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Length spectrum of geodesic spheres in a non-flat complex space form

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Abstract. We investigate the distribution of length of closed geodesics on geodesic spheres and tubes around complex hyperplane in a non-flat complex space form. The feature of the length spectrum of a geodesic sphere of radius r in a complex projective space of holomorphic sectional curvature 4 is quite different according as $\tan^2 r$ is rational or irrational. Each length spectrum is simple when $\tan^2 r$ is irrational, but when $\tan^2 r$ is rational, it is not necessarily simple and moreover the multiplicity is not uniformly bounded.

Introduction.

The aim of this paper is to study geodesics of geodesic spheres in a non-flat complex space form and investigate their length spectrum in detail. It is well-known that these spheres are typical examples of homogeneous spaces which are diffeomorphic to standard spheres. For a compact Riemannian symmetric space M, geodesics are well-understood: If it is of rank one, all geodesics are simple closed curves with the same length and they are congruent each other under the action of isometries of M, and if it is of rank greater than one, each geodesic lies on a totally geodesic flat submanifold of M. We are hence interested in geodesics of a compact non-symmetric Riemannian homogeneous space.

In this paper we restrict ourselves on geodesics of geodesic spheres in non-flat complex space forms, which are a complex projective space and a complex hyperbolic space. Geodesic spheres in non-flat complex space forms are nice objects in intrinsic geometry as well as extrinsic geometry (that is, submanifold theory).

From the viewpoint of intrinsic geometry, Weinstein [**W**] pointed out that geodesic spheres of sufficiently large radius in a complex projective space are

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examples of *Berger spheres*, so that these spheres are homogeneous Riemannian manifolds which are diffeomorphic to a sphere, whose sectional curvatures lie in the interval $[\delta K, K]$ for some $\delta \in (0, 1/9)$, and which have closed geodesics of length less than $2\pi/\sqrt{K}$. These tell us that odd-dimensional version of Klingenberg's lemma does not hold.

From the viewpoint of submanifold theory, they are the simplest real hypersurfaces in non-flat complex space forms. In a complex projective space, geodesic spheres are the only examples of real hypersurfaces with at most two distinct principal curvatures at each point (see [**CR**]). In a complex n-dimensional complex hyperbolic space, horospheres, geodesic spheres and tubes around totally geodesic complex $(n-1)$-dimensional complex hyperbolic space are the examples of such hypersurfaces. It is well-known that there exist no totally umbilic real hypersurfaces in non-flat complex space forms.

In this context it is natural to study geodesics of these hypersurfaces. We are interested in properties on lengths of all closed geodesics on these hypersurfaces. Our technique comes from submanifold theory. We study the extrinsic shape of geodesics on geodesic spheres by standing on the ambient non-flat complex space form. This gives us information about length of closed geodesics of such hypersurfaces. Once we obtain this information, our results come from classical number theory.

One of the most remarkable results for us is that for a geodesic sphere of radius r with irrational $\tan^2 r$ in a complex projective space of constant holomorphic sectional curvature 4, the length spectrum is simple (cf. Theorem 2.9). This means that two closed geodesics on this geodesic sphere M are congruent each other with respect to some isometry of M if and only if they have the same length. On the contrary, for a geodesic sphere of radius r with rational $\tan^2 r$ in a complex projective space, the multiplicity of length spectrum is not uniformly bounded. We also get similar results for length spectrum of geodesic spheres and tubes around complex hyperplane in a complex hyperbolic space (cf. Theorem 3.6).

Throughout this paper we suppose that a complex projective space $\boldsymbol{C}P^n$ is furnished with the standard metric of constant holomorphic sectional curvature 4 and a complex hyperbolic space $\boldsymbol{C}H^n$ is furnished with the standard metric of constant holomorphic sectional curvature -4.

1. Hypersurfaces with two distinct principal curvatures.

We summarize in this section some basic results which are useful in the following sections.

