $\|\ \|_A$. The topological linear space $\mathrm{Exp}(\boldsymbol{C}^n) = \mathrm{ind\,lim}_{A>0}\ E_A$ equipped with the inductive limit topology is our basic object to study. As is well known, $\mathrm{Exp}(\boldsymbol{C}^n)$ is a DFS space (a dual Fréchet-Schwartz space) and called the space of entire functions of exponential type. We denote the dual space of $\mathrm{Exp}(\boldsymbol{C}^n)$ by $\mathrm{Exp}'(\boldsymbol{C}^n)$. It is clear that $\mathrm{Exp}'(\boldsymbol{C}^n)$ becomes an FS space by the strong dual topology.

Moreover it is easily seen that $f + g$, $fg \in \mathrm{Exp}(\boldsymbol{C}^n)$ for any f, $g \in \mathrm{Exp}(\boldsymbol{C}^n)$, that is, $\mathrm{Exp}(\boldsymbol{C}^n)$ is a commutative algebra with respect to the usual sum and product of functions.

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For any $f, g \in \mathrm{Exp}(\boldsymbol{C}^n)$ and $T \in \mathrm{Exp}'(\boldsymbol{C}^n)$, we define gT by $(gT)(f) = T(gf)$. Since $\mathrm{Exp}(\boldsymbol{C}^n)$ is a commutative algebra and gT is a continuous linear functional, we have $gT \in \mathrm{Exp}'(\boldsymbol{C}^n)$.

Definition 1.1. For any $T \in \mathrm{Exp}'(\boldsymbol{C}^n)$, we define the Fourier-Borel transformation $\mathscr{F}$ by

$$\mathscr{F}(T)(z) = \langle T_\zeta, \exp(z \cdot \zeta)\rangle,$$

where $z \cdot \zeta = z_1\zeta_1 + \cdots + z_n\zeta_n$ for $z, \zeta \in \boldsymbol{C}^n$ and $\langle T, f\rangle$ is the dual pairing: $\langle T, f\rangle = T(f)$ for any $T \in \mathrm{Exp}'(\boldsymbol{C}^n)$ and $f \in \mathrm{Exp}(\boldsymbol{C}^n)$.

For a polynomial p on $\boldsymbol{C}^n$, we set the variety $V_p = \{z \in \boldsymbol{C}^n; p(z) = 0\}$. V_p is a closed set of $\boldsymbol{C}^n$. Thanks to the Oka-Cartan Theorem, the restriction mapping $r : \mathscr{O}(\boldsymbol{C}^n) \to \mathscr{O}(V_p)$ is surjective. Hence, we have the following exact sequence:

$$0 \to \mathscr{K}_p \xrightarrow{i} \mathscr{O}(\boldsymbol{C}^n) \xrightarrow{r} \mathscr{O}(V_p) \to 0,$$

where $\mathscr{K}_p = \{f \in \mathscr{O}(\boldsymbol{C}^n); f|_{V_p} = 0\}$ is a closed subspace of $\mathscr{O}(\boldsymbol{C}^n)$ and i is the canonical injection.

We define the space $\mathrm{Exp}(V_p)$ by the image of the space $\mathrm{Exp}(\boldsymbol{C}^n)$ of entire functions of exponential type under the restriction mapping r. The topology of $\mathrm{Exp}(V_p)$ is defined by the quotient topology of the restriction mapping r. We set $\mathscr{K}_p^E = \mathscr{K}_p \cap \mathrm{Exp}(\boldsymbol{C}^n)$. $\mathscr{K}_p^E$ is a closed subspace of $\mathrm{Exp}(\boldsymbol{C}^n)$. By definition, we have the exact sequence

$$0 \to \mathscr{K}_p^E \xrightarrow{i} \mathrm{Exp}(\boldsymbol{C}^n) \xrightarrow{r} \mathrm{Exp}(V_p) \to 0$$

and $\mathrm{Exp}(V_p) \cong \mathrm{Exp}(\boldsymbol{C}^n)/\mathscr{K}_p^E$. Hence $\mathrm{Exp}(V_p)$ is a DFS space, being a quotient space of a DFS space by a closed subspace.

