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Local representability as Dirichlet solutions

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Abstract. A bounded Euclidean domain R is said to be a Dirichlet domain if every quasibounded harmonic function on R is represented as a generalized Dirichlet solution on R. As a localized version of this, R is said to be locally a Dirichlet domain at a boundary point $y \in \partial R$ if there is a regular domain U containing y such that every quasibounded harmonic function on $U \cap R$ with vanishing boundary values on $\overline{R} \cap \partial U$ is represented as a generalized Dirichlet solution on $U \cap R$. The main purpose of this paper is to show that the following three statements are equivalent by pairs: R is a Dirichlet domain; R is locally a Dirichlet domain at every boundary point $y \in \partial R$; R is locally a Dirichlet domain at every boundary point $y \in \partial R$ except for points in a boundary set of harmonic measure zero. As an application it is shown that if every boundary point of R is graphic except for points in a boundary set of harmonic measure zero, then R is a Dirichlet domain, where a boundary point $y \in \partial R$ is said to be graphic if there are neighborhood V of y and an orthogonal (or polar) coordinate $x = (x', x^d)$ (or $x = r\xi$) such that $V \cap R$ is represented as one side of a graph of a continuous function $x^d = \varphi(x')$ (or $r = \varphi(\xi)$).

1. Introduction.

In studying a harmonic function u on a bounded domain R in the Euclidean space $\boldsymbol{R}^d$ of dimension $d \geq 2$, it is very convenient in many instances if u is represented as a generalized Dirichlet solution H_f^R of a resolutive boundary function f on ∂R even if the concrete properties of f are unknown. If u admits such a representation, then u must be quasibounded on R (cf. (4.3) below). However there are a domain R and a quasibounded harmonic function u on R such that u cannot be represented as a Dirichlet solution on R (cf. (4.4) below). In view of this situation we are interested in finding conditions on R under which every quasibounded harmonic function on R is represented as a generalized Dirichlet solution. Such a domain will be referred to as a *Dirichlet domain.* As a localization of this property, we say that a domain R is *locally* a Dirichlet domain at a boundary point $y \in \partial R$ with respect to a regular domain U containing y if every quasibounded harmonic function u on $R \cap U$ with vanishing boundary values on $\overline{R} \cap \partial U$ is represented as a Dirichlet solution $H_f^{R \cap U}$ on $R \cap U$ of a resolutive boundary function f on $\partial(R \cap U)$. Here a domain U whose boundary points are all regular in the sense of the generalized Dirichlet problem is said to be a regular domain. The primary purpose of this paper is to give an easily applicable practical criterion for a given bounded domain R to be a Dirichlet domain stated below as Theorem 1.3, which is an improvement of our former result in [**6**]. This will be derived from the following main assertion of the

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present paper that a bounded domain R is a Dirichlet domain if and only if R is locally a Dirichlet domain at every boundary point of R. Actually we will show by the following two theorems a bit more than the fact mentioned above. Namely, the first result is as follows.

Theorem 1.1. *If R is locally a Dirichlet domain at a boundary point y with respect to a regular domain U containing y, then R is locally a Dirichlet domain at y with respect to any regular domain V containing y with $V \subset U$. In particular, if R is a Dirichlet domain, then R is locally a Dirichlet domain at any boundary point y with respect to any regular domain V containing y.*

The above result is a more precise version of the statement that if R is a Dirichlet domain, then R is locally a Dirichlet domain at every boundary point of R. The converse of this, i.e. if R is locally a Dirichlet domain at every boundary point of R, then R is a Dirichlet domain, is also generalized as follows.

Theorem 1.2. *If R is locally a Dirichlet domain at every boundary point of R except for points in a boundary set of harmonic measure zero with respect to R, then R is a Dirichlet domain.*

A boundary point y of R is said to be *graphic* if there are an open neighborhood V of y, an orthogonal coordinate $x = (x', x^d)$ or a polar coordinate $x = r\xi$ $(r \geq 0,\ |\xi| = 1)$, and a continuous function $x^d = \varphi(x')$ or $r = \varphi(\xi) > 0$ such that $R \cap V = \{(x', x^d) : x^d < \varphi(x')\} \cap V$ or $R \cap V = \{r\xi : r < \varphi(\xi)\} \cap V$. If $y \in \partial R$ is not graphic, then we say that y is *nongraphic*. As an application of the above theorem 1.2, we will prove the following result.

Theorem 1.3. *If the set E of nongraphic boundary point of R is of harmonic measure zero with respect to R, then R is a Dirichlet domain.*

If $E = \varnothing$, then R is referred to as a *continuous domain* ([**4**]). We have shown in our former paper [**6**] that a continuous domain R is a Dirichlet domain. The above result is therefore a generalization of this result, i.e. an extension from the case $E = \varnothing$ to the case E may not be empty but at most has zero harmonic measure.

There are two interesting and important foregoing works [**1**] and [**5**] on the same theme as ours in which the term *Poissonian domain* is used for regions what we are calling Dirichlet domains. C. J. Bishop [**1**] gives a necessary and sufficient condition for a domain to be Poissonian which is strikingly interesting but seems to be not too practical as to derive our present criterion Theorem 1.3 above. Localization of the notion Poissonian is one of the central leading ideas in the work [**5**] by T. S. Mountford and S. C. Port in which closely related results to our Theorems 1.1 and 1.2 are found. In essence, our results are partly contained in their results and also partly contains them. Allowing the existence of exceptional sets of harmonic measure zero in our work (e.g. Theorem 1.2 above), which is actually a selling point of our present paper, is an essential distinction with which the work [**5**] is not concerned.

The paper consists of 10 sections including this Section 1 Introduction. In Section 2 Dirichlet solutions, some of basic properties of generalized solutions of Dirichlet problem are stated. Harmonic functions u admitting the Jordan decomposition $u = u^+ - u^-$ are considered in Section 3 Vector lattice $HP(W)$, and the equivalence of $u(y) = 0$ and $u^\pm(y) = 0$ at any regular boundary point y is established. The notion of Dirichlet domains is introduced and a criterion for being Dirichlet domains is given in Section 4 Quasiboundedness. In Section 5 Localization, the notion of Dirichlet domain and the related criterion are localized. After these preliminary discussions in Section 2–5, Theorem 1.1 is proved in Section 6 Proof of Theorem 1.1. As a useful tool to prove Theorem 1.2 in Section 8 Proof of Theorem 1.2, Wiener functions and their basic properties are explained in Section 7 Wiener potentials. After a geometric notion of graphic points for boundary points of domains is introduced in Section 9 Graphic points, we will prove Theorem 1.3 in the final Section 10 Proof of Theorem 1.3.

2. Dirichlet solutions.

Throughout this paper we denote by R a bounded domain in the Euclidean space $\boldsymbol{R}^d$ of dimension $d \geq 2$. We also denote by W a nonempty bounded open set in $\boldsymbol{R}^d$. In addition to the basic function space $H(W)$ of harmonic functions on W, i.e. C^2 solutions of the Laplace equation $\Delta u = 0$ on W, we consider the class $S(W)$ of superharmonic functions on W. In general in this paper, for a class $\mathscr{F}$ of some functions, we set $\mathscr{F}^+ := \{f \in \mathscr{F} : f \geq 0\}$. We also consider the class $\mathscr{P}(W)$ of potentials p on W characterized by that $p \in S(W)^+$ and the greatest harmonic minorant of p is zero on W. Hence 0 is included in $\mathscr{P}(W)$.

We consider the Dirichlet problem for a nonempty bounded open set W and a general boundary function f on ∂W with respect to the Laplace equation $\Delta u = 0$. We follow the standard procedure of Perron-Wiener-Brelot (cf. e.g. [**3**, pp. 156–176]): we denote by $\overline{\mathscr{V}}_f^W$ the class of lower bounded $s \in S(W)$ such that $\liminf_{x\to y} s(x) \geq f(y)$ for every $y \in \partial W$; we set $\underline{\mathscr{V}}_f^W := -\overline{\mathscr{V}}_{-f}^W$; we denote by $\overline{H}_f^W$ ($\underline{H}_f^W$, resp.) the lower (upper, resp.) envelope of $\overline{\mathscr{V}}_f^W$ ($\underline{\mathscr{V}}_f^W$, resp.), which is either harmonic or identically $\pm\infty$ in each component of W; we see that $\overline{H}_f^W \geq \underline{H}_f^W = -\overline{H}_{-f}^W$ and, if $\overline{H}_f^W = \underline{H}_f^W$ is harmonic, then the common function is denoted by H_f^W and f is said to be *resolutive*; H_f^W is referred to as the *Dirichlet solution* with the resolutive boundary function f; a point $y \in \partial W$ is said to be *regular* if $\lim_{x\to y} H_f^W(x) = f(y)$ for every resolutive boundary function f on ∂W continuous at y.

