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Isotropy subalgebras of elliptic orbits in semisimple Lie algebras, and the canonical representatives of pseudo-Hermitian symmetric elliptic orbits

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Abstract. We give a method of determining the centralizer of an elliptic element in a real semisimple Lie algebra $\mathfrak{g}$, in relation with the maximal compact subalgebra of $\mathfrak{g}$ and the compact dual of $\mathfrak{g}$. Moreover, we determine a special central element (called the H-element) of the isotropy subalgebra of each simple irreducible pseudo-Hermitian symmetric Lie algebra.

1. Introduction.

Let $\mathfrak{g}$ be a real semisimple Lie algebra. An element $X \in \mathfrak{g}$ is called semisimple, if the operator $\mathrm{ad}_{\mathfrak{g}}X$ is semisimple. A semisimple element X is called elliptic, if the eigenvalues of $\mathrm{ad}_{\mathfrak{g}}X$ are all purely imaginary. Let G be a connected Lie group with $\mathrm{Lie}\,G = \mathfrak{g}$. The adjoint orbit $\mathrm{Ad}(G)X$ through a semisimple element (resp. an elliptic element) X is called a semisimple orbit (resp. an elliptic orbit). $\mathrm{Ad}(G)X$ is expressed as the coset space $G/C_G(X)$, where $C_G(X)$ is the centralizer of X in G. Note that elliptic orbits of semisimple Lie groups can be characterized geometrically. Actually Dorfmeister-Guan [**5**] and Kobayashi-Ono [**11**], Kobayashi [**10**] have shown that a semisimple orbit $G/C_G(X)$ is elliptic if and only if it admits a G-invariant pseudo-Kähler metric.

Our main concern is how to determine the isotropy subgroup $C_G(X)$ for an arbitrary elliptic element X. We consider this problem in the Lie algebra level. So our problem is to determine the centralizer $\mathfrak{c}_{\mathfrak{g}}(X)$ of an arbitrary elliptic element $X \in \mathfrak{g}$. The first aim in this paper is to settle the problem. Let $\mathfrak{k}$ be a maximal compact subalgebra of $\mathfrak{g}$ containing X, and let $(\mathfrak{g}_u, \mathfrak{k})$ be the compact dual of orthogonal symmetric Lie algebra $(\mathfrak{g}, \mathfrak{k})$. Our main result is the structure theorem for $\mathfrak{c}_{\mathfrak{g}}(X)$ (cf. Theorem 3.4), which enables us to determine it in terms of the centralizer $\mathfrak{c}_{\mathfrak{k}}(X)$ in $\mathfrak{k}$ and the semisimple part and the center of the centralizer $\mathfrak{c}_{\mathfrak{g}_u}(X)$ in $\mathfrak{g}_u$. In Section 5, applying this structure theorem to $\mathfrak{sl}(4, \boldsymbol{R})$, we actually determine, up to inner automorphism, the centralizers of all possible elliptic elements in $\mathfrak{sl}(4, \boldsymbol{R})$ (cf. Proposition 5.1).

In 1957, Berger [**1**] has classified simple (affine) symmetric spaces. In [**1**], the notion of pseudo-Hermitian symmetric space was introduced. A symmetric space G/R is called pseudo-Hermitian, if it has an invariant complex structure J and an invariant pseudo-

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Hermitian metric (with respect to J). In 1971, Shapiro [**14**] has shown that for a pseudo-Hermitian symmetric space G/R, there exists an elliptic element $T \in \mathfrak{g} = \mathrm{Lie}\, G$ which satisfies two conditions (i) R is the centralizer $C_G(T)$ of T in G, and (ii) $\mathrm{ad}_{\mathfrak{g}}T$ induces the complex structure J. Following Satake's book [**13**], we prefer to call such an element T the "H-element" in pseudo-Hermitian symmetric Lie algebra $(\mathfrak{g}, \mathfrak{r})$, where $\mathfrak{r} = \mathrm{Lie}\, R$, although in [**13**] the terminology "H-element" is used only for a Hermitian symmetric Lie algebra.

Our second problem is to determine the H-elements in all simple irreducible pseudo-Hermitian symmetric Lie algebras. In the course of the realization of pseudo-Hermitian symmetric spaces of type K_ϵ as Siegel domains, Kaneyuki [**7**] solved the problem for that specified type of pseudo-Hermitian symmetric spaces. In Section 6, we completely settle this problem for all the twenty-nine simple irreducible pseudo-Hermitian symmetric Lie algebras (cf. Theorem 6.16). We describe the H-elements in terms of the dual basis for the simple roots. Our method depends on Theorem 3.4, and is different from Kaneyuki's.

In Section 2, we will collect the notation utilized through this paper. In Section 4, we will define involutive outer-automorphisms of compact simple Lie algebras precisely (cf. Lemmas 4.3 and 4.4) and make reference to the result of Murakami [**12**], which are used in Sections 5 and 6. This paper is organized as follows:

Section 1 Introduction.
Section 2 Definitions and notation.
Section 3 The structure of $\mathfrak{c}_{\mathfrak{g}}(T)$.
Section 4 Elementary facts about root theory.
Section 5 Determination of the centralizer $\mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(T)$.
Section 6 The H-elements in pseudo-Hermitian symmetric Lie algebras.

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2. Definitions and notation.

DEFINITION 2.1. Let $\mathfrak{g}$ be a real semisimple Lie algebra, let $\mathfrak{k}$ be a maximal compact subalgebra of $\mathfrak{g}$, and let

$$\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p} \quad (\subset \mathfrak{g}^{\boldsymbol{C}}) \tag{F1}$$

denote the Cartan decomposition. Then, a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}^{\boldsymbol{C}}$ can be given by

$$\mathfrak{g}_u = \mathfrak{k} \oplus \mathfrak{p} \quad (\subset \mathfrak{g}^{\boldsymbol{C}}), \tag{F2}$$

and there exists an involutive automorphism σ of $\mathfrak{g}_u$ such that

$$\begin{cases} \mathfrak{k} = \{K \in \mathfrak{g}_u \mid \sigma(K) = K\}, \\ \mathfrak{p} = \{P \in \mathfrak{g}_u \mid \sigma(P) = -P\}. \end{cases} \tag{F3}$$

In these settings, we say that compact semisimple Lie algebra $\mathfrak{g}_u$ is related with $(\mathfrak{g}, \mathfrak{k})$ as in the formulae (F1) and (F2), and say that compact symmetric pair $(\mathfrak{g}_u, \sigma)$ is related with $(\mathfrak{g}, \mathfrak{k})$ as in the formulae (F1), (F2) and (F3).

Definition 2.2 (Kobayashi [**9**] or [**10**]). Let $\mathfrak{g}$ be a real semisimple Lie algebra. An element $S \in \mathfrak{g}$ is called semisimple if $\mathrm{ad}_{\mathfrak{g}} S$ is a semisimple endomorphism of $\mathfrak{g}$. A semisimple element $T \in \mathfrak{g}$ is said to be elliptic if any eigenvalue of $\mathrm{ad}_{\mathfrak{g}} T$ is purely imaginary. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. The adjoint orbit $\mathrm{Ad}(G)S$ through semisimple element $S \in \mathfrak{g}$ is called a semisimple orbit. The adjoint orbit $\mathrm{Ad}(G)T$ through elliptic element $T \in \mathfrak{g}$ is said to be an elliptic orbit.

Notation 2.3. Throughout this paper, we utilize the following notation:

(n1) $\mathfrak{l}^C$: the complexification of real Lie algebra $\mathfrak{l}$.
(n2) $B_{\mathfrak{l}}$: the Killing form of Lie algebra $\mathfrak{l}$.
(n3) $\mathrm{ad}_{\mathfrak{l}}$: the adjoint representation of Lie algebra $\mathfrak{l}$.
(n4) $\mathfrak{c}_{\mathfrak{l}}(X)$: the centralizer of X in Lie algebra $\mathfrak{l}$, for $X \in \mathfrak{l}$.
(n5) $\mathfrak{l}_{\mathrm{ss}}$: the semisimple part of reductive Lie algebra $\mathfrak{l}$, namely $\mathfrak{l}_{\mathrm{ss}} = [\mathfrak{l}, \mathfrak{l}]$.
(n6) $\mathfrak{l}_{\mathrm{z}}$: the center part of reductive Lie algebra $\mathfrak{l}$.
(n7) $\mathrm{rk}\,\mathfrak{l}$: the rank of real reductive Lie algebra $\mathfrak{l}$.
(n8) $\mathfrak{m} \oplus \mathfrak{n}$: the direct sum of vector spaces $\mathfrak{m}$ and $\mathfrak{n}$.
(n9) $f|_A$: the restriction of mapping f to set A.
(n10) i: the imaginary unit, namely $i = \sqrt{-1}$.

If $\bar{\mathfrak{g}}$ is a complex semisimple Lie algebra and if $\bar{\mathfrak{h}}$ is a Cartan subalgebra of $\bar{\mathfrak{g}}$, then we specially utilize the following notation:

(n11) $\triangle(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$: the set of non-zero roots of $\bar{\mathfrak{g}}$ with respect to $\bar{\mathfrak{h}}$.
(n12) $\triangle^+(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$: the set of positive roots in $\triangle(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$ (with respect to some chosen order).
(n13) $\Pi_{\triangle(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})}$: the set of simple roots in $\triangle(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$ (with respect to some chosen order).

Let $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan decomposition (F1) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$. Then, we utilize the following notation:

(n14) $\mathfrak{t}^n$: an n-dimensional abelian subalgebra of $\mathfrak{g}$ which is contained in $\mathfrak{k}$.
(n15) $\boldsymbol{R}^n$: an n-dimensional abelian subalgebra of $\mathfrak{g}$ which is contained in $i\mathfrak{p}$.

3. The structure of $\mathfrak{c}_{\mathfrak{g}}(T)$.

Let $\mathfrak{g}$ be a real semisimple Lie algebra, let $\mathfrak{k}$ be a maximal compact subalgebra of $\mathfrak{g}$, and let $(\mathfrak{g}_u, \sigma)$ be the compact symmetric pair related with $(\mathfrak{g}, \mathfrak{k})$ as in the formulae (F1) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$, (F2) $\mathfrak{g}_u = \mathfrak{k} \oplus \mathfrak{p}$ $(\subset \mathfrak{g}^C)$, and (F3) $\mathfrak{k} = \{K \in \mathfrak{g}_u \mid \sigma(K) = K\}$, $\mathfrak{p} = \{P \in \mathfrak{g}_u \mid \sigma(P) = -P\}$. In Subsection 3.1, we verify that all elements of $\mathfrak{k}$ are elliptic ones of $\mathfrak{g}$. In Subsection 3.2, we investigate relation between $\mathfrak{c}_{\mathfrak{g}}(T)$ and $\mathfrak{c}_{\mathfrak{g}_u}(T)$, for $T \in \mathfrak{k}$. Finally in Subsection 3.3, we prove Theorem 3.4, and assert two Corollaries 3.6 and 3.8.

3.1.

Let $\tilde{\sigma}$ be a Cartan involution of $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ defined by $K + iP \mapsto K - iP$ for $K \in \mathfrak{k}$ and $P \in \mathfrak{p}$. Define a positive-definite inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ by $\langle X, Y \rangle := -B_{\mathfrak{g}}(X, \tilde{\sigma}(Y))$ for $X, Y \in \mathfrak{g}$. Then, it is obvious that $\langle [K, X], Y \rangle = -\langle X, [K, Y] \rangle$ for any $K \in \mathfrak{k}$ and $X, Y \in \mathfrak{g}$. Accordingly, each $K \in \mathfrak{k}$ is an elliptic element of $\mathfrak{g}$.

REMARK 3.1. For any elliptic element $T \in \mathfrak{g}$, there exists a maximal compact subalgebra of $\mathfrak{g}$ such that T belongs to it. Therefore, there exists an inner automorphism ψ of $\mathfrak{g}$ which maps T into $\mathfrak{k}$ (cf. Helgason [**6**, Theorem 7.2, p. 183]). Since $\mathfrak{c}_{\mathfrak{g}}(T)$ is isomorphic to $\mathfrak{c}_{\mathfrak{g}}(\psi(T))$ via ψ, one may assume that T belongs to a fixed, maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ from the beginning, as far as clarifying $\mathfrak{c}_{\mathfrak{g}}(T)$ up to inner automorphism of $\mathfrak{g}$.

3.2. Relation between $\mathfrak{c}_{\mathfrak{g}}(T)$ and $\mathfrak{c}_{\mathfrak{g}_u}(T)$.

For $T \in \mathfrak{k} = \mathfrak{g}_u \cap \mathfrak{g}$, we study $\mathfrak{c}_{\mathfrak{g}_u}(T)$ first, and investigate relation between $\mathfrak{c}_{\mathfrak{g}}(T)$ and $\mathfrak{c}_{\mathfrak{g}_u}(T)$ afterward.

3.2.1. Take any element $T \in \mathfrak{k}$. Since $\mathrm{ad}_{\mathfrak{g}_u} T$ is semisimple, $\mathfrak{c}_{\mathfrak{g}_u}(T)$ is a reductive Lie algebra. Thus, it can be decomposed as follows (recall Notation 2.3 (n5) and (n6) for $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}}$ and $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}}$):

$$\mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \oplus \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}}.$$

It follows from $\sigma(T) = T$ that $\sigma\big(\mathfrak{c}_{\mathfrak{g}_u}(T)\big) \subset \mathfrak{c}_{\mathfrak{g}_u}(T)$. Therefore, we obtain $\sigma\big(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}}\big) \subset \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}}$ and $\sigma\big(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}}\big) \subset \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}}$ because $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} = \big[\mathfrak{c}_{\mathfrak{g}_u}(T), \mathfrak{c}_{\mathfrak{g}_u}(T)\big]$. This provides

$$\mathfrak{c}_{\mathfrak{g}_u}(T) = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p}). \tag{3.2.1}$$

Here, we remark that

$$\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}} = \mathfrak{c}_{\mathfrak{g}_u}(T) \cap \mathfrak{k} = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}), \tag{3.2.2}$$
$$\mathfrak{c}_{\mathfrak{g}_u}(T) \cap \mathfrak{p} = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p}); \tag{3.2.3}$$

and that

the Killing form of $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p})$
is negative-definite. (3.2.4)

3.2.2. Now, let us investigate relation between $\mathfrak{c}_{\mathfrak{g}}(T)$ and $\mathfrak{c}_{\mathfrak{g}_u}(T)$.

LEMMA 3.2. *With the assumptions above; for $T \in \mathfrak{k}$, the following eight items hold:*

(i) $\mathfrak{c}_{\mathfrak{g}}(T) = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p})$.

(ii) *$\mathfrak{c}_{\mathfrak{g}}(T)$ is a reductive Lie algebra. Moreover, $\mathfrak{c}_{\mathfrak{g}}(T)_{\mathrm{ss}} = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p})$ and $\mathfrak{c}_{\mathfrak{g}}(T)_{\mathrm{z}} = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p})$.*

(iii) *If $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p} \neq \{0\}$, then $\big(\mathfrak{c}_{\mathfrak{g}}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}\big)$ is the non-compact dual, orthogonal symmetric Lie algebra of $\big(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}\big)$.*

(iv) *$\mathfrak{c}_{\mathfrak{k}}(T)$ is a maximal compact subalgebra of reductive Lie algebra $\mathfrak{c}_{\mathfrak{g}}(T)$.*

(v) $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k} \subset \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}$.

(vi) *$\mathfrak{c}_{\mathfrak{g}_u}(T)$ is the compact dual of reductive Lie algebra $\mathfrak{c}_{\mathfrak{g}}(T)$.*

(vii) *$\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}$ is the orthogonal complement of $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}$ in $\mathfrak{c}_{\mathfrak{k}}(T)$ (with respect to $B_{\mathfrak{g}_u}$).*

(viii) *If $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p} \neq \{0\}$, then $\big(\mathfrak{c}_{\mathfrak{g}}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}\big)$ is the non-compact dual of orthogonal symmetric Lie algebra $\big(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}\big)$. Here, $(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}$ denotes the orthogonal complement of $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}$ in $\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}$ with respect to $B_{\mathfrak{g}_u}$, namely*

$$(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}} := \big\{C \in \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}} \,\big|\, B_{\mathfrak{g}_u}(C, Y) = 0 \textit{ for all } Y \in \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}\big\}.$$

PROOF.

(i) For any $X \in \mathfrak{c}_{\mathfrak{g}}(T)$, one can describe it as $X = K + iP$ $(K \in \mathfrak{k}, P \in \mathfrak{p})$ because $X \in \mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$. Then $0 = [X, T] = [K, T] + i[P, T]$. Since $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ is Cartan decomposition, and since $T \in \mathfrak{k}$, we obtain $0 = [K, T] = [P, T]$. Accordingly, it follows that $K \in \mathfrak{c}_{\mathfrak{g}_u}(T) \cap \mathfrak{k}$ and $P \in \mathfrak{c}_{\mathfrak{g}_u}(T) \cap \mathfrak{p}$. This, together with (3.2.2) and (3.2.3), concludes that $X = K + iP \in (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p})$. Consequently, we have got $\mathfrak{c}_{\mathfrak{g}}(T) \subset (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p})$. The converse inclusion is immediate from direct computations. For the reasons, we have shown the first item.

(ii) It follows from (i) and (3.2.1) that $(\mathfrak{c}_{\mathfrak{g}}(T))^{\boldsymbol{C}} = (\mathfrak{c}_{\mathfrak{g}_u}(T))^{\boldsymbol{C}}$; and hence $\mathfrak{c}_{\mathfrak{g}}(T)$ is reductive. Since (i), and since $\big(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}}\big)^{\boldsymbol{C}} = \big((\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p})\big)^{\boldsymbol{C}}$ and $\big(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}}\big)^{\boldsymbol{C}} = \big((\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p})\big)^{\boldsymbol{C}}$, we perceive that $(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p})$ is the semisimple part of $\mathfrak{c}_{\mathfrak{g}}(T)$, and that $(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p})$ is the center part of $\mathfrak{c}_{\mathfrak{g}}(T)$—that is, $\mathfrak{c}_{\mathfrak{g}}(T)_{\mathrm{ss}} = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p})$ and $\mathfrak{c}_{\mathfrak{g}}(T)_{\mathrm{z}} = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p})$.

(iii) From (3.2.4) and $\big(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}}\big)^{\boldsymbol{C}} = \big((\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p})\big)^{\boldsymbol{C}}$, it is natural that the Killing form of $\mathfrak{c}_{\mathfrak{g}}(T)_{\mathrm{ss}} = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p})$ is negative-definite on $(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \times (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k})$, and positive-definite on $i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p}) \times i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p})$. Therefore, we see that $\mathfrak{c}_{\mathfrak{g}}(T)_{\mathrm{ss}} = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p})$ is the Cartan decomposition by involution $\tilde{\sigma}|_{\mathfrak{c}_{\mathfrak{g}}(T)_{\mathrm{ss}}}$, where $\tilde{\sigma}$ was defined in Subsection 3.1. Hence $\big(\mathfrak{c}_{\mathfrak{g}}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}\big)$ is an orthogonal symmetric Lie algebra when $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p} \neq \{0\}$. Furthermore, it is the non-compact dual of $\big(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}\big)$, because $\mathfrak{c}_{\mathfrak{g}}(T)_{\mathrm{ss}} = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p})$ and $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p})$.

(iv) Since (3.2.2) and (i), we confirm that $\mathfrak{c}_{\mathfrak{k}}(T) = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) = \mathfrak{c}_{\mathfrak{g}}(T) \cap \mathfrak{k}$; and so $\mathfrak{c}_{\mathfrak{k}}(T)$ is a maximal compact subalgebra of $\mathfrak{c}_{\mathfrak{g}}(T)$.

(v) It is obvious from (3.2.2) that $\mathfrak{c}_{\mathfrak{k}}(T) = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) \subset \mathfrak{c}_{\mathfrak{g}_u}(T)$, so that we obtain $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k} \subset \mathfrak{c}_{\mathfrak{k}}(T)$ and $\big[\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}, \mathfrak{c}_{\mathfrak{k}}(T)\big] \subset \big[\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}}, \mathfrak{c}_{\mathfrak{g}_u}(T)\big] = \{0\}$. Therefore, the item (v) holds.

(vi) The sixth item comes from (i) and (3.2.1).

(vii) It is natural from $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} = \big[\mathfrak{c}_{\mathfrak{g}_u}(T), \mathfrak{c}_{\mathfrak{g}_u}(T)\big]$ that $B_{\mathfrak{g}_u}(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}}) = \{0\}$, so that $(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})$ is an orthogonal decomposition with respect to $B_{\mathfrak{g}_u}$. Therefore we conclude that item (vii) holds, because of (3.2.2) $\mathfrak{c}_{\mathfrak{k}}(T) = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})$ and $B_{\mathfrak{g}_u}$ being negative-definite.

(viii) By virtue of (iii) we can get the last item, if $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k} = \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}$. Thus, let us prove that $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k} = \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}$. For any $X \in \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}$, it can be written as $X = S + C$ $(S \in \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}}, C \in \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}})$ because it follows from (3.2.2) that $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k} \subset \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}$. Since $B_{\mathfrak{g}_u}(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}}) = \{0\}$, we comprehend $B_{\mathfrak{g}_u}(X, \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) = \{0\}$; moreover, the item (v) and $B_{\mathfrak{g}_u}(\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}) = \{0\}$ mean that $B_{\mathfrak{g}_u}(S, \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) = \{0\}$. Hence, $B_{\mathfrak{g}_u}(C, \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) = B_{\mathfrak{g}_u}(X - S, \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) = \{0\}$; and so $C \in (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}$. Accordingly $X = S + C \in \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}$. Therefore, one perceives that $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k} \subset \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}$. Now, we will demonstrate that the converse inclusion also holds. Definition of $(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}$, combined with (v) and $B_{\mathfrak{g}_u}(\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}) = \{0\}$, implies that

$$\begin{aligned}
&B_{\mathfrak{g}_u}\big(\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}, \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}\big) \\
&\quad = B_{\mathfrak{g}_u}\big(\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}\big) \\
&\quad \subset B_{\mathfrak{g}_u}\big(\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}\big) \\
&\quad = \{0\}.
\end{aligned}$$

Consequently, it follows from (vii) that $\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}} \subset \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}$. For the reasons, we have proved that $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k} = \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}$. Hence, Lemma 3.2 has been shown. □

We will also verify the following lemma needed later:

LEMMA 3.3. *In the settings above*; *for $T \in \mathfrak{k}$, the following four items hold*:

(1) $\operatorname{rk} \mathfrak{g} = \operatorname{rk} \mathfrak{c}_{\mathfrak{g}}(T) = \operatorname{rk} \mathfrak{g}_u = \operatorname{rk} \mathfrak{c}_{\mathfrak{g}_u}(T)$.
(2) $\operatorname{rk} \mathfrak{k} = \operatorname{rk} \mathfrak{c}_{\mathfrak{k}}(T)$.
(3) *If* $\dim \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} = 1$, *then*

$$\begin{cases}
\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k} = \operatorname{span}_{\boldsymbol{R}}\{T\}, \\
\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p} = \{0\}, \\
\mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}).
\end{cases}$$

(4) *If* $\operatorname{rk} \mathfrak{g} = \operatorname{rk} \mathfrak{k}$, *then*

$$\begin{cases} \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p} = \{0\}, \\ \mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}), \\ \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} = \mathfrak{c}_{\mathfrak{g}}(T)_{\mathrm{z}} \qquad \big(= \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}\big). \end{cases}$$

PROOF. The items (1) and (2) are obvious because T is semisimple.