Let $\mathrm{Exp}'(V_p)$ be the dual space of $\mathrm{Exp}(V_p)$. The space $\mathrm{Exp}'(V_p)$ becomes an FS space by the strong dual topology, since $\mathrm{Exp}(V_p)$ is a DFS space. Because the restriction mapping $r : \mathrm{Exp}(\boldsymbol{C}^n) \to \mathrm{Exp}(V_p)$ is surjective, the transposed mapping ${}^t r : \mathrm{Exp}'(V_p) \to \mathrm{Exp}'(\boldsymbol{C}^n)$ is injective.

Let ∂p be a differential operator obtained by replacing each variable z_j with $\partial/\partial z_j$ in p. We set $\mathscr{O}_{\partial p}(\boldsymbol{C}^n) = \{f \in \mathscr{O}(\boldsymbol{C}^n); \partial p(f) = 0\}$. Since the mapping $\partial p : \mathscr{O}(\boldsymbol{C}^n) \to \mathscr{O}(\boldsymbol{C}^n)$ is continuous, $\mathscr{O}_{\partial p}(\boldsymbol{C}^n)$ is a closed subspace of the FS space $\mathscr{O}(\boldsymbol{C}^n)$. Thus $\mathscr{O}_{\partial p}(\boldsymbol{C}^n)$ is an FS space.

The purpose of this paper is to prove the following theorem;

THEOREM 1.2. *The composed mapping*

$$\mathscr{F} \circ {}^t r : \mathrm{Exp}'(V_p) \longrightarrow \mathscr{O}_{\partial p}(\boldsymbol{C}^n)$$

is a topological linear isomorphism, if and only if p is a reduced polynomial on $\boldsymbol{C}^n$*. We will abbreviate* $\mathscr{F} \circ {}^t r$ *to* $\mathscr{F}$*.*

Here we recall the definition of *reduced polynomial*. If the principal ideal (p) in the polynomial ring on $\boldsymbol{C}^n$ generated by a polynomial p is a reduced ideal, that is, $(p) = \sqrt{(p)}$, p is called a reduced polynomial. A reduced polynomial is nothing but a polynomial represented by a product of irreducible polynomials which has no multiplicity. An irreducible polynomial is obviously a reduced polynomial.

Before giving a proof, we review some known results which have been shown by Morimoto, Wada and Fujita. [**9**] is the general reference for these results.

For the polynomial $p(z) = z_1^2 + \cdots + z_n^2 + \lambda$ $(n \geq 2,\ \lambda \neq 0)$, we see that $\partial p = \Delta_z + \lambda$, where $\Delta_z = \partial^2/\partial z_1^2 + \cdots + \partial^2/\partial z_n^2$ is called the complex Laplacian. We put $\tilde{S}_\lambda := V_p$. $\tilde{S}_\lambda$ is isomorphic to the complex sphere $\tilde{S}^{n-1}$ defined by $\{z \in \boldsymbol{C}^n; z_1^2 + \cdots + z_n^2 = 1\}$. Since p is an irreducible polynomial, Theorem 1.2 implies

THEOREM 1.3 ([**3**] [**7**] [**8**]). *The Fourier-Borel transformation*

$$\mathrm{Exp}'(\tilde{S}_\lambda) \xrightarrow{\sim} \mathscr{O}_\lambda(\boldsymbol{C}^n)$$

is a topological linear isomorphism, where $\mathscr{O}_\lambda(\boldsymbol{C}^n)$ *is the space of eigenfunctions* $\{f \in \mathscr{O}(\boldsymbol{C}^n); (\Delta_z + \lambda)f = 0\}$ *with respect to the eigenvalue* $-\lambda$*.*

For the polynomial $p(z) = z_1^2 + \cdots + z_n^2$ $(n \geq 2)$, we see that $\partial p = \Delta_z$. We put $V_0 := V_p = \{z \in \boldsymbol{C}^n; z_1^2 + \cdots + z_n^2 = 0\}$. V_0 is called the complex light cone. p is an irreducible polynomial for $n \geq 3$ and still a reduced polynomial for $n = 2$. Hence, Theorem 1.2 implies

THEOREM 1.4 ([**3**] [**8**] [**10**] [**11**]). *The Fourier-Borel transformation*

$$\mathrm{Exp}'(V_0) \xrightarrow{\sim} \mathscr{O}_{\Delta_z}(\boldsymbol{C}^n)$$

is a topological linear isomorphism.

2. Isomorphism given by Fourier-Borel transformation.

The following theorem plays an important role in this section.

