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Reduction of generalized Calabi-Yau structures

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Abstract. A generalized Calabi-Yau structure is a geometrical structure on a manifold which generalizes both the concept of the Calabi-Yau structure and that of the symplectic one. In view of a result of Lin and Tolman in generalized complex cases, we introduce in this paper the notion of a generalized moment map for a compact Lie group action on a generalized Calabi-Yau manifold and construct a reduced generalized Calabi-Yau structure on the reduced space. As an application, we show some relationship between generalized moment maps and the Bergman kernels, and prove the Duistermaat-Heckman formula for a torus action on a generalized Calabi-Yau manifold.

1. Introduction.

Generalized Calabi-Yau structures introduced by Hitchin [**7**] were developed by Gualtieri [**4**] as a special case of generalized complex structures. It is a geometrical structure defined by a differential form, which generalizes both the concept of the Calabi-Yau structure – a non vanishing holomorphic form of the top degree – and that of the symplectic structure. In this paper, we consider a compact Lie group action on a generalized Calabi-Yau manifold.

A compact Lie group action on a generalized complex manifold was studied by Lin and Tolman in [**8**]. In [**8**], they introduced a notion of generalized moment maps for a compact Lie group action on a generalized complex manifold by generalizing the notion of moment maps for a compact Lie group action on a symplectic manifold. Using this definition, they constructed a generalized complex structure on the reduced space, which is natural up to a transformation by an exact B-field.

In the present paper, we apply the definition of a generalized moment map to a compact Lie group action on a generalized Calabi-Yau manifold, and construct a generalized Calabi-Yau structure on the reduced space. Moreover, we shall show that the reduced generalized Calabi-Yau structure is unique and has the same type as the original generalized Calabi-Yau structure (cf. Section 3).

Theorem A. *Let a compact Lie group G act on a generalized Calabi-Yau manifold (M, φ) in a Hamiltonian way with a generalized moment map $\mu : M \longrightarrow \mathfrak{g}^*$. If G acts freely on $\mu^{-1}(0)$, then the quotient space $M_0 = \mu^{-1}(0)/G$ is a smooth manifold, and inherits a unique generalized Calabi-Yau structure $\widetilde{\varphi}$ which satisfies*

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$$p_0^* \widetilde{\varphi} = i_0^* \varphi,$$

where $i_0 : \mu^{-1}(0) \longrightarrow M$ *is the inclusion and* $p_0 : \mu^{-1}(0) \longrightarrow M_0$ *is the natural projection. Moreover, for each* $p \in \mu^{-1}(0)$,

$$\mathrm{type}(\varphi_p) = \mathrm{type}(\widetilde{\varphi}_{[p]}).$$

The detailed definitions of the theorem are in Section 3. In particular, in the case that the generalized Calabi-Yau structure is induced by a symplectic structure, the reduced form is induced by the reduced symplectic form. In addition we construct an example of a Hamiltonian action on a generalized Calabi-Yau structure which is not induced by either a symplectic structure or a Calabi-Yau one. We then show some relationship between generalized moment maps and Bergman kernels (cf. Example 3.3.2 and 3.3.3 in Section 3).

We next consider that a generalized Calabi-Yau structure φ on a connected manifold M which has constant type k. Then there exists a natural volume form $dm = ((\sqrt{-1})^n/(2^{n-k}))\langle\varphi, \bar{\varphi}\rangle$ defined by φ, which generalizes the Liouville form on a symplectic manifold. Indeed, if φ is a generalized Calabi-Yau structure induced by a symplectic structure ω, then dm coincides with the Liouville form for the symplectic structure ω. Further by assuming that a compact torus T acts on M effectively. Under the assumptions, we shall show the Duistermaat-Heckman formula for the volume form dm (cf. Section 4).

Theorem B. *Let* (M, φ) *be a* $2n$*-dimensional connected generalized Calabi-Yau manifold which has constant type* k*, and suppose that compact* l*-torus* T *acts on* M *effectively and in a Hamiltonian way. In addition, we assume that the generalized moment map* μ *is proper. Then the pushforward* $\mu_*(dm)$ *of the natural volume form* dm *under* μ *is absolutely continuous with respect to the Lebesgue measure on* $\mathfrak{t}^*$ *and the Radon-Nikodym derivative* f *can be written by*

$$f(a) = \int_{M_a} dm_a = \mathrm{vol}(M_a)$$

for each regular value $a \in \mathfrak{t}^*$ *of* μ*, and* dm_a *denotes the measure defined by the natural volume form on the reduced space* $M_a = \mu^{-1}(a)/T$.

This paper is organized as follows. In Section 2 we introduce background materials and the definition of generalized Calabi-Yau structures. In Section 3 we define the notion of generalized moment maps for a Lie group action on a generalized Calabi-Yau manifold, and construct a generalized Calabi-Yau structure on the reduced space. In addition, we discuss some relations between generalized moment maps and Bergman kernels. At last Section, we proved the Duistermaat-Heckman formula for a Hamiltonian torus action on a generalized Calabi-Yau manifold.

2. Generalized Calabi-Yau structures.

In this section we recall the definition of generalized Calabi-Yau structures. For the detail, see [**4**] and [**7**].

2.1. Clifford algebras and the spin representation.

Let V be a real vector space of dimension n, and V^* be the dual space of V. Then the direct sum $V \oplus V^*$ admits a natural indefinite metric of signature (n, n) defined by

$$(X+\alpha,\ Y+\beta) = \frac{1}{2}(\beta(X) + \alpha(Y))$$

for $X+\alpha,\ Y+\beta \in V \oplus V^*$. Let $T(V \oplus V^*) = \oplus_{p=0}^{\infty}(\otimes^p(V \oplus V^*))$ be the tensor algebra of $V \oplus V^*$, and define $\mathscr{I}$ to be the two-sided ideal generated by $\{(X+\alpha)\otimes(X+\alpha) - (X+\alpha,\ X+\alpha) \mid X+\alpha \in V \oplus V^*\}$. Then we call the quotient algebra

$$CL(V \oplus V^*) = T(V \oplus V^*)/\mathscr{I}$$

the Clifford algebra of $V \oplus V^*$. For each $E, F \in CL(V \oplus V^*)$, $E \cdot F$ denotes the multiplication induced by the tensor product.

Consider the exterior algebra $\wedge^* V^*$ and a linear mapping $V \oplus V^* \longrightarrow \mathrm{End}(\wedge^* V^*)$ defined by

$$(X+\alpha)\cdot\varphi = \iota_X\varphi + \alpha\wedge\varphi.$$

Then we have

$$\begin{aligned}(X+\alpha)^2\cdot\varphi &= \iota_X(\alpha\wedge\varphi) + \alpha\wedge\iota_X\varphi \\ &= (\iota_X\alpha)\varphi \\ &= (X+\alpha,\ X+\alpha)\varphi,\end{aligned}$$

so it can be extended to a representation of the Clifford algebra $CL(V \oplus V^*) \longrightarrow \mathrm{End}(\wedge^* V^*)$. This is called the spin representation, and a element $\varphi \in \wedge^* V^*$ is called a spinor.

We define $Pin(V \oplus V^*)$ and $Spin(V \oplus V^*)$, subgroups of the group consists of invertible elements of $CL(V \oplus V^*)$ by

$$\begin{aligned}Pin(V \oplus V^*) &= \{E_1 \cdots E_k \mid k \in \boldsymbol{N}\cup\{0\},\ (E_i, E_i) = \pm 1\},\\ Spin(V \oplus V^*) &= \{E_1 \cdots E_{2k} \mid k \in \boldsymbol{N}\cup\{0\},\ (E_i, E_i) = \pm 1\}.\end{aligned}$$

we call $Pin(V \oplus V^*)$ the pin group, and $Spin(V \oplus V^*)$ the spin group. The following proposition says a geometrical meaning of the pin and spin group.