Let $\tilde{M}_n$ denote either a complex n-dimensional complex projective space $\boldsymbol{C}P^n$ or a complex n-dimensional complex hyperbolic space $\boldsymbol{C}H^n$. Let M^{2n-1} be an orientable real hypersurface of $\tilde{M}_n$ and $\mathcal{N}_M$ a unit normal vector field on M in $\tilde{M}_n$. The Riemannian connections $\tilde{\nabla}$ of $\tilde{M}_n$ and ∇ of M are related by

$$\tilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle \mathcal{N}_M \quad \text{and} \quad \tilde{\nabla}_X \mathcal{N}_M = -AX, \tag{1.1}$$

for vector fields X and Y tangent to M, where $\langle\,,\rangle$ denotes the Riemannian metric on M induced from the standard metric on $\tilde{M}_n$, and A is the shape operator of M in $\tilde{M}_n$. Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal curvature vectors*, respectively. It is known that M admits an almost contact metric structure $(\phi, \xi, \eta, \langle\,,\rangle)$ induced from the Kähler structure J of $\tilde{M}_n$, which satisfies

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \text{and} \quad \langle \phi X, \phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y),$$

where I denotes the identity map of the tangent bundle TM of M. It follows from the equalities (1.1) that

$$(\nabla_X \phi) Y = \eta(Y) AX - \langle AX, Y\rangle \xi \tag{1.2}$$

and

$$\nabla_X \xi = \phi AX. \tag{1.3}$$

The condition that the structure vector $\xi = -J\mathcal{N}_M$ is principal is quite natural. As was shown in [**NR**], for a real hypersurface M^{2n-1} in $\tilde{M}_n$ $(n \geq 2)$, if $A\xi = \alpha\xi$ holds with some function α on M then α is locally constant. In $\boldsymbol{CP}^n$, each real hypersurface M lying on a tube of constant radius r (> 0) around a complex submanifold of $\boldsymbol{CP}^n$ satisfies this condition on ξ (cf. [**CR**]). In $\boldsymbol{CH}^n$, each real hypersurface M lying on a tube around a complex submanifold or around a totally real submanifold of $\boldsymbol{CH}^n$ satisfies that condition (cf. [**M**]).

In this paper we study the following real hypersurfaces in non-flat complex space forms:

(I) geodesic spheres of radius r $(0 < r < \pi/2)$ in $\boldsymbol{CP}^n$,

(II) a) horospheres in $\boldsymbol{CH}^n$,

b) geodesic spheres of radius r $(0 < r < \infty)$ in $\boldsymbol{CH}^n$,

c) tubes of radius r $(0 < r < \infty)$ around totally geodesic complex hyperplane $\boldsymbol{CH}^{n-1}$ in $\boldsymbol{CH}^n$,

which are typical examples of real hypersurfaces with the condition that ξ is a principal curvature vector. It is known that each geodesic sphere of radius r $(0 < r < \pi/2)$ in $\boldsymbol{CP}^n$ is congruent to a tube of radius $\pi/2 - r$ around a totally geodesic complex hyperplane $\boldsymbol{CP}^{n-1}$ in $\boldsymbol{CP}^n$.

These hypersurfaces have two distinct constant principal curvatures. For a geodesic sphere M of radius r in $\boldsymbol{CP}^n$, we have

$$A\xi = (2\cot 2r)\xi, \quad \text{and} \quad Au = (\cot r)u,$$

for every tangent vector $u \in TM$ orthogonal to ξ. For real hypersurfaces in $\boldsymbol{C}H^n$ listed in (II), we have the following table for each tangent vector u orthogonal to ξ:

	horosphere	geodesic sphere of radius r	tube of radius r around $\boldsymbol{C}H^{n-1}$
$A\xi$	2ξ	$(2\coth 2r)\xi$	$(2\coth 2r)\xi$
Au	u	$(\coth r)u$	$(\tanh r)u$

These hypersurfaces listed in (I) and (II) are characterized as totally η-umbilic hypersurfaces in a non-flat complex space form. A real hypersurface M of $\tilde{M}_n$ $(n \geq 3)$ is called totally η-umbilic if its shape operator A is of the form $A = \alpha I + \beta\eta \otimes \xi$ for some functions α and β on M. When $\tilde{M}_n$ is a complex projective space, a totally η-umbilic hypersurface is locally congruent to a geodesic sphere, and when $\tilde{M}_n$ is a complex hyperbolic space, it is locally congruent to one of hypersurfaces listed in (II) (see [**M**], [**T2**]).