For a bounded domain R we denote by $d\omega_x^R$ the *harmonic measure* on ∂R characterized by $H_f^R(x) = \int_{\partial R} f d\omega_x^R$ for each $x \in R$ and each $f \in C(\partial R)$. We fix a point $x_0 \in R$ and use the simplified notation $d\omega = d\omega^R = d\omega_{x_0}^R$. There is a function $P(\cdot, x) \in L^1(\partial R, d\omega)$ (and actually a Borel function $P(\cdot, x)$ such that $c^{-1} \leq P(\cdot, x) \leq c$ $d\omega$-a.e. on ∂R with the Harnack constant c determined by x and x_0 and R) such that $d\omega_x^R = P(\cdot, x)d\omega$ on ∂R. Then a function f on ∂R is resolutive if and only if $f \in L^1(\partial R, d\omega)$ and in this case $H_f^R(x) = \int_{\partial R} f(y)P(y, x)d\omega(y)$ $(x \in R)$.

We use the following fact which is an easy consequence of the definitions of $\overline{H}_f^R$ and $\underline{H}_f^R$: for any function f on ∂R for which $\overline{H}_f^R$ ($\underline{H}_f^R$, resp.) is harmonic there is a Borel function g (and actually a decreasing (increasing, resp.) limit of lower (upper, resp.) semicontinuous functions) on ∂R such that $f \leq g \leq \|f; L^\infty(\partial R; d\omega)\| \leq \infty$ ($f \geq g \geq -\|f; L^\infty(\partial R; d\omega)\| \geq -\infty$, resp.) on ∂R with $\overline{H}_f^R = H_g^R$ ($\underline{H}_f^R = H_g^R$, resp.).

In particular we see that for any resolutive function f on ∂R there is a resolutive Borel function g on ∂R with $|g| \leq \|f; L^\infty(\partial R, d\omega)\| \leq \infty$ on ∂R and $g = f$ $d\omega$-a.e. on ∂R such that $H_f^R = H_g^R$.

We denote by $H_{ds}(W)$ the class of harmonic functions u expressible as Dirichlet solutions H_f^W with resolutive boundary functions f on ∂W so that in the case of a bounded domain R simply

$$H_{ds}(R) = \{H_f^R : f \in L^1(\partial R, d\omega)\}.$$

The operator $f \mapsto H_f^R : L^1(\partial R, d\omega) \to H_{ds}(R)$ is clearly surjective, linear, and positive. That it is injective and the inverse operator $H_f^R \mapsto f : H_{ds}(R) \to L^1(\partial R, d\omega)$ is linear and positive follows from the following result (see [**6**] for its proof):

Proposition 2.1. *If $\overline{H}_f^R$ and $\underline{H}_f^R$ are harmonic on R, then the inequalities*

$$\liminf_{x \to y} \underline{H}_f^R(x) \leq f(y) \leq \limsup_{x \to y} \overline{H}_f^R(x) \tag{2.1}$$

hold for $d\omega$-a.e. y in ∂R.

At the end of this section we state a technical lemma used later. We consider two regular domains G and U with nonempty intersections with R such that $U \subset G$ and a bounded nonnegative function f on $\boldsymbol{R}^d$ with spt $f \subset \overline{U} \cap \partial R$ such that $f|\partial(G \cap R)$ is a resolutive boundary function on $\partial(G \cap R)$ with respect to $G \cap R$ so that $f|\partial(U \cap R)$ is also a resolutive boundary function on $\partial(U \cap R)$ with respect to $U \cap R$. Hence if we write $H_f^{G \cap R}$ ($H_f^{U \cap R}$, resp.), then the boundary function f is understood to be $f|\partial(G \cap R)$ ($f|\partial(U \cap R)$, resp.). Since $H_f^{U \cap R}$ has vanishing boundary values on $R \cap \partial U$, $H_f^{U \cap R}$ is always understood to be subharmonic on $G \cap R$ by extending $H_f^{U \cap R}$ to $G \cap R$ as $H_f^{U \cap R} \equiv 0$ on $(G \setminus U) \cap R$. Observe that every point in $\partial U \cap \partial R$ is regular with respect to $U \cap R$ since U is regular. One way to see this is to observe that the restriction to $U \cap R$ of any barrier on U of any point z in $\partial U \cap \partial R$ is also a barrier on $U \cap R$ of the point z. Then we have the following result.

Lemma 2.1. *Suppose $H_f^{G \cap R}$ has vanishing boundary values on $\partial U \cap \partial R$. Then the least harmonic majorant of the subharmonic function $H_f^{U \cap R}$ on $G \cap R$ is $H_f^{G \cap R}$, or equivalently,*

$$H_f^{G \cap R} - H_f^{U \cap R} \in \mathscr{P}(G \cap R). \tag{2.2}$$

Proof. For simplicity we set $W := G \cap R$ and $V := U \cap R$. Define a Borel function φ on ∂V given by $\varphi = 0$ on $\overline{U} \cap \partial R$ and $\varphi = H_f^W$ on $R \cap \partial U$. As a bounded Borel function, φ is resolutive on ∂V with respect to V. We first note that

$$H_f^W | V = H_{f+\varphi}^V = H_f^V + H_\varphi^V.$$

In fact, take an arbitrary $s \in \overline{\mathscr{V}}_f^W$. Since $s \geq H_f^W$ on V, we have $s \in \overline{\mathscr{V}}_{f+\varphi}^V$ and a fortiori $H_f^W|V \geq \overline{H}_{f+\varphi}^V = H_{f+\varphi}^V$. Similarly $H_{-f}^W|V \geq \overline{H}_{-f-\varphi}^V = H_{-f-\varphi}^V$, which means $H_f^W|V \leq H_{f+\varphi}^V$. Thus we have deduced the above displayed identity.

In addition to H_f^V being extended to W by setting $H_f^V \equiv 0$ on $W \setminus V$, we also extend H_φ^V to W by setting $H_\varphi^V = H_f^W$ on $W \setminus V = (G \setminus U) \cap R$. Then by the above displayed identity we still have $H_f^W = H_f^V + H_\varphi^V$ on W so that

$$H_f^W - H_f^V = H_\varphi^V$$

on W. To deduce (2.2) we only have to show that the extended H_φ^V is a potential on W. Since every point in $R \cap \partial U$ is regular with respect to V, the extended $H_\varphi^V \in C(W)$. Since $H_f^W \geq H_\varphi^V \geq 0$ and H_f^W has vanishing boundary values on $\partial U \cap \partial R$, we see that $H_\varphi^V \in S(W)$ and H_φ^V has boundary values zero on the set of regular points in ∂W. This with the boundedness of H_φ^V assures that it is a potential on W. □

3. Vector lattice $\boldsymbol{HP(W)}$.

Besides a bounded domain R we also take a nonempty bounded open set W in $\boldsymbol{R}^d$ as before. Let u be a nonnegative subharmonic function on W and h be the least harmonic majorant of u on W. If h has the vanishing boundary value at a boundary point $z \in \partial W$, then the same is true of u but the converse is in general not true (if z is irregular). However we have the following result.

Lemma 3.1. *If $z \in \partial W$ is a regular boundary point with respect to W, then u has the vanishing boundary value at z if and only if h has the vanishing boundary value at z.*

Proof. We only have to show that h has the vanishing boundary value at z under the assumption that u has the vanishing boundary value at z. Let $U := B(z, r)$ be the open ball with radius $r > 0$ centered at z such that $0 \leq u \leq 1$ on $B(z, 2r) \cap W$ and let $v := h + 1$ on W. We set $V := U \cap W$. Consider a boundary function f on ∂V given by

$$f(y) := \limsup_{x \in V, x \to y} u(x)$$

for $y \in \overline{U} \cap \partial W$ and $f(y) := v(y)$ for $y \in W \cap \partial U$. Since

$$\liminf_{x \in V, x \to y} v(x) \geq 1 \geq f(y)$$

for $y \in \overline{U} \cap \partial W$ and

$$\liminf_{x \in V, x \to y} v(x) = v(y) = f(y)$$

for $y \in W \cap \partial U$, we must conclude that $v \in \overline{\mathscr{V}}_f^V$. Hence the Borel function f on ∂V is resolutive and $v \geq H_f^V$ on V. Therefore if we consider the function w on W such that $w := v$ on $W \setminus V$ and $w := H_f^V$ on V, then we see that $w \in S(W)$ since, in addition to $v \geq H_f^V$ on V, w is continuous on W and harmonic on $W \setminus \partial V$.

Next we show that $u \le w$ on W. It is clear on $W \setminus V$ because $u \le h < v = w$ on $W \setminus V$. Take an arbitrary $s \in \overline{\mathscr{V}}_f^V$. Then

$$\liminf_{x \in V, x \to y} s(x) \ge f(y) = \limsup_{x \in V, x \to y} u(x)$$

for every $y \in \overline{U} \cap \partial W$ and

$$\liminf_{x \in V, x \to y} s(x) \ge f(y) = v(y) > 1 \ge \limsup_{x \in V, x \to y} u(x)$$

for every $y \in W \cap \partial U$. Hence

$$\liminf_{x \in V, x \to y} (s(x) - u(x)) \ge 0$$

for every $y \in \partial V$, which implies that the lower bounded superharmonic function $s - u$ on V is nonnegative on V or $s \ge u$ on V. This yields $w = H_f^V \ge u$ on V. This completes the proof for $w \ge u$ on W.

