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Erratum to "The homotopy of spaces of maps between real projective spaces"

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The paper "The homotopy of spaces of maps between real projective spaces" ([**3**]) contains some mistakes in the computations of two fundamental groups. The mistakes occur in the proofs of Proposition 2.2, Theorems 5.1 and 5.3. Because all these results are summarized in Theorem 1.4, in this note we will only correct Theorem 1.4. First, in the proof of Lemma 2.2, it is stated that $\pi_1(\mathrm{P}V_{n+1,n}) \cong \boldsymbol{Z}/4$, but this is not true in general. In fact, the correct statement is

$$\pi_1(\mathrm{P}V_{n+1,n}) \cong \begin{cases} Z/4 & \text{if } n \equiv 1,2 \pmod 4, \\ \boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2 & \text{if } n \equiv 0,3 \pmod 4. \end{cases} \tag{0.1}$$

Since the order of $\pi_1(\mathrm{P}V_{n+1,n})$ is 4, this can be easily obtained by using the isomorphism (c.f. [**2**, Theorem 1.6])

$$H^1(\mathrm{P}V_{n+1,n}, \boldsymbol{Z}/2) \cong \begin{cases} \boldsymbol{Z}/2 & \text{if } n \equiv 1,2 \pmod 4, \\ \boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2 & \text{if } n \equiv 0,3 \pmod 4. \end{cases}$$

Next, in the proofs of Theorems 5.1 and 5.3, it is stated that $\pi_1(\mathrm{P}O(n+1)) \cong \boldsymbol{Z}/4$ but it is also not true in general. The correct statement is:

$$\pi_1(\mathrm{P}O(n+1)) \cong \begin{cases} \boldsymbol{Z}/4 & \text{if } n \equiv 1 \pmod 4, \\ \boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2 & \text{if } n \equiv 3 \pmod 4, \end{cases} \tag{0.2}$$

which easily follows from [**1**]. By a method analogous to the one used in [**3**] and by constructing explicit splittings, we obtain:

THEOREM 1.4. *Let* $1 \le m \le n$ *be integers and we write* $(\boldsymbol{Z}/2)^k = \boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2 \oplus \cdots \oplus \boldsymbol{Z}/2$ (*k-times*).

(i) *The induced homomorphisms*

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$$\begin{cases} \alpha_{m,n*} = \alpha^{\boldsymbol{R}}_{m,n*} : \pi_1(V_{n,m}) \to \pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)) \\ \beta_{m,n*} = \beta^{\boldsymbol{R}}_{m,n*} : \pi_1(\mathrm{P}V_{n+1,m+1}) \to \pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)) \end{cases}$$

are isomorphisms when $1 \leq m < n$ *or* $m = n = 1$, *and split monomorphisms if* $m = n \geq 2$.

(ii) *If* $m < n$, *there are isomorphisms*

$$\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)) \cong \begin{cases} \boldsymbol{Z} & \textit{if } (m,n) = (1,2), \\ 0 & \textit{if } 1 \leq m \leq n-2, \\ \boldsymbol{Z}/2 & \textit{if } m = n-1 \geq 2. \end{cases}$$

$$\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)) \cong \begin{cases} \boldsymbol{Z}/2 & \textit{if } 1 \leq m \leq n-2, \\ (\boldsymbol{Z}/2)^2 & \textit{if } m = n-1 \textit{ and } n \equiv 0,3 \pmod 4, \\ \boldsymbol{Z}/4 & \textit{if } m = n-1 \textit{ and } n \equiv 1,2 \pmod 4. \end{cases}$$

(iii) *If* $m = n$, *there are isomorphisms*

$$\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)) \cong \begin{cases} 0 & \textit{if } n = 1, \\ \boldsymbol{Z} & \textit{if } n = 2, \\ (\boldsymbol{Z}/2)^2 & \textit{if } n \geq 3. \end{cases}$$

$$\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)) \cong \begin{cases} \boldsymbol{Z} & \textit{if } n = 1, \\ (\boldsymbol{Z}/2)^2 & \textit{if } n = 2, \\ (\boldsymbol{Z}/2)^3 & \textit{if } n \geq 3 \textit{ and } n \equiv 0,3 \pmod 4, \\ \boldsymbol{Z}/4 \oplus \boldsymbol{Z}/2 & \textit{if } n \geq 5 \textit{ and } n \equiv 1,2 \pmod 4. \end{cases}$$

Proof. If we replace the false statement $\pi_1(\mathrm{P}V_{n+1,n}) = \boldsymbol{Z}/4$ by (0.1), the proof of Proposition 2.2 in [**3**] works in a completely analogous way and we obtain the assertions (i) and (ii). It remains to show (iii). Because the isomorphisms (0.1) and (0.2) are not used for the proof for the based case, it suffices to carry out the proof for the case $G_n = \pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n))$. The case $n = 1$ is trivial and if $n = 2$ exactly the same argument given as in the proof of Theorem 5.1 shows that the order of G_n is 4 and that G_n contains the subgroup $\pi_1(\mathrm{P}O(3)) = \pi_1(SO(3)) = \boldsymbol{Z}/2$ as a direct summand. Hence, $G_2 = (\boldsymbol{Z}/2)^2$.