Moreover, if a real hypersurface M of $\tilde{M}_n$ $(n \geq 2)$ is locally congruent to one of these hypersurfaces in (I) and (II), it satisfies the following ([**NR**]).

1) The structure tensor ϕ and the shape operator A of M in $\tilde{M}_n$ are commutative: $\phi A = A\phi$.
2) The covariant derivative of the shape operator A satisfies

$$(\nabla_X A)Y = \mp\{\langle \phi X, Y\rangle\xi + \eta(Y)\phi X\}, \tag{1.4}$$

where the double sign depends on the case that either $\tilde{M}_n$ is a complex projective space or a complex hyperbolic space.

We here make mention of the sectional curvature of hypersurfaces M of $\tilde{M}_n$ listed in (I) and (II). For orthonormal vectors $u, v \in T_xM$, the sectional curvature $\mathrm{Riem}(u,v)$ of the plane spanned by these vectors is calculated by

$$\mathrm{Riem}(u,v) = \pm(1 + 3\langle \phi u, v\rangle^2) + \langle Au, u\rangle\langle Av, v\rangle - \langle Au, v\rangle^2.$$

Here the double sign also depends on the case that either $\tilde{M}_n$ is a complex projective space or a complex hyperbolic space. Therefore the sectional curvature of a geodesic sphere of radius r in $\boldsymbol{C}P^n$ lies in the interval $[\cot^2 r, (4 + \cot^2 r)]$. For hypersurfaces in $\boldsymbol{C}H^n$ listed in (II), the sectional curvature lies in the interval $[-(4 - \coth^2 r), \coth^2 r]$ in the case of a geodesic sphere of radius r, lies in the interval $[-(4 - \tanh^2 r), \tanh^2 r]$ in the case of a tube around $\boldsymbol{C}H^{n-1}$, and lies in the interval $[-3, 1]$ in the case of a horosphere.

We devote the rest of this section to study the isometry group of real hypersurfaces in (I) and (II). For a non-zero tangent vector $v \in T_xM$ we denote by $\langle v\rangle$ the 1-dimensional linear subspace of T_xM spanned by v, and by $\langle v\rangle^{\perp}$ the orthogonal complement of $\langle v\rangle$ in T_xM.

Lemma 1.1. *Let M be a geodesic sphere in $\boldsymbol{CP}^n$. For any unit tangent vectors $u \in \langle\xi_x\rangle^{\perp}$, $v \in \langle\xi_y\rangle^{\perp}$ of M orthogonal to ξ at arbitrary points x, y, there exist isometries $\tilde{\varphi}^+, \tilde{\varphi}^-$ of $\boldsymbol{CP}^n$ with*

i) $\tilde{\varphi}^+(M) = \tilde{\varphi}^-(M) = M$ *and* $\tilde{\varphi}^+(x) = \tilde{\varphi}^-(x) = y$,

ii) $d\tilde{\varphi}^+_x(u) = d\tilde{\varphi}^-_x(u) = v$ *and* $d\tilde{\varphi}^+_x(\xi_x) = \xi_y$, $d\tilde{\varphi}^-_x(\xi_x) = -\xi_y$.

Proof. Let $\Pi : S^{2n+1} \to \boldsymbol{CP}^n$ denote the Hopf fibration of a unit sphere S^{2n+1} in $\boldsymbol{C}^{n+1}$ and $\hat{M}$ a hypersurface given by

$$\{w = (w_0, w_1, \dots, w_n) \in \boldsymbol{C}^{n+1} \mid |w_0| = \cos r, |w_1|^2 + \cdots + |w_n|^2 = \sin^2 r\}.$$

We see that $\Pi^{-1}(M)$ is isometric to $\hat{M}$.