(3) Since $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p})$, we see that $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k} = \mathrm{span}_{\boldsymbol{R}}\{T\}$ and $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p} = \{0\}$ in case of $\dim \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} = 1$. It follows from $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p} = \{0\}$ and (3.2.1) that $\mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})$.

(4) If $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p} \neq \{0\}$, then $\mathrm{rk}\, \mathfrak{c}_{\mathfrak{g}_u}(T) > \mathrm{rk}\, \mathfrak{c}_{\mathfrak{k}}(T)$ because of (3.2.1) and (3.2.2). Therefore, the items (1) and (2) show that $\mathrm{rk}\, \mathfrak{g} > \mathrm{rk}\, \mathfrak{k}$ when $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p} \neq \{0\}$. Thus by the contraposition, we conclude that $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p} = \{0\}$ if $\mathrm{rk}\, \mathfrak{g} = \mathrm{rk}\, \mathfrak{k}$. The rest of proof is immediate from $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p} = \{0\}$, (3.2.1) and Lemma 3.2-(ii). Consequently, we have verified Lemma 3.3. $\square$

3.3. Results.

Now, we will demonstrate the following (recall Notation 2.3 (n5) and (n6), for $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}}$, $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}}$, $\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}}$ and $\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}$):

THEOREM 3.4. *Let $\mathfrak{g}$ be a real semisimple Lie algebra, let $\mathfrak{k}$ be a maximal compact subalgebra of $\mathfrak{g}$, and let $\mathfrak{g}_u$ be the compact semisimple Lie algebra related with $(\mathfrak{g}, \mathfrak{k})$ as in the formulae* (F1) *and* (F2). *For any element $T \in \mathfrak{k}$, the structure of $\mathfrak{c}_{\mathfrak{g}}(T)$ is as follows*:

(A) *if $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})$, then*

$$\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{c}_{\mathfrak{k}}(T) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p});$$

(B) *if $\mathfrak{c}_{\mathfrak{k}}(T) \neq \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})$, then*

$$\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{s} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p}).$$

Here, $\mathfrak{s}$ is a semisimple Lie algebra such that $\big(\mathfrak{s}, \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}\big)$ is the non-compact dual of orthogonal symmetric Lie algebra $\big(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}\big)$, where $(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}$ is the orthogonal complement of $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}$ in $\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}$ with respect to $B_{\mathfrak{g}_u}$.

PROOF. We are going to prove Theorem 3.4 by use of the notation utilized in this section. Comparing $\mathfrak{c}_{\mathfrak{k}}(T)$ with $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})$, we see whether $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p} = \{0\}$ or not, because it follows from (3.2.2) that $\mathfrak{c}_{\mathfrak{k}}(T) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p}) = \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})$. Let us check two Cases (A) $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})$ and (B) $\mathfrak{c}_{\mathfrak{k}}(T) \neq \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})$, individually.

CASE (A): $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})$: In this case, $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p} = \{0\}$. Thus, Lemma 3.2-(i) means that $\mathfrak{c}_{\mathfrak{g}}(T) = (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{k}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p})$. Accordingly, we deduce

$$\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{c}_{\mathfrak{k}}(T) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p})$$

by virtue of (3.2.2).

CASE (B): $\mathfrak{c}_{\mathfrak{k}}(T) \neq \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})$: In this case, $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} \cap \mathfrak{p} \neq \{0\}$. Lemma 3.2-(viii) enables us to perceive that $\mathfrak{c}_{\mathfrak{g}}(T)_{\mathrm{ss}}$ is uniquely determined by duality $\big(\mathfrak{c}_{\mathfrak{g}}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}\big) \longleftrightarrow \big(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}\big)$. Therefore if $\mathfrak{s}$ denotes $\mathfrak{c}_{\mathfrak{g}}(T)_{\mathrm{ss}}$, then Lemma 3.2-(i) and -(ii) imply that

$$\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{s} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) \oplus i(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p}),$$

where $\mathfrak{s}$ is a semisimple Lie algebra such that $\big(\mathfrak{s}, \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}\big)$ is the non-compact dual, orthogonal symmetric Lie algebra of $\big(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}\big)$. For the reasons, we have completed the proof of Theorem 3.4. ☐

REMARK 3.5. Theorem 3.4 implies that it is possible to determine $\mathfrak{c}_{\mathfrak{g}}(T)$ with $T \in \mathfrak{k}$, if four structures of $\mathfrak{c}_{\mathfrak{k}}(T)$, $\mathfrak{c}_{\mathfrak{g}_u}(T)$, $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}$ and $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p}$ are clarified. These four structures can be clarified by using two root theories for $\mathfrak{k}$ and $\mathfrak{g}_u$ (see Section 5). Accordingly, Theorem 3.4 enables us to assert that $\mathfrak{c}_{\mathfrak{g}}(T)$ can be determined by using two root theories for $\mathfrak{k}$ and $\mathfrak{g}_u$.

Theorem 3.4 and Lemma 3.3-(3) lead the following:

COROLLARY 3.6. *With the same assumptions in Theorem* 3.4; *for an element* $T' \in \mathfrak{k}$ *with* $\dim \mathfrak{c}_{\mathfrak{g}_u}(T')_{\mathrm{z}} = 1$, *the structure of* $\mathfrak{c}_{\mathfrak{g}}(T')$ *is as follows*:

(A) *if* $\mathfrak{c}_{\mathfrak{k}}(T') = \mathfrak{c}_{\mathfrak{g}_u}(T')$, *then*

$$\mathfrak{c}_{\mathfrak{g}}(T') = \mathfrak{c}_{\mathfrak{k}}(T') \quad \big(= \mathfrak{c}_{\mathfrak{g}_u}(T')\big);$$

(B) *if* $\mathfrak{c}_{\mathfrak{k}}(T') \neq \mathfrak{c}_{\mathfrak{g}_u}(T')$, *then*

$$\mathfrak{c}_{\mathfrak{g}}(T') = \mathfrak{s} \oplus \mathrm{span}_{\boldsymbol{R}}\{T'\}.$$

Here, $\mathfrak{s}$ *is a semisimple Lie algebra such that* $\big(\mathfrak{s}, \mathfrak{c}_{\mathfrak{k}}(T')_{\mathrm{ss}} \oplus \mathrm{span}_{\boldsymbol{R}}\{T'\}^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T')_{\mathrm{z}}}\big)$ *is the non-compact dual of orthogonal symmetric Lie algebra* $\big(\mathfrak{c}_{\mathfrak{g}_u}(T')_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{k}}(T')_{\mathrm{ss}} \oplus \mathrm{span}_{\boldsymbol{R}}\{T'\}^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T')_{\mathrm{z}}}\big)$, *where* $\mathrm{span}_{\boldsymbol{R}}\{T'\}^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T')_{\mathrm{z}}}$ *is the orthogonal complement of* $\mathrm{span}_{\boldsymbol{R}}\{T'\}$ *in* $\mathfrak{c}_{\mathfrak{k}}(T')_{\mathrm{z}}$ *with respect to* $B_{\mathfrak{g}_u}$.

REMARK 3.7. Corollary 3.6 enables us to determine $\mathfrak{c}_{\mathfrak{g}}(T')$ by using the structures of $\mathfrak{c}_{\mathfrak{g}_u}(T')$ and $\mathfrak{c}_{\mathfrak{k}}(T')$, in case of $\dim \mathfrak{c}_{\mathfrak{g}_u}(T')_{\mathrm{z}} = 1$.

Theorem 3.4 and Lemma 3.3-(4) allow us to get the following:

COROLLARY 3.8. *With the same assumptions in Theorem* 3.4, *and with the assumption of* $\mathrm{rk}\,\mathfrak{g} = \mathrm{rk}\,\mathfrak{k}$; *for any element* $T \in \mathfrak{k}$, *the structure of* $\mathfrak{c}_{\mathfrak{g}}(T)$ *is as follows*:

(A) *if* $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{c}_{\mathfrak{g}_u}(T)$, *then*

$$\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{c}_{\mathfrak{k}}(T) \quad \bigl(= \mathfrak{c}_{\mathfrak{g}_u}(T)\bigr);$$

(B) *if* $\mathfrak{c}_{\mathfrak{k}}(T) \neq \mathfrak{c}_{\mathfrak{g}_u}(T)$, *then*

$$\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{s} \oplus \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}}.$$

Here, $\mathfrak{s}$ is a semisimple Lie algebra such that $\bigl(\mathfrak{s}, \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}\bigr)$ is the non-compact dual of orthogonal symmetric Lie algebra $\bigl(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}}, \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}\bigr)$, where $(\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}})^{\perp}_{\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}}$ is the orthogonal complement of $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}}$ in $\mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}}$ with respect to $B_{\mathfrak{g}_u}$.

Remark 3.9. By Corollary 3.8 we can completely determine $\mathfrak{c}_{\mathfrak{g}}(T)$ by using the structures of $\mathfrak{c}_{\mathfrak{g}_u}(T)$ and $\mathfrak{c}_{\mathfrak{k}}(T)$, in the case where $\mathfrak{g}$ is of inner type.

4. Elementary facts about root theory.

In this section, we will first review the relation between root theory for a complex semisimple Lie algebra and that for its compact real form (cf. Subsection 4.1), and next define involutive outer-automorphisms of compact simple Lie algebras (cf. Lemmas 4.3 and 4.4 in Subsection 4.2). Lastly in Subsection 4.3, we will make reference to the result of Murakami [**12**]. These arguments on this section are needed in Sections 5 and 6.

4.1. Weyl basis, and root-space decomposition.

Let $\bar{\mathfrak{g}}$ be a complex semisimple Lie algebra, and let $\bar{\mathfrak{h}}$ be a Cartan subalgebra of $\bar{\mathfrak{g}}$. Then, there exists a basis $\{X_\alpha \mid \alpha \in \triangle(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})\}$ of $\bar{\mathfrak{g}}$ (called Weyl basis) such that for all $\alpha, \beta \in \triangle(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$

$[X_\alpha, X_{-\alpha}] = H_\alpha, \quad [H, X_\alpha] = \alpha(H) \cdot X_\alpha$ for $H \in \bar{\mathfrak{h}}$;
$[X_\alpha, X_\beta] = 0$ if $\alpha + \beta \neq 0$ and $\alpha + \beta \notin \triangle(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$;
$[X_\alpha, X_\beta] = N_{\alpha,\beta} \cdot X_{\alpha+\beta}$ if $\alpha + \beta \in \triangle(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$,

where the real constants $N_{\alpha,\beta}$ satisfy $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ (cf. [**6**, Theorem 5.5, p. 176]). Here, for $\alpha \in \triangle(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$ we define $H_\alpha \in \bar{\mathfrak{h}}$ by $B_{\bar{\mathfrak{g}}}(H, H_\alpha) = \alpha(H)$ for all $H \in \bar{\mathfrak{h}}$. By using this Weyl basis, a compact real form $\mathfrak{g}_u$ of $\bar{\mathfrak{g}}$ can be given as follows:

$$\mathfrak{g}_u = i\bar{\mathfrak{h}}_{\boldsymbol{R}} \quad \oplus \bigoplus_{\alpha \in \triangle^+(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})} \mathrm{span}_{\boldsymbol{R}}\{X_\alpha - X_{-\alpha}\} \oplus \mathrm{span}_{\boldsymbol{R}}\{i(X_\alpha + X_{-\alpha})\} \tag{4.1.1}$$

(ref. the proof of Theorem 6.3 in [**6**, p. 181]), where $\bar{\mathfrak{h}}_{\boldsymbol{R}}$ is a real vector subspace of $\bar{\mathfrak{h}}$ defined by

$$\bar{\mathfrak{h}}_{\boldsymbol{R}} := \mathrm{span}_{\boldsymbol{R}}\{H_\alpha \mid \alpha \in \triangle(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})\} \quad \bigl(= \{H \in \bar{\mathfrak{h}} \mid \alpha(H) \in \boldsymbol{R} \text{ for all } \alpha \in \triangle(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})\}\bigr). \tag{4.1.2}$$

Remark 4.1. (1) $i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ is a maximal abelian subalgebra of $\mathfrak{g}_u$. (2) Decomposition (4.1.1) is the root-space decomposition of compact real form $\mathfrak{g}_u$ of $\bar{\mathfrak{g}}$ with respect to $i\bar{\mathfrak{h}}_{\boldsymbol{R}}$.

In this case, positive roots in $\triangle(\mathfrak{g}_u, i\bar{\mathfrak{h}}_{\boldsymbol{R}})$ coincide with ones in $\triangle(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$ multiplied by $-i$, namely

$$\triangle^+(\mathfrak{g}_u, i\bar{\mathfrak{h}}_{\boldsymbol{R}}) = \{ -i\alpha \mid \alpha \in \triangle^+(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})\}$$

(cf. Toda and Mimura [**15**]).

4.2. Outer automorphisms.

Involutive outer-automorphisms of $\mathfrak{g}_u$ need be precisely defined, in Sections 5 and 6. In order to obtain the goal, we will give two Lemmas 4.3 and 4.4.

Theorem 5.1 in Helgason [**6**, p. 421] and its proof enable us to demonstrate the following:

Lemma 4.2. *Let $\bar{\mathfrak{g}}$ be a complex semisimple Lie algebra, let $\bar{\mathfrak{h}}$ be a Cartan subalgebra of $\bar{\mathfrak{g}}$, let $\mathfrak{g}_u$ be a compact real form of $\bar{\mathfrak{g}}$ with decomposition* (4.1.1)*, and let ϕ be a real linear isomorphism of $i\bar{\mathfrak{h}}_{\boldsymbol{R}}$. Suppose that the transposed mapping of $\phi_{\boldsymbol{C}}$ satisfies*

$${}^t\phi_{\boldsymbol{C}}\big(\triangle(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})\big) = \triangle(\bar{\mathfrak{g}}, \bar{\mathfrak{h}}),$$

where $\phi_{\boldsymbol{C}}$ denotes the complex linear extension of ϕ to $\bar{\mathfrak{h}}$. Then, there exists an automorphism σ of $\bar{\mathfrak{g}}$ which satisfies three conditions

$$\text{(i) } \sigma(\mathfrak{g}_u) \subset \mathfrak{g}_u, \qquad \text{(ii) } \sigma|_{i\bar{\mathfrak{h}}_{\boldsymbol{R}}} = \phi,$$
$$\text{(iii) } \sigma(H_{\alpha_b}) = H_{{}^t\sigma^{-1}(\alpha_b)} \text{ and } \sigma(X_{\alpha_b}) = X_{{}^t\sigma^{-1}(\alpha_b)} \text{ for all } b \in \{1, \dots, r\}.$$

Moreover, σ is involutive if so is ϕ. Here, $\{\alpha_b\}_{b=1}^r$ denotes the set of simple roots in $\triangle(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$, and X_{α_b} are given in Subsection 4.1 *($H_{\alpha_b} = [X_{\alpha_b}, X_{-\alpha_b}]$).*

By means of Lemma 4.2 we will define involutive outer-automorphisms of $\mathfrak{g}_u = \mathfrak{su}(2l)$ and $\mathfrak{so}(2l)$, in Subsections 4.2.1 and 4.2.2.

4.2.1. Involutive outer-automorphism σ_1 of $\mathfrak{su}(2l)$, $l \geq 2$.

Let $\bar{\mathfrak{h}}$ be a Cartan subalgebra of the complex simple Lie algebra $\mathfrak{sl}(2l, \boldsymbol{C})$ of type A_{2l-1}. Fix a linear order in $\triangle(\mathfrak{sl}(2l, \boldsymbol{C}), \bar{\mathfrak{h}})$, and assume that $\{\alpha_a\}_{a=1}^{2l-1}$ is the set of simple roots in $\triangle(\mathfrak{sl}(2l, \boldsymbol{C}), \bar{\mathfrak{h}})$ whose Dynkin diagram is as follows:

$$\underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} - \cdots - \underset{\alpha_{2l-2}}{\circ} - \underset{\alpha_{2l-1}}{\circ}$$

(cf. Plate I in Bourbaki's book [**4**, p. 265]). Suppose that $\mathfrak{su}(2l)$ is situated in $\mathfrak{sl}(2l, \boldsymbol{C})$ as compact real form with decomposition (4.1.1). Now, let $\{Z_a\}_{a=1}^{2l-1}$ be the dual basis of $\{\alpha_a\}_{a=1}^{2l-1} = \Pi_{\triangle(\mathfrak{sl}(2l,\boldsymbol{C}),\bar{\mathfrak{h}})}$—that is, $\alpha_a(Z_b) = \delta_{ab}$. Then, it follows from (4.1.2) that $Z_a \in \bar{\mathfrak{h}}_{\boldsymbol{R}}$ for all $1 \leq a \leq 2l-1$, so that $\{Z_a\}_{a=1}^{2l-1}$ is a real basis of $\bar{\mathfrak{h}}_{\boldsymbol{R}}$. Let us define an involutive, real linear isomorphism ϕ_1 of $i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ by

$$\phi_1(iZ_a) := iZ_{2l-a} \qquad \text{for } 1 \leq a \leq 2l-1.$$

From $\triangle^+(\mathfrak{sl}(2l,\boldsymbol{C}),\bar{\mathfrak{h}}) = \{\sum_{m\leq k<n}\alpha_k \mid 1 \leq m < n \leq 2l\}$ and ${}^t\phi_{1\boldsymbol{C}}(\alpha_a) = \alpha_{2l-a}$, it is obvious that ${}^t\phi_{1\boldsymbol{C}}\bigl(\triangle(\mathfrak{sl}(2l,\boldsymbol{C}),\bar{\mathfrak{h}})\bigr) = \triangle(\mathfrak{sl}(2l,\boldsymbol{C}),\bar{\mathfrak{h}})$, where $\phi_{1\boldsymbol{C}}$ denotes the complex linear extension of ϕ_1 to $\bar{\mathfrak{h}}$. These arguments and Lemma 4.2 lead the following:

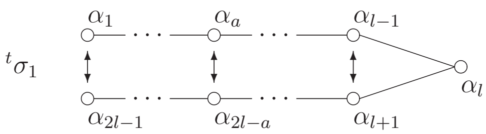
LEMMA 4.3. *With the settings in Subsection* 4.2.1; *there exists an involutive outer-automorphism σ_1 of $\mathfrak{sl}(2l,\boldsymbol{C})$ such that*

$$\text{(i) } \sigma_1\bigl(\mathfrak{su}(2l)\bigr) \subset \mathfrak{su}(2l), \qquad \text{(ii) } \sigma_1(iZ_a) = iZ_{2l-a},$$
$$\text{(iii) } \sigma_1(H_{\alpha_a}) = H_{{}^t\sigma_1(\alpha_a)} \text{ and } \sigma_1(X_{\alpha_a}) = X_{{}^t\sigma_1(\alpha_a)}$$

for all $1 \leq a \leq 2l-1$. In particular, a maximal abelian subalgebra $i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ of $\mathfrak{su}(2l)$ is decomposed as follows:

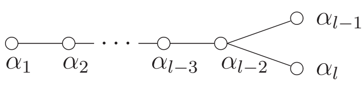
$$\bigl\{H \in i\bar{\mathfrak{h}}_{\boldsymbol{R}} \bigm| \sigma_1(H) = H\bigr\} = \mathrm{span}_{\boldsymbol{R}}\bigl\{i(Z_p + Z_{2l-p}),\, iZ_l \bigm| 1 \leq p \leq l-1\bigr\},$$
$$\bigl\{H \in i\bar{\mathfrak{h}}_{\boldsymbol{R}} \bigm| \sigma_1(H) = -H\bigr\} = \mathrm{span}_{\boldsymbol{R}}\bigl\{i(Z_p - Z_{2l-p}) \bigm| 1 \leq p \leq l-1\bigr\}.$$

Here, $\{Z_a\}_{a=1}^{2l-1}$ is the dual basis of $\{\alpha_a\}_{a=1}^{2l-1} = \Pi_{\triangle(\mathfrak{sl}(2l,\boldsymbol{C}),\bar{\mathfrak{h}})}$, and X_{α_a} are given in Subsection 4.1 *($H_{\alpha_a} = [X_{\alpha_a}, X_{-\alpha_a}]$).*



4.2.2. Involutive outer-automorphism σ_2 of $\mathfrak{so}(2l)$, $l \geq 4$.

Let $\bar{\mathfrak{h}}$ be a Cartan subalgebra of the complex simple Lie algebra $\mathfrak{so}(2l,\boldsymbol{C})$ of type D_l. Fix a linear order in $\triangle(\mathfrak{so}(2l,\boldsymbol{C}),\bar{\mathfrak{h}})$, and assume that $\{\alpha_b\}_{b=1}^{l}$ is the set of simple roots in $\triangle(\mathfrak{so}(2l,\boldsymbol{C}),\bar{\mathfrak{h}})$ whose Dynkin diagram is as follows:



(cf. Plate IV in [**4**, p. 271]). Suppose that $\mathfrak{so}(2l)$ is situated in $\mathfrak{so}(2l,\boldsymbol{C})$ as compact real form with decomposition (4.1.1). By discussions similar to those on Subsection 4.2.1, we are able to get Lemma 4.4.

LEMMA 4.4. *With the settings in Subsection* 4.2.2; *there exists an involutive outer-automorphism σ_2 of $\mathfrak{so}(2l,\boldsymbol{C})$ such that*

(i) $\sigma_2\big(\mathfrak{so}(2l)\big) \subset \mathfrak{so}(2l)$,

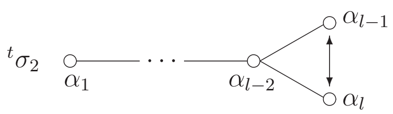
(ii) $\sigma_2(iZ_j) = iZ_j$ *for all* $1 \leq j \leq l-2$, $\sigma_2(iZ_{l-1}) = iZ_l$ *and* $\sigma_2(iZ_l) = iZ_{l-1}$;

(iii) $\sigma_2(H_{\alpha_b}) = H_{{}^t\sigma_2(\alpha_b)}$ *and* $\sigma_2(X_{\alpha_b}) = X_{{}^t\sigma_2(\alpha_b)}$ *for all* $1 \leq b \leq l$.