Since f is continuous on ∂V at z with $f(z) = 0$ and z is a regular boundary point of W and hence of V, we have

$$\lim_{x \to z} w(x) = \lim_{x \in V, x \to z} H_f^V(x) = f(z) = 0.$$

Clearly $w \ge h \ge u$ on W, the above displayed relation assures that $\lim_{x \to z} h(x) = 0$. $\square$

We denote by $HP(W)$ the space of every $u \in H(W)$ such that $|u|$ admits a harmonic majorant on W. Then we can consider the least harmonic majorant $u \vee v$ and the greatest harmonic minorant $u \wedge v$ of two functions u and v in $HP(W)$. With the usual linear structure as a subspace of $H(W)$ and lattice operations $\vee$ and $\wedge$, $HP(W)$ forms a vector lattice. By considering the positive part $u^+ := u \vee 0$ and the negative part $u^- := -(u \wedge 0)$, every $u \in HP(W)$ admits its Jordan decomposition $u = u^+ - u^-$. We will use the following result, which is a direct consequence of the above Lemma 3.1.

PROPOSITION 3.1. *Take a regular point $z \in \partial W$ and a $u \in HP(W)$. Then u has the vanishing boundary value at z if and only if both of u^+ and u^- have the vanishing boundary value at z.*

PROOF. Let h be the least harmonic majorant of the nonnegative subharmonic function $|u|$ on W. By Lemma 3.1, $|u|$ (and hence u) has the vanishing boundary value at z if and only if h has the vanishing boundary value at z. Since $u^{\pm} = (h \pm u)/2$, the desired conclusion follows. $\square$

Since $|H_f^W|$ of $H_f^W \in H_{ds}(W)$ admits a harmonic majorant $H_{|f|}^W$, we see that $H_{ds}(W) \subset HP(W)$. For two functions f and g we denote by $(f \cup g)(x) := \max(f(x), g(x))$ and $(f \cap g)(x) := \min(f(x), g(x))$. Then $L^1(\partial R, d\omega)$ forms a vector lattice with the usual linear operations and the lattice operations $\cup$ and $\cap$. One should not be confused by these $\cup$ and $\cap$ with $\vee$ and $\wedge$ used only for harmonic functions. Based upon the fact that $H_f^W \vee H_g^W = H_{f\cup g}^W$ and $H_f^W \wedge H_g^W = H_{f\cap g}^W$ on W, we see that $H_{ds}(W)$ is a vector sublattice of $HP(W)$ and moreover, by virtue of (2.1), we can conclude the following result.

PROPOSITION 3.2. *The spaces $H_{ds}(R)$ and $L^1(\partial R, d\omega)$ are isomorphic as vector lattices.*

4. Quasiboundedness.

A harmonic function u on a nonempty bounded open set W is *quasibounded* if $u \in HP(W)$ and

$$u = \lim_{\lambda \in \boldsymbol{R}, \lambda \uparrow \infty} (u \wedge \lambda) \vee (-\lambda) \tag{4.1}$$

almost uniformly on W, where $\boldsymbol{R} := \boldsymbol{R}^1$ is the field of real numbers. We denote by $H_{qb}(W)$ the space of every quasibounded harmonic function on W. Clearly (4.1) is equivalent to

$$u^{\pm} = \lim_{\lambda \in \boldsymbol{R}, \lambda \uparrow \infty} u^{\pm} \wedge \lambda, \tag{4.2}$$

where the convergence is almost uniform on W. Hence $H_{qb}(W)$ is a vector sublattice of $HP(W)$. If we denote by $HB(W)$ the space of bounded harmonic functions on W, then it is also a vector sublattice of $HP(W)$ and $HB(W) \subset H_{qb}(W) \subset HP(W)$.

Observe that $(H_f^W)^{\pm} = H_{f^{\pm}}^W$ on W for every resolutive function f on ∂W, where $f^+ := f \cup 0$ and $f^- := -(f \cap 0)$; $H_{f^{\pm}}^W \wedge \lambda = H_{f^{\pm}\cap\lambda}^W \uparrow H_{f^{\pm}}^W$ as $\lambda \uparrow \infty$. This means that

$$H_{ds}(W) \subset H_{qb}(W). \tag{4.3}$$

Let $X = B \setminus K$ with B the open unit ball $\boldsymbol{B}^d$ in $\boldsymbol{R}^d$ and $K = \{x = (x^1, \ldots, x^d) : x^1 \geq 0, x^d = 0\}$. We denote by B^+ the upper half ball, by π the plane $x^d = 0$, and by w the harmonic measure of $(\partial B^+ \setminus (\pi \setminus K)) \cap \pi$ on B^+. Then, by the reflection principle, w is extended to X across $(\pi \setminus K) \cap B$ as an antisymmetric function on X about $\pi \cap B$ and the cluster set of the extended w at each point of $(\pi \setminus \overline{(\pi \setminus K)}) \cap B$ is the two values set $\{1, -1\}$. Suppose the extended w on X is represented as $w = H_f^X$ with a resolutive boundary function f on $\partial X = \partial B \cup K$. Since $w = H_f^X$ has vanishing boundary values on $\partial B \setminus K$, by Proposition 2.1, we may assume that spt $f \subset K \cap \overline{B}$. Then $w = H_f^X$ must be symmetric about $\pi \cap B$ and the above cluster set has to be the one value set $\{1\}$. This contradiction shows that $w \in H_{qb}(X) \setminus H_{ds}(X)$ so that

$$H_{ds}(X) < H_{qb}(X), \tag{4.4}$$

where $<$ means the proper inclusion: $\subset$ and $\neq$. We are thus interested in finding conditions on R under which $H_{ds}(R) = H_{qb}(R)$ holds, and we are naturally led to give the following notion.

Definition 4.1. A bounded domain R in $\boldsymbol{R}^d$ is referred to as a *Dirichlet domain* if $H_{qb}(R) = H_{ds}(R)$.

In view of (4.3), the essence of the above condition is $H_{qb}(R) \subset H_{ds}(R)$. In checking whether a given R is a Dirichlet domain or not, the following criterion is convenient.

Lemma 4.1. *A domain R is a Dirichlet domain if and only if $HB(R)^+ \subset H_{ds}(R)$.*

Proof. We only have to show that $HB(R)^+ \subset H_{ds}(R)$ implies $H_{qb}(R) \subset H_{ds}(R)$. Since $H_{qb}(R)^+$ linearly generates $H_{qb}(R)$, it is sufficient to show $H_{qb}(R)^+ \subset H_{ds}(R)$ in order to conclude $H_{qb}(R) \subset H_{ds}(R)$. Take an arbitrary $u \in H_{qb}(R)^+$ and we are to show that $u \in H_{ds}(R)$. Note that $u = \lim_{i\uparrow\infty} u \wedge i$ on R. By $HB(R)^+ \subset H_{ds}(R)$, there is a resolutive function f_i on ∂R such that $u \wedge i = H^R_{f_i}$ on R for each $i = 1, 2, \ldots$. By Proposition 2.1, we may assume that the sequence $(f_i)_{i\geq 1}$ is increasing on ∂R. Hence we can define $f := \lim_{i\uparrow\infty} f_i$ on ∂R. In view of

$$\int_{\partial R} f_i d\omega = H^R_{f_i}(x_0) = (u \wedge i)(x_0) \leq u(x_0)$$

and the Lebesgue monotone convergence theorem, we see that

$$\int_{\partial R} f d\omega = \lim_{i\uparrow\infty} \int_{\partial R} f_i d\omega \leq u(x_0) < \infty$$

or $f \in L^1(\partial R, d\omega)$ so that f is resolutive on ∂R and H^R_f can be considered. Once again by the Lebesgue monotone convergence theorem we have

$$u = \lim_{i\uparrow\infty} u \wedge i = \lim_{i\uparrow\infty} H^R_{f_i} = \lim_{i\uparrow\infty} \int_{\partial R} f_i(y)P(y,\cdot)d\omega(y) = \int_{\partial R} f(y)P(y,\cdot)d\omega(y) = H^R_f,$$

i.e. $u = H^R_f$ on R so that $u \in H_{ds}(R)$, which is to be shown. $\square$

5. Localization.

For a bounded domain R of $\boldsymbol{R}^d$ $(d \geq 2)$ we take a regular domain U containing a point $y \in \partial R$. Since, in this section, we are mainly interested in the behavior of harmonic functions on $U \cap R$ at the part $U \cap \partial R$ of the boundary of $U \cap R$, eliminating the influence of the part $\overline{R} \cap \partial U$, we only consider those harmonic functions u on $U \cap R$ with vanishing boundary values on $\overline{R} \cap \partial U$. We denote by $H(U \cap R; \partial U)$ the class of such harmonic functions u. Suppose a $u \in H(U \cap R; \partial U)$ is represented as a Dirichlet solution $H^{U\cap R}_f$ on $U \cap R$ with resolutive boundary function f on $\partial(U \cap R)$. Then, by (2.1) applied to the restriction of f on each boundary of each component of $U \cap R$, f may be assumed to be zero on $\overline{R} \cap \partial U$ so that we may view that f is defined on $\boldsymbol{R}^d$ with $f = 0$ on $\boldsymbol{R}^d \setminus (U \cap \partial R)$. We denote by $H_{ds}(U \cap R; \partial U)$ the set of such Dirichlet solutions $H^{U\cap R}_f$ on $U \cap R$. Then we also have the identity $H_{ds}(U \cap R; \partial U) = H_{ds}(U \cap R) \cap H(U \cap R; \partial U)$. We denote by

$H_{qb}(U \cap R; \partial U) := H_{qb}(U \cap R) \cap H(U \cap R; \partial U)$. As a counterpart of (4.3) we easily see that

$$H_{ds}(U \cap R; \partial U) \subset H_{qb}(U \cap R; \partial U) \tag{5.1}$$

is valid. By using the example X in (4.4) and taking any small ball U with radius less than 1 centered at the origin $0 \in K \cap B \subset \partial X$, we can also conclude that

$$H_{ds}(U \cap X; \partial U) < H_{qb}(U \cap X; \partial U). \tag{5.2}$$

In view of these we are naturally led to localize the definition 4.1 as follows.