Theorem 2.1 (Martineau [**6**]). *The Fourier-Borel transformation*

$$\mathscr{F} : \mathrm{Exp}'(\boldsymbol{C}^n) \longrightarrow \mathscr{O}(\boldsymbol{C}^n)$$

is a topological linear isomorphism.

It is clear that for any $g \in \mathrm{Exp}(\boldsymbol{C}^n)$, the mapping

$$\tau_g : \mathrm{Exp}'(\boldsymbol{C}^n) \ni T \mapsto gT \in \mathrm{Exp}'(\boldsymbol{C}^n)$$

is linear and continuous. We set a subspace

$$\mathrm{Exp}'(\boldsymbol{C}^n)_p = \{T \in \mathrm{Exp}'(\boldsymbol{C}^n); \tau_p(T) = pT = 0\},$$

that is, $\mathrm{Exp}'(\boldsymbol{C}^n)_p = \ker \tau_p$. $\mathrm{Exp}'(\boldsymbol{C}^n)_p$ is an FS space as a closed subspace of the FS space $\mathrm{Exp}'(\boldsymbol{C}^n)$.

Owing to Martineau's theorem, we have the following proposition.

Proposition 2.2. *Let us denote the restriction of the Fourier-Borel transformation* $\mathscr{F}$ *to* $\mathrm{Exp}'(\boldsymbol{C}^n)_p$ *by the same notation. Then*

$$\mathscr{F} : \mathrm{Exp}'(\boldsymbol{C}^n)_p \longrightarrow \mathscr{O}_{\partial p}(\boldsymbol{C}^n)$$

is a topological linear isomorphism.

Proof. Obviously, the mappings $\tau_p : \mathrm{Exp}'(\boldsymbol{C}^n) \ni T \mapsto pT \in \mathrm{Exp}'(\boldsymbol{C}^n)$ and $\partial p : \mathscr{O}(\boldsymbol{C}^n) \to \mathscr{O}(\boldsymbol{C}^n)$ are continuous. Moreover it is easily seen that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Exp}'(\boldsymbol{C}^n) & \underset{\mathscr{F}}{\overset{\sim}{\longrightarrow}} & \mathscr{O}(\boldsymbol{C}^n) \\ \tau_p \Big\downarrow & \circlearrowleft & \Big\downarrow \partial p \\ \mathrm{Exp}'(\boldsymbol{C}^n) & \underset{\mathscr{F}}{\overset{\sim}{\longrightarrow}} & \mathscr{O}(\boldsymbol{C}^n). \end{array}$$

under the Fourier-Borel transformation $\mathscr{F}$. Hence, $\ker \tau_p \simeq \ker \partial p$. □

We set a subspace

$$\mathrm{Exp}'(\boldsymbol{C}^n; \mathscr{K}_p^E) = \{T \in \mathrm{Exp}'(\boldsymbol{C}^n); T|_{\mathscr{K}_p^E} = 0\},$$

where the mapping $T|_{\mathscr{K}_p^E}$ is the restriction of the linear mapping T on the subspace $\mathscr{K}_p^E$. It is obvious that $\mathrm{Exp}'(\boldsymbol{C}^n;\mathscr{K}_p^E)$ is a closed subspace of $\mathrm{Exp}'(\boldsymbol{C}^n)$. Indeed, let $i:\mathscr{K}_p^E\to\mathrm{Exp}(\boldsymbol{C}^n)$ be the canonical injection. Then we have $T|_{\mathscr{K}_p^E}={}^ti(T)$. Thus we have $\mathrm{Exp}'(\boldsymbol{C}^n;\mathscr{K}_p^E)=\ker{}^ti$. Therefore, $\mathrm{Exp}'(\boldsymbol{C}^n;\mathscr{K}_p^E)$ becomes an FS space.

Proposition 2.3.
(1). *The transposed mapping* ${}^tr:\mathrm{Exp}'(V_p)\to\mathrm{Exp}'(\boldsymbol{C}^n;\mathscr{K}_p^E)$ *is a topological linear isomorphism and* $\mathrm{Exp}'(\boldsymbol{C}^n;\mathscr{K}_p^E)$ *is a subspace of* $\mathrm{Exp}'(\boldsymbol{C}^n)_p$.
(2). *If* $\mathscr{K}_p^E$ *is the principal ideal of* $\mathrm{Exp}(\boldsymbol{C}^n)$ *generated by p, then we have*

$$\mathrm{Exp}'(\boldsymbol{C}^n;\mathscr{K}_p^E)=\mathrm{Exp}'(\boldsymbol{C}^n)_p.$$