Particularly, a maximal abelian subalgebra $i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ *of* $\mathfrak{so}(2l)$ *is decomposed as follows*:

$$\big\{H \in i\bar{\mathfrak{h}}_{\boldsymbol{R}} \bigm| \sigma_2(H) = H\big\} = \operatorname{span}_{\boldsymbol{R}}\big\{iZ_j,\, i(Z_{l-1} + Z_l) \bigm| 1 \leq j \leq l-2\big\},$$
$$\big\{H \in i\bar{\mathfrak{h}}_{\boldsymbol{R}} \bigm| \sigma_2(H) = -H\big\} = \operatorname{span}_{\boldsymbol{R}}\big\{i(Z_{l-1} - Z_l)\big\}.$$

Here $\{Z_b\}_{b=1}^l$ *is the dual basis of* $\{\alpha_b\}_{b=1}^l = \Pi_{\triangle(\mathfrak{so}(2l,\boldsymbol{C}),\bar{\mathfrak{h}})}$, *and* X_{α_b} *are given in Subsection* 4.1 $(H_{\alpha_b} = [X_{\alpha_b}, X_{-\alpha_b}])$.



Remark 4.5. The above decomposition of $i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ in Lemma 4.3 (resp. Lemma 4.4) will be utilized in Section 5 (resp. Section 6).

4.3. Cartan decompositions.

Let $\bar{\mathfrak{g}}$ be a complex (semi)simple Lie algebra, and let $\bar{\mathfrak{h}}$ be a Cartan subalgebra of $\bar{\mathfrak{g}}$. Let us fix a linear order in $\triangle(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$, and assume that the Dynkin diagram of $\Pi_{\triangle(\bar{\mathfrak{g}},\bar{\mathfrak{h}})}$ is one of the Dynkin diagrams in Bourbaki [**4**]. Then, two Lists of Murakami [**12**, p. 297 and p. 305] (also see Borel and de Siebenthal [**2**]) enable us to read off the following five items:

(i) Real semisimple Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}^{\boldsymbol{C}} = \bar{\mathfrak{g}}$.

(ii) Maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$.

(iii) Compact real form $\mathfrak{g}_u$ of $\bar{\mathfrak{g}}$ with decomposition (4.1.1).

(iv) An involutive automorphism σ of $\mathfrak{g}_u$ satisfying two conditions (1) it stabilizes a maximal abelian subalgebra $i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ of $\mathfrak{g}_u$, and (2) compact symmetric pair $(\mathfrak{g}_u, \sigma)$ is related with $(\mathfrak{g}, \mathfrak{k})$ as in the formulae (F1) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$, (F2) $\mathfrak{g}_u = \mathfrak{k} \oplus \mathfrak{p}$, and (F3) $\mathfrak{k} = \{K \in \mathfrak{g}_u \mid \sigma(K) = K\}$ and $\mathfrak{p} = \{P \in \mathfrak{g}_u \mid \sigma(P) = -P\}$.

(v) The set of simple roots in $\triangle(\mathfrak{k}, \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})$.

Here, we note that $\mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ is a maximal abelian subalgebra of compact Lie algebra $\mathfrak{k}$.

For example, the five items stated above are as follows (ref. [**12**, p. 305, type AI]):

Example 4.6. $\mathfrak{g} = \mathfrak{sl}(2l, \boldsymbol{R})$ with $l \geq 2$.

(ii) $\mathfrak{k} = \mathfrak{so}(2l)$.

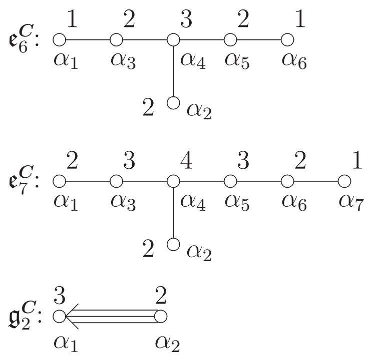
(iii) $\mathfrak{g}_u = \mathfrak{su}(2l) = i\bar{\mathfrak{h}}_{\boldsymbol{R}} \oplus \bigoplus_{\alpha \in \triangle^+(\mathfrak{sl}(2l,\boldsymbol{C}),\bar{\mathfrak{h}})} \operatorname{span}_{\boldsymbol{R}}\{X_\alpha - X_{-\alpha}\} \oplus \operatorname{span}_{\boldsymbol{R}}\{i(X_\alpha + X_{-\alpha})\}$.

(iv) $\sigma = \sigma_1 \circ \exp 2\pi i \mathrm{ad}_{\mathfrak{su}(2l)} h_l$,

(v) $\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})} = \{-i\tilde{\alpha}_1, \ldots, -i\tilde{\alpha}_{l-1}, -i(\tilde{\alpha}_{l-1} + \tilde{\alpha}_l)\}$, where σ_1 is given in Lemma 4.3 and $h_l \in \bar{\mathfrak{h}}_{\boldsymbol{R}}$ is defined by $\alpha_a(h_l) = \delta_{la}/2$ for $\alpha_a \in \Pi_{\triangle(\mathfrak{sl}(2l,\boldsymbol{C}),\bar{\mathfrak{h}})} = \{\alpha_b\}_{b=1}^{2l-1}$, and where $-i\tilde{\alpha}_k := -i\alpha_k|_{\mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}}$ for each $-i\alpha_k \in \Pi_{\triangle(\mathfrak{su}(2l),i\bar{\mathfrak{h}}_{\boldsymbol{R}})}$ and $1 \le k \le l$ (refer to Remark 4.1).

REMARK 4.7. Involutive outer automorphisms σ_1 in Lemma 4.3 and σ_2 in Lemma 4.4 are the same as θ_ρ utilized in Murakami's List [**12**, p. 305].

NOTICE 4.8. There are differences with respect to numbering of simple roots in the Dynkin diagrams of type E_6, E_7, E_8 and G_2, between Murakami [**12**] and Bourbaki [**4**].* Throughout this paper, we apply the numbering in Bourbaki [**4**] to our arguments. Thus, we utilize the following Dynkin diagrams of type E_6, E_7 and G_2:



and so we must rewrite List of Murakami [**12**, p. 297] as follows:

$\mathfrak{g}$	maximal root	h_i	$\mathfrak{k}_i$	$\mathfrak{g}_{\theta_i}$
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3$	h_1	$D_5 \times T$	EIII
	$+3\alpha_4 + 2\alpha_5 + \alpha_6$	h_3	$A_1 \times A_5$	EII
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3$	h_1	$A_1 \times D_6$	EVI
	$+4\alpha_4 + 3\alpha_5 + 2\alpha_6$	h_7	$E_6 \times T$	EVII
	$+\alpha_7$	h_2	A_7	EV
G_2	$3\alpha_1 + 2\alpha_2$	h_2	$A_1 \times A_1$	G

5. Determination of the centralizer $\mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(\boldsymbol{T})$.

This section is devoted to determining, up to inner automorphism, the centralizer of an arbitrary elliptic element T in $\mathfrak{sl}(4,\boldsymbol{R})$—that is, we will demonstrate Proposition 5.1 (on page 1148).

5.1.

First, let us introduce our settings. Let $\bar{\mathfrak{h}}$ be a Cartan subalgebra of $\mathfrak{sl}(4,\boldsymbol{C})$. Fix a linear order in $\triangle(\mathfrak{sl}(4,\boldsymbol{C}),\bar{\mathfrak{h}})$, and assume that $\{\alpha_t\}_{t=1}^3$ is the set of simple roots

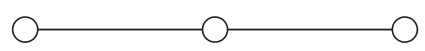
*Erratum: p. 289, line 9 on [**4**], read "α_1, $\alpha_1+\alpha_2$, $2\alpha_1+\alpha_2$, $3\alpha_1+\alpha_2$, $3\alpha_1+2\alpha_2$, α_2" instead of "α_1, $\alpha_1+\alpha_2$, $2\alpha_1+\alpha_2$, $3\alpha_1+\alpha_2$, $3\alpha_1+2\alpha_2$".

in $\triangle(\mathfrak{sl}(4,\boldsymbol{C}),\bar{\mathfrak{h}})$ whose Dynkin diagram is the Dynkin diagram utilized in Bourbaki [**4**, p. 265, Plate I]. Then, List of Murakami [**12**, p. 305, type AI] enables us to obtain the following five items:

(i) real simple Lie algebra $\mathfrak{g}=\mathfrak{sl}(4,\boldsymbol{R})$;
(ii) maximal compact subalgebra $\mathfrak{k}=\mathfrak{so}(4)$ of $\mathfrak{g}$;
(iii) compact real form $\mathfrak{g}_u=\mathfrak{su}(4)$ of $\mathfrak{sl}(4,\boldsymbol{C})$ with decomposition (4.1.1);
(iv) involutive automorphism $\sigma=\sigma_1\circ\exp 2\pi i\mathrm{ad}_{\mathfrak{su}(4)}h_2$ of $\mathfrak{g}_u$ satisfying two conditions (1) it stabilizes a maximal abelian subalgebra $i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ of $\mathfrak{g}_u$ and (2) compact symmetric pair $(\mathfrak{g}_u,\sigma)$ is related with $(\mathfrak{g},\mathfrak{k})$ as in the formulae (F1) $\mathfrak{g}=\mathfrak{k}\oplus i\mathfrak{p}$, (F2) $\mathfrak{g}_u=\mathfrak{k}\oplus\mathfrak{p}$, and (F3) $\mathfrak{k}=\{K\in\mathfrak{g}_u \mid \sigma(K)=K\}$ and $\mathfrak{p}=\{P\in\mathfrak{g}_u \mid \sigma(P)=-P\}$;
(v) the set of simple roots in $\triangle(\mathfrak{k},\mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})$

(see Example 4.6 for detail). Here, we remark that the set of positive roots in $\triangle(\mathfrak{g}_u,i\bar{\mathfrak{h}}_{\boldsymbol{R}})=\triangle(\mathfrak{su}(4),i\bar{\mathfrak{h}}_{\boldsymbol{R}})$ and the set of simple roots in $\triangle(\mathfrak{k},\mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})$ are as follows (cf. Remark 4.1):

$$\begin{aligned}\triangle^+(\mathfrak{g}_u,i\bar{\mathfrak{h}}_{\boldsymbol{R}})&=\{-i\alpha \mid \alpha\in\triangle^+(\mathfrak{sl}(4,\boldsymbol{C}),\bar{\mathfrak{h}})\}\\&=\left\{-i\sum_{m\le k<n}\alpha_k \;\middle|\; 1\le m<n\le 4\right\}\\&=\left\{\begin{matrix}-i\alpha_1, & -i(\alpha_1+\alpha_2), & -i(\alpha_1+\alpha_2+\alpha_3),\\ -i\alpha_2, & -i(\alpha_2+\alpha_3), & -i\alpha_3\end{matrix}\right\}.\end{aligned}\tag{5.1.1}$$

$\mathfrak{g}_u=\mathfrak{su}(4)$: 

$$\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})}=\{-i\tilde{\alpha}_1,\quad -i(\tilde{\alpha}_1+\tilde{\alpha}_2)\},\tag{5.1.2}$$

$\mathfrak{k}=\mathfrak{so}(4)$: 

where $-i\tilde{\alpha}_s:=-i\alpha_s|_{\mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}}$ for each $-i\alpha_s\in\triangle(\mathfrak{g}_u,i\bar{\mathfrak{h}}_{\boldsymbol{R}})$ and $1\le s\le 2$. In the settings, we are going to prove Proposition 5.1.

Proposition 5.1. *For any elliptic element $T\in\mathfrak{sl}(4,\boldsymbol{R})$, there exists an inner automorphism of $\mathfrak{sl}(4,\boldsymbol{R})$ which isomorphically maps $\mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(T)$ onto one of the following*:

$$\mathfrak{sl}(2,\boldsymbol{C})\oplus\mathfrak{t}^1,\qquad \mathfrak{t}^2\oplus\boldsymbol{R}^1,\qquad \mathfrak{sl}(2,\boldsymbol{R})\oplus\mathfrak{t}^1\oplus\boldsymbol{R}^1,\qquad \mathfrak{sl}(4,\boldsymbol{R}).$$

Here, the above four terms are given as follows:

$\mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(T)$	T
$\mathfrak{sl}(2,\boldsymbol{C})\oplus\mathfrak{t}^1$	$\mu_1\cdot i(Z_1-Z_2+Z_3)$ *or* $\mu_2\cdot iZ_2$
$\mathfrak{t}^2\oplus\boldsymbol{R}^1$	$\mu_1\cdot i(Z_1-Z_2+Z_3)+\mu_2\cdot iZ_2$ *with* $\mu_1\neq\mu_2$
$\mathfrak{sl}(2,\boldsymbol{R})\oplus\mathfrak{t}^1\oplus\boldsymbol{R}^1$	$\mu_1\cdot i(Z_1+Z_3)$
$\mathfrak{sl}(4,\boldsymbol{R})$	0

where $\{Z_t\}_{t=1}^3$ *is the dual basis of* $\{\alpha_t\}_{t=1}^3=\Pi_{\triangle(\mathfrak{sl}(4,\boldsymbol{C}),\bar{\mathfrak{h}})}$ *and* $\mu_1,\mu_2>0$.

Proof. The proof of this proposition requires many arguments. For the reason, we divide it into three processes. In the first process, we will provide a (positive) Weyl chamber $\mathfrak{W}^1_{\mathfrak{k}}$ with respect to $\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})}$ and we will verify that, Proposition 5.1 can be proved by studying four Cases (a) $T=\mu_1\cdot i(Z_1-Z_2+Z_3)$, (b) $T=\mu_1\cdot i(Z_1-Z_2+Z_3)+\mu_2\cdot iZ_2$, (c) $T=\mu_2\cdot iZ_2$ and (d) $T=0$ ($\mu_1,\mu_2>0$). In the second process, we will investigate Case (a) $T=\mu_1\cdot i(Z_1-Z_2+Z_3)$ and get $\mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(T)=\mathfrak{sl}(2,\boldsymbol{C})\oplus\mathfrak{t}^1$. Finally in the third process, we will study Case (b) $T=\mu_1\cdot i(Z_1-Z_2+Z_3)+\mu_2\cdot iZ_2$. However, Case (b) need be further divided into two Cases (b-1) $\mu_1\neq\mu_2$ and (b-2) $\mu_1=\mu_2$. We consider Case (b-1) (resp. (b-2)) and have $\mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(T)=\mathfrak{t}^2\oplus\boldsymbol{R}^1$ (resp. $\mathfrak{sl}(2,\boldsymbol{R})\oplus\mathfrak{t}^1\oplus\boldsymbol{R}^1$).

Notice 5.2. Without otherwise statements, we suppose that

$$\begin{aligned}&\mathfrak{g}=\mathfrak{sl}(4,\boldsymbol{R}),\qquad \mathfrak{g}_u=\mathfrak{su}(4),\qquad \mathfrak{k}=\mathfrak{so}(4),\\ &\sigma=\sigma_1\circ\exp 2\pi i\mathrm{ad}_{\mathfrak{su}(4)}h_2,\\ &\{Z_t\}_{t=1}^3\text{: the dual basis of }\{\alpha_t\}_{t=1}^3=\Pi_{\triangle(\mathfrak{sl}(4,\boldsymbol{C}),\bar{\mathfrak{h}})}\end{aligned}$$

on the proof of Proposition 5.1.

Process I. We aim to verify that, Proposition 5.1 can be proved by studying four Cases (a) $T=\mu_1\cdot i(Z_1-Z_2+Z_3)$, (b) $T=\mu_1\cdot i(Z_1-Z_2+Z_3)+\mu_2\cdot iZ_2$, (c) $T=\mu_2\cdot iZ_2$ and (d) $T=0$ ($\mu_1,\mu_2>0$).

Since $\sigma=\sigma_1\circ\exp 2\pi i\mathrm{ad}_{\mathfrak{su}(4)}h_2$ and $ih_2\in i\bar{\mathfrak{h}}_{\boldsymbol{R}}$, one deduces that $\sigma=\sigma_1$ on $i\bar{\mathfrak{h}}_{\boldsymbol{R}}$; and thus Lemma 4.3 implies that

$$\mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}=\mathrm{span}_{\boldsymbol{R}}\{i(Z_1+Z_3),\,iZ_2\},\tag{5.1.3}$$

$$\mathfrak{p}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}=\mathrm{span}_{\boldsymbol{R}}\{i(Z_1-Z_3)\}.\tag{5.1.4}$$

Now, let $\{iT_1,iT_2\}$ be the dual basis of $\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})}=\{-i\tilde{\alpha}_1,-i(\tilde{\alpha}_1+\tilde{\alpha}_2)\}$ (cf. (5.1.2)). From (5.1.3), it is clear that

$$\begin{cases}iT_1=i(Z_1-Z_2+Z_3),\\ iT_2=iZ_2\end{cases}\tag{5.1.5}$$

because $\{Z_t\}_{t=1}^3$ is the dual basis of $\{\alpha_t\}_{t=1}^3=\Pi_{\triangle(\mathfrak{sl}(4,\boldsymbol{C}),\bar{\mathfrak{h}})}$. Denote a Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})}$ by

$$\mathfrak{W}^1_{\mathfrak{k}} = \{H \in \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}} \mid -i\tilde{\alpha}_1(H) \geq 0,\ -i(\tilde{\alpha}_1 + \tilde{\alpha}_2)(H) \geq 0\}.$$

For any elliptic element $T \in \mathfrak{g}$, Remark 3.1 enables us to assume that $T \in \mathfrak{k}$. Then, since $\mathfrak{k} = \mathfrak{so}(4)$ is compact semisimple, there exists an inner automorphism of $\mathfrak{k}$ ($\subset \mathfrak{g}$) which maps T into $\mathfrak{W}^1_{\mathfrak{k}}$. Consequently, one may suppose that T belongs to $\mathfrak{W}^1_{\mathfrak{k}}$ from the beginning, as far as clarifying $\mathfrak{c}_{\mathfrak{g}}(T)$ up to inner automorphism of $\mathfrak{g} = \mathfrak{sl}(4, \boldsymbol{R})$. Accordingly, we suppose that $T \in \mathfrak{W}^1_{\mathfrak{k}}$ henceforth. This supposition allows us to describe T as follows:

$$T = \mu_1 \cdot iT_1 + \mu_2 \cdot iT_2 \ \big(= \mu_1 \cdot i(Z_1 - Z_2 + Z_3) + \mu_2 \cdot iZ_2\big) \text{ with } \mu_1, \mu_2 \geq 0 \tag{5.1.6}$$

since $\{iT_1, iT_2\}$ is the dual basis of $\{-i\tilde{\alpha}_1, -i(\tilde{\alpha}_1 + \tilde{\alpha}_2)\}$. For the coefficient of (5.1.6), the following four cases only occur:

(a) $\mu_1 > 0$ and $\mu_2 = 0$,
(b) $\mu_1, \mu_2 > 0$,
(c) $\mu_1 = 0$ and $\mu_2 > 0$,
(d) $\mu_1 = \mu_2 = 0$.

Consequently, this proposition can be proved by studying four Cases (a) $T = \mu_1 \cdot iT_1$ $\big(= \mu_1 \cdot i(Z_1 - Z_2 + Z_3)\big)$, (b) $T = \mu_1 \cdot iT_1 + \mu_2 \cdot iT_2$ $\big(= \mu_1 \cdot i(Z_1 - Z_2 + Z_3) + \mu_2 \cdot iZ_2\big)$, (c) $T = \mu_2 \cdot iT_2$ $\big(= \mu_2 \cdot iZ_2\big)$ and (d) $T = 0$ $(\mu_1, \mu_2 > 0)$. From now on, we devote ourselves to investigation of two Cases (a) and (b), because the other Cases (c) and (d) are similar.

PROCESS II. CASE (a): $T = \mu_1 \cdot iT_1 \in \mathfrak{W}^1_{\mathfrak{k}}$ $(\mu_1 > 0)$. First, let us study $\mathfrak{c}_{\mathfrak{k}}(T)$. Recall that $\Pi_{\triangle(\mathfrak{k}, \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})} = \{-i\tilde{\alpha}_1, -i(\tilde{\alpha}_1 + \tilde{\alpha}_2)\}$ (cf. (5.1.2)). For any root $\alpha = -n_1 \cdot i\tilde{\alpha}_1 - n_2 \cdot i(\tilde{\alpha}_1 + \tilde{\alpha}_2) \in \triangle(\mathfrak{k}, \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})$ $(n_1, n_2 \in \boldsymbol{Z})$, it is obvious that

$$\alpha(T) = \mu_1 \cdot n_1$$

since $\{iT_1, iT_2\}$ is the dual basis of $\{-i\tilde{\alpha}_1, -i(\tilde{\alpha}_1 + \tilde{\alpha}_2)\}$. Therefore, $\alpha = -n_1 \cdot i\tilde{\alpha}_1 - n_2 \cdot i(\tilde{\alpha}_1 + \tilde{\alpha}_2) \in \triangle(\mathfrak{k}, \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})$ is a root of $\mathfrak{c}_{\mathfrak{k}}(T)$ if and only if $\alpha(T) = 0$ if and only if $n_1 = 0$. Accordingly, it follows that

$$\Pi_{\triangle(\mathfrak{c}_{\mathfrak{k}}(T), \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})} = \{-i(\tilde{\alpha}_1 + \tilde{\alpha}_2)\}, \tag{a.i}$$

where we note that $\mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ is a maximal abelian subalgebra of not only $\mathfrak{k}$ but also $\mathfrak{c}_{\mathfrak{k}}(T)$ because $T \in \mathfrak{W}^1_{\mathfrak{k}} \subset \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}$. Lemma 3.3-(2) implies that $\operatorname{rk} \mathfrak{c}_{\mathfrak{k}}(T) = \operatorname{rk} \mathfrak{k} = \operatorname{rk} \mathfrak{so}(4) = 2$. This, together with (a.i), leads the following:

$$\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{su}(2) \oplus \mathfrak{t}^1. \tag{a.ii}$$

Next, let us determine $\mathfrak{c}_{\mathfrak{g}_u}(T)$. By virtue of (5.1.5), we can rewrite $T = \mu_1 \cdot iT_1$ as follows:

$$T = \mu_1 \cdot i(Z_1 - Z_2 + Z_3).$$

Therefore, for all positive roots in $\triangle(\mathfrak{g}_u, i\bar{\mathfrak{h}}_{\boldsymbol{R}})$ (see (5.1.1)), direct calculations give us

$$\begin{array}{llllll} -i\alpha_1(T) & = \mu_1, & -i(\alpha_1+\alpha_2)(T) & = 0, & -i(\alpha_1+\alpha_2+\alpha_3)(T) & = \mu_1, \\ -i\alpha_2(T) & = -\mu_1, & -i(\alpha_2+\alpha_3)(T) & = 0, & -i\alpha_3(T) & = \mu_1 \end{array} \tag{a.iii}$$

because $\{Z_t\}_{t=1}^3$ is the dual basis of $\Pi_{\triangle(\mathfrak{sl}(4,\boldsymbol{C}),\bar{\mathfrak{h}})} = \{\alpha_t\}_{t=1}^3$. This (a.iii) shows that

$$\begin{aligned} \triangle^+(\mathfrak{c}_{\mathfrak{g}_u}(T), i\bar{\mathfrak{h}}_{\boldsymbol{R}}) &= \{\alpha \in \triangle^+(\mathfrak{g}_u, i\bar{\mathfrak{h}}_{\boldsymbol{R}}) \mid \alpha(T) = 0\} \\ &= \{-i(\alpha_1+\alpha_2),\ -i(\alpha_2+\alpha_3)\} \end{aligned}$$

since $\mu_1 > 0$. In particular, it follows that

$$\Pi_{\triangle(\mathfrak{c}_{\mathfrak{g}_u}(T),\, i\bar{\mathfrak{h}}_{\boldsymbol{R}})} = \{-i(\alpha_1+\alpha_2),\ -i(\alpha_2+\alpha_3)\}$$

because $(\alpha_1+\alpha_2)$ and $(\alpha_2+\alpha_3)$ are linearly independent. The Dynkin diagram of $\Pi_{\triangle(\mathfrak{c}_{\mathfrak{g}_u}(T),\, i\bar{\mathfrak{h}}_{\boldsymbol{R}})}$ is as follows:

Case (a) $\quad \mathfrak{c}_{\mathfrak{g}_u}(T)$: $\underset{-i(\alpha_1+\alpha_2)}{\circ} \qquad\qquad \underset{-i(\alpha_2+\alpha_3).}{\circ}$

Consequently, since $\operatorname{rk} \mathfrak{c}_{\mathfrak{g}_u}(T) = \operatorname{rk} \mathfrak{g}_u = \operatorname{rk} \mathfrak{su}(4) = 3$, we obtain

$$\mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{t}^1 \tag{a.iv}$$

(cf. Lemma 3.3-(1)). Hence, Corollary 3.6, (a.ii) and (a.iv) imply that

$$\mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(T) = \mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{sl}(2,\boldsymbol{C}) \oplus \mathfrak{t}^1$$

in Case (a).