DEFINITION 5.1. A bounded domain R in $\boldsymbol{R}^d$ is said to be *locally a Dirichlet domain* at $y \in \partial R$ with respect to a regular domain U containing y if $H_{qb}(U \cap R; \partial U) = H_{ds}(U \cap R; \partial U)$.

By virtue of (5.1) the essence of the above condition is $H_{qb}(U \cap R; \partial U) \subset H_{ds}(U \cap R; \partial U)$. Let $HB(U \cap R; \partial U) := HB(U \cap R) \cap H(U \cap R; \partial U)$. As the counterpart of Lemma 4.1, we have the following criterion.

LEMMA 5.1. *A domain R is locally a Dirichlet domain at $y \in \partial R$ with respect to a regular domain U containing y if and only if $HB(U \cap R; \partial U)^+ \subset H_{ds}(U \cap R; \partial U)$.*

PROOF. We only have to show that $HB(U \cap R; \partial U)^+ \subset H_{ds}(U \cap R; \partial U)$ implies that $H_{qb}(U \cap R; \partial U) \subset H_{ds}(U \cap R; \partial U)$. Every point in $\overline{R} \cap \partial U$ is regular with respect to the open set $W := U \cap R$ because U is regular. One way to see this is to observe that the restriction to W of any barrier on U of any point z in $\overline{R} \cap \partial U$ is again a barrier on W of the point z. Hence, by Proposition 3.1, $H_{qb}(U \cap R; \partial U)^+$ linearly generates $H_{qb}(U \cap R; \partial U)$. Therefore it is sufficient to show $H_{qb}(U \cap R; \partial U)^+ \subset H_{ds}(U \cap R; \partial U)$ in order to conclude $H_{qb}(U \cap R; \partial U) \subset H_{ds}(U \cap R; \partial U)$. Take an arbitrary $u \in H_{qb}(U \cap R; \partial U)^+$ and we are to show that $u \in H_{ds}(U \cap R; \partial U)$. Note that $u = \lim_{i \uparrow \infty} u \wedge i$ on W. Since $0 \leq u \wedge i \leq u$ on W, we see that $u \wedge i \in HB(U \cap R; \partial U)^+$. By $HB(U \cap R; \partial U)^+ \subset H_{ds}(U \cap R; \partial U)$, there is a resolutive function f_i on ∂W such that $f_i | \overline{R} \cap \partial U = 0$ such that $u \wedge i = H_{f_i}^W$ on W for each $i = 1, 2, \ldots$. By Proposition 2.1 applied to each component of W, we may assume that the sequence $(f_i)_{i \geq 1}$ is increasing on ∂W. Hence we can define $f := \lim_{i \uparrow \infty} f_i$ on ∂W and $f | \overline{R} \cap \partial U = 0$. Arguing as in the proof of Lemma 4.1 upon $f | \partial W_j$ for each component W_j of W, we see that f is resolutive on ∂W and $u = H_f^W$ on W so that $u \in H_{ds}(U \cap R; \partial U)$, which is to be shown. □

6. Proof of Theorem 1.1.

We are ready to prove Theorem 1.1 stated in Section 1 Introduction. First we prove the first half of Theorem 1.1. Let R be a bounded domain, y a point in ∂R, and U a regular domain containing y. We assume that R is locally a Dirichlet domain at y with respect to U, or equivalently, by virtue of Lemma 5.1, that

$$HB(U \cap R; \partial U)^+ \subset H_{ds}(U \cap R; \partial U). \tag{6.1}$$

Take any regular domain V such that $y \in V \subset U$. We are to show that R is locally a Dirichlet domain at y with respect to V, or equivalently, again in view of Lemma 5.1, that

$$HB(V \cap R; \partial V)^+ \subset H_{ds}(V \cap R; \partial V). \tag{6.2}$$

In order to show (6.2) we take an arbitrary $v \in HB(V \cap R; \partial V)^+$. In the sequel we always consider v to be defined on $U \cap R$ by setting $v \equiv 0$ on $(U \setminus V) \cap R$. Since the original v has vanishing boundary values on $\overline{R} \cap \partial V$, we see that v is a nonnegative subharmonic function on $U \cap R$. Let u be the least harmonic majorant of v on $U \cap R$. By Lemma 3.1, u has vanishing boundary values on the set of regular points in $\partial(U \cap R) \setminus V$ along with v so that in particular u has vanishing boundary values on $\overline{R} \cap \partial U$. Hence we see that $u \in HB(U \cap R; \partial U)^+$ and a fortiori there exists a function f on $\boldsymbol{R}^d$ with spt $f \subset \overline{U} \cap \partial R$ such that $f|\partial(U \cap R)$ is resolutive and

$$u = H_f^{U \cap R} \tag{6.3}$$

on $U \cap R$. Since u has vanishing boundary values on the set of regular points in $(\overline{U} \setminus V) \cap \partial R$ and u is nonnegative and bounded on $U \cap R$, by applying (2.1) componentwise to $U \cap R$, we may assume that spt $f \subset \overline{V} \cap \partial R$ and f is bounded on $\boldsymbol{R}^d$. Then, as a bounded Borel function, $f|\partial(V \cap R)$ is resolutive and $H_f^{V \cap R}$ can be defined. We extend $H_f^{V \cap R}$ to $U \cap R$ by setting $H_f^{V \cap R} \equiv 0$ on $(U \setminus V) \cap R$. Since $f|R \cap \partial V = 0$ and every point in $R \cap \partial V$ is regular, $H_f^{V \cap R}$ thus extended is subharmonic on $U \cap R$. Then by Lemma 2.1 we see that

$$H_f^{U \cap R} - H_f^{V \cap R} \in \mathscr{P}(U \cap R). \tag{6.4}$$

As u is the least harmonic majorant of v on $U \cap R$ we have

$$u - v \in \mathscr{P}(U \cap R). \tag{6.5}$$

Since $v - H_f^{V \cap R} = (v - u) + (u - H_f^{U \cap R}) + (H_f^{U \cap R} - H_f^{V \cap R})$ on $U \cap R$, we see by (6.3) that

$$\left|v - H_f^{V \cap R}\right| \leq (u - v) + \left(H_f^{U \cap R} - H_f^{V \cap R}\right),$$

i.e. by (6.4) and (6.5) the subharmonic function $|v - H_f^{V \cap R}|$ on $U \cap R$ is dominated by the potential $(u - v) + (H_f^{U \cap R} - H_f^{V \cap R})$ on $U \cap R$. Hence $|v - H_f^{V \cap R}| \equiv 0$ on $U \cap R$ and in particular $v = H_f^{U \cap R}$ on $V \cap R$. In view of the boundary behavior of v on $\overline{R} \cap \partial V$, we conclude that $v \in H_{ds}(V \cap R; \partial V)$, i.e. (6.2) has been thus established.

To prove the last half of Theorem 1.1, we assume that R is a Dirichlet domain and take an arbitrary regular domain V containing an arbitrarily given point y in ∂R. Let U be any regular domain such that $U \supset \overline{R} \cup \overline{V}$. Then $U \cap R = R$ is locally a Dirichlet domain at $y \in U \cap \partial R$ with respect to U if and only if R is a Dirichlet domain. Hence in view of $y \in V \subset U$ we see that R is locally a Dirichlet domain at y with respect to V, which is to be shown. □

7. Wiener potentials.

Consider a real valued function f on a nonempty bounded open set W in $\boldsymbol{R}^d$. We denote by $\overline{\mathscr{W}}_f^W$ the class of functions $s \in S(W)$ having compact subsets $K_s \subset W$ such that $s \geq f$ on $W \setminus K_s$. We denote by $\overline{h}_f^W$ the lower envelope of the class $\overline{\mathscr{W}}_f^W$ and set $\underline{h}_f^W := -\overline{h}_{-f}^W$. The functions $\overline{h}_f^W$ and $\underline{h}_f^W$ are either harmonic or identically $\pm\infty$ on each component of W and satisfy $\overline{h}_f^W \geq \underline{h}_f^W$ on W. If $\overline{h}_f^W = \underline{h}_f^W$ on W and the common function is harmonic on W, then we say that f is *harmonizable* on W and we denote the common function by h_f^W (cf. [**2**, pp. 54–55]).

