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Weakly exact von Neumann algebras

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Abstract. The theory of exact C^*-algebras was introduced by Kirchberg and has been influential in recent development of C^*-algebras. A fundamental result on exact C^*-algebras is a local characterization of exactness. The notion of weakly exact von Neumann algebras was also introduced by Kirchberg. In this paper, we give a local characterization of weak exactness. As a corollary, we prove that a discrete group is exact if and only if its group von Neumann algebra is weakly exact. The proof naturally involves the operator space duality.

1. Introduction.

The theory of exact C^*-algebras was introduced and studied intensively by Kirchberg. We refer to [**17**] for general information on exact C^*-algebras. This influential theory has been playing a significant rôle in recent development of C^*-algebras. It is particularly important in the classification of C^*-algebras [**8**] [**14**] and in the theory of noncommutative topological entropy [**16**] [**2**] [**15**]. Hence it is natural to explore an analogue of this notion for von Neumann algebras.

Throughout this paper, we mean by $J \lhd B$ a closed two sided ideal J in a unital C^*-algebra B and denote the quotient map by $Q\colon B \to B/J$. Although it is not essential, we assume for simplicity that all C^*-algebras, except ideals, are unital. We denote by $\otimes$ the algebraic tensor product and by $\otimes_{\min}$ the minimal (or spatial) tensor product. Recall that a C^*-algebra A is said to be exact if

$$(A \otimes_{\min} B)/(A \otimes_{\min} J) = A \otimes_{\min} (B/J)$$

for any $J \lhd B$, or equivalently, for any $*$-representation $\pi\colon A \otimes_{\min} B \to \boldsymbol{B}(\mathscr{H})$ with $A \otimes J \subset \ker \pi$, the induced $*$-representation $\tilde{\pi}\colon A \otimes (B/J) \to \boldsymbol{B}(\mathscr{H})$ is continuous with respect to the minimal tensor norm. Since von Neumann algebras are not exact unless they are subhomogeneous, the correct notion of exactness for von Neumann algebras shall be the weak exactness introduced in [**8**]. Thus, for a von Neumann algebra $A = M$, we modify the above definition and require the $*$-representation π to be left normal, i.e., the restriction of π to M is normal.

Definition 1. A von Neumann algebra M is said to be *weakly exact* if for any $J \lhd B$ and any left normal $*$-representation

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$$\pi\colon M \otimes_{\min} B \to \boldsymbol{B}(\mathscr{H})$$

with $M \otimes J \subset \ker \pi$, the induced $*$-representation

$$\tilde{\pi}\colon M \otimes (B/J) \to \boldsymbol{B}(\mathscr{H})$$

is continuous with respect to the minimal tensor norm.

Kirchberg [**8**] proved that a von Neumann algebra M is weakly exact if it contains a weakly dense exact C^*-algebra A. Indeed, by Theorem 4 (10) in [**9**], the exact C^*-algebra A is locally reflexive [**1**] [**3**] i.e., the left normal embedding $A^{**} \otimes_{\min} C \subset (A \otimes_{\min} C)^{**}$ is continuous for any C. Let $J \lhd B$, π and $\tilde{\pi}$ be given as in Definition 1. Since A is exact, $\tilde{\pi}$ is continuous on $A \otimes_{\min} (B/J)$. By the local reflexivity of A, $\tilde{\pi}$ extends to a left normal $*$-representation ρ on $A^{**} \otimes_{\min} (B/J)$. But, since $\tilde{\pi}$ is left normal on $M \otimes (B/J)$, it coincides with ρ on $M \otimes (B/J)$ and hence is continuous on $M \otimes_{\min} (B/J)$. We will prove a partial converse of this result and a local characterization of weak exactness. Before stating the theorem, we recall that an operator system S is exact if

$$(S \otimes_{\min} B)/(S \otimes_{\min} J) = S \otimes_{\min} (B/J)$$

isometrically for any $J \lhd B$. We note that exact operator systems are characterized locally [**9**] [**12**] and they are locally reflexive (Corollary 4.8 in [**4**]). The term "u.c.p." stands for "unital completely positive".