Process III. Case (b): $T = \mu_1 \cdot iT_1 + \mu_2 \cdot iT_2 \in \mathfrak{W}^1_{\mathfrak{k}}$ $(\mu_1, \mu_2 > 0)$. In the first place, we will investigate $\mathfrak{c}_{\mathfrak{k}}(T)$. Since $\{iT_1, iT_2\}$ is the dual basis of $\Pi_{\triangle(\mathfrak{k}, \mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})} = \{-i\tilde{\alpha}_1, -i(\tilde{\alpha}_1+\tilde{\alpha}_2)\}$, we have $-i\tilde{\alpha}_1(T) = \mu_1 > 0$ and $-i(\tilde{\alpha}_1+\tilde{\alpha}_2)(T) = \mu_2 > 0$; and hence $T \in \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ is a regular element of $\mathfrak{k} = \mathfrak{so}(4)$, so that

$$\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}.$$

It is obvious from (5.1.3) that $\mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ is a 2-dimensional abelian subalgebra of $\mathfrak{k}$. Accordingly we deduce that

$$\begin{cases} \mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{t}^2, \\ \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{ss}} = \{0\}, \quad \mathfrak{c}_{\mathfrak{k}}(T)_{\mathrm{z}} = \mathfrak{t}^2. \end{cases} \tag{b.i}$$

In the second place, let us consider $\mathfrak{c}_{\mathfrak{g}_u}(T)$. By virtue of (5.1.5), one can rewrite $T = \mu_1 \cdot iT_1 + \mu_2 \cdot iT_2$ as follows:

$$T = i\big(\mu_1 \cdot Z_1 - (\mu_1 - \mu_2) \cdot Z_2 + \mu_1 \cdot Z_3\big). \tag{b.ii}$$

Thus, for all roots in $\triangle^+(\mathfrak{g}_u, i\bar{\mathfrak{h}}_{\boldsymbol{R}})$ (see (5.1.1)), we provide

$$\begin{aligned} -i\alpha_1(T) &= \mu_1, & -i(\alpha_1+\alpha_2)(T) &= \mu_2, \\ -i(\alpha_1+\alpha_2+\alpha_3)(T) &= \mu_1+\mu_2, & -i\alpha_2(T) &= -\mu_1+\mu_2, \\ -i(\alpha_2+\alpha_3)(T) &= \mu_2, & -i\alpha_3(T) &= \mu_1 \end{aligned} \tag{b.iii}$$

because $\{Z_t\}_{t=1}^3$ is the dual basis of $\Pi_{\triangle(\mathfrak{sl}(4,\boldsymbol{C}),\bar{\mathfrak{h}})} = \{\alpha_t\}_{t=1}^3$. This (b.iii) means that

$$\begin{aligned} \triangle^+(\mathfrak{c}_{\mathfrak{g}_u}(T), i\bar{\mathfrak{h}}_{\boldsymbol{R}}) &= \big\{\alpha \in \triangle^+(\mathfrak{g}_u, i\bar{\mathfrak{h}}_{\boldsymbol{R}}) \bigm| \alpha(T) = 0\big\} \\ &= \begin{cases} \emptyset & \text{if } \mu_1 \neq \mu_2, \\ \big\{-i\alpha_2\big\} & \text{if } \mu_1 = \mu_2. \end{cases} \end{aligned} \tag{b.iv}$$

From (b.iv), we separate Case (b) “$T = \mu_1 \cdot iT_1 + \mu_2 \cdot iT_2$ $(\mu_1, \mu_2 > 0)$” into the following two cases:

(b-1) $\mu_1 \neq \mu_2$ and $\mu_1, \mu_2 > 0$;
(b-2) $\mu_1 = \mu_2 > 0$.

CASE (b-1): $T = \mu_1 \cdot iT_1 + \mu_2 \cdot iT_2 \in \mathfrak{W}^1_{\mathfrak{k}}$ ($\mu_1 \neq \mu_2$ and $\mu_1, \mu_2 > 0$): It follows from (b.iv) that $\Pi_{\triangle(\mathfrak{c}_{\mathfrak{g}_u}(T), i\bar{\mathfrak{h}}_{\boldsymbol{R}})} = \emptyset$. Thus, $\mathfrak{c}_{\mathfrak{g}_u}(T) = i\bar{\mathfrak{h}}_{\boldsymbol{R}} = (\mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}) \oplus (\mathfrak{p} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})$. Thereby, from (5.1.3) and (5.1.4) we conclude that

$$\begin{cases} \mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{t}^3, \\ \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} = \{0\}, \quad \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} = \mathfrak{t}^3, \\ \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k} = \mathfrak{t}^2 \quad \big(= \mathrm{span}_{\boldsymbol{R}}\{i(Z_1+Z_3), iZ_2\}\big), \\ \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p} = \mathfrak{t}^1 \quad \big(= \mathrm{span}_{\boldsymbol{R}}\{i(Z_1-Z_3)\}\big). \end{cases} \tag{b-1.i}$$

Consequently, Theorem 3.4-(A), (b.i) and (b-1.i) allow us to get

$$\mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(T) = \mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{t}^2 \oplus \boldsymbol{R}^1$$

in Case (b-1).

Case (b-2): $T = \mu_1 \cdot iT_1 + \mu_2 \cdot iT_2 \in \mathfrak{W}^1_{\mathfrak{k}}$ $(\mu_1 = \mu_2 > 0)$: By (b.iv) we see that $\Pi_{\triangle(\mathfrak{c}_{\mathfrak{g}_u}(T), i\bar{\mathfrak{h}}_{\boldsymbol{R}})} = \{-i\alpha_2\}$. Thus, Lemma 3.3-(1) and $\operatorname{rk}\mathfrak{g}_u = \operatorname{rk}\mathfrak{su}(4) = 3$ imply that $\mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{su}(2) \oplus \mathfrak{t}^2$, so that

$$\begin{cases} \mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{su}(2) \oplus \mathfrak{t}^2, \\ \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} = \mathfrak{su}(2), \quad \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} = \mathfrak{t}^2. \end{cases} \tag{b-2.i}$$

Now, let us prove Lemma 5.3.

Lemma 5.3. *With the assumptions and notation stated above*; *in Case* (b-2), *the structures of* $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}$ *and* $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p}$ *are as follows*:

$$\begin{aligned} \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k} &= \operatorname{span}_{\boldsymbol{R}}\{i(Z_1 + Z_3)\}, \\ \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p} &= \operatorname{span}_{\boldsymbol{R}}\{i(Z_1 - Z_3)\}. \end{aligned}$$

Proof. Since $\mu_1 = \mu_2$ and (b.ii), one obtains $T = \mu_1 \cdot i(Z_1 + Z_3)$. Therefore $i(Z_1 + Z_3)$ belongs to $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}}$ because $\mu_1 > 0$ and T is an element of $\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}}$. On the other hand, it is natural from (5.1.3) that $i(Z_1 + Z_3) \in \mathfrak{k}$. These deduce that

$$i(Z_1 + Z_3) \in \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}. \tag{b-2.ii}$$

Now, we want to show

$$i(Z_1 - Z_3) \in \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p}. \tag{b-2.iii}$$

Let us apply the root-space decomposition (4.1.1) to $\mathfrak{c}_{\mathfrak{g}_u}(T)$. Since $\triangle^+(\mathfrak{c}_{\mathfrak{g}_u}(T), i\bar{\mathfrak{h}}_{\boldsymbol{R}}) = \{\alpha \in \triangle^+(\mathfrak{g}_u, i\bar{\mathfrak{h}}_{\boldsymbol{R}}) \mid \alpha(T) = 0\} = \{-i\alpha_2\}$ and $T = \mu_1 \cdot i(Z_1 + Z_3) \in i\bar{\mathfrak{h}}_{\boldsymbol{R}}$, we perceive that

$$\mathfrak{c}_{\mathfrak{g}_u}(T) = i\bar{\mathfrak{h}}_{\boldsymbol{R}} \oplus \operatorname{span}_{\boldsymbol{R}}\{X_{\alpha_2} - X_{-\alpha_2}\} \oplus \operatorname{span}_{\boldsymbol{R}}\{i(X_{\alpha_2} + X_{-\alpha_2})\}$$

(recall Remark 4.1). Thus, any element $X \in \mathfrak{c}_{\mathfrak{g}_u}(T)$ can be written as $X = H + \lambda \cdot (X_{\alpha_2} - X_{-\alpha_2}) + \nu \cdot i(X_{\alpha_2} + X_{-\alpha_2})$ $(H \in i\bar{\mathfrak{h}}_{\boldsymbol{R}};\ \lambda, \nu \in \boldsymbol{R})$. It follows from (5.1.4) that $i(Z_1 - Z_3) \in i\bar{\mathfrak{h}}_{\boldsymbol{R}}$; and hence

$$\begin{aligned} \big[i(Z_1 - Z_3),\ X\big] &= \big[i(Z_1 - Z_3),\ H + \lambda \cdot (X_{\alpha_2} - X_{-\alpha_2}) + \nu \cdot i(X_{\alpha_2} + X_{-\alpha_2})\big] \\ &= \big[i(Z_1 - Z_3),\ \lambda \cdot (X_{\alpha_2} - X_{-\alpha_2}) + \nu \cdot i(X_{\alpha_2} + X_{-\alpha_2})\big] \\ &= \lambda \cdot \alpha_2(Z_1 - Z_3) \cdot i(X_{\alpha_2} + X_{-\alpha_2}) - \nu \cdot \alpha_2(Z_1 - Z_3) \cdot (X_{\alpha_2} - X_{-\alpha_2}) \\ &= 0 \end{aligned}$$

because $\{Z_t\}_{t=1}^3$ is the dual basis of $\Pi_{\triangle(\mathfrak{sl}(4,\boldsymbol{C}),\bar{\mathfrak{h}})} = \{\alpha_t\}_{t=1}^3$. This implies that $i(Z_1 - Z_3) \in \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}}$. On the other hand, we obtain $i(Z_1 - Z_3) \in \mathfrak{p}$ since (5.1.4). For the reasons, we have shown (b-2.iii). It is obvious from (b-2.i) that $\dim\big((\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k}) \oplus (\mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p})\big) = \dim \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} = 2$. This, together with (b-2.ii) and (b-2.iii), concludes Lemma 5.3. □

By Lemma 5.3 and (b-2.i), we deduce that

$$\begin{cases} \mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{su}(2) \oplus \mathfrak{t}^2, & \\ \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{ss}} = \mathfrak{su}(2), & \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} = \mathfrak{t}^2, \\ \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{k} = \mathfrak{t}^1 & \big(= \mathrm{span}_{\boldsymbol{R}}\{i(Z_1 + Z_3)\}\big), \\ \mathfrak{c}_{\mathfrak{g}_u}(T)_{\mathrm{z}} \cap \mathfrak{p} = \mathfrak{t}^1 & \big(= \mathrm{span}_{\boldsymbol{R}}\{i(Z_1 - Z_3)\}\big). \end{cases} \tag{b-2.iv}$$

Thus, Theorem 3.4-(B), together with (b.i) and (b-2.iv), means that

$$\mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(T) = \mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{sl}(2, \boldsymbol{R}) \oplus \mathfrak{t}^1 \oplus \boldsymbol{R}^1$$

in Case (b-2).

Now, let us collect the results obtained above. For $T = \mu_1 \cdot iT_1 + \mu_2 \cdot iT_2$ $\big(= \mu_1 \cdot i(Z_1 - Z_2 + Z_3) + \mu_2 \cdot iZ_2\big) \in \mathfrak{W}^1_{\mathfrak{k}}$ (see (5.1.6)), we investigated Case (a) by means of Process II and Case (b) by means of Process III. Then, we had

$$\mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(T) = \begin{cases} \mathfrak{sl}(2, \boldsymbol{C}) \oplus \mathfrak{t}^1 & \text{in Case (a) } \mu_1 > 0 \text{ and } \mu_2 = 0, \\ \mathfrak{t}^2 \oplus \boldsymbol{R}^1 & \text{in Case (b-1) } \mu_1, \mu_2 > 0 \text{ and } \mu_1 \neq \mu_2, \\ \mathfrak{sl}(2, \boldsymbol{R}) \oplus \mathfrak{t}^1 \oplus \boldsymbol{R}^1 & \text{in Case (b-2) } \mu_1 = \mu_2 > 0. \end{cases}$$

By investigations similar to those into Cases (a) and (b), we are able to confirm that

$$\mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(T) = \begin{cases} \mathfrak{sl}(2, \boldsymbol{C}) \oplus \mathfrak{t}^1 & \text{in Case (c) } \mu_1 = 0 \text{ and } \mu_2 > 0, \\ \mathfrak{sl}(4, \boldsymbol{R}) & \text{in Case (d) } \mu_1 = \mu_2 = 0 \end{cases}$$

(see Cases (a), (b), (c) and (d) on page 1150). Consequently, we have completed the proof of Proposition 5.1. □

Remark 5.4. Let us explain that the four terms in Proposition 5.1 are concretely obtained by use of matrices

$\mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(T)$	T (matrix)
$\mathfrak{sl}(2, \boldsymbol{C}) \oplus \mathfrak{t}^1$	$\lambda_1 \cdot (-E_{14} - E_{23} + E_{32} + E_{41})$ or $\lambda_2 \cdot (E_{14} - E_{23} + E_{32} - E_{41})$
$\mathfrak{t}^2 \oplus \boldsymbol{R}^1$	$(-\lambda_1 + \lambda_2) \cdot (E_{14} - E_{41}) - (\lambda_1 + \lambda_2) \cdot (E_{23} - E_{32})$ with $\lambda_1 \neq \lambda_2$
$\mathfrak{sl}(2, \boldsymbol{R}) \oplus \mathfrak{t}^1 \oplus \boldsymbol{R}^1$	$\lambda_1 \cdot (-E_{23} + E_{32})$
$\mathfrak{sl}(4, \boldsymbol{R})$	0

where E_{ab} denotes the matrix of degree 4 whose (c, d)-th entry is $\delta_{ca} \cdot \delta_{bd}$, and where $\lambda_1, \lambda_2 > 0$.

For the purpose, we will have matrices corresponding to the dual basis $\{Z_t\}_{t=1}^3$ of

$\{\alpha_t\}_{t=1}^3 = \Pi_{\triangle(\mathfrak{sl}(4,\boldsymbol{R}),\bar{\mathfrak{h}})}$ (i.e. (5.1.9)). To do so, it is necessary to concretely determine a compact simple Lie algebra $\mathfrak{g}_u$ which satisfies two conditions (I) it is a compact real form (4.1.1) of $\mathfrak{sl}(4, \boldsymbol{C})$, and (II) symmetric pair $(\mathfrak{g}_u, \sigma)$ is related with $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}(4, \boldsymbol{R}), \mathfrak{so}(4))$ as in the formulae (F1) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$, (F2) $\mathfrak{g}_u = \mathfrak{k} \oplus \mathfrak{p}$ and (F3) $\mathfrak{k} = \{K \in \mathfrak{g}_u \mid \sigma(K) = K\}$ and $\mathfrak{p} = \{P \in \mathfrak{g}_u \mid \sigma(P) = -P\}$. Here $\sigma = \sigma_1 \circ \exp 2\pi i \mathrm{ad}_{\mathfrak{su}(4)} h_2$ is the involution given in the first part of this section.

Denote by $\bar{\mathfrak{h}}'$ the set of diagonal matrices in $\mathfrak{sl}(4, \boldsymbol{C})$—that is,

$$\bar{\mathfrak{h}}' = \left\{ H' = \mathrm{diag}(\xi_1, \xi_2, \xi_3, \xi_4) \,\middle|\, \sum_{a=1}^{4} \xi_a = 0 \right\}.$$

Then, $\bar{\mathfrak{h}}'$ is a Cartan subalgebra of $\mathfrak{sl}(4, \boldsymbol{C})$, and it follows that $[H', E_{ab}] = (\xi_a - \xi_b) \cdot E_{ab}$ for $H' = \mathrm{diag}(\xi_1, \xi_2, \xi_3, \xi_4) \in \bar{\mathfrak{h}}'$ $(1 \le a, b \le 4)$; so that a root $\alpha'_{ab} : \bar{\mathfrak{h}}' \to \boldsymbol{C}$ is defined by

$$\alpha'_{ab}(H') = \xi_a - \xi_b \quad \text{for } H' = \mathrm{diag}(\xi_1, \xi_2, \xi_3, \xi_4) \in \bar{\mathfrak{h}}'$$

(cf. Helgason [**6**, pp. 186–187]). In this case, the set of simple roots in $\triangle(\mathfrak{sl}(4, \boldsymbol{C}), \bar{\mathfrak{h}}') = \{\alpha'_{cd} \mid 1 \le c \ne d \le 4\}$ can be

$$\Pi_{\triangle(\mathfrak{sl}(4,\boldsymbol{C}),\bar{\mathfrak{h}}')} = \{\alpha'_{12},\ \alpha'_{23},\ \alpha'_{34}\}.$$

Remark that a Weyl basis $\{X_{\alpha'_{cd}} \mid \alpha'_{cd} \in \triangle(\mathfrak{sl}(4, \boldsymbol{C}), \bar{\mathfrak{h}}')\}$ of $\mathfrak{sl}(4, \boldsymbol{C})$ (cf. Subsection 4.1) and the dual basis $\{Z'_1, Z'_2, Z'_3\}$ of $\Pi_{\triangle(\mathfrak{sl}(4,\boldsymbol{C}),\bar{\mathfrak{h}}')} = \{\alpha'_{12}, \alpha'_{23}, \alpha'_{34}\}$ are as follows:

$$X_{\alpha'_{cd}} = E_{cd} \quad \text{for } 1 \le c \ne d \le 4;$$
$$\begin{cases} Z'_1 = \mathrm{diag}(\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}), \\ Z'_2 = \mathrm{diag}(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), \\ Z'_3 = \mathrm{diag}(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}). \end{cases} \tag{5.1.7}$$

Now, let $\mathfrak{g}'_u$ be a compact real form (4.1.1) of $\mathfrak{sl}(4, \boldsymbol{C})$ provided by this Weyl basis $\{X_{\alpha'_{cd}} \mid \alpha'_{cd} \in \triangle(\mathfrak{sl}(4, \boldsymbol{C}), \bar{\mathfrak{h}}')\}$. Then, $\mathfrak{g}'_u$ accords with $\mathfrak{su}(4) = \{A \in \mathfrak{sl}(4, \boldsymbol{C}) \mid {}^t\bar{A} = -A\}$. Denote the canonical decomposition of symmetric pair $(\mathfrak{g}'_u, \sigma)$ by

$$\mathfrak{g}'_u = \mathfrak{k}' \oplus \mathfrak{p}',$$

where $\mathfrak{k}'$ (resp. $\mathfrak{p}'$) is the $+1$ (resp. -1)-eigenspace of σ in $\mathfrak{g}'_u = \mathfrak{su}(4)$. This gives us a real form $\mathfrak{g}'$ of $\mathfrak{sl}(4, \boldsymbol{C})$ defined by

$$\mathfrak{g}' := \mathfrak{k}' \oplus i\mathfrak{p}'.$$

Notice that the above $\mathfrak{g}' = \mathfrak{k}' \oplus i\mathfrak{p}'$ (resp. $\mathfrak{k}'$) is isomorphic to $\mathfrak{sl}(4, \boldsymbol{R}) = \{B \in \mathfrak{gl}(4, \boldsymbol{R}) \mid \mathrm{Tr}B = 0\}$ (resp. $\mathfrak{so}(4) = \{C \in \mathfrak{gl}(4, \boldsymbol{R}) \mid {}^tC = -C\}$), but does not accord with it. Indeed,