We denote by $\mathscr{W}(W)$ the class of bounded continuous functions f on W which are harmonizable on every open subset of W. Each function in $\mathscr{W}(W)$ is said to be a (bounded continuous) *Wiener function* on W and $\mathscr{W}(W)$ is referred to as the *Wiener algebra* on W. It is known that $\mathscr{W}(W)$ forms a Banach algebra having the multiplicative identity 1 equipped with the supremum norm and also $\mathscr{W}(W)$ is a vector lattice under the lattice operations $\cup$ and $\cap$ (cf. [**7**, pp. 223–227]). We once more stress that $\mathscr{W}(W)$ is closed under multiplication and complete with respect to the uniform convergence. The operator $f \mapsto h_f^W : \mathscr{W}(W) \to HB(W)$ is seen to be a homomorphism as vector lattices. Thus we have e.g. $h_{f\cap g}^W = h_f^W \wedge h_g^W$ on W.

We denote by $C_b(W)$ the class of bounded continuous functions on W. It is not difficult to see that any function f in $S(W) \cap C_b(W)$ or in $C(\overline{W})$ is harmonizable on any open subset V of W. In fact, the proof for the former case is found in [**6**]. In the latter case, let $f \in C(\overline{W})$ and U be a bounded open set with $\overline{W} \subset U$. As is well known there are sequences $(s_i')_{i\geq 1}$ and $(s_i'')_{i\geq 1}$ in $S(U) \cap C(U)$ such that $(s_i)_{i\geq 1}$ with $s_i = s_i' - s_i''$ converges uniformly to f on $\overline{W}$. By the above, s_i' and s_i'' and hence s_i belong to $\mathscr{W}(W)$. By the completeness of $\mathscr{W}(W)$ in the uniform convergence we conclude that $f \in \mathscr{W}(W)$. Hence we have the following relation:

$$(S(W) \cap C_b(W)) \cup C(\overline{W}) \subset \mathscr{W}(W). \tag{7.1}$$

The above relation (7.1) implies the following (see [**4**]): if $u \in HB(W)$ and $\varphi \in C_0^\infty(\boldsymbol{R}^d)$, then $\varphi u \in \mathscr{W}(W)$.

A function $f \in \mathscr{W}(W)$ is said to be a (bounded continuous) *Wiener potential* on W if $h_f^W = 0$ on W. We denote by $\mathscr{W}_0(W)$ the totality of Wiener potentials $f \in \mathscr{W}(W)$. A necessary and sufficient condition for an $f \in \mathscr{W}(W)$ to belong to $\mathscr{W}_0(W)$ is that there is a potential p on W such that $|f| \leq p$ on W (cf. [**2**, p. 56]). The set $\mathscr{W}_0(W)$ forms a closed (with respect to the uniform convergence on W) ideal of $\mathscr{W}(W)$ (i.e. if $\varphi \in \mathscr{W}_0(W)$ and $f \in \mathscr{W}(W)$, then $\varphi f \in \mathscr{W}_0(W)$) and at the same time a vector sublattice of $\mathscr{W}(W)$. The direct sum decomposition

$$\mathscr{W}(W) = HB(W) \oplus \mathscr{W}_0(W) \tag{7.2}$$

is referred to as the *Wiener decomposition*: any $f \in \mathscr{W}(W)$ is uniquely decomposed as the sum $u + \varphi$ ($u \in HB(W), \varphi \in \mathscr{W}_0(W)$). Here u is nothing but $u = h_f^W$. We use this in the following fashion: if u and v are in $HB(W)$ and $q \in \mathscr{W}_0(W)$ such that $u = v + q$ on W, then $u = v$ on W. We will use the following result.

Lemma 7.1 (Constantinescu-Cornea [**2**, p. 64]). *Let $f \in \mathscr{W}(W)$ and W' be an open subset of W such that $f|W' \in \mathscr{W}_0(W')$ and $f|W \setminus W' = 0$. Then $f \in \mathscr{W}_0(W)$.*

Let V be an open subset of W. We say that V is *regular in* W if every point in $W \cap \partial V$ is regular with respect to V. Take a regular open subset V in a bounded open set W and a function $u \in \mathscr{W}(W)$. We denote by H_u^V the Dirichlet solution H_f^V on V, where f is the boundary function on ∂V such that $f = u$ on $W \cap \partial V$ and $f = 0$ on $(\partial V) \setminus W$. Since V is regular in W, the boundary values of $H_u^V := H_f^V$ on $W \cap \partial V$ is u and therefore if we extend H_u^V to W by setting $H_u^V = u$ on $W \setminus V$, then $H_u^V \in C_b(W)$. Actually we know that $H_u^V \in \mathscr{W}(W)$ for any $u \in \mathscr{W}(W)$ and for any regular open set V in W (cf. [**2**, p. 57]). As a corresponding assertion for h_u^V we have the following result.

Lemma 7.2. *Let V be a regular open subset in a bounded open set W. If $u \in \mathscr{W}(W)$, then $h_u^V \in \mathscr{W}(W)$, where h_u^V is extended to W by setting $h_u^V = u$ on $W \setminus V$.*

Proof. Set $v := u - H_u^V$. Then $v \in \mathscr{W}(W)$ and $v|W \setminus V = 0$ since $H_u^V \in \mathscr{W}(W)$ and $H_u^V = u$ on $W \setminus V$. Let χ be 0 on $W \cap \partial V$ and $\chi = \sup_W |v|$ on $(\partial V) \setminus W$. Take an arbitrary $s \in \overline{\mathscr{V}}_\chi^V$ and an arbitrary positive number $\varepsilon > 0$. Then clearly $s + \varepsilon \geq v$ on V except for a compact subset of V so that $s + \varepsilon \in \overline{\mathscr{W}}_v^V$. Hence $s + \varepsilon \geq h_v^V$ on V. Thus $H_\chi^V + \varepsilon \geq h_v^V$ and thus, on letting $\varepsilon \downarrow 0$, $H_\chi^V \geq h_v^V$ on V. Similarly, $H_\chi^V \geq h_{-v}^V$ on V and finally $|h_v^V| \leq H_\chi^V$. Since every point in $W \cap \partial V$ is regular and $\chi = 0$ there continuously, we see that h_v^V on V has boundary values zero on $W \cap \partial V$. Hence $h_v^V \in C_b(W)$ if h_v^V is extended to W by setting $h_v^V = 0$ on $W \setminus V$. Let h be the least harmonic majorant of $|h_v^V|$ on V. By Lemma 3.1, h has vanishing boundary values on $W \cap \partial V$. If we extend h to W by setting $h = 0$ on $W \setminus V$, then h is subharmonic on W and hence, by (7.1), $h \in \mathscr{W}(W)$. Similarly $h - h_v^V \in H(V)^+$ and $h - h_v^V = 0$ on $W \setminus V$ imply that $h - h_v^V$ is subharmonic on W so that $h - h_v^V \in \mathscr{W}(W)$. Hence $h_v^V = h - (h - h_v^V) \in \mathscr{W}(W)$. From $v = u - H_u^V$ it follows that $h_v^V = h_u^V - H_u^V$ or $h_u^V = h_v^V + H_u^V$ so that $h_u^V \in \mathscr{W}(W)$. $\square$

8. Proof of Theorem 1.2.

We prove here Theorem 1.2 stated in Section 1 Introduction. By the assumption of Theorem 1.2, there exists a subset E of ∂R with harmonic measure zero relative to R such that for every $y \in \partial R \setminus E$ there is a regular domain U_y containing y such that R is locally a Dirichlet domain at y with respect to U_y. We fix an arbitrary positive integer m. Since the harmonic measure $d\omega = d\omega_{x_0}^R$ $(x_0 \in R)$ on ∂R is regular and $\int_E d\omega = 0$, there is an open set G in $\boldsymbol{R}^d$ such that $G \supset E$ and $\int_{G \cap \partial R} d\omega < 1/m$. Then

$$w(x) := \int_{G \cap \partial R} P(x, y) d\omega(y) \leq \frac{c(x)}{m} \quad (x \in R), \tag{8.1}$$

where $P(x, \cdot) d\omega = d\omega_x^R$ and $c(x)$ is the Harnack constant for $\{x, x_0\}$ with respect to the domain R. The function $w(x)$ is the harmonic measure of $G \cap \partial R$ on R evaluated at $x \in R$ and we also have the expression $w = H_{\chi_{G \cap \partial R}}^R$, where $\chi_{G \cap \partial R}$ is the characteristic function on $\boldsymbol{R}^d$ of the set $G \cap \partial R$.