PROPOSITION 2.1.1 ([**1**], [**4**]). *The pin group and the spin group have following short exact sequences.*

$$1 \longrightarrow \boldsymbol{Z}/2\boldsymbol{Z} \longrightarrow Pin(V \oplus V^*) \longrightarrow O(V \oplus V^*) \longrightarrow 1$$
$$1 \longrightarrow \boldsymbol{Z}/2\boldsymbol{Z} \longrightarrow Spin(V \oplus V^*) \longrightarrow SO(V \oplus V^*) \longrightarrow 1$$

Let $Spin_0(V \oplus V^*)$ denote the identity component of $Spin(V \oplus V^*)$. Then $\wedge^* V^*$ has a $Spin_0(V \oplus V^*)$-invariant bilinear form defined by

$$\langle \varphi, \psi \rangle = (\sigma(\varphi) \wedge \psi)_n,$$

where $()_n$ indicates taking the n-th degree component of the form, and $\sigma : \wedge^* V^* \longrightarrow \wedge^* V^*$ is an anti-homomorphism on $\wedge^* V^*$ defined by

$$\sigma(\varphi_1 \wedge \cdots \wedge \varphi_k) = \varphi_k \wedge \cdots \wedge \varphi_1$$

for each $\varphi_1, \cdots, \varphi_k \in \wedge^1 V^*$.

2.2. Pure spinors and generalized Calabi-Yau structures on a vector space.

Given a spinor $\varphi \in \wedge^* V^*$, we define the annihilator of φ by

$$E_\varphi = \{X + \alpha \in V \oplus V^* \big| \ (X + \alpha) \cdot \varphi = 0\}.$$

We also define the annihilator $E_\varphi \subset (V \oplus V^*) \otimes \boldsymbol{C}$ of a complex spinor $\varphi \in \wedge^* V^* \otimes \boldsymbol{C}$ in a similar way. Since an element $X + \alpha \in E_\varphi$ satisfies

$$\begin{aligned}(X + \alpha, X + \alpha)\varphi &= (X + \alpha)^2 \cdot \varphi \\ &= 0,\end{aligned}$$

we see that if φ is a non-zero spinor or a complex spinor, then E_φ is isotropic with respect to the natural metric on $(V \oplus V^*) \otimes \boldsymbol{C}$. In particular, we have $\dim E_\varphi \leq n$.

DEFINITION 2.2.1. A spinor $\varphi \in \wedge^* V^*$ is called pure if E_φ is maximally isotropic, which means that has the dimension equal to n. A complex spinor $\varphi \in \wedge^* V^* \otimes \boldsymbol{C}$ with the maximal isotropic subspace E_φ is called a complex pure spinor.

REMARK 2.2.2. It is known that if $\varphi \in \wedge^* V^*$ is a pure spinor, then $\varphi \in \wedge^{\mathrm{ev/od}} V^*$, where

$$\wedge^{\mathrm{ev}} V^* = \wedge^0 V^* \oplus \wedge^2 V^* \oplus \cdots,$$
$$\wedge^{\mathrm{od}} V^* = \wedge^1 V^* \oplus \wedge^3 V^* \oplus \cdots,$$

and $\varphi \in \wedge^{\mathrm{ev/od}} V^*$ means that φ belongs in either $\wedge^{\mathrm{ev}} V^*$ or $\wedge^{\mathrm{od}} V^*$.

EXAMPLE 2.2.3. The spinor $1 \in \wedge^0 V^*$ is pure, since $E_1 = V$.

EXAMPLE 2.2.4. A non-zero vector $\varphi \in \wedge^n V^*$ is also pure. The annihilator is $E_\varphi = V^*$.

EXAMPLE 2.2.5. If φ is a pure spinor on V and B is a 2-form, then

$$\exp(B)\varphi = \left(1 + B + \frac{1}{2!}B^2 + \cdots\right) \wedge \varphi$$

is also pure. The annihilator is $E_{\exp(B)\varphi} = \{X + \alpha + \iota_X B \big| \; X + \alpha \in E_\varphi\}$, where E_φ is the annihilator of φ.

Gualtieri shows in his thesis [**4**] that every pure spinor can be written by a complex 2-form and a decomposable complex form as follows.

FACT 2.2.6 ([**4**]). Let φ be a complex pure spinor on V. Then there exists a complex 2-form $B + \sqrt{-1}\omega \in \wedge^2 V^* \otimes \boldsymbol{C}$ and a complex k-form Ω such that

$$\varphi = \exp(B + \sqrt{-1}\omega)\Omega.$$

Moreover, Ω can be written

$$\Omega = \theta^1 \wedge \cdots \wedge \theta^k$$

by some 1-forms $\theta^1, \cdots, \theta^k \in \wedge^1 V^* \otimes \boldsymbol{C}$, and ω is nondegenerate on a subspace $W = \{X \in V \big| \; \iota_X \Omega = 0\}$.

The degree of the form Ω is called the type of the complex pure spinor φ and written by type(φ).

Now we give the definition of a generalized Calabi-Yau structure on a real vector space V.

DEFINITION 2.2.7. Let V be a real vector space of dimension $n = 2m$. A generalized Calabi-Yau structure on V is a complex pure spinor $\varphi \in \wedge^{\mathrm{ev/od}} V^* \otimes \boldsymbol{C}$ which satisfies that $\langle \varphi, \bar{\varphi} \rangle \neq 0$.

Fact 2.2.6 tells us if there exists a complex pure spinor on V which satisfies $\langle \varphi, \bar{\varphi} \rangle \neq 0$, then V must be even dimensional. The condition $\langle \varphi, \bar{\varphi} \rangle \neq 0$ has the following geometrical meaning.

FACT 2.2.8 ([**1**]). Let φ and ψ be pure spinors. Then they satisfy $\langle \varphi, \psi \rangle \neq 0$ if and only if their annihilators E_φ and E_ψ satisfy $E_\varphi \cap E_\psi = \{0\}$.

For the proof, see III.2.4 in [**1**].

EXAMPLE 2.2.9. For a symplectic form ω on V, we put

$$\varphi_\omega = \exp\sqrt{-1}\omega.$$

Then we have $E_{\varphi_\omega} = \{X - \sqrt{-1}\iota_X\omega \mid X \in V \otimes \boldsymbol{C}\}$ and $\dim E_{\varphi_\omega} = n$. Since ω is non-degenerate, we have $\langle\varphi_\omega, \bar{\varphi}_\omega\rangle = ((-2\sqrt{-1})^m/m!)\omega^m \neq 0$. Hence φ_ω is a generalized Calabi-Yau structure on V. The type of φ_ω is equal to 0.

Example 2.2.10. If V has a complex structure J, then for the $\sqrt{-1}$-eigenspace $V^{1,0}$ of $J^* : V^* \otimes \boldsymbol{C} \longrightarrow V^* \otimes \boldsymbol{C}$, $\wedge^m V^{1,0}$ is one-dimensional complex vector space. Let Ω be a non-zero vector in $\wedge^m V^{1,0}$. Then, we have $E_\Omega = V_{0,1} \oplus V^{1,0}$ and $\langle\Omega, \bar{\Omega}\rangle = (-1)^m \Omega \wedge \bar{\Omega} \neq 0$. So Ω is a generalized Calabi-Yau structure on V. The type of Ω is equal to m.