Theorem 1. *Let M be a von Neumann algebra with a separable predual. Then, the following are equivalent.*

(1) *The von Neumann algebra M is weakly exact.*
(2) *For any finite dimensional operator system E in M, there exist nets of u.c.p. maps $\varphi_i\colon E \to \boldsymbol{M}_{n(i)}$ and $\psi_i\colon \varphi_i(E) \to M$ such that the net $\{\psi_i \circ \varphi_i\}$ converges to id_E in the point-ultrastrong topology.*
(3) *There exist an exact operator system S and normal u.c.p. maps $\varphi\colon M \to S^{**}$ and $\psi\colon S^{**} \to M$ such that $\psi \circ \varphi = \mathrm{id}_M$.*

We note that the implications $1 \Leftrightarrow 2 \Leftarrow 3$ hold without separability assumption on M. It would be interesting to know whether we can take S in the condition 3 to be an exact C^*-algebra (and moreover ψ to be a normal $*$-homomorphism). Kirchberg [**8**] asked whether every von Neumann algebra is weakly exact. The following corollary solves this problem negatively. (See [**6**] for existence of non-exact groups.) It is left open whether the ultrapower R^ω of the hyperfinite type II_1 factor R is weakly exact or not.

Corollary 2. *Let Γ be a discrete group. Then, the group von Neumann algebra $L\Gamma$ is weakly exact if and only if Γ is exact.*

We recall that a discrete group Γ is exact if and only if its reduced group C^*-algebra $C^*_{\mathrm{r}}\Gamma$ is exact [**10**]. Since $C^*_{\mathrm{r}}\Gamma$ is weakly dense in $L\Gamma$, the "if" part of the corollary follows from Kirchberg's theorem mentioned above. It is not hard to show that weak exactness passes to a von Neumann subalgebra which is the range of a normal conditional expectation. Hence, every von Neumann subalgebra of a weakly exact finite von Neumann algebra is again weakly exact. It follows that if Γ is exact and Λ is not, then $L\Lambda \not\hookrightarrow L\Gamma$. Moreover, exactness is a measure equivalence invariant.

2. Proofs.

It is well-known that exactness for a C^*-algebra is equivalent to the property C' (cf. [**1**] [**3**]). We prove a von Neumann algebra analogue of this fact. Recall that for von Neumann algebras M and N, a $*$-representation π of $M \otimes N$ is said to be binormal if both $\pi|_M$ and $\pi|_N$ are normal.

PROPOSITION 3. *A von Neumann algebra M is weakly exact if and only if for any C^*-algebra B and any left normal $*$-representation π: $M \otimes_{\min} B \to \boldsymbol{B}(\mathscr{H})$, the binormal extension $\hat{\pi}$: $M \otimes B^{**} \to \boldsymbol{B}(\mathscr{H})$ is continuous with respect to the minimal tensor norm.*

PROOF. We first prove the "if" part. Let $J \triangleleft B$ and π: $M \otimes_{\min} B \to \boldsymbol{B}(\mathscr{H})$ be as in Definition 1. Let p be the central projection which supports the normal $*$-homomorphism Q^{**}: $B^{**} \to (B/J)^{**}$ so that we may identify $(B/J)^{**}$ with pB^{**}. We denote the canonical inclusion by

$$\psi\colon B/J \to (B/J)^{**} = pB^{**} \subset B^{**}.$$

By the assumption, π extends to a binormal $*$-homomorphism $\hat{\pi}$ on $M \otimes_{\min} B^{**}$. Since $M \otimes J \subset \pi$, we have $\hat{\pi}(1 \otimes p) = 1$. We claim that $\tilde{\pi}$ coincides with $\hat{\pi} \circ (\mathrm{id} \otimes \psi)$ and hence continuous on $M \otimes_{\min} (B/J)$. Indeed, for any $a \in M$ and $x \in B$, we have

$$(\hat{\pi} \circ (\mathrm{id} \otimes \psi))(a \otimes Q(x)) = \hat{\pi}(a \otimes px) = \pi(a \otimes x) = \tilde{\pi}(a \otimes Q(x)).$$

We next prove the "only if" part. Let a C^*-algebra B and a left normal $*$-representation π: $M \otimes B \to \boldsymbol{B}(\mathscr{H})$ be given. For a directed set I, we set

$$B_I = \left\{ (x(i))_{i \in I} \in \prod_{i \in I} B : \text{ultrastrong}^*\text{-}\lim_{i \in I} x(i) \text{ exists in } B^{**} \right\}.$$