From now on, assume $n \geq 3$. Exactly the same argument as given in the proof of Theorem 5.3 in [**3**] shows that G_n is an abelian group of order 8. Since $\pi_1(\beta_{n,n})$ is a split monomorphism, G_n contains the subgroup $\pi_1(\mathrm{P}O(n+1))$ as a direct summand. Hence, if $n \equiv 1 \pmod 2$, since the order of $\pi_1(\mathrm{P}O(n+1))$ is 4, there is an isomorphism $G_n \cong \pi_1(\mathrm{P}O(n+1)) \oplus \boldsymbol{Z}/2$ and the assertion is proved by using (0.2). Next, consider the case $n \equiv 0 \pmod 2$. Because, G_n also contains the subgroup $\pi_1(\mathrm{P}O(n+1)) = \pi_1(SO(n+1)) = \boldsymbol{Z}/2$ as a direct summand, $G_n \cong \boldsymbol{Z}/4 \oplus \boldsymbol{Z}/2$ or $G_n \cong (\boldsymbol{Z}/2)^3$. If we recall the induced long exact sequence of $(\dagger)_2$ ([**3**, p. 1180]), this reduced to the short exact sequence

$$0 \to \boldsymbol{Z}/2 \to G_n \xrightarrow{r_{n*}} \pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^{n-1}, \boldsymbol{R}\mathrm{P}^n)) = H_n \to 0. \tag{0.3}$$

If $n \equiv 2 \pmod 4$, since $H_n = \boldsymbol{Z}/4$, there is an isomorphism $G_n \cong \boldsymbol{Z}/4 \oplus \boldsymbol{Z}/2$.

Finally consider the case $n = 4l \equiv 0 \pmod 4$. Let $p_n : SO(n+1) \to \mathrm{P}V_{n+1,n}$ denote the double covering and recall the commutative diagram

$$\begin{array}{ccc} \boldsymbol{Z}/2 \cdot \iota = \pi_1(SO(n+1)) & \xrightarrow{p_{n*}} & \pi_1(\mathrm{P}V_{n+1,n}) \\ {\scriptstyle \beta_{n,n_*}} \downarrow & & \downarrow {\scriptstyle \beta_{n-1,n_*}} \cong \\ G_n = \pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)) & \xrightarrow{r_{n*}} & \pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^{n-1}, \boldsymbol{R}\mathrm{P}^n)) = H_n. \end{array}$$

Let $q_n : O(n+1) \to \mathrm{P}V_{n+1,n}$ be a natural projection and define the map $A_n : S^1 \to O(n+1)$ by $A_n(e^{i\theta}) = \underbrace{A(\theta) \oplus \cdots \oplus A(\theta)}_{2l \text{ times}} \oplus (-1)$, where we set $A(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$. Because $\det(A_n(\theta)) = -1$ for any θ, by using (0.1) we can show that $\pi_1(\mathrm{P}V_{n+1,n}) = \boldsymbol{Z}/2 \cdot p_{n*}(\iota) \oplus \boldsymbol{Z}/2 \cdot q_{n*}(A_n) \cong (\boldsymbol{Z}/2)^2$. Because β_{n-1,n_*} is an isomorphism, we have $H_n = \boldsymbol{Z}/2 \cdot \beta_{n-1,n_*}(p_{n*}(\iota)) \oplus \boldsymbol{Z}/2 \cdot \beta_{n-1,n_*}(q_{n*}(A_n)) \cong (\boldsymbol{Z}/2)^2$.

On the other hand, since β_{n,n_*} is a split monomorphism, there is a subgroup G_n' such that $G_n = \boldsymbol{Z}/2 \cdot \beta_{n,n_*}(\iota) \oplus G_n'$, and (0.3) reduces to the short exact sequence $0 \to \boldsymbol{Z}/2 \to G_n' \xrightarrow{r} \boldsymbol{Z}/2 \cdot \beta_{n-1,n_*}(q_{n*}(A_n)) \to 0$ $(r = r_{n*}|G_n')$.

If we define the map $s_n : S^1 \to \mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)$ by the matrix multiplication $s_n(e^{i\theta})([x_0 : \cdots : x_n]) = [x_0 : \cdots : x_n] \cdot A_n(\theta)$, we can prove the equality $r_{n*}(s_n) = \beta_{n-1,n_*}(q_{n*}(A_n)) \neq 0$. Thus, $s_n \neq 0$ in G_n. Because $s_n \circ s_n(e^{i\theta}) = \mathrm{id}$ for any θ and $\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)$ is an H-space with multiplication induced from the composition of maps, the order of s_n is at most 2. Hence, the order of s_n is just 2. If we define the homomorphism $s : H_n \to G_n$ by $s(\beta_{n-1,n_*}(p_{n*}(\iota))) = \beta_{n,n_*}(\iota)$ and $s(\beta_{n-1,n_*}(q_{n*}(A_n))) = s_n$, it is easy to see that it gives the splitting of (0.3). Therefore, we have an isomorphism $G_n \cong \boldsymbol{Z}/2 \cdot s_n \oplus \boldsymbol{Z}/2 \cdot \beta_{n,n_*}(\iota) \oplus \boldsymbol{Z}/2 \cong (\boldsymbol{Z}/2)^3$. □

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