Example 2.2.11. Let φ be a generalized Calabi-Yau structure on V. For each $B \in \wedge^2 V^*$, the previous example shows that $\exp(B)\varphi$ is pure. Moreover, the bilinear form gives $\langle\exp(B)\varphi, \overline{\exp(B)\varphi}\rangle = \langle\varphi, \bar{\varphi}\rangle \neq 0$. Hence $\exp(B)\varphi$ is also a generalized Calabi-Yau structure on V. The type of $\exp(B)\varphi$ coincides with that of φ.

Example 2.2.12. If φ_1 and φ_2 are two generalized Calabi-Yau structures on two vector spaces V_1 and V_2, and p_1, p_2 are the projections from the direct sum $V_1 \oplus V_2$. Then $\varphi = p_1^*\varphi_1 \wedge p_2^*\varphi_2$ is a generalized Calabi-Yau structure on the product. The type of φ is equal to the sum type(φ_1) + type(φ_2).

2.3. Generalized Calabi-Yau structures on a manifold.

Let M be a smooth manifold of dimension $2n$, and consider the direct sum $TM \oplus T^*M$ of the tangent bundle and the cotangent bundle. Then there is an indefinite metric on the vector bundle $TM \oplus T^*M$ defined by $(X + \alpha, Y + \beta) = \frac{1}{2}(\beta(X) + \alpha(Y))$.

Definition 2.3.1 ([**7**]). A generalized Calabi-Yau structure on a manifold M is a closed differential form $\varphi \in \Omega^{\mathrm{ev/od}} \otimes \boldsymbol{C}$ which satisfies the following conditions.

- For each $p \in M$, φ_p is a complex pure spinor on $(T_pM \oplus T_p^*M) \otimes \boldsymbol{C}$.
- At each point, $\langle\varphi, \bar{\varphi}\rangle \neq 0$.

Remark 2.3.2. Generalized Calabi-Yau structures were defined by Hitchin in [**7**]. If a generalized Calabi-Yau structure φ is given, then the annihilator E_φ defines a generalized complex structure in the sense of Hitchin [**7**]. This shows that a generalized Calabi-Yau manifold is a special case of a generalized complex manifold. For the detail, see Proposition 1 in [**7**].

Example 2.3.3. Let M be a $2n$-dimensional symplectic manifold with the symplectic form ω, and put

$$\varphi_\omega = \exp\sqrt{-1}\omega.$$

Then we have $E_{\varphi_\omega} = \{X - \sqrt{-1}\iota_X\omega \mid X \in T \otimes \boldsymbol{C}\}$ and $\langle\varphi_\omega, \bar{\varphi}_\omega\rangle = ((-2\sqrt{-1})^n/n!)\omega^n \neq 0$. Since ω is closed, φ_ω is also closed. Hence φ_ω is a generalized Calabi-Yau structure on M.

EXAMPLE 2.3.4. Let M be an n-dimensional complex manifold with a non-vanishing holomorphic n-form Ω. Then Ω is pure since $E_\Omega = T_{0,1} \oplus T^{1,0}$. In addition, the bilinear form gives $\langle \Omega, \bar{\Omega} \rangle = (-1)^n \Omega \wedge \bar{\Omega}$, which is non-vanishing. Since Ω is closed, Ω is a generalized Calabi-Yau structure on M.

EXAMPLE 2.3.5. If B is a closed 2-form on a generalized Calabi-Yau manifold (M, φ), then $\exp(B)\varphi$ is also a closed form. By the previous example, $\exp(B)\varphi$ is pure and $\langle \exp(B)\varphi, \overline{\exp(B)\varphi} \rangle \neq 0$ at each point. So $\exp(B)\varphi$ is also a generalized Calabi-Yau structure on M. This is called the B-field transform of φ.

EXAMPLE 2.3.6. If (M_1, φ_1) and (M_2, φ_2) are two generalized Calabi-Yau manifolds and p_1, p_2 are the projections from the product manifold $M_1 \times M_2$. Then $\varphi = p_1^* \varphi_1 \wedge p_2^* \varphi_2$ is a generalized Calabi-Yau structure on the product. In particular, a product manifold of generalized Calabi-Yau manifolds is also a generalized Calabi-Yau manifold.

The local expression of a generalized Calabi-Yau structure is given by the following proposition by Gualtieri [**4**]. This helps us to prove the Duistermaat-Heckman formula later.

FACT 2.3.7 ([**4**]). An element of a generalized Calabi-Yau manifold (M, φ) is said to be regular if it has a neighborhood where the type of φ is constant. If $p \in M$ is regular, then for sufficiently small neighborhood U_p of p, there exists a complex 2-form $B + \sqrt{-1}\omega \in \Omega^2(U_p) \otimes \boldsymbol{C}$ such that

$$\varphi = \exp(B + \sqrt{-1}\omega)\varphi_k \text{ on } U_p,$$

where k is the type of φ_p. Moreover, φ_k can be written

$$\varphi_k = \theta^1 \wedge \cdots \wedge \theta^k$$

by some 1-forms $\theta^1, \cdots, \theta^k \in \Omega^1(U_p) \otimes \boldsymbol{C}$.

3. Reduction of generalized Calabi-Yau structures.

3.1. Generalized moment maps.

In this section we define the notion of generalized moment maps for a compact Lie group action on a generalized Calabi-Yau manifold, and construct a generalized Calabi-Yau structure on the reduced space. The definition of generalized moment maps for generalized complex cases is given by Lin and Tolman [**8**].

DEFINITION 3.1.1. Let a compact Lie group G with its Lie algebra $\mathfrak{g}$ act on a generalized Calabi-Yau manifold (M, φ) preserving φ. A generalized moment map is a smooth function $\mu : M \longrightarrow \mathfrak{g}^*$ which satisfies

- μ is G-equivariant, and
- $\xi_M - \sqrt{-1}d\mu^\xi$ lies in E_φ for all $\xi \in \mathfrak{g}$, where ξ_M denotes the induced vector field on M and μ^ξ is the smooth function defined by $\mu^\xi(p) = \mu(p)(\xi)$.

A G-action which preserves the generalized Calabi-Yau structure φ is called Hamiltonian if a generalized moment map exists.

Here are some examples of generalized moment maps.

Example 3.1.2. Let G act on a symplectic manifold (M,ω) preserving ω, and $\mu : M \longrightarrow \mathfrak{g}^*$ be a moment map. Then G also preserves the generalized Calabi-Yau structure $\varphi_\omega = \exp \sqrt{-1}\omega$, and μ is also a generalized moment map.

Example 3.1.3. Let G act on a connected Calabi-Yau n-fold (M,Ω), where Ω is a non-vanishing holomorphic n-form. If the G-action is Hamiltonian, ξ_M must be anti-holomorphic for all $\xi \in \mathfrak{g}$. However induced vector fields must be real, so we have $\xi_M = 0$. In particular, the G-action is trivial and the generalized moment map is regarded as a linear functional on the Lie algebra $\mathfrak{g}$.

Example 3.1.4. If G acts on two generalized Calabi-Yau manifolds (M_1,φ_1) and (M_2,φ_2), preserving both φ_1 and φ_2. Let μ_1 and μ_2 are generalized moment maps for these actions. Then the diagonal action of G on the product manifold $M_1 \times M_2$ preserves the generalized Calabi-Yau structure $\varphi = p_1^*\varphi_1 \wedge p_2^*\varphi_2$, where p_1 and p_2 are the projections from the product $M_1 \times M_2$. Moreover $\mu = \mu_1 \circ p_1 + \mu_2 \circ p_2$ is a generalized moment map for this action.