For simplicity we only treat the case $n = 2$ and $M = \Pi(\hat{M})$. At a point $w = (\cos r, \lambda e^{\sqrt{-1}\theta}, \mu e^{\sqrt{-1}\psi}) \in \hat{M}$, $\lambda^2 + \mu^2 = \sin^2 r$, the tangent space of $\hat{M}$ is represented as

$$\begin{aligned} T_w\hat{M} &= \{(w, v) \in \{w\} \times \boldsymbol{C}^3 \mid \mathrm{Re}(w_0\overline{v_0}) = \mathrm{Re}(w_1\overline{v_1} + w_2\overline{v_2}) = 0\} \\ &= \{(w, (\sqrt{-1}a, \sqrt{-1}b\lambda e^{\sqrt{-1}\theta} - \mu\alpha e^{-\sqrt{-1}\psi}, \sqrt{-1}b\mu e^{\sqrt{-1}\psi} + \lambda\alpha e^{-\sqrt{-1}\theta})) \\ &\qquad\qquad \mid a, b \in \boldsymbol{R}, \alpha \in \boldsymbol{C}\}, \end{aligned}$$

where $\mathrm{Re}(\alpha)$ denotes the real part of a complex number α. Let $T_wS^5 = \mathscr{H}_w \oplus \mathscr{V}_w$ denote the horizontal and vertical decomposition of the tangent space of S^5 at w with respect to the Hopf fibration. We denote by $(T_w\hat{M})^{\perp}$ the orthogonal complement of $T_w\hat{M}$ in T_wS^5. We find that the horizontal lift $\hat{\mathscr{N}_w} \in (T_w\hat{M})^{\perp} \cap \mathscr{H}_w$ of the unit normal $\mathscr{N}_M$ at $\Pi(w)$ is of the form

$$\hat{\mathscr{N}_w} = (w, (-\sin r, \lambda \cot r e^{\sqrt{-1}\theta}, \mu \cot r e^{\sqrt{-1}\psi})).$$

We set $\hat{\xi}_w = -J\hat{\mathscr{N}_w}$ and denote by $\langle\hat{\xi}_w\rangle$ the real linear subspace spanned by $\hat{\xi}_w$. Here we denote the complex structure of $\boldsymbol{C}^3$ also by J. The horizontal part $\langle\hat{\xi}_w\rangle^{\perp} \cap \mathscr{H}_w$ of the orthogonal complement subspace $\langle\hat{\xi}_w\rangle^{\perp}$ in T_wS^5 is represented as $\{(w, (0, -\mu\alpha e^{-\sqrt{-1}\psi}, \lambda\alpha e^{-\sqrt{-1}\theta})) \mid \alpha \in \boldsymbol{C}\}$.

Put $z = (\cos r, \sin r, 0)$ $(\in \hat{M})$. For $\alpha \in \boldsymbol{C}$ with $|\alpha| = 1$ the unitary matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & (\lambda/\sin r)e^{\sqrt{-1}\theta} & (-\mu\alpha/\sin r)e^{-\sqrt{-1}\psi} \\ 0 & (\mu/\sin r)e^{\sqrt{-1}\psi} & (\lambda\alpha/\sin r)e^{-\sqrt{-1}\theta} \end{pmatrix}$$

induces an isometry $\hat{\varphi}_\alpha$ of S^5 such that

i) $\hat{\varphi}_\alpha(z) = w$ and $\hat{\varphi}_\alpha(\hat{M}) = \hat{M}$,

ii) $(d\hat{\varphi}_\alpha)_z(\hat{\mathscr{N}}_z) = \hat{\mathscr{N}}_w$ and $(d\hat{\varphi}_\alpha)_z(\langle\hat{\xi}_z\rangle^\perp \cap \mathscr{H}_z) = \langle\hat{\xi}_w\rangle^\perp \cap \mathscr{H}_w$,

iii) $(d\hat{\varphi}_\alpha)_z((z,(0,0,1))) = (w,(0,-\mu\alpha e^{-\sqrt{-1}\psi}/\sin r, \lambda\alpha e^{-\sqrt{-1}\theta}/\sin r))$.

This guarantees the existence of an isometry $\tilde{\varphi}^+$ of $\boldsymbol{CP}^n$ with desirable conditions. We can similarly construct an isometry $\tilde{\varphi}^-$ and get the conclusion. □

Such a result also holds in a complex hyperbolic space.