Proof. (1). It is easily seen that the transposed mapping ${}^tr$ is linear, continuous and injective. Indeed, for any $S\in\mathrm{Exp}'(V_p)$ and $f\in\mathscr{K}_p^E$, we have

$$\langle{}^tr(S),f\rangle=\langle S,r(f)\rangle=0.$$

This implies that

$${}^tr(\mathrm{Exp}'(V_p))\subset\mathrm{Exp}'(\boldsymbol{C}^n;\mathscr{K}_p^E).$$

Let T be an element of $\mathrm{Exp}'(\boldsymbol{C}^n;\mathscr{K}_p^E)$. Since $r:\mathrm{Exp}(\boldsymbol{C}^n)\to\mathrm{Exp}(V_p)$ is surjective and $\ker r\subset\ker T$, there exists a unique linear mapping $S:\mathrm{Exp}(V_p)\to\boldsymbol{C}$ such that $T=S\circ r$. If U is an open subset of $\boldsymbol{C}$, then $r(T^{-1}(U))=S^{-1}(U)$ since r is surjective. On the other hand, because r is an open mapping, S is a continuous mapping. Hence the mapping S belongs to $\mathrm{Exp}'(V_p)$. Moreover, since ${}^tr(S)=T$, we obtain the surjectivity of ${}^tr$. By the closed graph theorem for FS spaces, we get the first assertion. The second assertion is clear from the definitions of $\mathrm{Exp}'(\boldsymbol{C}^n)_p$ and $\mathscr{K}_p^E$.

(2). If $\mathscr{K}_p^E$ is the principal ideal of $\mathrm{Exp}(\boldsymbol{C}^n)$ generated by a polynomial p, then for $f\in\mathscr{K}_p^E$ there exists a function $g\in\mathrm{Exp}(\boldsymbol{C}^n)$ such that $f=pg$. So, if $T\in\mathrm{Exp}'(\boldsymbol{C}^n)_p$ and $f\in\mathscr{K}_p^E$, then $T(f)=T(pg)=pT(g)=0$ and hence $T\in\mathrm{Exp}'(\boldsymbol{C}^n;\mathscr{K}_p^E)$. □

From Propositions 2.2 and 2.3, we have the following corollary.

Corollary 2.4. *Let p be a polynomial on* $\boldsymbol{C}^n$. *If* $\mathscr{K}_p^E$ *is the principal ideal of* $\mathrm{Exp}(\boldsymbol{C}^n)$ *generated by p, then the composed mapping*

$$\mathscr{F} \circ {}^t r : \mathrm{Exp}'(V_p) \to \mathscr{O}_{\partial p}(\boldsymbol{C}^n)$$

is a topological linear isomorphism. We will abbreviate $\mathscr{F} \circ {}^t r$ *to* $\mathscr{F}$.

3. Proof of Theorem 1.2.

In this section, we shall prove Theorem 1.2. We need some lemmas and propositions for a proof.

First of all, we consider the exponential growth in one variable case. Let p be a polynomial defined by $p(z) = a_0 z^d + a_1 z^{d-1} + \cdots + a_d$ on $\boldsymbol{C}$. We fix a complex number ξ and a positive number r. Owing to the Pólya-Szegö result [**12**, p. 86, problem 66], we can find a positive number $0 < \rho \le r$ such that

$$2|a_0|\left(\frac{r}{4}\right)^d \le |p(\zeta)| \quad \text{for any } \zeta \in \{z \in \boldsymbol{C}; |z-\xi| = \rho\}.$$

Let f be a holomorphic function on $\{z \in \boldsymbol{C}; |z - \xi| \le r\}$ satisfying

$$|p(z)f(z)| \le Me^{A|z|} \quad \text{for some } A > 0,\ M \ge 0.$$

Applying the maximal principle of holomorphic functions to f, we find $\zeta_0 \in \{z \in \boldsymbol{C}; |z-\xi| = \rho\}$ such that

$$|p(\zeta_0)f(\xi)| \le |p(\zeta_0)f(\zeta_0)|.$$

Thus we have the following Lemma, putting $c = 4^d/2r^d$.