3.2. Generalized Calabi-Yau structure on the reduced space.

Let a compact Lie group G act on a generalized Calabi-Yau manifold (M,φ) in a Hamiltonian way with a generalized moment map $\mu : M \longrightarrow \mathfrak{g}^*$. Suppose that G acts freely on $\mu^{-1}(0)$. Then 0 is a regular value and the quotient space

$$M_0 = \mu^{-1}(0)/G$$

is a manifold. The purpose of 3.2 is to prove Theorem A in Introduction. By restricting to an appropriate neighborhood of $\mu^{-1}(0)$, we may assume that G acts freely on M. The following lemmas are required for the proof of the theorem.

Lemma 3.2.1. *Under the assumptions above, let $\mathfrak{g}_M$ be the subbundle of TM generated by the fundamental vector fields ξ_M for $\xi \in \mathfrak{g}$, and $d\mu$ be the subbundle of T^*M generated by the differential $d\mu^\eta$ for $\eta \in \mathfrak{g}$. Then we have*

(1) $T_p\mu^{-1}(0) = (d\mu)_p^0$,
(2) $\ker(p_{0*})_p = (\mathfrak{g}_M)_p$, *and*
(3) $T_{[p]}M_0 \cong T_p\mu^{-1}(0)/(\mathfrak{g}_M)_p = (d\mu)_p^0/(\mathfrak{g}_M)_p$,

where $p \in \mu^{-1}(0)$ and $(d\mu)_p^0 = \{X \in T_pM \,|\, (d\mu^\xi)_p(X) = 0\ (\xi \in \mathfrak{g})\}$ is the annihilator of $(d\mu)_p$.

Proof. For each $\xi \in \mathfrak{g}$, the smooth function μ^ξ vanishes on $\mu^{-1}(0)$. So $(d\mu^\xi)(X) = 0$ for all $X \in T_p\mu^{-1}(0)$. This implies that $T_p\mu^{-1}(0) \subset (d\mu)_p^0$. In addition, because $\dim T_p\mu^{-1}(0) = \dim(d\mu)_p^0$, the first claim holds. Since $(\mathfrak{g}_M)_p \subset \ker(p_{0*})_p$ and

$p_0 : \mu^{-1}(0) \longrightarrow M_0$ is a submersion, the second claim holds. Now it is easy to see the last claim. □

The following lemma will help us to prove that the reduced form does not vanish anywhere.

LEMMA 3.2.2. *Under the assumptions above, let* $\pi : (TM \oplus T^*M) \otimes \boldsymbol{C} \longrightarrow TM \otimes \boldsymbol{C}$ *be the natural projection. Then we have*

$$\dim_{\boldsymbol{C}}(T_p\mu^{-1}(0) \otimes \boldsymbol{C}) \cap \pi(E_\varphi)_p = \dim_{\boldsymbol{C}} \pi(E_\varphi)_p - \dim G$$

for each $p \in \mu^{-1}(0)$.

PROOF. For a subspace $W \subset (TM \oplus T^*M)_p \otimes \boldsymbol{C}$, we denote by $W^\perp$ the annihilator of W with respect to the natural metric on $(TM \oplus T^*M)_p \otimes \boldsymbol{C}$. Then, since E_φ is maximal isotropic, we have

$$E_\varphi = E_\varphi^\perp, \text{ and } W^\perp \cap (E_\varphi)_p = (W + (E_\varphi)_p)^\perp.$$

If $X \in (T_p\mu^{-1}(0) \otimes \boldsymbol{C}) \cap \pi(E_\varphi)_p$, then it satisfies that

$$X \in \pi(E_\varphi)_p, \text{ and } d\mu^\xi(X) = 0$$

for each $\xi \in \mathfrak{g}$. Thus we have $X \in \pi((\mathfrak{g}_M \otimes \boldsymbol{C})^\perp \cap E_\varphi)_p$. Conversely, if $X \in \pi((\mathfrak{g}_M \otimes \boldsymbol{C})^\perp \cap E_\varphi)_p$, then we also have $d\mu^\xi(X) = 0$ for each $\xi \in \mathfrak{g}$. So we have $X \in (T_p\mu^{-1}(0) \otimes \boldsymbol{C}) \cap \pi(E_\varphi)_p$. This shows that

$$(T_p\mu^{-1}(0) \otimes \boldsymbol{C}) \cap \pi(E_\varphi)_p = \pi((\mathfrak{g}_M \otimes \boldsymbol{C})^\perp \cap E_\varphi)_p.$$

Since the kernel of $\pi : (TM \oplus T^*M)_p \otimes \boldsymbol{C} \longrightarrow T_pM \otimes \boldsymbol{C}$ is equal to $T_p^*M \otimes \boldsymbol{C}$, we have

$$\begin{aligned}(\mathfrak{g}_M \otimes \boldsymbol{C})_p^\perp \cap (E_\varphi)_p \cap T_p^*M \otimes \boldsymbol{C} &= ((\mathfrak{g}_M \otimes \boldsymbol{C}) + E_\varphi)_p^\perp \cap T_p^*M \otimes \boldsymbol{C} \\ &= \pi((\mathfrak{g}_M \otimes \boldsymbol{C}) + E_\varphi)_p^0 \\ &= \pi(E_\varphi)_p^0,\end{aligned}$$

and thus

$$\dim_{\boldsymbol{C}} \pi((\mathfrak{g}_M \otimes \boldsymbol{C})^\perp \cap E_\varphi)_p = \dim_{\boldsymbol{C}}(\mathfrak{g}_M \otimes \boldsymbol{C})_p^\perp \cap (E_\varphi)_p - \dim_{\boldsymbol{C}} \pi(E_\varphi)_p^0.$$

In addition, by $(\mathfrak{g}_M \otimes \boldsymbol{C})_p \cap (E_\varphi)_p = \{0\}$, we obtain the dimension

$$\begin{aligned}\dim_{\boldsymbol{C}}(\mathfrak{g}_M \otimes \boldsymbol{C})_p^\perp \cap (E_\varphi)_p &= \dim_{\boldsymbol{C}}((\mathfrak{g}_M \otimes \boldsymbol{C}) + E_\varphi)_p^\perp \\ &= \dim M - \dim G.\end{aligned}$$

Hence we have

$$\begin{aligned}\dim_{\boldsymbol{C}}\big(T_p\mu^{-1}(0)\otimes \boldsymbol{C}\big)\cap \pi(E_\varphi)_p &= \dim_{\boldsymbol{C}} \pi((\mathfrak{g}_M\otimes \boldsymbol{C})^\perp \cap (E_\varphi))_p \\ &= \dim_{\boldsymbol{C}}(\mathfrak{g}_M\otimes \boldsymbol{C})_p^\perp \cap (E_\varphi)_p - \dim_{\boldsymbol{C}} \pi(E_\varphi)_p^0 \\ &= \dim_{\boldsymbol{C}} \pi(E_\varphi)_p - \dim G,\end{aligned}$$

this completes the proof. □

PROOF OF THEOREM A. For each $p\in\mu^{-1}(0)$, we denote by $(\varphi_s)_p$ the s-th degree component of $\varphi_p\in\wedge^{\mathrm{ev/od}}T_p^*M\otimes \boldsymbol{C}$. Then, by the definition of the generalized moment map, we have