LEMMA 1.2. *Let M be one of a horosphere, a geodesic sphere, and a tube around totally geodesic $\boldsymbol{CH}^{n-1}$ in $\boldsymbol{CH}^n$. For any unit tangent vectors $u \in \langle\xi_x\rangle^\perp$, $v \in \langle\xi_y\rangle^\perp$ of M orthogonal to ξ at arbitrary points x, y, there exist isometries $\tilde{\varphi}^+, \tilde{\varphi}^-$ of $\boldsymbol{CH}^n$ with*

i) $\tilde{\varphi}^+(M) = \tilde{\varphi}^-(M) = M$ *and* $\tilde{\varphi}^+(x) = \tilde{\varphi}^-(x) = y$,

ii) $d\tilde{\varphi}^+_x(u) = d\tilde{\varphi}^-_x(u) = v$ *and* $d\tilde{\varphi}^+_x(\xi_x) = \xi_y$, $d\tilde{\varphi}^-_x(\xi_x) = -\xi_y$.

PROOF. Let $\Pi : H_1^{2n+1} \to \boldsymbol{CH}^n$ denote the Hopf fibration of an anti-de Sitter space

$$H_1^{2n+1} = \{w = (w_0, w_1, \ldots, w_n) \in \boldsymbol{C}^{n+1} \mid -|w_0|^2 + |w_1|^2 + \cdots + |w_n|^2 = -1\}$$

in $\boldsymbol{C}^{n+1}$. We denote by $\hat{M}$ a hypersurface given by one of the following;

$$\left\{ w = (w_0, w_1, \ldots, w_n) \in \boldsymbol{C}^{n+1} \;\middle|\; \begin{array}{c} -|w_0|^2 + |w_1|^2 + \cdots + |w_n|^2 = -1, \\ |w_0 - w_1| = 1 \end{array} \right\},$$

$$\{w = (w_0, w_1, \ldots, w_n) \in \boldsymbol{C}^{n+1} \mid |w_0| = \cosh r, |w_1|^2 + \cdots + |w_n|^2 = \sinh^2 r\},$$

$$\left\{ w = (w_0, w_1, \ldots, w_n) \in \boldsymbol{C}^{n+1} \;\middle|\; \begin{array}{c} -|w_0|^2 + |w_1|^2 + \cdots + |w_{n-1}|^2 = -\cosh^2 r, \\ |w_n| = \sinh r \end{array} \right\}.$$

Then $\Pi^{-1}(M)$ is isometric to $\hat{M}$ according as M is a horosphere, a geodesic sphere of radius r or a tube of radius r around a complex hyperplane in $\boldsymbol{CH}^n$, respectively.

For the cases that M is a geodesic sphere or a tube around a complex hyperplane, our proof is just the same as of Lemma 1.1. So we here make mention of the case that M is a horosphere. For simplicity we only treat the case $n = 2$ and $M = \Pi(\hat{M})$. At a point $w = (w_0, w_1, w_2) \in \hat{M}$ the tangent space is represented as

$$T_w\hat{M} = \left\{ (w, v) \in \{w\} \times \boldsymbol{C}^3 \;\middle|\; \begin{array}{c} \mathrm{Re}(-w_0\overline{v_0} + w_1\overline{v_1} + w_2\overline{v_2}) = 0 \\ \mathrm{Re}((w_0 - w_1)\overline{v_0} - (w_0 - w_1)\overline{v_1}) = 0 \end{array} \right\},$$

and the horizontal lift $\hat{\mathcal{N}}_w \in (T_w\hat{M})^{\perp} \cap \mathscr{H}_w$ of the unit normal $\mathcal{N}_M$ at $\Pi(w)$ is of the form $\hat{\mathcal{N}}_w = (w, (-w_1, w_0 - 2w_1, -w_2, \dots, -w_n))$.