Since $\partial R \setminus G$ is compact, we can easily find a finite system of points $y_1, \ldots, y_n \in$

$\partial R\backslash G$ and a finite system of open sets $U_0, U_1, \ldots, U_n, U_{n+1}$ in $\boldsymbol{R}^d$ satisfying the following 7 conditions: $U_1, \ldots, U_n$ are regular domains and $y_i \in U_i$ $(1 \le i \le n)$; R is locally a Dirichlet domain at y_i with respect to U_i $(1 \le i \le n)$; $U_{n+1} = G$; $\overline{U}_0 \subset R$; $\bigcup_{1\le i\le n+1} U_i \supset \partial R$; $\bigcup_{0\le i\le n+1} U_i \supset \overline{R}$; there is a partition $\{\varphi_i\}_{0\le i\le n+1}$ of unity by functions $\varphi_i \in C_0^\infty(\boldsymbol{R}^d)$ with $0 \le \varphi_i \le 1$ on $\boldsymbol{R}^d$ such that the support of φ_i spt $\varphi_i \subset U_i$ $(0 \le i \le n+1)$ and $\sum_{0\le i\le n+1} \varphi_i = 1$ on $\overline{R}$. For simplicity we set $V_i = R \cap U_i$ $(1 \le i \le n)$.

To show that R is a Dirichlet domain, we take an arbitrary $u \in HB(R)^+$. By Lemma 4.1, we only have to show that $u \in H_{ds}(R)$: we are to find a resolutive function f on ∂R such that $u = H_f^R$ on R. Without loss of generality we can assume that $0 \le u \le 1$ on R. Observe that

$$u = 1 \cdot u = \Bigg(\sum_{0\le i\le n+1} \varphi_i\Bigg) u = \sum_{0\le i\le n+1} \varphi_i u$$

on R. On setting $u_i := \varphi_i u$, we see by the remark right after (7.1) that $u_i \in \mathscr{W}(R)$ $(0 \le i \le n+1)$ so that we have the decomposition

$$u = \sum_{0\le i\le n+1} u_i \qquad (u_i \in \mathscr{W}(R),\ \text{spt } u_i \subset V_i). \tag{8.2}$$

Fix an arbitrary $1 \le i \le n$ and, for simplicity, we set $v := u_i$, $U := U_i$, $V := V_i := R\cap U_i$ for the time being. Take a regular domain $U' = U_i'$ such that $y_i \in U'$, $\overline{U'} \subset U$, and spt $v \subset U'$. Let $\chi := \chi_{U'\cap\partial V}$ be the characteristic function on $\boldsymbol{R}^d$ of $U'\cap\partial V$. Let $s \in \overline{\mathscr{V}}_\chi^V$ be arbitrary and choose any positive number $\varepsilon > 0$. Clearly $s + \varepsilon \ge v$ on V except for a compact subset of V. Then $s + \varepsilon \in \overline{\mathscr{W}}_v^V$ and hence $s + \varepsilon \ge h_v^V$ or $H_\chi^V + \varepsilon \ge h_v^V$. By letting $\varepsilon \downarrow 0$ we obtain on V that

$$0 \le h_v^V \le H_\chi^V. \tag{8.3}$$

Since every point in $R \cap \partial U$ is regular and $\chi = 0$ there, H_χ^V has zero boundary values on $R \cap \partial U$, so does h_v^V as a consequence of (8.3). Extend h_v^V to R by setting $h_v^V = 0$ on $R \setminus V$. Then h_v^V is subharmonic on R and thus, by (7.1), $h_v^V \in \mathscr{W}(R)$. This also follows directly from Lemma 7.2. Since $v \in \mathscr{W}(R)$, we see that $v - h_v^V \in \mathscr{W}(R)$. On the other hand, $v - h_v^V \in \mathscr{W}_0(V)$ and $v - h_v^V = 0$ on $R \setminus V$. By Lemma 7.1, we see that $v - h_v^V \in \mathscr{W}_0(R)$. Then with the trivial relation $v - h_v^R \in \mathscr{W}_0(R)$, we deduce that

$$h_v^R - h_v^V = \big(v - h_v^V\big) - \big(v - h_v^R\big) \in \mathscr{W}_0(R).$$

Since this is true for every i, we can conclude that

$$h_{u_i}^R - h_{u_i}^{V_i} \in \mathscr{W}_0(R) \qquad (1 \le i \le n). \tag{8.4}$$

In view of the assumption that R is locally a Dirichlet domain at y_i with respect to $U = U_i$, there exists a resolutive Borel function $g = g_i$ such that $h_v^V = H_g^V$, i.e.

$$h_{u_i}^{V_i} = H_{g_i}^{V_i} \qquad (1 \le i \le n). \tag{8.5}$$

By (8.3) we see that $0 \le H_g^V \le H_\chi^V$ on V. Hence by Proposition 2.1 applied to each component of V we may assume that

$$0 \le g_i \le \chi_{U_i' \cap \partial V_i} \tag{8.6}$$

on ∂V_i $(1 \le i \le n)$. Hence we can view $g = g_i$ is a bounded Borel function defined on $\boldsymbol{R}^d$ with spt $g \subset U \cap \partial R$. Hence $g|\partial R$ is resolutive on ∂R and H_g^R can be considered. Since every point in $\partial R \cap \partial U$ is regular and g is continuous on $\partial R \cap \partial U$ and $g = 0$ there, H_g^R has the vanishing boundary values on $\partial R \cap \partial U$. By Lemma 2.1, $H_g^R - H_g^V \in \mathscr{P}(R)$, so that we have deduced

$$H_{g_i}^R - H_{g_i}^{V_i} \in \mathscr{W}_0(R) \qquad (1 \le i \le n). \tag{8.7}$$

We next examine $h_{u_0}^R$ and $h_{u_{n+1}}^R$. Choose any $s \in \overline{\mathscr{V}}_0^R$ and any positive number $\varepsilon > 0$. Since $s + \varepsilon \ge u_0$ on $R \setminus \overline{U}_0$ and hence $s + \varepsilon \in \overline{\mathscr{W}}_{u_0}^R$ so that $s + \varepsilon \ge h_{u_0}^R \ge 0$. Then $H_0^R + \varepsilon \ge h_{u_0}^R \ge 0$ and on letting $\varepsilon \downarrow 0$ we conclude that

$$h_{u_0}^R \equiv 0 \tag{8.8}$$

on R. Next, let $\gamma := \chi_{U_{n+1} \cap \partial R} = \chi_{G \cap \partial R}$, the characteristic function on $\boldsymbol{R}^d$ of $G \cap \partial R$. Choose any $s \in \overline{\mathscr{V}}_\gamma^R$ and any positive number $\varepsilon > 0$. We see that $s + \varepsilon \ge u_{n+1}$ on R except for a compact subset of R so that $s + \varepsilon \in \overline{\mathscr{W}}_{u_{n+1}}^R$. Then $s + \varepsilon \ge h_{u_{n+1}}^R \ge 0$ and hence $H_\gamma^R + \varepsilon \ge h_{u_{n+1}}^R$ on R. On letting $\varepsilon \downarrow 0$, we deduce $H_\gamma^R \ge h_{u_{n+1}}^R$ on R. Since $H_\gamma^R = H_{\chi_{G \cap \partial R}}^R = w$, we conclude that

$$h_{u_{n+1}}^R(x) \le w(x) \qquad (x \in R). \tag{8.9}$$

We consider the function $\varphi \in \mathscr{W}_0(R)$ and the bounded Borel function f_m on $\boldsymbol{R}^d$ given by

$$\varphi := \sum_{1 \le i \le n} \left(h_{u_i}^R - h_{u_i}^{V_i}\right) + \sum_{1 \le i \le n} \left(H_{g_i}^{V_i} - H_{g_i}^R\right), \tag{8.10}$$

which is certainly in $\mathscr{W}_0(R)$ by (8.4) and (8.7), and

$$f_m := \sum_{1 \le i \le n} g_i, \tag{8.11}$$

which is bounded and Borel as desired by the choice of g_i and (8.6) $(1 \le i \le n)$. By (8.2), (8.8), and (8.5) we see that

$$u = h_u^R = \sum_{0\le i\le n+1} h_{u_i}^R = h_{u_0}^R + \sum_{1\le i\le n} h_{u_i}^R + h_{u_{n+1}}^R = \sum_{1\le i\le n} h_{u_i}^R + h_{u_{n+1}}^R$$
$$= \sum_{1\le i\le n} H_{g_i}^R + \sum_{1\le i\le n} \left(H_{g_i}^{V_i} - H_{g_i}^R\right) + \sum_{1\le i\le n} \left(h_{u_i}^R - h_{u_i}^{V_i}\right) + h_{u_{n+1}}^R$$

on R. This with (8.10) and (8.11) implies that $u = H_{f_m}^R + h_{u_{n+1}}^R + \varphi$ on R. By (7.2): the uniqueness of the Wiener decomposition, we conclude that

$$u = H_{f_m}^R + h_{u_{n+1}}^R \tag{8.12}$$

on R. This in particular implies that $0 \le H_{f_m}^R \le u \le 1$, or $0 \le H_{f_m}^R \le 1$, on R and by Proposition 2.1

$$0 \le f_m \le 1 \qquad (m = 1, 2, \ldots) \tag{8.13}$$

$d\omega$-a.e. on ∂R or as functions in $L^\infty(\partial R, d\omega)$.