Since adjoint operation and multiplication are jointly ultrastrong*-continuous on a bounded subset, B_I is a C*-algebra and the map

$$\sigma\colon B_I \ni (x(i))_{i \in I} \mapsto \text{ultrastrong}^*\text{-}\lim_{i \in I} x(i) \in B^{**}$$

is a continuous $*$-homomorphism. Because of Kaplansky's density theorem, we may assume that σ is surjective. Consider the $*$-representation ρ: $M \otimes B_I \to \boldsymbol{B}(\mathscr{H})$ defined by

$$\rho\left(\sum_{k=1}^{n} a_k \otimes (x_k(i))_{i\in I}\right) = \text{ultrastrong}^*\text{-}\lim_i \pi\left(\sum_{k=1}^{n} a_k \otimes x_k(i)\right).$$

We observe that ρ is continuous on $M \otimes_{\min} B_I \subset \prod_{i\in I}(M \otimes_{\min} B)$. Since M is weakly exact, ρ is left normal and $M \otimes \ker\sigma \subset \ker\rho$, the induced $*$-representation $\tilde{\rho}$ is continuous on $M \otimes_{\min} B^{**}$. It is not too hard to see that $\hat{\pi} = \tilde{\rho}$ and hence $\hat{\pi}$ is continuous on $M \otimes_{\min} B^{**}$. □

The advantage of the above formulation is that B in the statement need not be a C^*-algebra. We make this point more precise. Let X be a C^*-algebra or an operator space. Let $p \in M^{**}$ be the central support of the identity representation of M so that we may identify M with pM^{**}. Then, the canonical binormal embedding $M^{**} \otimes X^{**} \subset (M \otimes_{\min} X)^{**}$ gives rise to a (non-unital) binormal embedding

$$\Theta_X\colon M \otimes X^{**} = pM^{**} \otimes X^{**} \hookrightarrow (M \otimes_{\min} X)^{**},$$

i.e., for $z = \sum a_k \otimes x_k \in M \otimes X$, we have $\Theta_X(z) = \sum pa_k \otimes x_k$. Now, assume that M is weakly exact and let B be a C*-algebra. Since Θ_B is a binormal $*$-homomorphism which is continuous on $M \otimes_{\min} B$, it follows from Proposition 3 that Θ_B is continuous (isometric) on $M \otimes_{\min} B^{**}$. This implies that Θ_X is isometric on $M \otimes_{\min} X^{**}$ for any operator space X. Indeed, when $X \subset B$, we have the commuting diagram;

$$\begin{array}{ccc} M \otimes X^{**} & \xrightarrow{\Theta_X} & (M \otimes_{\min} X)^{**} \\ \downarrow & & \downarrow \\ M \otimes_{\min} B^{**} & \xrightarrow{\Theta_B} & (M \otimes_{\min} B)^{**} \end{array}$$

where the bottom and the vertical inclusions are all isometric. It follows that for any $z \in M \otimes X^{**}$ with $\|z\|_{\min} \le 1$, there exists a net $\{z_i\}$ in $M \otimes X$ with $\|z_i\|_{\min} \le 1$ which converges to $\Theta_X(z) \in (M \otimes X)^{**}$ in the weak*-topology. We observe that the net $\{z_i\}$ converges to z in the $\sigma(M \otimes X^{**}, M_* \otimes X^*)$-topology.

We use the operator space duality to restate the above result. Recall that the operator space duality says that for $z = \sum a_k \otimes x_k \in M \otimes X^{**}$ and the corresponding (finite rank) map $T_z\colon X^* \to M$ defined by

$$T_z\colon X^* \ni f \mapsto \sum \langle f, x_k\rangle\, a_k \in M,$$

we have $\|T_z\|_{\text{cb}} = \|z\|_{\min}$ (cf. [**5**] [**13**]). We note that T_z is weak*-continuous if and only if z comes from $M \otimes X$ (rather than $M \otimes X^{**}$). Thus we arrive at the following corollary. The term "c.c." stands for "completely contractive".

COROLLARY 4. *Let M be a weakly exact von Neumann algebra and X be an operator space. Then, for any finite rank c.c. map $\varphi\colon X^* \to M$, there exists a net $\{\varphi_i\}$ of weak*-continuous finite rank c.c. maps $\varphi_i\colon X^* \to M$ which converges to φ in the point-ultraweak topology.*

We are now in position to prove the theorem.