$\mathfrak{k}'$ and $i\mathfrak{p}'$ are as follows:

$$\mathfrak{k}' = \left\{ \begin{pmatrix} b_{11} \cdot i & a_{12} + b_{12} \cdot i & a_{13} + b_{13} \cdot i & 0 \\ -a_{12} + b_{12} \cdot i & b_{22} \cdot i & 0 & a_{13} + b_{13} \cdot i \\ -a_{13} + b_{13} \cdot i & 0 & -b_{22} \cdot i & a_{12} + b_{12} \cdot i \\ 0 & -a_{13} + b_{13} \cdot i & -a_{12} + b_{12} \cdot i & -b_{11} \cdot i \end{pmatrix} \middle| \, a_{pq}, b_{st} \in \boldsymbol{R} \right\},$$

$$i\mathfrak{p}' = \left\{ \begin{pmatrix} -c_{11} & -c_{12} + d_{12} \cdot i & -c_{13} + d_{13} \cdot i & -c_{14} + d_{14} \cdot i \\ -c_{12} - d_{12} \cdot i & c_{11} & -c_{23} + d_{23} \cdot i & c_{13} - d_{13} \cdot i \\ -c_{13} - d_{13} \cdot i & -c_{23} - d_{23} \cdot i & c_{11} & c_{12} - d_{12} \cdot i \\ -c_{14} - d_{14} \cdot i & c_{13} + d_{13} \cdot i & c_{12} + d_{12} \cdot i & -c_{11} \end{pmatrix} \middle| \, c_{uv}, d_{xy} \in \boldsymbol{R} \right\}.$$

For the reason, we define an automorphism φ of $\mathfrak{sl}(4, \boldsymbol{C})$ by

$$\varphi := \mathrm{Ad}(g_0), \tag{5.1.8}$$

where g_0 is an element of $GL(4, \boldsymbol{C})$ such that

$$g_0 = \begin{pmatrix} 0 & 1 & -i & 0 \\ -1 & 0 & 0 & -i \\ i & 0 & 0 & 1 \\ 0 & i & -1 & 0 \end{pmatrix}.$$

This automorphism φ satisfies that $\varphi(\mathfrak{g}') = \mathfrak{sl}(4, \boldsymbol{R})$ and $\varphi(\mathfrak{k}') = \mathfrak{so}(4)$. Therefore, $\mathfrak{g}_u := \varphi(\mathfrak{g}'_u)$ is a compact simple Lie algebra satisfying the two conditions (I) it is a compact real form (4.1.1) of $\mathfrak{sl}(4, \boldsymbol{C})$ provided by Weyl basis $\{X_{\alpha_{cd}} \mid \alpha_{cd} \in \triangle(\mathfrak{sl}(4, \boldsymbol{C}), \bar{\mathfrak{h}})\}$, and (II) symmetric pair $(\mathfrak{g}_u, \sigma)$ is related with $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}(4, \boldsymbol{R}), \mathfrak{so}(4))$ as in the formulae (F1), (F2) and (F3), where we identify σ with $\varphi \circ \sigma \circ \varphi^{-1}$. Here, $\bar{\mathfrak{h}} := \varphi(\bar{\mathfrak{h}}')$, $\alpha_{cd} := \alpha'_{cd} \circ \varphi^{-1}$ and $X_{\alpha_{cd}} := \varphi(X_{\alpha'_{cd}})$ $(1 \leq c \neq d \leq 4)$. Notice that $\{Z_1 := \varphi(Z'_1), Z_2 := \varphi(Z'_2), Z_3 := \varphi(Z'_3)\}$ is the dual basis of $\Pi_{\triangle(\mathfrak{sl}(4,\boldsymbol{C}),\bar{\mathfrak{h}})} = \{\alpha_{12}, \alpha_{23}, \alpha_{34}\}$ and is the following:

$$\begin{cases} Z_1 = \frac{1}{4}\mathrm{diag}(-1, 1, 1, -1) + \frac{i}{2}(E_{23} - E_{32}), \\ Z_2 = \frac{i}{2}(-E_{14} + E_{23} - E_{32} + E_{41}), \\ Z_3 = -\frac{1}{4}\mathrm{diag}(-1, 1, 1, -1) + \frac{i}{2}(E_{23} - E_{32}) \end{cases} \tag{5.1.9}$$

(see (5.1.7) and (5.1.8)). By using this dual basis $\{Z_t\}_{t=1}^{3}$, the arguments on the end of the proof of Proposition 5.1 lead the following:

$$\text{(a)}\begin{cases} \mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(T) = \mathfrak{sl}(2,\boldsymbol{C}) \oplus \mathfrak{t}^1, \\ T = (\mu_1/2) \cdot (-E_{14} - E_{23} + E_{32} + E_{41}); \end{cases}$$
$$\text{(b-1)}\begin{cases} \mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(T) = \mathfrak{t}^2 \oplus \boldsymbol{R}^1, \\ T = (-\mu_1 + \mu_2)/2 \cdot (E_{14} - E_{41}) - (\mu_1 + \mu_2)/2 \cdot (E_{23} - E_{32}) \text{ with } \mu_1 \neq \mu_2; \end{cases}$$
$$\text{(b-2)}\begin{cases} \mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(T) = \mathfrak{sl}(2,\boldsymbol{R}) \oplus \mathfrak{t}^1 \oplus \boldsymbol{R}^1, \\ T = \mu_1 \cdot (-E_{23} + E_{32}); \end{cases}$$
$$\text{(c)}\begin{cases} \mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(T) = \mathfrak{sl}(2,\boldsymbol{C}) \oplus \mathfrak{t}^1, \\ T = (\mu_2/2) \cdot (E_{14} - E_{23} + E_{32} - E_{41}); \end{cases}$$
$$\text{(d)}\begin{cases} \mathfrak{c}_{\mathfrak{sl}(4,\boldsymbol{R})}(T) = \mathfrak{sl}(4,\boldsymbol{R}), \\ T = 0 \end{cases}$$

$(\mu_1, \mu_2 > 0)$. Therefore, we have obtained the four terms in Proposition 5.1 by means of matrices E_{ab}.

5.2. Other results.

Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ be a real semisimple Lie algebra, where all $\mathfrak{g}_a$ are simple ideals of $\mathfrak{g}$ $(1 \leq a \leq n)$. Take any elliptic element $T \in \mathfrak{g}$, and write it as $T = T^1 + \cdots + T^n$ $(T^a \in \mathfrak{g}_a)$. Then, one sees that $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{c}_{\mathfrak{g}_1}(T^1) \oplus \cdots \oplus \mathfrak{c}_{\mathfrak{g}_n}(T^n)$, and that each T^a is an elliptic element of $\mathfrak{g}_a$ because $\mathrm{ad}_{\mathfrak{g}} T(\mathfrak{g}_a) \subset \mathfrak{g}_a$ and $\mathrm{ad}_{\mathfrak{g}} T|_{\mathfrak{g}_a} = \mathrm{ad}_{\mathfrak{g}_a} T^a$. Consequently, study on the structure of $\mathfrak{c}_{\mathfrak{g}}(T)$ can be reduced to that of $\mathfrak{c}_{\mathfrak{g}_a}(T^a)$. For every real simple Lie algebra $\mathfrak{g}_a$ and any elliptic element $T^a \in \mathfrak{g}_a$, we can determine $\mathfrak{c}_{\mathfrak{g}_a}(T^a)$ up to inner automorphism of $\mathfrak{g}_a$, by utilizing arguments on this section. For example, we get the following:

PROPOSITION 5.5. *For any elliptic element $T \in \mathfrak{g}_{2(2)}$, there exists an inner automorphism of $\mathfrak{g}_{2(2)}$ which isomorphically maps $\mathfrak{c}_{\mathfrak{g}_{2(2)}}(T)$ onto one of the following*:

$$\mathfrak{su}(2) \oplus \mathfrak{t}^1, \qquad \mathfrak{t}^2, \qquad \mathfrak{sl}(2,\boldsymbol{R}) \oplus \mathfrak{t}^1, \qquad \mathfrak{g}_{2(2)}.$$

Here, $\mathfrak{g}_{2(2)}$ is determined by involution $\sigma = \exp 2\pi i \mathrm{ad}_{\mathfrak{g}_2} h_2$ (see List of Murakami [**12**, p. 297, type G] *and our Notice* 4.8) *and the above four terms are given as follows*:

$\mathfrak{c}_{\mathfrak{g}_{2(2)}}(T)$	T
$\mathfrak{su}(2) \oplus \mathfrak{t}^1$	$\lambda_1 \cdot i(2Z_1 - 3Z_2)$ *or* $\lambda_2 \cdot iZ_2$
$\mathfrak{t}^2$	$\lambda_1 \cdot i(2Z_1 - 3Z_2) - \lambda_2 \cdot iZ_2$ *with* $3\lambda_1 \neq 3\lambda_2, \lambda_2$
$\mathfrak{sl}(2,\boldsymbol{R}) \oplus \mathfrak{t}^1$	$\lambda_1 \cdot i(Z_1 - 2Z_2)$ *or* $\lambda_1 \cdot i(Z_1 - 3Z_2)$
$\mathfrak{g}_{2(2)}$	0

where $\{Z_1, Z_2\}$ is the dual basis of $\Pi_{\triangle(\mathfrak{g}_2^{\boldsymbol{C}}, \bar{\mathfrak{h}})} = \{\alpha_1, \alpha_2\}$ and $\lambda_1, \lambda_2 > 0$.

PROPOSITION 5.6. *For any elliptic element $T \in \mathfrak{su}^*(2l)$, there exists an inner automorphism of $\mathfrak{su}^*(2l)$ which isomorphically maps $\mathfrak{c}_{\mathfrak{su}^*(2l)}(T)$ onto one of the following*:

$$\bigoplus_{a=1}^{k}\mathfrak{sl}(j_a-j_{a-1},\boldsymbol{C})\oplus\mathfrak{sl}(l-j_k,\boldsymbol{C})\oplus\mathfrak{t}^{k+1}\oplus\boldsymbol{R}^k,$$
$$\bigoplus_{a=1}^{k}\mathfrak{sl}(j_a-j_{a-1},\boldsymbol{C})\oplus\mathfrak{su}^*(2l-2j_k)\oplus\mathfrak{t}^{k}\oplus\boldsymbol{R}^k,$$

where $l\geq 2$, $0\leq k\leq l-1$, $1\leq j_1<\cdots<j_k\leq l-1$, *and* $j_0:=0$. *Here* $\mathfrak{su}^*(2l)$ *is determined by involution* σ_1 *in Lemma* 4.3 (*see List of type* AII *in* [**12**, p. 305]; *also see Remark* 4.7) *and the above terms are given as follows*:

$$\begin{cases}\mathfrak{c}_{\mathfrak{su}^*(2l)}(T)=\bigoplus_{a=1}^{k}\mathfrak{sl}(j_a-j_{a-1},\boldsymbol{C})\oplus\mathfrak{sl}(l-j_k,\boldsymbol{C})\oplus\mathfrak{t}^{k+1}\oplus\boldsymbol{R}^k,\\ T=\sum_{p=1}^{k}\nu_{j_p}\cdot i(Z_{j_p}+Z_{2l-j_p})+\nu_l\cdot iZ_l;\end{cases}$$
$$\begin{cases}\mathfrak{c}_{\mathfrak{su}^*(2l)}(T)=\bigoplus_{a=1}^{k}\mathfrak{sl}(j_a-j_{a-1},\boldsymbol{C})\oplus\mathfrak{su}^*(2l-2j_k)\oplus\mathfrak{t}^{k}\oplus\boldsymbol{R}^k,\\ T=\sum_{p=1}^{k}\nu_{j_p}\cdot i(Z_{j_p}+Z_{2l-j_p});\end{cases}$$

where $\{Z_j\}_{j=1}^{2l-1}$ *denotes the dual basis of* $\Pi_{\triangle(\mathfrak{sl}(2l,\boldsymbol{C}),\bar{\mathfrak{h}})}=\{\alpha_j\}_{j=1}^{2l-1}$ *and* $\nu_{j_1},\dots,\nu_{j_k},\nu_l>0$.

6. The $\boldsymbol{H}$-elements in pseudo-Hermitian symmetric Lie algebras.

Our aim in this section is to determine the H-element in each simple irreducible pseudo-Hermitian symmetric Lie algebra. First, let us introduce the notion of simple irreducible pseudo-Hermitian symmetric Lie algebra.

Definition 6.1 (Shapiro [**14**]). A simple symmetric Lie algebra $(\mathfrak{g},\mathfrak{r})$ is called irreducible pseudo-Hermitian if $\mathfrak{g}^{\boldsymbol{C}}$ is also simple, and if there exists an elliptic element $T\in\mathfrak{g}$ such that $\mathfrak{c}_{\mathfrak{g}}(T)$ coincides with the isotropy subalgebra $\mathfrak{r}$. It is said to be reducible pseudo-Hermitian, if $\mathfrak{g}$ admits a structure of complex Lie algebra and $\mathfrak{r}$ is not semisimple. Remark that every semisimple pseudo-Hermitian symmetric Lie algebra is a finite direct sum of simple irreducible or simple reducible pseudo-Hermitian symmetric Lie algebras (cf. [**14**]).

We will illustrate the way of finding the H-elements in two simple irreducible pseudo-Hermitian symmetric Lie algebras $(\mathfrak{g},\mathfrak{r})=(\mathfrak{sp}(3,\boldsymbol{R}),\mathfrak{su}(2,1)\oplus\mathfrak{t}^1)$ and $(\mathfrak{so}(3,5),\mathfrak{so}(3,3)\oplus\mathfrak{t}^1)$. These two examples correspond to the two cases where a maximal compact subalgebra of $\mathfrak{g}$ admits a non-trivial center (see Subsection 6.1) and admits no centers (see Subsection 6.2). Lastly in Subsection 6.3, we will accomplish our aim (see Theorem 6.16 on page 1171).

For the sake of Subsections 6.1.4 and 6.2.4, we are going to prove Lemma 6.2.

LEMMA 6.2. *Let $(\mathfrak{g}, \mathfrak{r})$ be a semisimple pseudo-Hermitian symmetric Lie algebra, let T be a semisimple element of $\mathfrak{g}$ satisfying $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{r}$, and let $\mathfrak{h}^{\boldsymbol{C}}$ be a Cartan subalgebra of $\mathfrak{g}^{\boldsymbol{C}}$ such that $T \in \mathfrak{h}^{\boldsymbol{C}}$. Then, the following three items* (i), (ii) *and* (iii) *hold*:

(i) $\mathfrak{g}$ *is decomposed as*

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{q} \ \bigl(= \mathfrak{c}_{\mathfrak{g}}(T) \oplus [T, \mathfrak{g}]\bigr),$$

where $\mathfrak{q} := [T, \mathfrak{g}]$. In particular, $\mathfrak{r}$ and $\mathfrak{q}$ satisfy $[\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r}$, $[\mathfrak{r}, \mathfrak{q}] \subset \mathfrak{q}$ and $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{r}$; and thus the above decomposition is the canonical one of symmetric Lie algebra $(\mathfrak{g}, \mathfrak{r})$.

(ii) *The following three conditions* (c1), (c2) *and* (c3) *are mutually equivalent*:

(c1) *T is the H-element in $(\mathfrak{g}, \mathfrak{r})$;*
(c2) $\mathrm{ad}_{\mathfrak{g}^{\boldsymbol{C}}} T|_{[T, \mathfrak{g}^{\boldsymbol{C}}]}$ *is a complex structure of $[T, \mathfrak{g}^{\boldsymbol{C}}]$;*
(c3) *$\beta(T) = \pm i$ for all $\beta \in \{\beta \in \triangle(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{h}^{\boldsymbol{C}}) \mid \beta(T) \neq 0\}$.*

(iii) *If T is the H-element in $(\mathfrak{g}, \mathfrak{r})$, then an inner automorphism $\rho := \exp \pi \mathrm{ad}_{\mathfrak{g}} T$ of $\mathfrak{g}$ is involutive and the $+1$ (resp. -1)-eigenspace of ρ in $\mathfrak{g}$ coincides with $\mathfrak{r}$ (resp. $\mathfrak{q}$).*

PROOF.
(i) Since $(\mathfrak{g}, \mathfrak{r})$ is symmetric Lie algebra, there exists an involutive automorphism $\tilde{\rho}$ of $\mathfrak{g}$ such that

$$\mathfrak{g}_{+1} = \mathfrak{r} \text{ and } \mathfrak{g} = \mathfrak{g}_{+1} \oplus \mathfrak{g}_{-1}, \tag{6.0.1}$$

where we denote by $\mathfrak{g}_{\pm 1}$ the ± 1-eigenspace of $\tilde{\rho}$ in $\mathfrak{g}$. Then, it is natural that $[\mathfrak{g}_{+1}, \mathfrak{g}_{+1}] \subset \mathfrak{g}_{+1}$, $[\mathfrak{g}_{+1}, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{-1}$ and $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{+1}$. Therefore, the item (i) holds if $\mathfrak{g}_{-1} = [T, \mathfrak{g}]$ $(= \mathfrak{q})$. Hence we will be devoted to showing that $\mathfrak{g}_{-1} = [T, \mathfrak{g}]$ from now on. The non-degeneracy of $B_{\mathfrak{g}}$, combined with $B_{\mathfrak{g}}(\mathfrak{g}_{+1}, \mathfrak{g}_{-1}) = \{0\}$ and $\mathfrak{g}_{+1} = \mathfrak{r}$, implies that

$$\mathfrak{g}_{-1} = \{X \in \mathfrak{g} \mid B_{\mathfrak{g}}(X, R) = 0 \text{ for all } R \in \mathfrak{r}\}.$$

Accordingly, it follows from $\mathfrak{r} = \mathfrak{c}_{\mathfrak{g}}(T)$ that

$$[T, \mathfrak{g}] \subset \mathfrak{g}_{-1}. \tag{6.0.2}$$

Since T is semisimple, $\mathfrak{g}$ is decomposed as $\mathfrak{g} = \mathfrak{c}_{\mathfrak{g}}(T) \oplus [T, \mathfrak{g}]$. Besides from $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{r} = \mathfrak{g}_{+1}$, one obtains

$$\mathfrak{g} = \mathfrak{g}_{+1} \oplus [T, \mathfrak{g}]. \tag{6.0.3}$$

Consequently, we have $\mathfrak{g}_{-1} = [T, \mathfrak{g}]$ by (6.0.1), (6.0.2) and (6.0.3). Thus, the item (i) holds.

(ii) (c1)$\leftrightarrow$(c2): In the first place, let us verify that two conditions (c1) and (c2) are equivalent to each other. The hypothesis of $\mathfrak{r} = \mathfrak{c}_{\mathfrak{g}}(T)$, together with (i), enables us to confirm that

$$\begin{aligned} &T \text{ is the } H\text{-element in } (\mathfrak{g}, \mathfrak{r}) \text{ if and only if} \\ &\quad \mathrm{ad}_{\mathfrak{g}} T|_{\mathfrak{q}} \text{ is a complex structure of } \mathfrak{q} = [T, \mathfrak{g}]. \end{aligned} \tag{6.0.4}$$

Furthermore, we see that

$$\begin{aligned} &\mathrm{ad}_{\mathfrak{g}} T|_{\mathfrak{q}} \text{ is a complex structure of } \mathfrak{q} = [T, \mathfrak{g}] \text{ if and only if} \\ &\quad \mathrm{ad}_{\mathfrak{g}^{\boldsymbol{C}}} T|_{[T, \mathfrak{g}^{\boldsymbol{C}}]} \text{ is a complex structure of } [T, \mathfrak{g}^{\boldsymbol{C}}] \end{aligned} \tag{6.0.5}$$

because a vector space $[T, \mathfrak{g}^{\boldsymbol{C}}]$ coincides with the complex vector subspace of $\mathfrak{g}^{\boldsymbol{C}}$ generated by $[T, \mathfrak{g}]$. Consequently, it follows from (6.0.4) and (6.0.5) that two conditions (c1) and (c2) are equivalent to each other.

(c2)$\leftrightarrow$(c3): In the second place, we will prove that two conditions (c2) and (c3) are equivalent to each other. First, let us clarify the structure of $[T, \mathfrak{g}^{\boldsymbol{C}}]$. Define $\triangle_0(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{h}^{\boldsymbol{C}})$ and $\triangle_1(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{h}^{\boldsymbol{C}})$ by

$$\begin{cases} \triangle_0(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{h}^{\boldsymbol{C}}) := \{\zeta \in \triangle(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{h}^{\boldsymbol{C}}) \mid \zeta(T) = 0\}, \\ \triangle_1(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{h}^{\boldsymbol{C}}) := \{\beta \in \triangle(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{h}^{\boldsymbol{C}}) \mid \beta(T) \neq 0\}. \end{cases}$$

Then, the root-space decomposition of $\mathfrak{g}^{\boldsymbol{C}}$ with respect to $\mathfrak{h}^{\boldsymbol{C}}$ is rewritten as follows:

$$\begin{aligned} \mathfrak{g}^{\boldsymbol{C}} &= \mathfrak{h}^{\boldsymbol{C}} \oplus \bigoplus_{\alpha \in \triangle(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{h}^{\boldsymbol{C}})} \mathrm{span}_{\boldsymbol{C}}\{X_\alpha\} \\ &= \mathfrak{h}^{\boldsymbol{C}} \oplus \bigoplus_{\zeta \in \triangle_0(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{h}^{\boldsymbol{C}})} \mathrm{span}_{\boldsymbol{C}}\{X_\zeta\} \oplus \bigoplus_{\beta \in \triangle_1(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{h}^{\boldsymbol{C}})} \mathrm{span}_{\boldsymbol{C}}\{X_\beta\} \\ &= \mathfrak{c}_{\mathfrak{g}^{\boldsymbol{C}}}(T) \oplus \bigoplus_{\beta \in \triangle_1(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{h}^{\boldsymbol{C}})} \mathrm{span}_{\boldsymbol{C}}\{X_\beta\}, \end{aligned}$$

where X_α, $\alpha \in \triangle(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{h}^{\boldsymbol{C}})$, are given in Subsection 4.1. Therefore, we perceive that

$$[T, \mathfrak{g}^{\boldsymbol{C}}] = \bigoplus_{\beta \in \triangle_1(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{h}^{\boldsymbol{C}})} \mathrm{span}_{\boldsymbol{C}}\{X_\beta\} \tag{6.0.6}$$

because $[T, X_\beta] = \beta(T) \cdot X_\beta \neq 0$ for all $\beta \in \triangle_1(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{h}^{\boldsymbol{C}})$ and semisimple element $T \in \mathfrak{h}^{\boldsymbol{C}}$ splits $\mathfrak{g}^{\boldsymbol{C}}$ into $\mathfrak{c}_{\mathfrak{g}^{\boldsymbol{C}}}(T) \oplus [T, \mathfrak{g}^{\boldsymbol{C}}]$. Since (6.0.6) and since $(\mathrm{ad}_{\mathfrak{g}^{\boldsymbol{C}}} T)^2(X_\beta) = (\beta(T))^2 \cdot X_\beta$, we conclude that $\mathrm{ad}_{\mathfrak{g}^{\boldsymbol{C}}} T|_{[T, \mathfrak{g}^{\boldsymbol{C}}]}$ is a complex structure of $[T, \mathfrak{g}^{\boldsymbol{C}}]$ if and only if $\beta(T) = \pm i$ for all $\beta \in \triangle_1(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{h}^{\boldsymbol{C}})$. Accordingly, two conditions (c2) and (c3) are equivalent to each other. For the reasons, three conditions (c1), (c2) and (c3) are mutually equivalent.