Lemma 3.1. *Fix a polynomial* p *on* $\boldsymbol{C}$, *an element* $\xi \in \boldsymbol{C}$ *and an* $r > 0$. *Suppose* f *is a holomorphic function on* $\{z \in \boldsymbol{C}; |z - \xi| \le r\}$ *satisfying* $|p(z)f(z)| \le Me^{A|z|}$ *for some* $A > 0$ *and* $M \ge 0$. *Then we have* $|a_0 f(\xi)| \le ce^{A(|\xi|+r)}M$, *where* c *is a positive constant depending only on* p *and* r.

Next, we shall extend Lemma 3.1 to the n-variable case. Let p be a non-zero polynomial on $\boldsymbol{C}^n$ and fix any $\xi = (\xi_1, \dots, \xi_n)$ in $\boldsymbol{C}^n$ and any $r > 0$. Suppose f is a holomorphic function on the polydisk $\{z = (z_1, \dots, z_n) \in \boldsymbol{C}^n; |z_i - \xi_i| \le r\ (1 \le i \le n)\}$ satisfying $|p(z)f(z)| \le Me^{A|z|}$ for some $A > 0$ and $M \ge 0$. First, we fix $z' = (z_1, \dots, z_{n-1})$ in $\{z' \in \boldsymbol{C}^{n-1}; |z_i - \xi_i| \le r\ (1 \le i \le n-1)\}$, and regard p as a polynomial of the single variable z_n with degree d. Let c be a positive constant $4^d/2r^d$. By Lemma 3.1, we have $|\tilde{p}(z')f(z', \xi_n)| \le ce^{A(|z'|+|\xi_n|+r)}M$, where $\tilde{p}(z')$ be the coefficient of p with respect to z_n^d. Here we used the inequality

$|p(z', z_n)f(z', z_n)| \leq Me^{A(|z'|+|z_n|)}$. By iteration, there exists a positive constant $\hat{c}$ depending only on p and r such that

$$|f(\xi)| \leq \hat{c}e^{A(|\xi_1|+\cdots+|\xi_n|+nr)}M.$$

Applying the Cauchy-Schwarz inequality $(1 \cdot |\xi_1| + \cdots + 1 \cdot |\xi_n|)^2 \leq n|\xi|^2$, we obtain the following lemma.

LEMMA 3.2. *Fix a non-zero polynomial p on $\boldsymbol{C}^n$, an element $\xi = (\xi_1, \ldots, \xi_n) \in \boldsymbol{C}^n$ and an $r > 0$. Suppose f is a holomorphic function on the polydisk $\{z = (z_1, \ldots, z_n) \in \boldsymbol{C}^n; |z_i - \xi_i| \leq r \ (1 \leq i \leq n)\}$ satisfying $|p(z)f(z)| \leq Me^{A|z|}$ for some $A > 0$ and $M \geq 0$. Then $|f(\xi)|e^{-\sqrt{n}A|\xi|} \leq \hat{c}e^{nrA}M$, where $\hat{c}$ is a positive constant depending only on p and r.*

Now, we recall the definition of $E_A = \{f \in \mathscr{O}(\boldsymbol{C}^n); \|f\|_A < \infty\}$, where $\|f\|_A = \sup_{z \in \boldsymbol{C}^n}\{|f(z)|e^{-A|z|}\}$. We have the following proposition about global exponential growth in $\boldsymbol{C}^n$ by Lemma 3.2.

PROPOSITION 3.3. *Fix a non-zero polynomial p on $\boldsymbol{C}^n$ and an $A > 0$. Suppose F is an entire function satisfying $\|pF\|_A < \infty$. Then $\|F\|_{\sqrt{n}A} \leq c_A\|pF\|_A$, where c_A is a positive constant depending only on p and A.*

PROOF. We fix an $r > 0$. Since $|p(z)F(z)| \leq e^{A|z|}\|pF\|_A$ for any $z \in \boldsymbol{C}^n$, by Lemma 3.2 there exists a positive constant $\hat{c}$ depending only on p and r such that

$$|F(z)| \leq \hat{c}e^{nrA}e^{\sqrt{n}A|z|}\|pF\|_A.$$

Setting $c_A = \hat{c}e^{nrA}$, which depends on p, A and the fixed positive constant r, we have

$$\|F\|_{\sqrt{n}A} = \sup_{z \in \boldsymbol{C}^n} |F(z)|e^{-\sqrt{n}A|z|} \leq c_A\|pF\|_A.$$

□

We have the following proposition by Proposition 3.3.