PROOF OF THEOREM 1.

Ad1⇒2: Let $M \subset \boldsymbol{B}(\mathscr{H})$ be a weakly exact von Neumann algebra and $E \subset M$ be a finite dimensional operator system. Fix an increasing sequence $\{\mathscr{H}_k\}$ of finite dimensional subspaces of $\mathscr{H}$ with a dense union and denote by $\Phi_k\colon E \to \boldsymbol{B}(\mathscr{H}_k) = \boldsymbol{M}_{n(k)}$ the compression corresponding to $\mathscr{H}_k$. We note that Φ_k is a linear isomorphism onto $E_k = \Phi_k(E)$ if k is sufficiently large. Consider a complete isometry given by

$$\Phi\colon E \ni a \mapsto (\Phi_k(a))_k \in \prod_k E_k.$$

We note that $\prod E_k \subset \prod \boldsymbol{M}_{n(k)}$ is an ultraweakly closed operator subspace (and hence is a dual operator space). There exists a right inverse of Φ given by

$$\Psi\colon \prod_k E_k \ni (x_k)_k \mapsto \text{ultraweak-}\lim_{k\to\omega} \Phi_k^{-1}(x_k) \in E \subset M,$$

where ω is a fixed free ultrafilter on $\boldsymbol{N}$. Indeed, Ψ is a well-defined c.c. map with $\Psi \circ \Phi = \mathrm{id}_E$. By Corollary 4, there exists a net of ultraweakly continuous c.c. maps $\psi_i\colon \prod E_k \to M$ such that $\lim_i \psi_i = \Psi$ in the point-ultraweak topology. Since each ψ_i is ultraweakly continuous, if we set

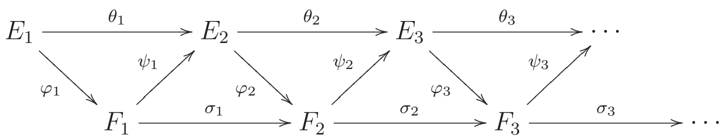
$$\varphi_m\colon E \ni a \mapsto (\Phi_1(a), \dots, \Phi_m(a), 0, 0, \dots) \in \prod_k E_k,$$

then we have

$$\lim_i \lim_m \psi_i \circ \varphi_m = \lim_i \psi_i \circ \Phi = \Psi \circ \Phi = \mathrm{id}_E.$$

We note that $\varphi_m(E) \subset \bigoplus_{k=1}^m E_k \subset \bigoplus_{k=1}^m \boldsymbol{M}_{n(k)}$. Passing to a convex combination, we may assume that the convergence is with respect to the point-ultrastrong topology. This proves the condition 2 except that ψ_i's may not be u.c.p. One can fix this problem as follows. First, using a standard cut and paste method, one may assume that ψ_i is approximately unital in norm. Then, one invokes Theorem 2.5 in [**3**].

Ad2⇒3: Let M be a weakly exact von Neumann algebra with a separable predual and A be a ultraweakly-dense norm-separable C*-subalgebra in M. We denote by $|\cdot|_\sigma$ a seminorm which defines the ultrastrong topology on the unit ball of M. Using the condition 2 recursively, we can construct sequences of finite dimensional operator systems and connecting u.c.p. maps



such that

(1) $E_1 \subset E_2 \subset \cdots \subset M$ and the norm-closure of $\bigcup E_k$ contains A,
(2) $F_n \subset \boldsymbol{M}_{n(k)}$ for some $n(k) \in \boldsymbol{N}$,
(3) the diagram commutes and $|\theta_k(a) - a|_\sigma < 2^{-k}\|a\|$ for $a \in E_k$.

We define the operator system S as the inductive limit of (F_k, σ_k); let

$$S = Q(\tilde{S}) \subset \prod \boldsymbol{M}_{n(k)} / \bigoplus \boldsymbol{M}_{n(k)}$$

where $Q\colon \prod \boldsymbol{M}_{n(k)} \to \prod \boldsymbol{M}_{n(k)} / \bigoplus \boldsymbol{M}_{n(k)}$ is the quotient map and $\tilde{S}$ is the norm closure of

$$\left\{ (x_k)_k \in \prod_{k=1}^{\infty} \boldsymbol{M}_{n(k)} : x_k \in F_k \text{ and } x_{k+1} = \sigma_k(x_k) \text{ eventually} \right\} \subset \prod_{k=1}^{\infty} \boldsymbol{M}_{n(k)}.$$