(iii) If T is the H-element in $(\mathfrak{g}, \mathfrak{r})$, then it follows from (6.0.4) that $(\mathrm{ad}_{\mathfrak{g}} T)^2(Q) = -Q$ for every $Q \in \mathfrak{q}$. Hence, we deduce that

$$\begin{aligned}\rho(Q) &= \exp \pi \mathrm{ad}_{\mathfrak{g}} T(Q) = \sum_{l \geq 0} \frac{1}{l\,!} (\pi \mathrm{ad}_{\mathfrak{g}} T)^l (Q) \\ &= \sum_{m \geq 0} \frac{1}{2m!} (\pi \mathrm{ad}_{\mathfrak{g}} T)^{2m} (Q) + \sum_{n \geq 0} \frac{1}{(2n+1)!} (\pi \mathrm{ad}_{\mathfrak{g}} T)^{2n+1} (Q) \\ &= \sum_{m \geq 0} (-1)^m \cdot \frac{\pi^{2m}}{2m!} \cdot Q + \sum_{n \geq 0} (-1)^n \cdot \frac{\pi^{2n+1}}{(2n+1)!} \cdot [T, Q] \\ &= \cos \pi \cdot Q + \sin \pi \cdot [T, Q] \\ &= -Q\end{aligned}$$

for all $Q \in \mathfrak{q}$. On the other hand, it is clear that

$$\rho(R) = \exp \pi \mathrm{ad}_{\mathfrak{g}} T(R) = R \text{ for any } R \in \mathfrak{r}$$

since $\mathfrak{r} = \mathfrak{c}_{\mathfrak{g}}(T)$. These, combined with (i), imply that ρ is an involutive automorphism of $\mathfrak{g}$, and that $\mathfrak{r} = \{X \in \mathfrak{g} \mid \rho(X) = X\}$ and $\mathfrak{q} = \{Y \in \mathfrak{g} \mid \rho(Y) = -Y\}$. Consequently, we have proved Lemma 6.2. $\square$

NOTICE 6.3. If T is the H-element in pseudo-Hermitian symmetric Lie algebra $(\mathfrak{g}, \mathfrak{r})$, then $-T$ is also its H-element. Therefore, the H-element has irregularity with respect to $\pm$-sign. We determine the H-element T in $(\mathfrak{g}, \mathfrak{r})$ up to $\pm$-sign.

Now, let us illustrate the way of finding the H-element T in simple irreducible pseudo-Hermitian symmetric Lie algebra $(\mathfrak{sp}(3, \boldsymbol{R}), \mathfrak{su}(2,1) \oplus \mathfrak{t}^1)$.

REMARK 6.4. On discussions stated in Subsections 6.1 and 6.2; simple Lie algebra $\mathfrak{g}$, maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, compact symmetric pair $(\mathfrak{g}_u, \sigma)$ related with $(\mathfrak{g}, \mathfrak{k})$ as in the formulae (F1), (F2) and (F3), and so forth are determined by List of Murakami [**12**] (read Subsection 4.3).

6.1. $(\mathfrak{g}, \mathfrak{r}) = (\mathfrak{sp}(3, \boldsymbol{R}), \mathfrak{su}(2,1) \oplus \mathfrak{t}^1)$.

In Subsection 6.1.1, we will give two necessary conditions (N-1.1) and (N-1.2) for $T \in \mathfrak{k}$ to satisfy $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{su}(2,1) \oplus \mathfrak{t}^1$. In order to easily find $T \in \mathfrak{k}$ with $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{su}(2,1) \oplus \mathfrak{t}^1$, we will prove Lemmas 6.7 and 6.8 in Subsection 6.1.2. By direct calculations, we will attempt to find $T \in \mathfrak{k}$ with $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{su}(2,1) \oplus \mathfrak{t}^1$ in Subsection 6.1.3. Finally in Subsection 6.1.4, we will demonstrate that element T, found in Subsection 6.1.3, is the H-element in $(\mathfrak{sp}(3, \boldsymbol{R}), \mathfrak{su}(2,1) \oplus \mathfrak{t}^1)$.

NOTICE 6.5. In Subsection 6.1, we assume that

$$\mathfrak{g} = \mathfrak{sp}(3, \boldsymbol{R}), \qquad \mathfrak{g}_u = \mathfrak{sp}(3), \qquad \mathfrak{k} = \mathfrak{su}(3) \oplus \mathfrak{t}^1,$$
$$\{Z_a\}_{a=1}^3\text{: the dual basis of } \{\alpha_a\}_{a=1}^3 = \Pi_{\triangle(\mathfrak{sp}(3,\boldsymbol{C}),\bar{\mathfrak{h}})}.$$

6.1.1. Two necessary conditions.

First, we suppose that $T \in \mathfrak{k}$ satisfies $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{su}(2,1) \oplus \mathfrak{t}^1$, and we will obtain two necessary conditions (N-1.1) and (N-1.2). For $\mathfrak{su}(2,1) \oplus \mathfrak{t}^1$, one deduces that

$$\mathfrak{su}(3) \oplus \mathfrak{t}^1 \text{ is its compact dual,} \tag{6.1.1}$$

$$\mathfrak{su}(2) \oplus \mathfrak{t}^2 \text{ is its maximal compact subalgebra.} \tag{6.1.2}$$

Since $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{su}(2,1) \oplus \mathfrak{t}^1$, the following is immediate from (6.1.1) and Lemma 3.2-(vi):

$$\mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{su}(3) \oplus \mathfrak{t}^1. \tag{N-1.1}$$

On the other hand, Lemma 3.2-(iv) and (6.1.2) imply that

$$\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{su}(2) \oplus \mathfrak{t}^2. \tag{N-1.2}$$

Accordingly, we have got two necessary conditions (N-1.1) and (N-1.2).

REMARK 6.6. Corollary 3.6 implies that these two conditions (N-1.1) and (N-1.2) are also a sufficient condition for $T \in \mathfrak{k}$ to satisfy $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{su}(2,1) \oplus \mathfrak{t}^1$.

6.1.2. Existence zone.

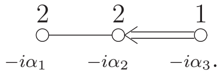
In order to find an element $T \in \mathfrak{k}$ with $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{su}(2,1) \oplus \mathfrak{t}^1$, we want to restrict its existence zone. By the arguments mentioned below, one will be able to expect that an element $T = \sum_{a=1}^{3} \lambda_a \cdot iZ_a \in i\bar{\mathfrak{h}}_{\boldsymbol{R}} \subset \mathfrak{k}$ with "$\lambda_1, \lambda_2 \geq 0$" satisfies $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{su}(2,1) \oplus \mathfrak{t}^1$, where $\bar{\mathfrak{h}}$ denotes a Cartan subalgebra of $\mathfrak{g}^{\boldsymbol{C}} = \mathfrak{sp}(3, \boldsymbol{C})$. Note that $i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ is a maximal abelian subalgebra of $\mathfrak{g}_u = \mathfrak{sp}(3)$ and of $\mathfrak{k} = \mathfrak{su}(3) \oplus \mathfrak{t}^1$.

In the first place, let us enumerate the set of positive roots in $\triangle(\mathfrak{g}_u, i\bar{\mathfrak{h}}_{\boldsymbol{R}})$; the set of simple roots in $\triangle(\mathfrak{k}_{\mathrm{ss}}, \mathfrak{k}_{\mathrm{ss}} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})$ (see Notation 2.3 (n5) for $\mathfrak{k}_{\mathrm{ss}}$); and the Dynkin diagrams of $\Pi_{\triangle(\mathfrak{g}_u, i\bar{\mathfrak{h}}_{\boldsymbol{R}})}$ and $\Pi_{\triangle(\mathfrak{k}, \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})}$ (ref. List of Murakami [**12**, p. 297, type CI]).

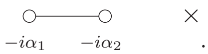
$$\begin{aligned} \triangle^+(\mathfrak{g}_u, i\bar{\mathfrak{h}}_{\boldsymbol{R}}) &= \{-i\alpha \mid \alpha \in \triangle^+(\mathfrak{sp}(3, \boldsymbol{C}), \bar{\mathfrak{h}})\} \\ &= \left\{ \begin{array}{ll} -i\alpha_1, & -i(\alpha_1 + \alpha_2), \\ -i(\alpha_1 + 2\alpha_2 + \alpha_3), & -i\alpha_2, \\ -i(\alpha_1 + \alpha_2 + \alpha_3), & -i(\alpha_2 + \alpha_3), \\ -i(2\alpha_1 + 2\alpha_2 + \alpha_3), & -i(2\alpha_2 + \alpha_3), \\ -i\alpha_3 & \end{array} \right\}. \end{aligned} \tag{6.1.3}$$

$$\Pi_{\triangle(\mathfrak{k}_{\mathrm{ss}},\, \mathfrak{k}_{\mathrm{ss}} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})} = \{-i\alpha_1|_{\mathfrak{k}_{\mathrm{ss}} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}},\ -i\alpha_2|_{\mathfrak{k}_{\mathrm{ss}} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}}\}. \tag{6.1.4}$$

$\mathfrak{g}_u = \mathfrak{sp}(3)$:



$\mathfrak{k} = \mathfrak{su}(3) \oplus \mathfrak{t}^1$:



Here, we recollect Remark 4.1, and refer to Plate III in [**4**, p. 269].[†] In the second place, let us demonstrate the following:

Lemma 6.7. *With the assumptions in Subsection* 6.1; *for any* $T \in \mathfrak{k} = \mathfrak{su}(3) \oplus \mathfrak{t}^1$, *there exists an inner automorphism* ψ *of* $\mathfrak{g} = \mathfrak{sp}(3, \boldsymbol{R})$ *such that*

$$\psi(T_{\mathrm{ss}}) \in \mathfrak{W}^2_{\mathfrak{k}_{\mathrm{ss}}} \quad \textit{and} \quad \psi(T_{\mathrm{z}}) = T_{\mathrm{z}}.$$

Here T_{ss} (*resp.* T_{z}) *denotes the* $\mathfrak{k}_{\mathrm{ss}}$ (*resp.* $\mathfrak{k}_{\mathrm{z}}$)*-component of* $T \in \mathfrak{k}$, *and* $\mathfrak{W}^2_{\mathfrak{k}_{\mathrm{ss}}}$ *denotes a Weyl chamber with respect to* $\Pi_{\triangle(\mathfrak{k}_{\mathrm{ss}}, \mathfrak{k}_{\mathrm{ss}} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})}$ (*see* (6.1.4)) *defined by*

$$\mathfrak{W}^2_{\mathfrak{k}_{\mathrm{ss}}} = \bigl\{H' \in \mathfrak{k}_{\mathrm{ss}} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}} \bigm| -i\alpha_1(H') \geq 0,\ -i\alpha_2(H') \geq 0\bigr\}.$$

Proof. Since $T_{\mathrm{ss}} \in \mathfrak{k}_{\mathrm{ss}}$, and since $\mathfrak{k}_{\mathrm{ss}}$ is compact semisimple, there exists an element $K' \in \mathfrak{k}_{\mathrm{ss}}$ such that $\exp \mathrm{ad}_{\mathfrak{k}_{\mathrm{ss}}} K'(T_{\mathrm{ss}}) \in \mathfrak{W}^2_{\mathfrak{k}_{\mathrm{ss}}}$. Since $\mathfrak{k}_{\mathrm{ss}} \subset \mathfrak{g}$, we can define an inner automorphism ψ of $\mathfrak{g}$ by $\psi := \exp \mathrm{ad}_{\mathfrak{g}} K'$. Then, it is clear that $\psi(T_{\mathrm{ss}}) \in \mathfrak{W}^2_{\mathfrak{k}_{\mathrm{ss}}}$, and that $\psi(T_{\mathrm{z}}) = T_{\mathrm{z}}$ because T_{z} belongs to the center of $\mathfrak{k}$. Therefore, we have completed the proof of Lemma 6.7. $\square$

Lemma 6.7 means that an element T, which we want to find, should exist in the following set:

$$\begin{gathered} \bigl\{H = H_{\mathrm{ss}} + H_{\mathrm{z}} \in i\bar{\mathfrak{h}}_{\boldsymbol{R}} \bigm| -i\alpha_1(H_{\mathrm{ss}}) \geq 0,\ -i\alpha_2(H_{\mathrm{ss}}) \geq 0\bigr\} \\ \Bigl(= \bigl\{H = H_{\mathrm{ss}} + H_{\mathrm{z}} \in i\bar{\mathfrak{h}}_{\boldsymbol{R}} \bigm| H_{\mathrm{ss}} \in \mathfrak{W}^2_{\mathfrak{k}_{\mathrm{ss}}}\bigr\}\Bigr), \end{gathered} \tag{6.1.5}$$

where H_{ss} (resp. H_{z}) is the $\mathfrak{k}_{\mathrm{ss}}$ (resp. $\mathfrak{k}_{\mathrm{z}}$)-component of $H \in i\bar{\mathfrak{h}}_{\boldsymbol{R}} \subset \mathfrak{k}$. We here note that $\mathfrak{k}_{\mathrm{z}} \subset i\bar{\mathfrak{h}}_{\boldsymbol{R}}$.

Now, put an element $T \in i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ as $\sum_{a=1}^{3} \lambda_a \cdot iZ_a$ ($\lambda_a \in \boldsymbol{R}$). In the third place, we will research a condition for $T = \sum_{a=1}^{3} \lambda_a \cdot iZ_a \in i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ to belong to the set (6.1.5). We perceive that

$$\mathfrak{k}_{\mathrm{ss}} = \{K \in \mathfrak{k} = \mathfrak{su}(3) \oplus \mathfrak{t}^1 \mid B_{\mathfrak{g}_u}(K, K'') = 0 \text{ for all } K'' \in \mathfrak{k}_{\mathrm{z}}\} \tag{6.1.6}$$

because $B_{\mathfrak{g}_u}$ is negative-definite, $\mathfrak{k} = \mathfrak{k}_{\mathrm{ss}} \oplus \mathfrak{k}_{\mathrm{z}}$ is the direct sum and $B_{\mathfrak{g}_u}(\mathfrak{k}_{\mathrm{ss}}, \mathfrak{k}_{\mathrm{z}}) = \{0\}$. By use of (6.1.6), we will prove Lemma 6.8.

[†]Erratum: p. 269, line 10 on [**4**], read "$2\epsilon_i = 2\sum_{i \leq k < l} \alpha_k + \alpha_l$" instead of "$2\epsilon_i = \sum_{i \leq k < l} \alpha_k + \alpha_l$".

Lemma 6.8. *With the settings in Subsection* 6.1; *for* $T=\sum_{a=1}^{3}\lambda_a\cdot iZ_a\in i\bar{\mathfrak{h}}_{\boldsymbol{R}}$*, it belongs to the set* (6.1.5) *if and only if* "$\lambda_1,\lambda_2\geq 0$."

Proof. We confirm that

$$\mathfrak{k}_{\mathrm{z}}=\mathrm{span}_{\boldsymbol{R}}\{iZ_3\} \tag{6.1.7}$$

because $\mathfrak{k}=\mathfrak{su}(3)\oplus\mathfrak{t}^1$ is determined by involutive inner automorphism $\sigma=\exp 2\pi i\mathrm{ad}_{\mathfrak{g}_u}h_3$ of $\mathfrak{g}_u=\mathfrak{sp}(3)$ (see List of Murakami [**12**, p. 297, type CI]), $h_3\in\bar{\mathfrak{h}}_{\boldsymbol{R}}$ is defined by $\alpha_a(h_3)=\delta_{3a}/2$ for $\alpha_a\in\Pi_{\triangle(\mathfrak{sp}(3,\boldsymbol{C}),\bar{\mathfrak{h}})}=\{\alpha_a\}_{a=1}^3$, namely $h_3=(1/2)\cdot Z_3$, and the coefficient of $-i\alpha_3$ with respect to maximal root in $\triangle^+(\mathfrak{g}_u,i\bar{\mathfrak{h}}_{\boldsymbol{R}})$ is one (recall (6.1.3)). Let us rewrite $T=\sum_{a=1}^{3}\lambda_a\cdot iZ_a$ as follows:

$$T=\sum_{b=1}^{2}\lambda_b\left(iZ_b-\frac{B_{\mathfrak{g}_u}(iZ_b,iZ_3)}{B_{\mathfrak{g}_u}(iZ_3,iZ_3)}iZ_3\right)+\left(\lambda_3+\sum_{b=1}^{2}\lambda_b\frac{B_{\mathfrak{g}_u}(iZ_b,iZ_3)}{B_{\mathfrak{g}_u}(iZ_3,iZ_3)}\right)iZ_3. \tag{$*$}$$

Since $iZ_a\in i\bar{\mathfrak{h}}_{\boldsymbol{R}}\subset\mathfrak{k}$, and since (6.1.6) and (6.1.7), we see that the first term and the other term of right-hand side in equation $(*)$ are an element of $\mathfrak{k}_{\mathrm{ss}}$ and of $\mathfrak{k}_{\mathrm{z}}$, respectively. Hence, it follows that $T=\sum_{a=1}^{3}\lambda_a\cdot iZ_a$ belongs to the set (6.1.5) if and only if "$\lambda_1,\lambda_2\geq 0$" because $\{Z_a\}_{a=1}^3$ is the dual basis of $\Pi_{\triangle(\mathfrak{sp}(3,\boldsymbol{C}),\bar{\mathfrak{h}})}=\{\alpha_a\}_{a=1}^3$. For the reasons, we have obtained the conclusion. $\square$

6.1.3. The way of finding an element $T\in\mathfrak{k}$ with $\mathfrak{c}_{\mathfrak{g}}(T)=\mathfrak{su}(2,1)\oplus\mathfrak{t}^1$.

Suppose that $T\in\mathfrak{k}=\mathfrak{su}(3)\oplus\mathfrak{t}^1$ satisfies $\mathfrak{c}_{\mathfrak{g}}(T)=\mathfrak{su}(2,1)\oplus\mathfrak{t}^1$. Then, we confirmed that it satisfied two conditions (N-1.1) $\mathfrak{c}_{\mathfrak{g}_u}(T)=\mathfrak{su}(3)\oplus\mathfrak{t}^1$ and (N-1.2) $\mathfrak{c}_{\mathfrak{k}}(T)=\mathfrak{su}(2)\oplus\mathfrak{t}^2$ (see Subsection 6.1.1). Moreover, we concluded that there existed an inner automorphism of $\mathfrak{g}=\mathfrak{sp}(3,\boldsymbol{R})$ which mapped T to an element $\sum_{a=1}^{3}\lambda_a\cdot iZ_a\in i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ with "$\lambda_1,\lambda_2\geq 0$" (see Subsection 6.1.2). For the reasons, we will search the set $\left\{\sum_{a=1}^{3}\lambda_a\cdot iZ_a\in i\bar{\mathfrak{h}}_{\boldsymbol{R}}\;\middle|\;\lambda_1,\lambda_2\geq 0\right\}$ for an element T which satisfies two conditions (N-1.1) and (N-1.2). The search depends only on direct calculations, but it is not too hard.

Let T be an element $\sum_{a=1}^{3}\lambda_a\cdot iZ_a\in i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ with "$\lambda_1,\lambda_2\geq 0$" (recall that $\{Z_a\}_{a=1}^3$ is the dual basis of $\Pi_{\triangle(\mathfrak{sp}(3,\boldsymbol{C}),\bar{\mathfrak{h}})}=\{\alpha_a\}_{a=1}^3$). Necessary condition (N-1.2) $\mathfrak{c}_{\mathfrak{k}}(T)=\mathfrak{su}(2)\oplus\mathfrak{t}^2$ implies that

$$\text{either case "}\lambda_1>0\text{ and }\lambda_2=0\text{" or case "}\lambda_1=0\text{ and }\lambda_2>0\text{" only occurs} \tag{6.1.8}$$

because $\mathfrak{c}_{\mathfrak{k}}(T)=\mathfrak{t}^3$ (resp. $\mathfrak{su}(3)\oplus\mathfrak{t}^1$) if $\lambda_1,\lambda_2>0$ (resp. $\lambda_1=\lambda_2=0$) (see Dynkin diagram of $\mathfrak{k}=\mathfrak{su}(3)\oplus\mathfrak{t}^1$ on page 1163). It follows from (N-1.1) $\mathfrak{c}_{\mathfrak{g}_u}(T)=\mathfrak{su}(3)\oplus\mathfrak{t}^1$ and (6.1.8) that

$$\lambda_3\leq 0$$

because $\mathfrak{c}_{\mathfrak{g}_u}(T)=\mathfrak{su}(2)\oplus\mathfrak{t}^2$ if "$\lambda_1>0,\ \lambda_2=0$ and $\lambda_3>0$," or if "$\lambda_1=0,\ \lambda_2>0$ and $\lambda_3>0$" (see Dynkin diagram of $\mathfrak{g}_u=\mathfrak{sp}(3)$ on page 1163). Furthermore, we see that

$$\lambda_3 < 0 \tag{6.1.9}$$

by virtue of (N-1.1) and (6.1.8). It is natural that $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{c}_{\mathfrak{g}}(\gamma \cdot T)$ for any non-zero scalar $\gamma \in \boldsymbol{R}$; and thus, since (6.1.8) and (6.1.9), one may assume that

$$\text{(I) "}\lambda_1 > 0,\ \lambda_2 = 0 \text{ and } \lambda_3 = -1\text{" or (II) "}\lambda_1 = 0,\ \lambda_2 > 0 \text{ and } \lambda_3 = -1.\text{"}$$

Case (I) $T = i(\lambda_1 \cdot Z_1 - Z_3)$ $(\lambda_1 > 0)$: In this case, it is clear that T satisfies the condition (N-1.2), namely $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{su}(2) \oplus \mathfrak{t}^2$.

$$\mathfrak{c}_{\mathfrak{k}}(T),\ T = i(\lambda_1 \cdot Z_1 - Z_3): \qquad \underset{-i\alpha_2}{\circ}$$