$$\iota_{\xi_M}\varphi_s-\sqrt{-1}d\mu^\xi\wedge\varphi_{s-2}=0$$

for each $\xi\in\mathfrak{g}$. Moreover, the identity $T_p\mu^{-1}(0)=(d\mu)_p^0$ in Lemma 3.2.1 tells us that the $(s-1)$-form $\iota_{(\xi_M)_p}(\varphi_s)_p$ vanishes on $T_p\mu^{-1}(0)$. So by identifying the tangent space $T_{[p]}M_0$ with $T_p\mu^{-1}(0)/(\mathfrak{g}_M)_p$ (see Lemma 3.2.1, (3)), we obtain a well-defined complex s-form $(\tilde\varphi_s)_{[p]}$ on $T_{[p]}M_0$ by

$$(\tilde\varphi_s)_{[p]}([X_1],\cdots,[X_s])=(i_0^*\varphi_s)_p(X_1,\cdots,X_s),$$

where $X_1,\cdots,X_s\in T_p\mu^{-1}(0)$. Thus we have a complex form $(\tilde\varphi)_{[p]}\in\wedge^{\mathrm{ev/od}}T_{[p]}^*M_0\otimes\boldsymbol{C}$ defined by

$$(\tilde\varphi)_{[p]}=(\tilde\varphi_k)_{[p]}+(\tilde\varphi_{k+2})_{[p]}+\cdots,$$

where k is the type of φ_p. G-invariance of the form φ tells us that the definition of $(\widetilde\varphi)_{[p]}$ does not depend on a representative $p\in\mu^{-1}(0)$. So we get the reduced form $\widetilde\varphi\in\Omega^{\mathrm{ev/od}}\otimes\boldsymbol{C}$. It is clear that $\widetilde\varphi$ satisfies that $p_0^*\widetilde\varphi=i_0^*\varphi$ and $d\widetilde\varphi=0$.

Next we shall show that $(\widetilde\varphi)_{[p]}\neq0$. It is sufficient to show that $(i_0^*\varphi_k)_p\neq0$. Suppose that $\dim M=2n$ and $\dim G=l$. Then Lemma 3.2.2 tells us

$$\dim_{\boldsymbol{C}}(T_p\mu^{-1}(0)\otimes\boldsymbol{C})\cap\pi(E_\varphi)_p=2n-k-l.$$

So we can take a basis

$$e_1,\cdots,e_{2n-k-l},u_1,\cdots,u_k,v_1,\cdots,v_l$$

of $T_pM\otimes\boldsymbol{C}$, where $\{e_1,\cdots,e_{2n-k-l},u_1,\cdots,u_k\}$ is a basis of $T_p\mu^{-1}(0)\otimes\boldsymbol{C}$, and $\{e_1,\cdots,e_{2n-k-l},v_1,\cdots,v_l\}$ is a basis of $\pi(E_\varphi)$. Since $(\varphi_k)_p\neq0$, so we have

$$(\varphi_k)_p(u_1,\cdots,u_k)\neq0.$$

This shows that $(i_0^*\varphi_k)_p \neq 0$.

Now we say that an element $\tilde{X}+\tilde{\alpha} \in (TM_0 \oplus T^*M_0)_{[p]} \otimes \boldsymbol{C}$ satisfies the compatibility condition if there exists $X \in T_p\mu^{-1}(0) \otimes \boldsymbol{C}$ and $\alpha \in T_p^*M \otimes \boldsymbol{C}$ such that $(p_{0*})_p X = \tilde{X}$, $p_0^*\tilde{\alpha} = i_0^*\alpha$, and that $(i_{0*})_p(X) + \alpha \in (E_\varphi)_p$. We denote by E_0 the set of elements $\tilde{X}+\tilde{\alpha} \in (TM_0 \oplus T^*M_0)_{[p]} \otimes \boldsymbol{C}$ which satisfy the compatibility condition. Then, for each $\tilde{X}+\tilde{\alpha} \in E_0$, we have

$$\begin{aligned} p_0^*(\iota_{\tilde{X}}\tilde{\varphi} + \tilde{\alpha} \wedge \tilde{\varphi}) &= i_0^*(\iota_{(i_{0*})X}\varphi + \alpha \wedge \varphi) \\ &= 0. \end{aligned}$$

So we can see $E_0 \subset E_{\tilde{\varphi}}$ because p_0 is a submersion. Moreover, since $E_{\tilde{\varphi}}$ is isotropic, we have $\dim_{\boldsymbol{C}} E_0 \leq \dim_{\boldsymbol{C}} E_{\tilde{\varphi}} \leq 2(n-l)$. Let us show the equality $\dim_{\boldsymbol{C}} E_0 = 2(n-l)$. Since $\dim_{\boldsymbol{C}}(T_p\mu^{-1}(0) \otimes \boldsymbol{C}) \cap \pi(E_\varphi)_p = 2n-k-l$, we can take

$$X_1+\alpha_1, \cdots, X_{2n-l-k}+\alpha_{2n-l-k} \in E_\varphi,$$

which are linearly independent and $X_i \in T_p\mu^{-1}(0) \cap \pi(E_\varphi)_p$ for $i = 1, \cdots, 2n-l-k$. Since

$$\iota_{\xi_M}\alpha_i = (\alpha_i, \xi_M) = (X_i + \alpha_i, \xi_M - d\mu^\xi) = 0$$

for each $\xi \in \mathfrak{g}$, α_i descends to a form $\tilde{\alpha}_i \in \wedge^{\mathrm{ev/od}} T^*_{[p]} M_0 \otimes \boldsymbol{C}$. If we take

$$\tilde{X}_i = (p_{0*})_p X,$$

then we have $\tilde{X}_i + \tilde{\alpha}_i \in E_0$. Furthermore, since $\ker(p_{0*})_p = (\mathfrak{g}_M)_p$ has dimension l, and it is contained in $T_p\mu^{-1}(0) \cap \pi(E_\varphi)_p$, so we may assume that

$$\tilde{X}_1 + \tilde{\alpha}_1, \cdots, \tilde{X}_{2(n-l)-k} + \tilde{\alpha}_{2(n-l)-k}$$

are linearly independent.

On the other hand, by Fact 2.2.6, we can take $\theta^1, \cdots, \theta^k \in T_p^*M \otimes \boldsymbol{C}$ which satisfy

$$(\varphi_k)_p = \theta^1 \wedge \cdots \wedge \theta^k.$$

Then, since $(\varphi_k)_p$ satisfies $\iota_{\xi_M}(\varphi_k)_p = 0$ for each $\xi \in \mathfrak{g}$, so does θ^i for $i = 1, \cdots, k$. Hence θ^i descends to a 1-form $\tilde{\theta}^i \in \wedge^{\mathrm{ev/od}} T^*_{[p]} M_0 \otimes \boldsymbol{C}$. Then $\tilde{\theta}^i \in E_0$, and

$$\begin{aligned} p_0^*(\tilde{\theta}^1 \wedge \cdots \wedge \tilde{\theta}^k) &= i_0^*(\theta^1 \wedge \cdots \wedge \theta^k) \\ &= i_0^*((\varphi_k)_p) \\ &\neq 0. \end{aligned}$$

This shows that $\tilde{\theta}^1, \cdots, \tilde{\theta}^k$ are linearly independent. Thus we have

$$\dim_{\mathbf{C}} E_0 = 2(n-l), \text{ and } E_0 = E_{\tilde{\varphi}},$$

in particular $E_{\tilde{\varphi}}$ is maximal isotropic.

Furthermore, since E_φ does not have a real vector except for 0, neither does E_0. So we also have

$$(E_{\widetilde{\varphi}})_{[p]} \cap (\bar{E}_{\widetilde{\varphi}})_{[p]} = \{0\}.$$

This shows that $\widetilde{\varphi}$ is a generalized Calabi-Yau structure on M_0.