We note that S coincides with the quotient of $\tilde{S}$ by the complete M-ideal $\bigoplus \boldsymbol{M}_{n(k)}$. Since $\tilde{S}$ is exact and locally reflexive, so is S. Let $\Phi\colon \bigcup E_k \to S^{**}$ be a cluster point of the sequence

$$\Phi_m\colon \bigcup E_k \ni a \mapsto Q((\sigma_{k-1} \circ \cdots \circ \sigma_m \circ \varphi_m(a))_{k=m+1}^{\infty}) \in S$$

and let $\tilde{\psi}\colon \tilde{S} \to M$ be a cluster point of the sequence

$$\tilde{\psi}_m\colon \tilde{S} \ni (x_k)_k \mapsto \psi_m(x_m) \in M.$$

In both cases, cluster points are taken with respect to the point-weak* topology. It is easy to see that $\tilde{\psi} = \psi \circ Q$ for some u.c.p. map $\psi\colon S \to M$. The unique weak*-continuous extension of ψ on S^{**} is still denoted by ψ. Then, for any $a \in \bigcup E_n$, we have that

$$\begin{aligned}
|a - \psi \circ \Phi(a)|_\sigma &\leq \limsup_m |a - \psi \circ \Phi_m(a)|_\sigma \\
&= \limsup_m |a - \tilde{\psi}((\sigma_{k-1} \circ \cdots \circ \sigma_m \circ \varphi_m(a))_{k=m+1}^{\infty})|_\sigma \\
&\leq \limsup_m \limsup_k |a - \psi_k \circ \sigma_{k-1} \circ \cdots \circ \sigma_m \circ \varphi_m(a)|_\sigma \\
&= \limsup_m \limsup_k |a - \theta_k \circ \cdots \circ \theta_m(a)|_\sigma \\
&\leq \limsup_m \limsup_k \sum_{j=m}^{k} 2^{-j}\|a\| = 0.
\end{aligned}$$

It follows that $\psi \circ \Phi$ is the identity map on $\bigcup E_k$. We first extend Φ on the norm closure of $\bigcup E_k$ by norm continuity and then further extend $\Phi|_A$ to $\bar{\Phi}\colon A^{**} \to S^{**}$ by ultraweak continuity. Then, $\psi \circ \bar{\Phi}\colon A^{**} \to M$ is a normal u.c.p. map such that $\psi \circ \bar{\Phi}|_A = \mathrm{id}_A$. Hence, the restriction φ of $\bar{\Phi}$ to $M \subset A^{**}$ satisfies $\psi \circ \varphi = \mathrm{id}_M$.

Ad3⇒1: We omit the proof because it is almost same as that of Kirchberg's theorem cited in the remarks preceding Theorem 1. □

PROOF OF COROLLARY 2. Let Γ be a discrete group such that $L\Gamma$ is weakly exact. To prove exactness of Γ, we use the criterion given in [**11**]. Let a finite subset $\mathscr{E} \subset \Gamma$ and $\varepsilon > 0$ be given. By Arveson's extension theorem, Theorem 1 and its proof (or Paulsen's trick), there exist a finite subset $\mathscr{F} \subset \Gamma$ and a u.c.p. map $\psi\colon \boldsymbol{B}(\ell_2\mathscr{F}) \to \boldsymbol{B}(\ell_2\Gamma)$ such that if we denote by $\varphi\colon L\Gamma \to \boldsymbol{B}(\ell_2\mathscr{F})$ the compression, then $\psi \circ \varphi(\lambda(s)) \in L\Gamma$ and $\|\lambda(s) - \psi \circ \varphi(\lambda(s))\|_2 < \varepsilon$ for every $s \in \mathscr{E}$. As in [**7**], we consider the positive definite kernel defined by

$$u(s,t) = \langle \psi \circ \varphi(\lambda(st^{-1}))\delta_t, \delta_s \rangle$$

for $s, t \in \Gamma$. We have that $u(s,t) \neq 0$ only if $st^{-1} \in \mathscr{F}\mathscr{F}^{-1}$ and that $\|u(s,t) - 1\| < \varepsilon$ if $st^{-1} \in \mathscr{E}$. This proves exactness of Γ. □

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