For any root in $\triangle^+(\mathfrak{g}_u, i\bar{\mathfrak{h}}_{\boldsymbol{R}})$ (see (6.1.3)), direct calculations give us

$$\begin{aligned} -i\alpha_1(T) &= \lambda_1, & -i(\alpha_1 + \alpha_2)(T) &= \lambda_1, \\ -i(\alpha_1 + 2\alpha_2 + \alpha_3)(T) &= \lambda_1 - 1, & -i\alpha_2(T) &= 0, \\ -i(\alpha_1 + \alpha_2 + \alpha_3)(T) &= \lambda_1 - 1, & -i(\alpha_2 + \alpha_3)(T) &= -1, \\ -i(2\alpha_1 + 2\alpha_2 + \alpha_3)(T) &= 2\lambda_1 - 1, & -i(2\alpha_2 + \alpha_3)(T) &= -1, \\ -i\alpha_3(T) &= -1 \end{aligned} \tag{6.1.10}$$

because $T = i(\lambda_1 \cdot Z_1 - Z_3)$ and $\{Z_a\}_{a=1}^3$ is the dual basis of $\{\alpha_a\}_{a=1}^3$. This (6.1.10) means that the set $\triangle^+(\mathfrak{c}_{\mathfrak{g}_u}(T), i\bar{\mathfrak{h}}_{\boldsymbol{R}}) = \{\alpha \in \triangle^+(\mathfrak{g}_u, i\bar{\mathfrak{h}}_{\boldsymbol{R}}) \mid \alpha(T) = 0\}$ consists of three-elements if $\lambda_1 = 1$, two-elements if $\lambda_1 = 1/2$, and one-element if $\lambda_1 \neq 1, 1/2$ (because $\lambda_1 > 0$). Therefore by necessary condition (N-1.1) $\mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{su}(3) \oplus \mathfrak{t}^1$, we anticipate that $\lambda_1 = 1$. Suppose that $T = i(Z_1 - Z_3)$, namely $\lambda_1 = 1$. Then, it is obvious from (6.1.10) that

$$\begin{aligned} \triangle^+(\mathfrak{c}_{\mathfrak{g}_u}(T), i\bar{\mathfrak{h}}_{\boldsymbol{R}}) &= \bigl\{\alpha \in \triangle^+(\mathfrak{g}_u, i\bar{\mathfrak{h}}_{\boldsymbol{R}}) \bigm| \alpha(T) = 0\bigr\} \\ &= \left\{ \begin{array}{ll} -i(\alpha_1 + 2\alpha_2 + \alpha_3), & -i\alpha_2, \\ -i(\alpha_1 + \alpha_2 + \alpha_3) & \end{array} \right\}. \end{aligned}$$

Accordingly, we get $\Pi_{\triangle(\mathfrak{c}_{\mathfrak{g}_u}(T), i\bar{\mathfrak{h}}_{\boldsymbol{R}})} = \{-i\alpha_2; -i(\alpha_1 + \alpha_2 + \alpha_3)\}$. Therefore, the Dynkin diagram of $\Pi_{\triangle(\mathfrak{c}_{\mathfrak{g}_u}(T), i\bar{\mathfrak{h}}_{\boldsymbol{R}})}$ is as follows:

$$\mathfrak{c}_{\mathfrak{g}_u}(T),\ T = i(Z_1 - Z_3): \qquad \underset{-i\alpha_2}{\circ} \text{———————} \underset{-i(\alpha_1 + \alpha_2 + \alpha_3).}{\circ}$$

This, together with $\operatorname{rk} \mathfrak{c}_{\mathfrak{g}_u}(T) = \operatorname{rk} \mathfrak{g}_u = \operatorname{rk} \mathfrak{sp}(3) = 3$, shows that $\mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{su}(3) \oplus \mathfrak{t}^1$ (cf. Lemma 3.3-(1)); and so $T = i(Z_1 - Z_3)$ satisfies the condition (N-1.1). Therefore, we conclude that element $T = i(Z_1 - Z_3) \in i\bar{\mathfrak{h}}_{\boldsymbol{R}} \subset \mathfrak{k}$ satisfies two conditions (N-1.1) $\mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{su}(3) \oplus \mathfrak{t}^1$ and (N-1.2) $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{su}(2) \oplus \mathfrak{t}^2$. Thus by using Corollary 3.6, the centralizer of $T = i(Z_1 - Z_3)$ in $\mathfrak{g} = \mathfrak{sp}(3, \boldsymbol{R})$ coincides with $\mathfrak{su}(2,1) \oplus \mathfrak{t}^1$. Consequently, we have found an elliptic element $T = i(Z_1 - Z_3) \in \mathfrak{g}$ with $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{su}(2,1) \oplus \mathfrak{t}^1$.

REMARK 6.9. An element $T = i(Z_2 - Z_3)$ in Case (II) satisfies $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{su}(2,1) \oplus \mathfrak{t}^1$, too. Thereby, we can obtain $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{su}(2,1) \oplus \mathfrak{t}^1$ by using not only $T = i(Z_1 - Z_3)$ but also $T = i(Z_2 - Z_3)$ (see Cases (I) and (II) on page 1165).

6.1.4. The H-element $T = i(Z_1 - Z_3)$ in $(\mathfrak{g}, \mathfrak{r}) = (\mathfrak{sp}(3, \boldsymbol{R}), \mathfrak{su}(2,1) \oplus \mathfrak{t}^1)$.

In Subsection 6.1.3, we verified that element $T = i(Z_1 - Z_3) \in i\bar{\mathfrak{h}}_{\boldsymbol{R}} \subset \mathfrak{k}$ satisfied $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{su}(2,1) \oplus \mathfrak{t}^1$. In this subsection, we will demonstrate that $T = i(Z_1 - Z_3)$ is the H-element in $(\mathfrak{sp}(3, \boldsymbol{R}), \mathfrak{su}(2,1) \oplus \mathfrak{t}^1)$.

Define $\triangle_1^+(\mathfrak{sp}(3, \boldsymbol{C}), \bar{\mathfrak{h}})$ by

$$\triangle_1^+(\mathfrak{sp}(3, \boldsymbol{C}), \bar{\mathfrak{h}}) := \{\beta \in \triangle^+(\mathfrak{sp}(3, \boldsymbol{C}), \bar{\mathfrak{h}}) \mid \beta(T) \neq 0\}.$$

Then, Lemma 6.2-(ii) allows us to conclude that $T = i(Z_1 - Z_3) \in i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ is the H-element in $(\mathfrak{sp}(3, \boldsymbol{R}), \mathfrak{su}(2,1) \oplus \mathfrak{t}^1)$ if $\beta(T) = \pm i$ for all $\beta \in \triangle_1^+(\mathfrak{sp}(3, \boldsymbol{C}), \bar{\mathfrak{h}})$. Therefore, we will show that $\beta(T) = \pm i$ for all $\beta \in \triangle_1^+(\mathfrak{sp}(3, \boldsymbol{C}), \bar{\mathfrak{h}})$. It follows from (6.1.3) and $T = i(Z_1 - Z_3)$ that

$$\triangle_1^+(\mathfrak{sp}(3, \boldsymbol{C}), \bar{\mathfrak{h}}) = \left\{ \begin{array}{ll} \alpha_1, & \alpha_1 + \alpha_2, \\ \alpha_2 + \alpha_3, & 2\alpha_1 + 2\alpha_2 + \alpha_3, \\ 2\alpha_2 + \alpha_3, & \alpha_3 \end{array} \right\}$$

because $\{Z_a\}_{a=1}^3$ is the dual basis of $\{\alpha_a\}_{a=1}^3 = \Pi_{\triangle(\mathfrak{sp}(3,\boldsymbol{C}),\bar{\mathfrak{h}})}$. This provides

$$\beta(T) = i \text{ or } -i$$

for any $\beta \in \triangle_1^+(\mathfrak{sp}(3, \boldsymbol{C}), \bar{\mathfrak{h}})$. For the reasons, we have shown that $T = i(Z_1 - Z_3)$ is the H-element in $(\mathfrak{sp}(3, \boldsymbol{R}), \mathfrak{su}(2,1) \oplus \mathfrak{t}^1)$.

Summarizing the statements in Subsection 6.1, we get the following table:

PROPOSITION 6.10.

$(\mathfrak{g},\mathfrak{r})$	$(\mathfrak{sp}(3, \boldsymbol{R}),\ \mathfrak{su}(2,1) \oplus \mathfrak{t}^1)$
T	$i(Z_1 - Z_3)$
$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(2) \oplus \mathfrak{t}^2$
$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{su}(3) \oplus \mathfrak{t}^1$
$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{su}(2,1) \oplus \mathfrak{t}^1$

6.2. $(\mathfrak{g}, \mathfrak{r}) = (\mathfrak{so}(3,5), \mathfrak{so}(3,3) \oplus \mathfrak{t}^1)$.

Let us recall Remark 6.4. We will illustrate the way of finding the H-element T in simple irreducible pseudo-Hermitian symmetric Lie algebra $(\mathfrak{so}(3,5), \mathfrak{so}(3,3) \oplus \mathfrak{t}^1)$. Our arguments on this subsection are similar to those on Subsection 6.1—that is, we obtain two necessary conditions (N-2.1) and (N-2.2) for $T \in \mathfrak{k}$ to satisfy $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{so}(3,3) \oplus \mathfrak{t}^1$ (in

Subsection 6.2.1); and show Lemmas 6.12 and 6.13, in order to easily find an element $T \in \mathfrak{k}$ which satisfies (N-2.1) and (N-2.2) (in Subsection 6.2.2). Moreover in Subsection 6.2.3, direct computations enable us to find $T \in \mathfrak{k}$ with $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{so}(3,3) \oplus \mathfrak{t}^1$. Lastly in Subsection 6.2.4, we verify that element T, found in Subsection 6.2.3, is the H-element in $(\mathfrak{so}(3,5), \mathfrak{so}(3,3) \oplus \mathfrak{t}^1)$.

NOTICE 6.11. In Subsection 6.2, we utilize the following settings:

$$\mathfrak{g} = \mathfrak{so}(3,5), \qquad \mathfrak{g}_u = \mathfrak{so}(8), \qquad \mathfrak{k} = \mathfrak{so}(3) \oplus \mathfrak{so}(5),$$
$$\{Z_j\}_{j=1}^4\text{: the dual basis of } \{\alpha_j\}_{j=1}^4 = \Pi_{\triangle(\mathfrak{so}(8,\boldsymbol{C}),\bar{\mathfrak{h}})}.$$

6.2.1. Two necessary conditions.

Let T be an element of $\mathfrak{k}$ such that $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{so}(3,3) \oplus \mathfrak{t}^1$. Two necessary conditions which T should satisfy are as follows:

$$\mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{so}(6) \oplus \mathfrak{t}^1, \tag{N-2.1}$$
$$\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{t}^1 \tag{N-2.2}$$

(apply arguments on Subsection 6.1.1).

6.2.2. Existence zone.

In order to find an element $T \in \mathfrak{k}$ with $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{so}(3,3) \oplus \mathfrak{t}^1$, we want to restrict its existence zone. By discussions stated below, we will search the set $\{\nu_1 \cdot iZ_1 + \nu_2 \cdot iZ_2 + \nu_3 \cdot i(Z_3 + Z_4) \in \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}} \mid \nu_1, (\nu_2 + \nu_3),\ \nu_3 \geq 0\}$ for an element T which satisfies two conditions (N-2.1) and (N-2.2), in Subsection 6.2.3.

First, let us notice that $\mathfrak{k} = \mathfrak{so}(3) \oplus \mathfrak{so}(5)$ is determined by an involution $\sigma = \sigma_2 \circ \exp 2\pi i \mathrm{ad}_{\mathfrak{so}(8)} h_2$ of $\mathfrak{g}_u = \mathfrak{so}(8)$ (cf. List of type DI in the paper [**12**] on page 305; Remark 4.7). Here, σ_2 is defined in Lemma 4.4 and $h_2 \in \bar{\mathfrak{h}}_{\boldsymbol{R}}$ is defined by $\alpha_j(h_2) = \delta_{2j}/2$ for $\alpha_j \in \Pi_{\triangle(\mathfrak{so}(8,\boldsymbol{C}),\bar{\mathfrak{h}})} = \{\alpha_j\}_{j=1}^4$. Lemma 4.4 means that

$$\mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}} = \mathrm{span}_{\boldsymbol{R}}\{iZ_1,\ iZ_2,\ i(Z_3 + Z_4)\} \tag{6.2.1}$$

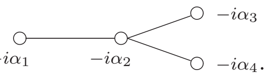
because $\sigma = \sigma_2$ on $i\bar{\mathfrak{h}}_{\boldsymbol{R}}$.

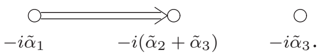
Now, let us enumerate the set of positive roots in $\triangle(\mathfrak{g}_u, i\bar{\mathfrak{h}}_{\boldsymbol{R}})$, the set of simple roots in $\triangle(\mathfrak{k}, \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})$, and their Dynkin diagrams (cf. List of Murakami [**12**, p. 305, type DI]).

$$\triangle^+(\mathfrak{g}_u, i\bar{\mathfrak{h}}_{\boldsymbol{R}}) = \{-i\alpha \mid \alpha \in \triangle^+(\mathfrak{so}(8, \boldsymbol{C}), \bar{\mathfrak{h}})\}$$
$$= \left\{ \begin{array}{lll} -i(\alpha_1+\alpha_2), & -i(\alpha_1+\alpha_2+\alpha_3), & -i(\alpha_2+\alpha_3), \\ -i\alpha_3, & -i(\alpha_1+2\alpha_2+\alpha_3+\alpha_4), & -i\alpha_1, \\ -i(\alpha_1+\alpha_2+\alpha_3+\alpha_4), & -i\alpha_2, & -i(\alpha_1+\alpha_2+\alpha_4), \\ -i(\alpha_2+\alpha_3+\alpha_4), & -i(\alpha_2+\alpha_4), & -i\alpha_4 \end{array} \right\}. \tag{6.2.2}$$

$$\Pi_{\triangle(\mathfrak{k},\, \mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})} = \{-i\tilde{\alpha}_1,\ -i\tilde{\alpha}_3,\ -i(\tilde{\alpha}_2+\tilde{\alpha}_3)\}, \tag{6.2.3}$$

where $-i\tilde{\alpha}_b := -i\alpha_b|_{\mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}}$ for $-i\alpha_b \in \triangle(\mathfrak{g}_u, i\bar{\mathfrak{h}}_{\boldsymbol{R}})$ $(1 \leq b \leq 3)$. Here, recall Remark 4.1, and see Plate IV in [**4**, p. 271].‡

$\mathfrak{g}_u = \mathfrak{so}(8)$: 

$\mathfrak{k} = \mathfrak{so}(3) \oplus \mathfrak{so}(5)$: 

Let $\{iT_1, iT_2, iT_3\}$ be the dual basis of $\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})} = \{-i\tilde{\alpha}_1, -i\tilde{\alpha}_3, -i(\tilde{\alpha}_2+\tilde{\alpha}_3)\}$. Then, it follows from (6.2.1) that

$$\begin{cases} iT_1 = iZ_1, \\ iT_2 = i(-Z_2+Z_3+Z_4), \\ iT_3 = iZ_2 \end{cases} \tag{6.2.4}$$

because $\{Z_j\}_{j=1}^4$ is the dual basis of $\Pi_{\triangle(\mathfrak{so}(8,\boldsymbol{C}),\bar{\mathfrak{h}})} = \{\alpha_j\}_{j=1}^4$.

Next, we give the following:

Lemma 6.12. *With the assumptions in Subsection* 6.2; *let* $\mathfrak{W}_{\mathfrak{k}}^3$ *be a Weyl chamber with respect to* $\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})}$ (*see* (6.2.3)) *defined by*

$$\mathfrak{W}_{\mathfrak{k}}^3 = \left\{ H \in \mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}} \;\middle|\; \begin{array}{l} -i\tilde{\alpha}_1(H) \geq 0,\ -i\tilde{\alpha}_3(H) \geq 0, \\ -i(\tilde{\alpha}_2+\tilde{\alpha}_3)(H) \geq 0 \end{array} \right\}.$$

For any element of $\mathfrak{k} = \mathfrak{so}(3)\oplus\mathfrak{so}(5)$, *there exists an inner automorphism of* $\mathfrak{g} = \mathfrak{so}(3,5)$ *which maps it into* $\mathfrak{W}_{\mathfrak{k}}^3$.

Proof. Refer to the proof of Lemma 6.7. □

‡Erratum: p. 271, line 8 on [**4**], read "$\epsilon_i - \epsilon_j = \sum_{i\leq k<l}\alpha_k$" instead of "$\epsilon_i - \epsilon_j = \sum_{i<k<l}\alpha_k$".

For any element $\nu_1 \cdot iZ_1 + \nu_2 \cdot iZ_2 + \nu_3 \cdot i(Z_3 + Z_4)$ of $\mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ (see (6.2.1)), we comprehend

$$\begin{aligned} &\nu_1 \cdot iZ_1 + \nu_2 \cdot iZ_2 + \nu_3 \cdot i(Z_3 + Z_4) \\ &= \nu_1 \cdot iT_1 + \nu_3 \cdot iT_2 + (\nu_2 + \nu_3) \cdot iT_3 \end{aligned} \tag{6.2.5}$$

by virtue of (6.2.4). Since (6.2.5), and since $\{iT_b\}_{b=1}^3$ is the dual basis of $\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})}$, we obtain Lemma 6.13.

LEMMA 6.13. *In the settings in Subsection* 6.2; *for an element* $\nu_1 \cdot iZ_1 + \nu_2 \cdot iZ_2 + \nu_3 \cdot i(Z_3 + Z_4)$ *of* $\mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}$, *it belongs to* $\mathfrak{W}_{\mathfrak{k}}^3$ *if and only if* "$\nu_1, (\nu_2 + \nu_3), \nu_3 \geq 0$." *Here,* $\mathfrak{W}_{\mathfrak{k}}^3$ *is defined in Lemma* 6.12.

6.2.3. The way of finding an element $T \in \mathfrak{k}$ with $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{so}(3,3) \oplus \mathfrak{t}^1$.

Let T be an element of $\mathfrak{k}$ such that $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{so}(3,3) \oplus \mathfrak{t}^1$. Then, it satisfies two necessary conditions (N-2.1) and (N-2.2) (see Subsection 6.2.1). On the other hand, each element of $\mathfrak{k}$ can be mapped into $\mathfrak{W}_{\mathfrak{k}}^3$ (recall Lemma 6.12). Therefore, we are going to search the set of elements $\nu_1 \cdot iZ_1 + \nu_2 \cdot iZ_2 + \nu_3 \cdot i(Z_3 + Z_4) \in \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ with "$\nu_1, (\nu_2 + \nu_3), \nu_3 \geq 0$" for an element T which satisfies two conditions (N-2.1) and (N-2.2) (see Lemma 6.13).

REMARK 6.14. Suppose that $T \in \mathfrak{k}$ satisfies two necessary conditions (N-2.1) $\mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{so}(6) \oplus \mathfrak{t}^1$ and (N-2.2) $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{t}^1$. Then, Corollary 3.6 implies that $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{so}(3,3) \oplus \mathfrak{t}^1$. Consequently, two conditions (N-2.1) and (N-2.2) become a sufficient condition for $T \in \mathfrak{k}$ to satisfy $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{so}(3,3) \oplus \mathfrak{t}^1$.

Let $T = \nu_1 \cdot iZ_1 + \nu_2 \cdot iZ_2 + \nu_3 \cdot i(Z_3 + Z_4)$ be an element of $\mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ with "$\nu_1, (\nu_2 + \nu_3), \nu_3 \geq 0$." Taking Dynkin diagram of $\mathfrak{k}$ into consideration (see Dynkin diagram of $\mathfrak{k} = \mathfrak{so}(3) \oplus \mathfrak{so}(5)$ on page 1168), we perceive that

$$\triangle^+(\mathfrak{k}, \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}) = \left\{ \begin{array}{ll} -i\tilde{\alpha}_3, & -i(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3), \\ -i\tilde{\alpha}_1, & -i(\tilde{\alpha}_1 + 2(\tilde{\alpha}_2 + \tilde{\alpha}_3)), \\ -i(\tilde{\alpha}_2 + \tilde{\alpha}_3) & \end{array} \right\}.$$

Thus, for all roots in $\triangle^+(\mathfrak{k}, \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}})$, direct computations tell us

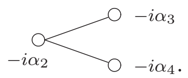
$$\begin{aligned} -i\tilde{\alpha}_3(T) &= \nu_3, \\ -i(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3)(T) &= \nu_1 + \nu_2 + \nu_3, \\ -i\tilde{\alpha}_1(T) &= \nu_1, \\ -i\big(\tilde{\alpha}_1 + 2(\tilde{\alpha}_2 + \tilde{\alpha}_3)\big)(T) &= \nu_1 + 2\nu_2 + 2\nu_3, \\ -i(\tilde{\alpha}_2 + \tilde{\alpha}_3)(T) &= \nu_2 + \nu_3 \end{aligned} \tag{6.2.6}$$

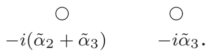
because $-i\tilde{\alpha}_b = -i\alpha_b|_{\mathfrak{k}\cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}}$ $(1 \leq b \leq 3)$, $T = \nu_1 \cdot iZ_1 + \nu_2 \cdot iZ_2 + \nu_3 \cdot i(Z_3 + Z_4) \in \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ and $\{Z_j\}_{j=1}^4$ is the dual basis of $\Pi_{\triangle(\mathfrak{so}(8,\boldsymbol{C}),\bar{\mathfrak{h}})} = \{\alpha_j\}_{j=1}^4$. Let us suppose that $(\nu_2 + \nu_3) > 0$.

Then, both ν_1 and ν_3 must be zero because of $\nu_1, \nu_3 \geq 0$, (6.2.6) and the necessary condition (N-2.2) $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{t}^1$. Hence $T = \nu_1 \cdot iZ_1 + \nu_2 \cdot iZ_2 + \nu_3 \cdot\ i(Z_3 + Z_4) = \nu_2 \cdot iZ_2$; and therefore $\mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{so}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{t}^1$ (see Dynkin diagram of $\mathfrak{g}_u = \mathfrak{so}(8)$ on page 1168). Consequently, the necessary condition (N-2.1) $\mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{so}(6) \oplus \mathfrak{t}^1$ can not hold if $(\nu_2 + \nu_3) > 0$. For the reasons, we will consider the case $(\nu_2 + \nu_3) = 0$ from now on. It is immediate from $(\nu_2 + \nu_3) = 0$ and (6.2.6) that

$$\begin{aligned} -i\tilde{\alpha}_3(T) &= \nu_3, \\ -i(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3)(T) &= \nu_1, \\ -i\tilde{\alpha}_1(T) &= \nu_1, \\ -i\big(\tilde{\alpha}_1 + 2(\tilde{\alpha}_2 + \tilde{\alpha}_3)\big)(T) &= \nu_1, \\ -i(\tilde{\alpha}_2 + \tilde{\alpha}_3)(T) &= 0. \end{aligned} \tag{6.2.7}$$

Therefore, we conclude that $\nu_1 > 0$ and $\nu_3 = 0$ by the necessary condition (N-2.2) $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{t}^1$. Indeed, the other cases can not satisfy the necessary condition (N-2.2) because $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{so}(3) \oplus \mathfrak{so}(5)$ if $\nu_1 = \nu_3 = 0$; $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{so}(5) \oplus \mathfrak{t}^1$ if $\nu_1 = 0$ and $\nu_3 > 0$; and $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{so}(3) \oplus \mathfrak{t}^2$ if $\nu_1, \nu_3 > 0$. Accordingly, it follows from $\nu_1 > 0, (\nu_2 + \nu_3) = \nu_3 = 0$ that $T = \nu_1 \cdot iZ_1 + \nu_2 \cdot iZ_2 + \nu_3 \cdot i(Z_3 + Z_4) = \nu_1 \cdot iZ_1$ with $\nu_1 > 0$. This element $T = \nu_1 \cdot iZ_1 \in \mathfrak{k}$ $(\nu_1 > 0)$ satisfies two conditions (N-2.1) $\mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{so}(6) \oplus \mathfrak{t}^1$ and (N-2.2) $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{t}^1$.