The last claim is clear because $\operatorname{type}(\varphi_p) = \operatorname{type}((\widetilde{\varphi})_{[p]}) = k$. □

REMARK 3.2.3. The reduction for other levels can be done by taking the coadjoint orbit. The detailed statement is as follows. Let a compact Lie group G act on a generalized Calabi-Yau manifold (M, φ) Hamiltonian way with a generalized moment map $\mu : M \longrightarrow \mathfrak{g}^*$. For each $a \in \mathfrak{g}^*$, $\mathscr{O}_a$ denotes the coadjoint orbit of a. Suppose that G acts on $\mu^{-1}(\mathscr{O}_a)$ freely. Then the quotient space $M_a = \mu^{-1}(\mathscr{O}_a)/G$ is a manifold and has unique generalized Calabi-Yau structure $\widetilde{\varphi}$ which satisfies that

$$p_a^* \widetilde{\varphi} = i_a^* \varphi$$

and

$$\operatorname{type}(\varphi_p) = \operatorname{type}(\widetilde{\varphi}_{[p]})$$

for all $p \in \mu^{-1}(\mathscr{O}_a)$, where $i_a : \mu^{-1}(\mathscr{O}_a) \longrightarrow M$ is the inclusion and $p_a : \mu^{-1}(\mathscr{O}_a) \longrightarrow M_a$ is the natural projection. In addition, we have $\dim M_a = \dim M + \dim \mathscr{O}_a - 2 \dim G$.

EXAMPLE 3.2.4. Let G act on a symplectic manifold (M, ω) preserving ω, and let $\mu : M \longrightarrow \mathfrak{g}^*$ be a moment map. Then G also acts on (M, φ_ω) Hamiltonian way and μ is a generalized moment map. Moreover if we assume that G acts freely on $\mu^{-1}(0)$, then we get the reduced symplectic structure $\widetilde{\omega}$ and the reduced generalized Calabi-Yau structure $\widetilde{\varphi}_\omega$ on the reduced space M_0. Then $\widetilde{\varphi}_\omega$ coincides with the generalized Calabi-Yau structure $\varphi_{\tilde{\omega}}$ induced by the reduced symplectic structure $\widetilde{\omega}$.

EXAMPLE 3.2.5. Let G act on a Calabi-Yau manifold (M, Ω). If the G-action is Hamiltonian, then the action is trivial and the generalized moment map μ is regarded as a linear functional on the Lie algebra $\mathfrak{g}$. So the reduced space M_0 coincides with either M or the empty set.

REMARK 3.2.6. Lin and Tolman showed the existence of a generalized complex structure on the reduced space in [**8**]. The generalized complex structure induced by the reduced generalized Calabi-Yau structure coincides with the reduced generalized complex structure from the generalized complex structure induced by the original generalized Calabi-Yau structure.

3.3. Relationship to Bergman kernels.

We introduce a Hamiltonian action on a generalized Calabi-Yau structure which is not induced from either a symplectic structure or a Calabi-Yau one here. Let $D \subset \boldsymbol{C}^{m+n}$ be a Reinhardt bounded domain, that is, a bounded domain which the standard action of $(m+n)$-dimensional torus T^{m+n} on $\boldsymbol{C}^{m+n}$ leaves D invariant. For each $w = (w_1, \cdots, w_m) \in \boldsymbol{C}^m$, D_w denotes the slice of D at w,

$$D_w = \big\{(z_1, \cdots, z_{m+n}) \in D \;\big|\; z_j = w_j \quad (j = 1, \cdots, m)\big\}.$$

If the slice D_w is not empty, we can regard D_w as a Reinhardt bounded domain in $\boldsymbol{C}^n$ naturally. Let

$$K_w(z) = K_w(z, z) : D_w \longrightarrow \boldsymbol{R}$$

be the Bergman kernel function of D_w, and $\Omega_w = ((\sqrt{-1})/2)\partial\bar{\partial}\log K_w$ be the Kähler form of the Bergman metric on D_w. Then the natural action of S^1 on D_w preserves Ω_w, and

$$\mu_w = -\frac{1}{2}\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}\big(\log K_w\big)$$

is a moment map for this action. Note that the function μ_w is real and S^1-invariant since the real function $\log K_w$ is S^1-invariant and the fundamental vector field ξ induced by the S^1-action is given by

$$\xi = \sqrt{-1}\sum_{j=1}^{n}\bigg\{z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j}\bigg\}.$$

Now we assume that the Bergman kernel K_w depends smoothly on w. Then we can define a smooth function K on D by

$$K(w, z) = K_w(z) : D \longrightarrow \boldsymbol{R},$$

and a complex form φ on D by

$$\varphi = dw_1 \wedge \cdots \wedge dw_m \wedge \exp\sqrt{-1}\Omega,$$

where $\Omega = (\sqrt{-1}/2)\partial\bar{\partial}\log K$. It is easy to see that the complex form φ is a generalized Calabi-Yau structure on D, and the S^1-action on D defined by

$$e^{\sqrt{-1}\theta}(w_1, \cdots, w_m, z_1, \cdots, z_n) = (w_1, \cdots, w_m, e^{\sqrt{-1}\theta}z_1, \cdots, e^{\sqrt{-1}\theta}z_n),$$

preserves φ.

Theorem 3.3.1. *Let μ be a smooth function on D defined by*

$$\mu(w_1,\cdots,w_m,z_1,\cdots,z_n)=\mu_w(z_1,\cdots,z_n)=-\frac{1}{2}\sum_{j=1}^{n}z_j\frac{\partial}{\partial z_j}\big(\log K\big).$$

Then the function μ is a generalized moment map for the S^1 action on D defined above.

Proof. Let ξ be the fundamental vector field for this action. Then S^1-invariance of the function $\log K$ implies that μ is a S^1-invariant real-valued function. By simple calculation, we have

$$\begin{aligned}\iota_\xi\Omega\left(\frac{\partial}{\partial z_i}\right)&=\Omega\left(\sqrt{-1}\sum_{j=1}^{n}\left\{z_j\frac{\partial}{\partial z_j}-\bar{z}_j\frac{\partial}{\partial\bar{z}_j}\right\},\frac{\partial}{\partial z_i}\right)\\&=\sqrt{-1}\sum_{j=1}^{n}\bar{z}_j\left(\frac{\sqrt{-1}}{2}\frac{\partial^2}{\partial z_i\partial\bar{z}_j}(\log K)\right)\\&=\frac{\partial}{\partial z_i}\left(-\frac{1}{2}\sum_{j=1}^{n}\bar{z}_j\frac{\partial}{\partial\bar{z}_j}(\log K)\right)\\&=\frac{\partial}{\partial z_i}\left(-\frac{1}{2}\sum_{j=1}^{n}z_j\frac{\partial}{\partial z_j}(\log K)\right)\\&=\frac{\partial\mu}{\partial z_i},\end{aligned}$$

and $\iota_\xi\Omega\big(\frac{\partial}{\partial w_i}\big)=\frac{\partial\mu}{\partial w_i}$ similarly. Hence we have $d\mu=\iota_\xi\Omega$, and we can check easily that μ is a generalized moment map for this action. □