PROPOSITION 3.4. *Let p be a polynomial on $\boldsymbol{C}^n$. The continuous map $\sigma_p : \mathrm{Exp}(\boldsymbol{C}^n) \ni f \mapsto pf \in \mathrm{Exp}(\boldsymbol{C}^n)$ is a closed mapping.*

PROOF. Since the proposition is clear if $p \equiv 0$, we may assume that p is a non-zero polynomial. Let Z be a closed subset of $\mathrm{Exp}(\boldsymbol{C}^n)$. We take a sequence $\{pf_m\}$ in $\sigma_p(Z)$ such that $pf_m \to g \ (m \to \infty)$ for some $g \in \mathrm{Exp}(\boldsymbol{C}^n)$. By the property of the inductive limit topology, there exists some $A > 0$ such that

$pf_m \to g$ $(m \to \infty)$ in E_A. On the other hand, by Proposition 3.3, we can see that $\{f_m\}$ is a Cauchy sequence in the Banach space $E_{\sqrt{n}A}$, and hence we find a unique element f in $E_{\sqrt{n}A}$ such that $f_m \to f$. In addition, since Z is closed, $f \in Z \cap E_{\sqrt{n}A}$. Hence, $pf_m \to pf$ in $E_{\sqrt{n}A+1}$ and $pf = g$, because $\mathrm{Exp}(\boldsymbol{C}^n)$ is a Hausdorff space. Thus the sequence $\{pf_m\}$ is convergent in $\sigma_p(Z)$. Therefore $\sigma_p(Z)$ is closed. □

Proof of Theorem 1.2. Let p be a reduced polynomial on $\boldsymbol{C}^n$ and f an entire function such that $f|_{V_p} = 0$. Owing to Rückert Nullstellensatz [**1**], there exists an entire function g such that $f = pg$. (There exists locally such a function near V_p by Rückert Nullstellensatz, which coincides with the holomorphic function f/p on $\boldsymbol{C}^n - V_p$.) Further, if $f \in \mathrm{Exp}(\boldsymbol{C}^n)$, then $g \in \mathrm{Exp}(\boldsymbol{C}^n)$ by Proposition 3.3. Thus, $\mathscr{K}_p^E = \langle p \rangle$ and $\mathrm{Exp}'(V_p) \cong \mathscr{O}_{\partial p}(\boldsymbol{C}^n)$ by Corollary 2.4, where $\langle p \rangle$ is the principal ideal of $\mathrm{Exp}(\boldsymbol{C}^n)$ generated by p, that is, the subspace $\{fp; f \in \mathrm{Exp}(\boldsymbol{C}^n)\}$.

Conversely, if p is not a reduced polynomial, we can find some irreducible polynomial p_1 such that $p = p_1^2 p_2$. Set $q = p_1 p_2$. Obviously, $V_p = V_q$ and $\langle p \rangle \subsetneq \langle q \rangle \subset \mathscr{K}_p^E$. By Proposition 3.4, $\langle p \rangle$ and $\langle q \rangle$ are closed subspaces of the DFS space $\mathrm{Exp}(\boldsymbol{C}^n)$, and each space is a DFS space. We can choose a non-zero continuous linear map $S : \langle q \rangle \to \boldsymbol{C}$ such that $S|_{\langle p \rangle} = 0$. Indeed, for example, for $v \in V_{p_1}$, we define a linear map $T_v : \langle q \rangle \to \boldsymbol{C}$ by $T_v(fq) = f(v)$ for $f \in \mathrm{Exp}(\boldsymbol{C}^n)$. Fix an $A > 0$. By Proposition 3.3, there exists a positive constant c_A such that $|T_v(fq)| = |f(v)| \le c_A e^{\sqrt{n}A|v|} \|fq\|_A$. This means that T_v is a continuous map. It is clear that $T_v \neq 0$ and $T_v|_{\langle p \rangle} = 0$.

Applying Hahn-Banach's Theorem, we have $\hat{S} \in \mathrm{Exp}'(\boldsymbol{C}^n)$ satisfying $\hat{S}|_{\langle q \rangle} = S$. It is clear that $\hat{S} \in \mathrm{Exp}'(\boldsymbol{C}^n)_p$ and $\hat{S} \notin \mathrm{Exp}'(\boldsymbol{C}^n; \mathscr{K}_p^E)$. Thus, $\mathrm{Exp}'(V_p) \not\cong \mathscr{O}_{\partial p}(\boldsymbol{C}^n)$ by Propositions 2.2 and 2.3. □

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