$\mathfrak{c}_{\mathfrak{g}_u}(T)$, $T = \nu_1 \cdot iZ_1$:


$\mathfrak{c}_{\mathfrak{k}}(T)$, $T = \nu_1 \cdot iZ_1$:


Consequently, Remark 6.14 implies that $T = \nu_1 \cdot iZ_1$ $(\nu_1 > 0)$ satisfies $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{so}(3,3) \oplus \mathfrak{t}^1$. Thus, we have found an elliptic element $T = \nu_1 \cdot iZ_1$ of $\mathfrak{g} = \mathfrak{so}(3,5)$ satisfying $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{so}(3,3) \oplus \mathfrak{t}^1$.

6.2.4. The H-element $T = iZ_1$ in $(\mathfrak{g}, \mathfrak{r}) = (\mathfrak{so}(3,5), \mathfrak{so}(3,3) \oplus \mathfrak{t}^1)$.

In Subsection 6.2.3, we got an element $T = \nu_1 \cdot iZ_1 \in \mathfrak{k} \cap i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ with $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{so}(3,3) \oplus \mathfrak{t}^1$ $(\nu_1 > 0)$. In this subsection, we will verify that $T = iZ_1$ is the H-element in $(\mathfrak{so}(3,5), \mathfrak{so}(3,3) \oplus \mathfrak{t}^1)$.

Define $\triangle_1^+(\mathfrak{so}(8, \boldsymbol{C}), \bar{\mathfrak{h}})$ by

$$\triangle_1^+(\mathfrak{so}(8, \boldsymbol{C}), \bar{\mathfrak{h}}) := \big\{\beta \in \triangle^+(\mathfrak{so}(8, \boldsymbol{C}), \bar{\mathfrak{h}}) \;\big|\; \beta(T) \neq 0\big\}.$$

Then, Lemma 6.2-(ii) means that $T = iZ_1 \in i\bar{\mathfrak{h}}_{\boldsymbol{R}}$ is the H-element in $(\mathfrak{so}(3,5), \mathfrak{so}(3,3) \oplus \mathfrak{t}^1)$ if $\beta(T) = \pm i$ for every $\beta \in \triangle_1^+(\mathfrak{so}(8, \boldsymbol{C}), \bar{\mathfrak{h}})$. Hence, the rest of our arguments are devoted to showing that $\beta(T) = \pm i$ for every $\beta \in \triangle_1^+(\mathfrak{so}(8, \boldsymbol{C}), \bar{\mathfrak{h}})$. Since (6.2.2) and $T = iZ_1$, we have

$$\triangle_1^+(\mathfrak{so}(8,\boldsymbol{C}),\bar{\mathfrak{h}}) = \left\{ \begin{array}{ll} \alpha_1+\alpha_2, & \alpha_1+\alpha_2+\alpha_3, \\ \alpha_1+2\alpha_2+\alpha_3+\alpha_4, & \alpha_1, \\ \alpha_1+\alpha_2+\alpha_3+\alpha_4, & \alpha_1+\alpha_2+\alpha_4 \end{array} \right\},$$

and thus

$$\beta(T) = i \text{ for all } \beta \in \triangle_1^+(\mathfrak{so}(8,\boldsymbol{C}),\bar{\mathfrak{h}})$$

because $\{Z_j\}_{j=1}^4$ is the dual basis of $\Pi_{\triangle(\mathfrak{so}(8,\boldsymbol{C}),\bar{\mathfrak{h}})} = \{\alpha_j\}_{j=1}^4$. Consequently, $T = iZ_1$ is the H-element in $(\mathfrak{so}(3,5),\mathfrak{so}(3,3)\oplus\mathfrak{t}^1)$.

Summarizing the statements in Subsection 6.2, we obtain the following table:

Proposition 6.15.

$(\mathfrak{g},\mathfrak{r})$	$(\mathfrak{so}(3,5),\ \mathfrak{so}(3,3)\oplus\mathfrak{t}^1)$
T	iZ_1
$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{so}(3)\oplus\mathfrak{so}(3)\oplus\mathfrak{t}^1$
$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{so}(6)\oplus\mathfrak{t}^1$
$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{so}(3,3)\oplus\mathfrak{t}^1$

6.3. Results.

In Subsections 6.1 and 6.2, we have discussed the way of finding the H-elements T in two simple irreducible pseudo-Hermitian symmetric Lie algebras $(\mathfrak{g},\mathfrak{r})$. The way gives us the following Table I which exhausts all simple irreducible pseudo-Hermitian symmetric Lie algebras:

Theorem 6.16.

Table I.

1	$(\mathfrak{g},\mathfrak{r})$	$\left(\mathfrak{su}^*(2l),\mathfrak{sl}(l,\boldsymbol{C})\oplus\mathfrak{t}^1\right) : l\geq 2$
	T	iZ_l
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(l)\oplus\mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{su}(l)\oplus\mathfrak{su}(l)\oplus\mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{sl}(l,\boldsymbol{C})\oplus\mathfrak{t}^1$
2	$(\mathfrak{g},\mathfrak{r})$	$\left(\mathfrak{sl}(2l,\boldsymbol{R}),\mathfrak{sl}(l,\boldsymbol{C})\oplus\mathfrak{t}^1\right) : l\geq 2$
	T	iZ_l
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(l)\oplus\mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{su}(l)\oplus\mathfrak{su}(l)\oplus\mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{sl}(l,\boldsymbol{C})\oplus\mathfrak{t}^1$

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Continued.

3[§]	$(\mathfrak{g},\mathfrak{r})$	$\big(\mathfrak{su}(l+1-j,j),\mathfrak{su}(m,k)\oplus\mathfrak{su}(l+1-j-m,j-k)\oplus\mathfrak{t}^1\big)$: $l\geq 3,\ 1\leq j\leq[\frac{l-1}{2}]+1,\ 1\leq k\leq[\frac{j-2}{2}]+1,\ 1\leq m\leq[\frac{l-j-1}{2}]+1$
	T	$i(Z_k-Z_j+Z_{j+m})$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(k)\oplus\mathfrak{su}(j-k)\oplus\mathfrak{su}(m)\oplus\mathfrak{su}(l+1-j-m)\oplus\mathfrak{t}^3$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{su}(m+k)\oplus\mathfrak{su}(l+1-m-k)\oplus\mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{su}(m,k)\oplus\mathfrak{su}(l+1-j-m,j-k)\oplus\mathfrak{t}^1$
4	$(\mathfrak{g},\mathfrak{r})$	$\big(\mathfrak{su}(l+1-j,j),\mathfrak{su}(j-k)\oplus\mathfrak{su}(l+1-j,k)\oplus\mathfrak{t}^1\big)$: $l\geq 3,\ 1\leq j\leq[\frac{l-1}{2}]+1,\ 1\leq k\leq j-1$
	T	$i(Z_k-Z_j)$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(j-k)\oplus\mathfrak{su}(k)\oplus\mathfrak{su}(l+1-j)\oplus\mathfrak{t}^2$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{su}(j-k)\oplus\mathfrak{su}(l+1+k-j)\oplus\mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{su}(j-k)\oplus\mathfrak{su}(l+1-j,k)\oplus\mathfrak{t}^1$
5	$(\mathfrak{g},\mathfrak{r})$	$\big(\mathfrak{su}(l+1-j,j),\mathfrak{su}(m)\oplus\mathfrak{su}(l+1-m-j,j)\oplus\mathfrak{t}^1\big)$: $l\geq 2,\ 1\leq j\leq[\frac{l-1}{2}]+1,\ 1\leq m\leq l-j$
	T	$i(Z_j-Z_{j+m})$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(m)\oplus\mathfrak{su}(j)\oplus\mathfrak{su}(l+1-m-j)\oplus\mathfrak{t}^2$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{su}(m)\oplus\mathfrak{su}(l+1-m)\oplus\mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{su}(m)\oplus\mathfrak{su}(l+1-m-j,j)\oplus\mathfrak{t}^1$
6	$(\mathfrak{g},\mathfrak{r})$	$\big(\mathfrak{su}(l+1-j,j),\mathfrak{su}(j)\oplus\mathfrak{su}(l+1-j)\oplus\mathfrak{t}^1\big)$: $l\geq 1,\ 1\leq j\leq[\frac{l-1}{2}]+1$
	T	iZ_j
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(j)\oplus\mathfrak{su}(l+1-j)\oplus\mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{su}(j)\oplus\mathfrak{su}(l+1-j)\oplus\mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{su}(j)\oplus\mathfrak{su}(l+1-j)\oplus\mathfrak{t}^1$
7	$(\mathfrak{g},\mathfrak{r})$	$\big(\mathfrak{sp}(l-j,j),\mathfrak{su}(l-j,j)\oplus\mathfrak{t}^1\big):l\geq 3,\ 1\leq j\leq[\frac{l-1}{2}]+1$
	T	$i(Z_j-Z_l)$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(j)\oplus\mathfrak{su}(l-j)\oplus\mathfrak{t}^2$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{su}(l)\oplus\mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{su}(l-j,j)\oplus\mathfrak{t}^1$
8	$(\mathfrak{g},\mathfrak{r})$	$\big(\mathfrak{sp}(l,\boldsymbol{R}),\mathfrak{su}(l-k,k)\oplus\mathfrak{t}^1\big):l\geq 3,\ 1\leq k\leq[\frac{l-2}{2}]+1$
	T	$i(Z_k-Z_l)$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(k)\oplus\mathfrak{su}(l-k)\oplus\mathfrak{t}^2$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{su}(l)\oplus\mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{su}(l-k,k)\oplus\mathfrak{t}^1$

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[§]Erratum: p. 297, line 4 on [**12**], read “$A_{i-1}\times A_{l-i}\times T$” instead of “$A_i\times A_{l-i-1}\times T$”.

Continued.

9	$(\mathfrak{g}, \mathfrak{r})$	$\big(\mathfrak{sp}(l, \boldsymbol{R}), \mathfrak{su}(l) \oplus \mathfrak{t}^1\big) : l \geq 3$
	T	iZ_l
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(l) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{su}(l) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{su}(l) \oplus \mathfrak{t}^1$
10	$(\mathfrak{g}, \mathfrak{r})$	$\big(\mathfrak{so}(2l+1-2j, 2j), \mathfrak{so}(2l+1-2j, 2j-2) \oplus \mathfrak{t}^1\big) :$ $l \geq 2,\ 1 \leq j \leq l$
	T	$i(Z_{j-1} - Z_j)$ *where* $Z_0 := 0$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{so}(2j-2) \oplus \mathfrak{so}(2l-2j+1) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{so}(2l-1) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{so}(2l+1-2j, 2j-2) \oplus \mathfrak{t}^1$
11	$(\mathfrak{g}, \mathfrak{r})$	$\big(\mathfrak{so}(2l-2j+1, 2j),\ \mathfrak{so}(2l-2j-1, 2j) \oplus \mathfrak{t}^1\big)$: $l \geq 2,\ 1 \leq j \leq l-1$
	T	$i(Z_j - Z_{j+1})$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{so}(2j) \oplus \mathfrak{so}(2l-2j-1) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{so}(2l-1) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{so}(2l-1-2j, 2j) \oplus \mathfrak{t}^1$
12	$(\mathfrak{g}, \mathfrak{r})$	$\big(\mathfrak{so}(2l-2j, 2j), \mathfrak{so}(2l-2j, 2j-2) \oplus \mathfrak{t}^1\big) : l \geq 4,\ 1 \leq j \leq [\frac{l}{2}]$
	T	$i(Z_{j-1} - Z_j)$ *where* $Z_0 := 0$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{so}(2j-2) \oplus \mathfrak{so}(2l-2j) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{so}(2l-2) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{so}(2l-2j, 2j-2) \oplus \mathfrak{t}^1$
13	$(\mathfrak{g}, \mathfrak{r})$	$\big(\mathfrak{so}(2l-2j, 2j), \mathfrak{so}(2l-2j-2, 2j) \oplus \mathfrak{t}^1\big) : l \geq 4,\ 1 \leq j \leq [\frac{l}{2}]$
	T	$i(Z_j - Z_{j+1})$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{so}(2j) \oplus \mathfrak{so}(2l-2j-2) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{so}(2l-2) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{so}(2l-2j-2, 2j) \oplus \mathfrak{t}^1$
14	$(\mathfrak{g}, \mathfrak{r})$	$\big(\mathfrak{so}(2l-2j-1, 2j+1), \mathfrak{so}(2l-2j-1, 2j-1) \oplus \mathfrak{t}^1\big) :$ $l \geq 4,\ 1 \leq j \leq [\frac{l}{2}]$
	T	iZ_1
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{so}(2j-1) \oplus \mathfrak{so}(2l-2j-1) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{so}(2l-2) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{so}(2l-2j-1, 2j-1) \oplus \mathfrak{t}^1$
15	$(\mathfrak{g}, \mathfrak{r})$	$\big(\mathfrak{so}(2l-2j-1, 2j+1),\ \mathfrak{so}(2l-2j-3, 2j+1) \oplus \mathfrak{t}^1\big)$: $l \geq 4,\ 0 \leq j \leq \min\{l-3, [\frac{l}{2}]\}$
	T	$i(Z_j - Z_{j+1})$ *where* $Z_0 := 0$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{so}(2j+1) \oplus \mathfrak{so}(2l-2j-3) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{so}(2l-2) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{so}(2l-2j-3, 2j+1) \oplus \mathfrak{t}^1$

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16	$(\mathfrak{g}, \mathfrak{r})$	$\big(\mathfrak{so}(2l-2j, 2j), \mathfrak{su}(l-j, j) \oplus \mathfrak{t}^1\big) : l \geq 4,\ 1 \leq j \leq [\frac{l}{2}]$
	T	$i(Z_j - Z_l)$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(j) \oplus \mathfrak{su}(l-j) \oplus \mathfrak{t}^2$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{su}(l) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{su}(l-j, j) \oplus \mathfrak{t}^1$
17	$(\mathfrak{g}, \mathfrak{r})$	$\big(\mathfrak{so}^*(2l), \mathfrak{su}(l-k, k) \oplus \mathfrak{t}^1\big) : l \geq 4,\ 1 \leq k \leq [\frac{l-2}{2}] + 1$
	T	$i(Z_k - Z_l)$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(k) \oplus \mathfrak{su}(l-k) \oplus \mathfrak{t}^2$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{su}(l) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{su}(l-k, k) \oplus \mathfrak{t}^1$
18	$(\mathfrak{g}, \mathfrak{r})$	$\big(\mathfrak{so}^*(2l), \mathfrak{su}(l) \oplus \mathfrak{t}^1\big) : l \geq 4$
	T	iZ_l
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(l) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{su}(l) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{su}(l) \oplus \mathfrak{t}^1$
19	$(\mathfrak{g}, \mathfrak{r})$	$\big(\mathfrak{so}^*(2l), \mathfrak{so}^*(2l-2) \oplus \mathfrak{t}^1\big) : l \geq 4$
	T	iZ_1
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(l-1) \oplus \mathfrak{t}^2$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{so}(2l-2) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{so}^*(2l-2) \oplus \mathfrak{t}^1$
20	$(\mathfrak{g}, \mathfrak{r})$	$\big(\mathfrak{e}_{6(2)}, \mathfrak{so}^*(10) \oplus \mathfrak{t}^1\big)$
	T	$i(Z_1 - Z_3)$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(5) \oplus \mathfrak{t}^2$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{so}(10) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{so}^*(10) \oplus \mathfrak{t}^1$
21	$(\mathfrak{g}, \mathfrak{r})$	$\big(\mathfrak{e}_{6(2)}, \mathfrak{so}(6,4) \oplus \mathfrak{t}^1\big)$
	T	$i(Z_2 - Z_3)$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(4) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{so}(10) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{so}(6,4) \oplus \mathfrak{t}^1$
22	$(\mathfrak{g}, \mathfrak{r})$	$\big(\mathfrak{e}_{6(-14)}, \mathfrak{so}(8,2) \oplus \mathfrak{t}^1\big)$
	T	iZ_6
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{so}(8) \oplus \mathfrak{t}^2$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{so}(10) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{so}(8,2) \oplus \mathfrak{t}^1$

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23	$(\mathfrak{g}, \mathfrak{r})$	$\left(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) \oplus \mathfrak{t}^1\right)$
	T	iZ_1
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{so}(10) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{so}(10) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{so}(10) \oplus \mathfrak{t}^1$
24	$(\mathfrak{g}, \mathfrak{r})$	$\left(\mathfrak{e}_{6(-14)}, \mathfrak{so}^*(10) \oplus \mathfrak{t}^1\right)$
	T	$i(Z_1 - Z_3)$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(5) \oplus \mathfrak{t}^2$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{so}(10) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{so}^*(10) \oplus \mathfrak{t}^1$
25	$(\mathfrak{g}, \mathfrak{r})$	$\left(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(2)} \oplus \mathfrak{t}^1\right)$
	T	$i(Z_1 - Z_2)$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{e}_6 \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{e}_{6(2)} \oplus \mathfrak{t}^1$
26	$(\mathfrak{g}, \mathfrak{r})$	$\left(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{t}^1\right)$
	T	$i(Z_1 - Z_7)$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{so}(10) \oplus \mathfrak{t}^2$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{e}_6 \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{e}_{6(-14)} \oplus \mathfrak{t}^1$
27	$(\mathfrak{g}, \mathfrak{r})$	$\left(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(2)} \oplus \mathfrak{t}^1\right)$
	T	$i(Z_2 - Z_6)$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{e}_6 \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{e}_{6(2)} \oplus \mathfrak{t}^1$
28	$(\mathfrak{g}, \mathfrak{r})$	$\left(\mathfrak{e}_{7(-25)}, \mathfrak{e}_6 \oplus \mathfrak{t}^1\right)$
	T	iZ_7
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{e}_6 \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{e}_6 \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{e}_6 \oplus \mathfrak{t}^1$
29	$(\mathfrak{g}, \mathfrak{r})$	$\left(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{t}^1\right)$
	T	$i(Z_6 - Z_7)$
	$\mathfrak{c}_{\mathfrak{k}}(T)$	$\mathfrak{so}(10) \oplus \mathfrak{t}^2$
	$\mathfrak{c}_{\mathfrak{g}_u}(T)$	$\mathfrak{e}_6 \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(T)$	$\mathfrak{e}_{6(-14)} \oplus \mathfrak{t}^1$

Remark 6.17. Let us comment on the above table of pseudo-Hermitian symmetric Lie algebra $(\mathfrak{g}, \mathfrak{r}) = (\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(T))$.

(i) All simple irreducible pseudo-Hermitian symmetric Lie algebras in Table I are taken from Berger's classification of (affine) symmetric spaces (cf. Berger [**1**]).

(ii) The H-element T in $(\mathfrak{g}, \mathfrak{r})$ has irregularity with respect to $\pm$-sign (cf. Notice 6.3).

(iii) $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{g}_u$ are determined by two Lists of Murakami [**12**, p. 297 and p. 305] (read Subsection 4.3). However, we utilize Dynkin diagrams given in the book of Bourbaki [**4**] (recall Notice 4.8).

(iv) $\{Z_a\}_a$ denotes the dual basis of $\{\alpha_a\}_a$ whose Dynkin diagram given in Bourbaki [**4**], where $\{\alpha_a\}_a$ is the set of simple roots, and where the type of Dynkin diagram is determined by that of $\mathfrak{g}^{\boldsymbol{C}}$.

(v) Denote by ρ an inner automorphism $\exp \pi \mathrm{ad}_{\mathfrak{g}} T$ of $\mathfrak{g}$. Then ρ is an involutive automorphism of $\mathfrak{g}$ such that $\mathfrak{r} = \{X \in \mathfrak{g} \mid \rho(X) = X\}$ (cf. Lemma 6.2). In particular, ρ is commutative with a Cartan involution $\tilde{\sigma}$ of $\mathfrak{g}$ defined in Subsection 3.1, because $T \in \mathfrak{k}$ and $\rho = \exp \pi \mathrm{ad}_{\mathfrak{g}} T$.

(vi) Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{q}$ be the canonical decomposition of symmetric pair $(\mathfrak{g}, \rho)$. Define a linear transformation I of $\mathfrak{q}$ and a skew-symmetric form ω on $\mathfrak{q}$ by $I := \mathrm{ad}_{\mathfrak{g}} T|_{\mathfrak{q}}$ and by $\omega(X, Y) := B_{\mathfrak{g}}(T, [X, Y])$ for $X, Y \in \mathfrak{q}$, respectively. Then I is an $\mathrm{ad}_{\mathfrak{g}}\mathfrak{r}$-invariant complex structure of $\mathfrak{q}$, and ω is an $\mathrm{ad}_{\mathfrak{g}}\mathfrak{r}$-invariant symplectic form on $\mathfrak{q}$. Moreover, ω is I-invariant.

(vii) Pair $\mathfrak{g}_u$ with $\mathfrak{c}_{\mathfrak{g}_u}(T)$. Then, pairs $(\mathfrak{g}_u, \mathfrak{c}_{\mathfrak{g}_u}(T))$ exhaust all simple Hermitian symmetric Lie algebras of compact type, and pairs $(\mathfrak{g}_u^{\boldsymbol{C}}, \mathfrak{c}_{\mathfrak{g}_u^{\boldsymbol{C}}}(T)) = (\mathfrak{g}_u^{\boldsymbol{C}}, (\mathfrak{c}_{\mathfrak{g}_u}(T))^{\boldsymbol{C}})$ exhaust all simple reducible pseudo-Hermitian symmetric Lie algebras. Moreover, T is the H-element in $(\mathfrak{g}_u, \mathfrak{c}_{\mathfrak{g}_u}(T))$ and in $(\mathfrak{g}_u^{\boldsymbol{C}}, \mathfrak{c}_{\mathfrak{g}_u^{\boldsymbol{C}}}(T))$.

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