Example 3.3.2. Let D be an $(m+n)$-dimensional polydisc,

$$D=(D^1)^{m+n}=\{(z_1,\cdots,z_{m+n})\mid |z_j|<1\quad(j=1,\cdots,m+n)\}.$$

For each $w\in(D^1)^m=\{(w_1,\cdots,w_m)\in\boldsymbol{C}^m\mid |w_j|<1\quad(j=1,\cdots,m)\}$, D_w denote the slice of D at w,

$$D_w=\{(z_1,\cdots,z_n)\in\boldsymbol{C}^n\mid |z_j|<1\quad(j=1,\cdots,n)\}.$$

Then D_w is a polydisc on $\boldsymbol{C}^n$, and

$$K_w=\frac{1}{\pi^n}\frac{1}{\prod_{j=1}^{n}(1-|z_j|^2)^2}$$

is the Bergman kernel function of D_w. Since the Bergman kernel K_w does not depend on w,

$$K(w,z) = K_w(z) : D \longrightarrow \boldsymbol{R},$$

is a smooth function on D, and thus we get a generalized Calabi-Yau structure on D,

$$\varphi = dw_1 \wedge \cdots \wedge dw_m \wedge \exp \sqrt{-1}\Omega,$$

where $\Omega = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log K$. The natural S^1-action defined above preserves φ, and we have a generalized moment map μ for this action,

$$\mu = -\sum_{j=1}^{n} \frac{|z_j|^2}{1-|z_j|^2}.$$

On the other hand, since the total space D and the parameter space $(D^1)^m$ are also Reinhardt bounded domains, they have Kähler forms induced by their Bergman kernels. So they have also generalized Calabi-Yau structures induced by their Kähler forms, and they are preserved by the natural S^1-actions on them. By simple calculations, we get moment maps for their actions,

$$\mu_D = -\left(\sum_{i=1}^{m} \frac{|w_i|^2}{1-|w_i|^2} + \sum_{j=1}^{n} \frac{|z_j|^2}{1-|z_j|^2}\right)$$

on D, and

$$\mu_{D^m} = -\sum_{i=1}^{m} \frac{|w_i|^2}{1-|w_i|^2}$$

on D^m. Then they satisfy the following additive relation;

$$\mu_D = \mu_{D^m} + \mu.$$

EXAMPLE 3.3.3. Let D be an $(m+n)$-dimensional complex ball

$$D = D^{m+n} = \left\{ (w_1, \cdots, w_m, z_1, \cdots, z_n) \in \boldsymbol{C}^{m+n} \;\middle|\; \sum_{j=1}^{m} |w_j|^2 + \sum_{j=1}^{n} |z_j|^2 < 1 \right\}.$$

For each $w \in D^m = \{(w_1, \cdots, w_m) \in \boldsymbol{C}^m \mid \sum_{j=1}^{m} |w_j|^2 < 1\}$, D_w denote the slice of D at w,

$$D_w = \left\{ (z_1, \cdots, z_n) \in \boldsymbol{C}^n \;\middle|\; \sum_{j=1}^{n} |z_j|^2 < 1 - \sum_{j=1}^{m} |w_j|^2 \right\}.$$

Then D_w is also a complex ball on $\boldsymbol{C}^n$, and

$$K_w = \frac{n!}{\pi^n} \frac{1 - \sum_{j=1}^m |w_j|^2}{(1 - \sum_{j=1}^m |w_j|^2 - \sum_{j=1}^n |z_j|^2)^{n+1}}$$

is the Bergman kernel function of D_w. Since the Bergman kernel K_w depends smoothly on w,

$$K(w, z) = K_w(z) : D \longrightarrow \boldsymbol{R},$$

is a smooth function on D, and thus we get a generalized Calabi-Yau structure on D,

$$\varphi = dw_1 \wedge \cdots \wedge dw_m \wedge \exp \sqrt{-1}\Omega,$$

where $\Omega = \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log K$. The natural S^1-action on D preserves φ, and we have a generalized moment map μ for this action,

$$\mu = -\frac{n+1}{2} \frac{1 - \sum_{j=1}^m |w_j|^2}{1 - (\sum_{j=1}^m |w_j|^2 + \sum_{j=1}^n |z_j|^2)}.$$

As in the case of the previous example, we have moment maps for the natural actions of S^1 on D and D^m which are derived from their Bergman kernels,

$$\mu_D = -\frac{m+n+1}{2} \frac{1}{1 - (\sum_{j=1}^m |w_j|^2 + \sum_{j=1}^n |z_j|^2)}$$

on D, and

$$\mu_{D^m} = -\frac{m+1}{2} \frac{1}{1 - \sum_{j=1}^m |w_j|^2}$$

on D^m. They have the following multiplicative relation;

$$\mu_D = -\frac{2(m+n+1)}{(m+1)(n+1)} \mu_{D^m} \cdot \mu.$$

4. The Duistermaat-Heckman formula.

4.1. The Duistermaat-Heckman measures and the reduced volumes.

Let (M, φ) be a $2n$-dimensional connected generalized Calabi-Yau manifold which has constant type k, and suppose that compact l-torus T acts on M effectively and in a Hamiltonian way. In addition, we assume that the generalized moment map μ is proper. Then we have a natural volume form

$$dm = \frac{(\sqrt{-1})^n}{2^{n-k}} \langle \varphi, \bar{\varphi} \rangle.$$

The volume form dm defines a measure on M. Our second purpose is to prove the Duistermaat-Heckman formula in this case.

Let $\mathfrak{t}$ denote the Lie algebra of T, and $\mathfrak{t}^*_{\mathrm{reg}}$ denote the subset of $\mathfrak{t}^*$ consisting of the regular values of μ. If $a \in \mathfrak{t}^*$ is a regular value of μ and $p \in \mu^{-1}(a)$, then the stabilizer group

$$T_p = \{ g \in T \big| \ g \cdot p = p \}$$

is finite. So if T-action on $\mu^{-1}(a)$ is not free, the quotient space $M_a = \mu^{-1}(a)/T$ is an orbifold. In this case, There exists a complex differential form on M_a which, in each local representation is a generalized Calabi-Yau structure on $\boldsymbol{R}^{2(n-l)}$, and satisfies

$$p_a^* \widetilde{\varphi} = i_a^* \varphi,$$

where $i_a : \mu^{-1}(a) \longrightarrow M$ is the inclusion and $p_a : \mu^{-1}(a) \longrightarrow M_a$ is the natural projection. We call it a generalized Calabi-Yau structure on an orbifold M_a.

Since μ is proper, $\mathfrak{t}^*_{\mathrm{reg}}$ is a dense open subset, and $\mathfrak{t}^* \setminus \mathfrak{t}^*_{\mathrm{reg}}$ has measure 0 because of Sard's theorem. The following lemma is due to Appendix B in [**5**].

LEMMA 4.1.1. *Suppose that M is connected and T acts on M effectively. Then the set M_{free} on which T acts freely is equal to the complement of a locally finite union of submanifolds of codimension ≥ 2. In particular M_{free} is open, connected, dense, and $M \setminus M_{\mathrm{free}}$ has measure 0. Also $(\mu_*)_p$ is surjective for all $p \in M_{\mathrm{free}}$.*

Now we consider the normalized Haar measure dt on T. Then the measure dt induces the Lebesgue measure dX on its Lie algebra $\mathfrak{t}$, and we obtain the dual Lebesgue measure $d\zeta$ on $\mathfrak{t}^*$. The assumption that μ is proper implies that the pushforward $\mu_*(dm)$ of dm under μ defines a measure in $\mathfrak{t}^*$. We call it the Duistermaat-Heckman measure. In view of Lemma 4.1.1, we obtain $M \setminus M_{\mathrm{free}}$ has measure 0 and $\mu|_{M_{\mathrm{free}}} : M_{\mathrm{free}} \longrightarrow \mathfrak{t}^*$ is a submersion. This shows that $\mu_*(dm)$ is absolutely continuous with respect to the Lebesgue measure $d\zeta$. So there exists a Borel measurable function f on $\mathfrak{t}^*$ which satisfies

$$\mu_*(dm) = f d\zeta.$$

The corresponding Duistermaat-Heckman formula is stated in Theorem B in Introduction. For the proof, we need the following lemma.