By (8.12) and (8.9), we see that $0 \le u - H_{f_m}^R = h_{u_{n+1}}^R \le w$ on R and a fortiori, by (8.1), we deduce

$$0 \le u(x) - H_{f_m}^R(x) \le \frac{c(x)}{m} \qquad (x \in R;\ m = 1, 2, \ldots). \tag{8.14}$$

By (8.13) the sequence $(f_m)_{m\ge1}$ is contained in the unit ball of the Banach space $L^\infty(\partial R, d\omega) = (L^1(\partial R, d\omega))^*$, the dual space of $L^1(\partial R, d\omega)$. By the Alaoglu theorem, there is a net $(f_{m(\iota)})_\iota$ and an $f \in L^\infty(\partial R, d\omega)$ such that $m(\iota)$ are positive integers with $m(\iota) \to \infty$ as integers and $f_{m(\iota)} \to f$ in the weak * topology in $L^\infty(\partial R, d\omega)$. Hence, since $P(x, \cdot) \in L^1(\partial R, d\omega)$ (and in reality $P(x, \cdot) \in L^\infty(\partial R, d\omega)$) (cf. Section 2), we see that

$$H_{f_{m(\iota)}}^R(x) = \int_{\partial R} P(x, y) f_{m(\iota)}(y) d\omega(y) \to \int_{\partial R} P(x, y) f(y) d\omega(y) = H_f^R(x)$$

for every $x \in R$. By (8.14) we have

$$0 \le u(x) - H_{f_{m(\iota)}}^R(x) \le \frac{c(x)}{m(\iota)},$$

and, on taking the limit in the above, we see that $u(x) = H_f^R(x)$ $(x \in R)$, i.e. $u = H_f^R$ on R.

The proof of Theorem 1.2 is herewith complete. □

9. Graphic points.

We call a coordinate system on $\boldsymbol{R}^d$ a *Cartesian coordinate* (*polar coordinate*, resp.) if it is obtained from the standard Cartesian coordinate (i.e. orthogonal coordinate)

$x = (x^1, \dots, x^d) = (x', x^d)$ (the standard polar coordinate $x = r\xi = (r, \xi)$ with $r \geq 0$ and $|\xi| = 1$, resp.) by translation and rotation on $\boldsymbol{R}^d$.

For a Cartesian coordinate $x = (x^1, \dots, x^d) = (x', x^d)$ we use the following notation: for any positive number $a > 0$ we set

$$\beta(a) = \left\{x' \in \boldsymbol{R}^{d-1} : |x'| = \sqrt{(x^1)^2 + \cdots + (x^{d-1})^2} < a\right\}$$

and, for any $a, b > 0$ and any real number c we consider a cylinder

$$U(c; a, b) := \{x = (x', x^d) : x' \in \beta(a), |x^d - c| < b\}.$$

The point with coordinate $(0', c)$ will be referred to as the *center* of the cylinder $U(c; a, b)$, where $0' = (0, \dots, 0)$ is the origin of $\boldsymbol{R}^{d-1}$ in this coordinate.

For a polar coordinate $x = r\xi = (r, \xi)$ we use the similar notation as above: for any $a \in (0, 2)$ we set

$$\beta(a) := \{\xi \in S^{d-1} : |\xi - e_1| < a\}$$

with $S^{d-1} = \{\xi \in \boldsymbol{R}^d : |\xi| = 1\}$ and $e_1 = (1, 0, \dots, 0) \in S^{d-1}$ and, for any $0 < a < 2$ and $c > 0$ and $b \in (0, c)$ we consider a sectorial ring

$$U(c; a, b) := \{x = r\xi : \xi \in \beta(a), |r - c| < b\},$$

for which the point with coordinate ce_1 is referred to as its *center*.

Consider a bounded domain R in $\boldsymbol{R}^d$. A point $p \in \partial R$ is said to be a *Cartesian graphic point* for R if there exist a Cartesian coordinate $x = (x', x^d)$ for which the coordinate of p is $(0', c)$ $(c > 0)$ and there exist two positive numbers a and b and a continuous function $x^d = \varphi(x')$ defined on $\beta(a)$ such that $\sup_{\beta(a)} |\varphi - c| < b$ and

$$U(c; a, b) \cap R = \{x = (x', x^d) : x' \in \beta(a), c - b < x^d < \varphi(x')\}$$

and at the same time we have

$$U(c; a, b) \cap \partial R = \{x = (x', x^d) : x' \in \beta(a), x^d = \varphi(x')\}.$$

In this case the neighborhood $U(c; a, b)$ of p is said to be an *admissible neighborhood* of p and the function $x^d = \varphi(x')$ is called the *local representing function* of R (or ∂R) associated with $U(c; a, b)$.

A point $p \in \partial R$ is said to be a *polar graphic point* for R if there exist a polar coordinate $x = r\xi$ for which the coordinate of p is ce_1 $(c > 0)$ and there exist two numbers $0 < a < 2$ and $0 < b < c$ and a continuous function $r = \varphi(\xi) > 0$ defined on $\beta(a)$ such that $\sup_{\beta(a)} |\varphi - c| < b$ and

$$U(c; a, b) \cap R = \{x = r\xi : \xi \in \beta(a), c - b < r < \varphi(\xi)\}$$

and in this case it holds that

$$U(c;a,b)\cap\partial R=\{x=r\xi:\xi\in\beta(a),r=\varphi(\xi)\}.$$

In such a case the neighborhood $U(c;a,b)$ of p is referred to as an *admissible neighborhood* of p and the function $r=\varphi(\xi)$ is said to be the *local representing function* of R (or ∂R) associated with $U(c;a,b)$.

In either of the above two parallel definitions of graphic points we assumed that φ is continuous. However this is not a thing to be assumed but the consequence of the very definitions: if we simply assume that φ is merely a single valued function in either of the above two definitions, then we can prove that φ must be automatically continuous. It should also be noted that there are a lot of examples of R and $p\in\partial R$ such that p is a Cartesian (polar, resp.) graphic point but not polar (Cartesian, resp.) graphic point; that there exist many examples of R such that ∂R contains both of Cartesian graphic points and polar graphic points. We state here an open problem. It may be of some interest in connection with Theorem 1.3 to resolve the question, one step further than the above remark, whether there exists a domain R such that ∂R contains a subset of positive harmonic measure consisting of points which are Cartesian (polar, resp.) graphic points but not polar (Cartesian, resp.) graphic points.

Finally a point p in ∂R is referred to simply as a *graphic point* for R if p is either a Cartesian graphic point or polar graphic point. A point $q\in\partial R$ is said to be a *nongraphic point* if q is not a graphic point. We denote by $E=E_R$ the set of nongraphic points in ∂R. If $p\in\partial R$ is a graphic point and $U(c;a,b)$ is its admissible neighborhood, then every point in $U(c;a,b)\cap\partial R$ is a graphic point so that $\partial R\setminus E$ is open and E is closed (compact) in ∂R. A bounded domain R with $E=\varnothing$ is called a *continuous domain* (cf. e.g. [**4**]). Convex domains, star shaped domains, C^1 domains or more generally Lipschitz domains are examples of continuous domains.

At the end of this section we add an explanation why we consider graphic points both in the Cartesian and also in the polar coordinate. Let u be a harmonic function on the upper half space $\boldsymbol{R}^d_+:=\{(x',x_d):x_d>0\}$ (the unit ball $\boldsymbol{B}^d:=\{|x|<1\}$, resp.) in $\boldsymbol{R}^d$, i.e. $u\in H(\boldsymbol{R}^d_+)$ ($u\in H(\boldsymbol{B}^d)$, resp.). The function

$$x=(x',x_d)\mapsto u_c(x):=u(x',x_d+c)\ (c>0)\quad (x\mapsto u_r(x):=u(rx)\ (r<1),\ \text{resp.})$$

is also harmonic on the closure of $\boldsymbol{R}^d_+$ ($\boldsymbol{B}^d$, resp.) based on the translation (dilation, resp.) invariance of harmonicity. If

$$u\mid\partial\boldsymbol{R}^d_+=\lim_{c\downarrow 0}u_c\mid\partial\boldsymbol{R}^d_+\quad (u\mid\partial\boldsymbol{B}^d=\lim_{r\uparrow 1}u_r\mid\partial\boldsymbol{B}^d,\text{resp.})$$

can be defined in some sense, then, by using u_c (u_r, resp.) in place of u, u may be treated as if it is continuous on the closure of $\boldsymbol{R}^d_+$ ($\boldsymbol{B}^d$, resp.). This trick is often used effectively and conveniently. The intention we introduce Cartesian (polar, resp.) graphic points is to make the localized version of the above technique corresponding to the translation (dilation, resp.) method applicable. For this reason we naturally need to consider graphic points not only in the Cartesian coordinate but also in the polar coordinate. In connection with the above explanation one might suspect that subdomains of $\boldsymbol{R}^{m+n}$ of the product form $X \times Y$ with X Cartesian Dirichlet domains in $\boldsymbol{R}^m$ and with Y polar Dirichlet domains in $\boldsymbol{R}^n$ are also worth to consider. However the harmonicity of $u(x,y)$ with respect to $(x,y) \in X \times Y$ has in general nothing to do with the harmonicity of $x \mapsto u(x,y)$ on X and also that of $y \mapsto u(x,y)$ on Y not like the case of analytic functions of several complex variables. Because of this the above expectation seems to be hopeless.