LEMMA 4.1.2. *For each regular point $p \in M$ of the generalized moment map μ, there exists a neighborhood U_p of p and a complex 2-form $B + \sqrt{-1}\omega \in \Omega^2(U_p) \otimes \boldsymbol{C}$ such that $\varphi = \exp(B + \sqrt{-1}\omega)\varphi_k$ on U_p, and $\iota_{\xi_M}\omega = d\mu^\xi$ for all $\xi \in \mathfrak{t}$.*

Proof. By Fact 2.3.7, there exists a neighborhood U_p and a complex 2-form $\widetilde{B}+\sqrt{-1}\widetilde{\omega} \in \Omega^2(U_p)\otimes \boldsymbol{C}$ such that $\varphi = \exp(\widetilde{B}+\sqrt{-1}\widetilde{\omega})\varphi_k$ on U_p. Moreover, there exists a local frame $\theta^1,\cdots,\theta^{2n}$ of $\wedge^1 T^*M$ such that $\varphi_k = \theta^1\wedge\cdots\wedge\theta^k$ on U_p. So we may assume that $\widetilde{B}+\sqrt{-1}\widetilde{\omega}$ can be written

$$\widetilde{B}+\sqrt{-1}\widetilde{\omega} = \sum_{i,j>k} c_{ij}\theta^i\wedge\theta^j,$$

where c_{ij} is a smooth complex function on U_p. In addition, since p is a regular point, so $(\mathfrak{t}_M)_p$ has dimension l. Hence we may assume that $\mathfrak{t}_M$ has dimension l on U_p.

Now consider the dual basis $\{X_1,\cdots,X_{2n}\}$ of $\{\theta^1,\cdots,\theta^{2n}\}$, and take an arbitrary Riemannian metric on M. Then we can define a complex 1-forms $\eta^1,\cdots,\eta^k$ on U_p defined by

$$\eta^i(\xi_M) = \sqrt{-1}d\mu^\xi(X_i)$$

for $\xi_M \in \mathfrak{t}_M$, and vanishes on the orthogonal complement of $\mathfrak{t}_M$. Then we define a complex 2-form $B+\sqrt{-1}\omega$ on U_p by

$$B+\sqrt{-1}\omega = \widetilde{B}+\sqrt{-1}\widetilde{\omega} + \sum_{s=1}^{k}\eta^s\wedge\theta^s.$$

It is clear that $\varphi = \exp(B+\sqrt{-1}\omega)\varphi_k$ on U_p and

$$\begin{aligned}
\iota_{\xi_M}(B+\sqrt{-1}\omega)(X_i) &= \left(\sum_{i,j>k} c_{ij}\theta^i\wedge\theta^j\right)(\xi_M,X_i) + \left(\sum_{s=1}^{k}\eta^s\wedge\theta^s\right)(\xi_M,X_i)\\
&= \left(\sum_{s=1}^{k}\eta^s\wedge\theta^s\right)(\xi_M,X_i)\\
&= \sum_{s=1}^{k}\eta^s(\xi_M)\theta^s(X_i)\\
&= \eta^i(\xi_M)\\
&= \sqrt{-1}d\mu^\xi(X_i),
\end{aligned}$$

for each $\xi\in\mathfrak{t}$ and $i=1,\cdots,k$. On the other hand, since $\xi_M - \sqrt{-1}d\mu^\xi \in E_\varphi$ for each $\xi\in\mathfrak{t}$, so we have

$$(\iota_{\xi_M}(B+\sqrt{-1}\omega) - \sqrt{-1}d\mu^\xi)\wedge\varphi_k = 0.$$

Thus for $i = k+1,\cdots,2n$, we obtain

$$\begin{aligned}0 &= \iota_{X_i}\Big((\iota_{\xi_M}(B+\sqrt{-1}\omega) - \sqrt{-1}d\mu^{\xi}) \wedge \varphi_k\Big) \\ &= (\iota_{\xi_M}(B+\sqrt{-1}\omega) - \sqrt{-1}d\mu^{\xi})(X_i)\varphi_k,\end{aligned}$$

and

$$\iota_{\xi_M}(B+\sqrt{-1}\omega)(X_i) = \sqrt{-1}d\mu^{\xi}(X_i).$$

This shows that

$$\iota_{\xi_M}(B+\sqrt{-1}\omega) = \sqrt{-1}d\mu^{\xi},$$

and in particular we have $\iota_{\xi_M}\omega = d\mu^{\xi}$. □

Proof of Theorem B. Let $a \in \mathfrak{t}^*_{\mathrm{reg}}$ be an arbitrary regular value of μ and U be a convex neighborhood of a contained in $\mathfrak{t}^*_{\mathrm{reg}}$. Since $\mathfrak{t}^*_{\mathrm{reg}}$ is an open set of $\mathfrak{t}^*$, there exists such a neighborhood. Now consider a T-invariant connection for the fibration $\mu : \mu^{-1}(U) \longrightarrow U$. For each $p \in \mu^{-1}(U)$, draw the horizontal curves lying over the straight lines through a and $b = \mu(p)$. This defines a T-equivariant projection $\Phi : \mu^{-1}(U) \longrightarrow \mu^{-1}(a)$ such that for each $b \in U$ the restriction $\Phi\big|_{\mu^{-1}(b)} : \mu^{-1}(b) \longrightarrow \mu^{-1}(a)$ is a T-equivariant diffeomorphism and

$$\mu \times \Phi : \mu^{-1}(U) \longrightarrow U \times \mu^{-1}(a)$$

is a trivialization. Using this trivialization and Fubini theorem, we have that $f(a)$ is equal to the volume of $\mu^{-1}(a)$ with respect to the quotient of dm by $\mu^* d\zeta$. In addition, by Lemma 4.1.2 $dm/\mu^* d\zeta$ is locally given by the $(2n-l)$-form

$$i_a^*(\varphi_k \wedge \bar{\varphi}_k) \wedge \frac{1}{(n-k-l)!}(i_a^*\omega)^{n-k-l} \wedge \eta,$$

where ω is a 2-form given by the lemma above and η is an l-form which on the T-orbits takes the value ± 1 on an l-tuple $(X_1, \cdots, X_l)$ such that $dX(X_1, \cdots, X_l) = 1$.

Note that the complement of $p_a(M_{\mathrm{free}} \cap \mu^{-1}(a)) = (M_{\mathrm{free}})_a$ has measure 0 for the projection $p_a : \mu^{-1}(a) \longrightarrow M_a$ because the complement of $(M_{\mathrm{free}})_a$ is equal to the image of a finite union of submanifolds (or suborbifolds) of $\mu^{-1}(a)$ of codimension ≥ 2. Since $p_a : M_{\mathrm{free}} \cap \mu^{-1}(a) \longrightarrow (M_{\mathrm{free}})_a$ is a principle T-fibration and $\mathrm{vol}(T) = 1$, we get that the volume of $M_{\mathrm{free}} \cap \mu^{-1}(a)$ is equal to the volume of $(M_{\mathrm{free}})_a$ with respect to the measure dm_a induced by the reduced generalized Calabi-Yau structure on M_a. Because the complement of $(M_{\mathrm{free}})_a$ has measure 0, we have proved the formula. □

Remark 4.1.3. For the density function f, one can show that f is a piecewise polynomial of degree at most $n - l - k$. Moreover, in the case that M is compact, the localization formula holds by applying the Atiyah-Bott-Berline-Vergne localization theorem. Detailed statements and proofs can be seen in [**9**].

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