10. Proof of Theorem 1.3.

In this final section we prove Theorem 1.3 stated in Section 1 Introduction as an application of Theorem 1.2 also stated in the same place. We take as before a bounded domain R in $\boldsymbol{R}^d$. The essence of the proof lies in the following result.

Lemma 10.1. *If $p \in \partial R$ is a graphic point and $U := U(c;a,b)$ is any admissible neighborhood of p (which is of course a regular domain), then R is locally a Dirichlet domain at p with respect to U.*

Proof. There is a $\gamma \in (0,b)$ such that even $U(c;a,b-\gamma)$ is admissible, which we fix throughout the proof. According to whether p is a Cartesian or polar graphic point, the coordinate of the center p of $U = U(c;a,b)$ is $(0',c)$ in the case of Cartesian coordinate or ce_1 in the case of polar coordinate. In view of Lemma 5.1, we only have to show that $HB(U \cap R; \partial U)^+ \subset H_{ds}(U \cap R; \partial U)$, i.e. for any $v \in HB(U \cap R; \partial U)^+$ we only have to find a resolutive function f on $\partial(U \cap R)$ such that $v = H_f^{U \cap R}$ on $U \cap R$. Here without loss of generality we may assume that $0 \le v \le 1$ on $U \cap R$. For simplicity we set $V := U \cap R$, which is a domain. We define V_i and v_i for $i \ge i_0$ below separately according as p is a Cartesian graphic point or p is a polar graphic point.

In the case p is a Cartesian graphic point we set $i_0 := 1 + [1/\gamma]$, where $[\ \cdot\ ]$ is the Gaussian symbol. For each $i \ge i_0$ we set $V_i := \{x + (1/i)e_d : x \in V\}$, where $e_d = (0,\dots,0,1) \in \boldsymbol{R}^d$. Consider a function $v_i \in H(V_i \cap R) \cap C(\overline{V})$ given by $v_i(x) := v(x - (1/i)e_d)$ for $x \in V_i \cap \overline{R}$ and $v_i(x) = 0$ for $x \in \overline{V} \setminus V_i$. In the case p is a polar graphic point we set $i_0 := 1 + [(c-b)/\gamma]$. For each $i \ge i_0$ we set $V_i := \{(1+(1/i))x : x \in V\}$. Consider a function $v_i \in H(V_i \cap R) \cap C(\overline{V})$ given by $v_i(x) := v((1+(1/i))^{-1}x)$ for $x \in V_i \cap \overline{R}$ and $v_i(x) = 0$ for $\overline{V} \setminus V_i$. In either case we set $w_i := H_{f_i}^V$, where $f_i := v_i|\partial V \in C(\partial V)$ and satisfies $0 \le f_i \le 1$ on ∂V.

We denote by $d\omega_x^V$ the harmonic measure on ∂V evaluated at $x \in V$; fixing an arbitrary point $x_0 \in V$ we set $d\omega = d\omega_{x_0}^V$; there is a function $P(\cdot,x) \in L^\infty(\partial V, d\omega)$ such that $d\omega_x^V = P(\cdot,x)d\omega$ (cf. Section 2). We denote, as in the last part of Section 8, by $(L^1(\partial V, d\omega))^*$ the dual space of $L^1(\partial V, d\omega)$, which is nothing but $L^\infty(\partial V, d\omega)$. By the Alaoglu theorem, the closed unit ball B in $L^\infty(\partial V, d\omega) = (L^1(\partial V, d\omega))^*$ is weakly * compact, which is characterized by the fact that every generalized sequence in B has a cluster point in B. Hence the particular generalized sequence $(f_i)_{i \ge i_0}$ in B has a cluster point f in B. We will show that

$$v(x) = H_f^V(x) \qquad (x \in V). \tag{10.1}$$

For this purpose we only have to show that

$$\left|v(x) - H_f^V(x)\right| < \varepsilon \tag{10.2}$$

for every positive number $\varepsilon > 0$ and any point $x \in V$ chosen arbitrarily and then fixed in advance.

If $i \geq i_0$ is sufficiently large, then $x \in V_i \cap R$ and hence $v_i(x) - v(x) = v(x-(1/i)e_d) - v(x)$ or $v((1+(1/i))^{-1}x) - v(x)$, according as p is a Cartesian or polar graphic point, tends to zero as $i \uparrow \infty$ by the continuity of v on V. Therefore there is a number $j_1 > i_0$ such that

$$|v_i(x) - v(x)| < \frac{\varepsilon}{3} \qquad (i \geq j_1). \tag{10.3}$$

We take an auxiliary function $w := H_\varphi^V$, where the boundary function φ on ∂V is given by $\varphi|R \cap \partial V = 0$ and $\varphi|\overline{V} \cap \partial R = 1$. Then w has boundary values zero on $R \cap \partial V$ and one at every regular point in $\partial V \setminus \overline{R \cap \partial V}$. Clearly $0 \leq w \leq 1$ on V. Choose a sufficiently large ball $B(p,\rho) := \{x : |x-p| < \rho\} \supset \overline{R}$ and a function $q \in H(B(p,\rho) \setminus (\partial R \cap \partial V))^+$ such that q has boundary values 0 on $\partial B(p,\rho)$ and $+\infty$ at each irregular point of $\partial R \cap \partial V$ with respect to V. By examining the boundary values of the harmonic function $w - w_i + \delta q$ on ∂V for an arbitrary positive number δ we see that $w + \delta q \geq w_i$ on V so that $w \geq w_i$ on V by letting $\delta \downarrow 0$. In particular $0 \leq w_i \leq \sup_{\overline{V} \setminus V_i} w$ on $\overline{V} \setminus V_i$. Once again by examining the boundary values on $\partial(V_i \cap R)$ of the harmonic function $w_i - v_i - \sup_{\overline{V} \setminus V_i} w - \delta q$ on $V_i \cap R$ for an arbitrary positive number δ, we see that it is nonpositive on $V_i \cap R$ and then on V so that, by letting $\delta \downarrow 0$,

$$0 \leq w_i - v_i \leq \sup_{\overline{V} \setminus V_i} w$$

on V. The rightmost term of the above tends to zero as $i \uparrow \infty$. Therefore we can find a number $j_2 > i_0$ such that

$$|w_i(x) - v_i(x)| < \frac{\varepsilon}{3} \qquad (i \geq j_2). \tag{10.4}$$

Finally, since f is a cluster point of the generalized sequence $(f_i)_{i \geq i_0}$ in the dual space $(L^1(\partial V, d\omega))^* = L^\infty(\partial V, d\omega)$, for the number $j_3 := \max(j_1, j_2) > i_0$, there is a number $i_1 > j_3$ such that f_{i_1} is contained in the given neighborhood of f in the dual space determined by $P(\cdot, x) \in L^1(\partial V, d\omega)$ and the positive number $\varepsilon/3$, i.e.

$$\left| \int_{\partial V} P(y,x) f_{i_1}(y) d\omega(y) - \int_{\partial V} P(y,x) f(y) d\omega(y) \right| < \frac{\varepsilon}{3},$$

which means that

$$\left|w_{i_1}(x) - H_f^V(x)\right| < \frac{\varepsilon}{3}. \tag{10.5}$$

Since $i_1 > j_1$ and j_2, from (10.3) with $i = i_1$, (10.4) with $i = i_1$, and the above (10.5), the desired relation (10.2) follows so that we can conclude (10.1): $v = H_f^{U \cap R}$ on $U \cap R$ with $f \in L^\infty(\partial(U \cap R), d\omega) \subset L^1(\partial(U \cap R), d\omega)$, which is resolutive on $\partial(U \cap R)$. $\square$

Proof of Theorem 1.3. Suppose that the set E of nongraphic boundary points of a bounded domain R in $\boldsymbol{R}^d$ is of harmonic measure zero with respect to R. Then by Lemma 10.1, R is locally a Dirichlet domain at every point $y \in \partial R \setminus E$ (i.e. every $y \in \partial R$ except for points in the set E of harmonic measure zero) with respect to its admissible neighborhood $U(c_y; a_y, b_y)$. Then Theorem 1.2 assures that R is a Dirichlet domain, which was to be shown. $\square$

References

[1] C. J. Bishop, A characterization of Poisson domains, Ark. Mat., **29** (1991), 1–24.

[2] C. Constantinescu und A. Cornea, Ideale Ränder Riemannscher Flächen, Springer, 1963.

[3] L. L. Helms, Introduction to Potential Theory, Wiley-Interscience, 1969.

[4] Y. Ishikawa and M. Nakai, Regular and stable points in Dirichlet problem, Proc. Japan Acad., Ser. A Math. Sci., **73** (1997), 1–4.

[5] T. S. Mountford and S. C. Port, Representations of bounded harmonic functions, Ark. Mat., **29** (1991), 107–126.

[6] M. Nakai, Harmonic functions expressible as Dirichlet solutions, Kodai Math. J., **22** (1999), 116–130.

[7] L. Sario and M. Nakai, Classification Theory of Riemann Surfaces, Springer, 1970.

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