Put $z = (1, 0, 0)$ $(\in \hat{M})$. For $\alpha \in \boldsymbol{C}$ with $|\alpha| = 1$ the matrix

$$\begin{pmatrix} w_0 & -w_1 & \alpha\overline{w_2}/(\overline{w_0} - \overline{w_1}) \\ w_1 & w_0 - 2w_1 & \alpha\overline{w_2}/(\overline{w_0} - \overline{w_1}) \\ w_2 & -w_2 & \alpha \end{pmatrix}$$

induces an isometry $\hat{\varphi}_\alpha$ of H_1^5 such that

i) $\hat{\varphi}_\alpha(z) = w$ and $\hat{\varphi}_\alpha(\hat{M}) = \hat{M}$,

ii) $(d\hat{\varphi}_\alpha)_z(\hat{\mathcal{N}}_z) = \hat{\mathcal{N}}_w$ and $(d\hat{\varphi}_\alpha)_z(\langle\hat{\xi}_z\rangle^{\perp} \cap \mathscr{H}_z) = \langle\hat{\xi}_w\rangle^{\perp} \cap \mathscr{H}_w$,

iii) $(d\hat{\varphi}_\alpha)_z((z, (0, 0, 1)) = (w, (\alpha\overline{w_2}/(\overline{w_0} - \overline{w_1}), \alpha\overline{w_2}/(\overline{w_0} - \overline{w_1}), \alpha) \in \langle\hat{\xi}_w\rangle^{\perp} \cap \mathscr{H}_z$.

Hence we obtain the conclusion in this case. □

2. Geodesics on geodesic spheres in a complex projective space.

In this section we study geodesics on geodesic spheres in a complex projective space. A smooth curve σ is said to be *closed* if there exists t_0 $(\neq 0)$ with $\sigma(t + t_0) = \sigma(t)$ for all t. The minimum positive t_0 with this property is called the *length* of a closed curve σ and is denoted by length(σ). Two smooth curves σ_1, σ_2 on a Riemannian manifold N are said to be *congruent* if there exist an isometry φ of N and a constant T with $\sigma_2(t) = \varphi \circ \sigma_1(t + T)$ for every t. One of our goal in this section is to show that on each geodesic sphere in a complex projective space there exist infinitely many congruency classes of closed geodesics and the set of lengths is an unbounded discrete subset of the real line $\boldsymbol{R}$.

In order to investigate geodesics on geodesic spheres in $\boldsymbol{C}P^n$ we study their extrinsic shape in $\boldsymbol{C}P^n$. We start with recalling some terminology for helices on a Kähler manifold. A smooth curve $\gamma = \gamma(t)$ parametrized by its arclength t on a Riemannian manifold N with Riemannian connection ∇ is called a *helix of proper order d* if there exist an orthonormal frame $\{X_1 = \dot{\gamma}, X_2, \dots, X_d\}$ along γ and positive constants $\kappa_1, \dots, \kappa_{d-1}$ which satisfy the system of ordinary differential equations

$$\nabla_{\dot{\gamma}} X_i(t) = -\kappa_{i-1} X_{i-1}(t) + \kappa_i X_{i+1}(t), \quad i = 1, \dots, d, \tag{2.1}$$

where X_0, X_{d+1} are null vector fields, and $\nabla_{\dot{\gamma}}$ denotes the covariant differentiation along γ. The constants κ_i $(1 \leq i \leq d-1)$ and the orthonormal frame $\{X_1, \dots, X_d\}$ are called the *curvatures* and the *Frenet frame* of γ, respectively. A curve is called a *helix of order d* if it is a helix of proper order h $(\leq d)$. For a helix of order d which is of proper order h $(\leq d)$, we use the convention in (2.1) that $\kappa_i = 0$ $(h \leq i \leq d-1)$ and $X_i = 0$ $(h+1 \leq i \leq d)$. A helix of order 1 is nothing but a geodesic and a helix of order 2 is called a *circle*.

When N is a Kähler manifold with complex structure J and Riemannian metric $\langle\, , \rangle$, we define *complex torsions* τ_{ij} of a helix γ on N of order d by

$$\tau_{ij}(t) = \langle X_i(t), JX_j(t)\rangle \quad \text{for } 1 \leq i < j \leq d,$$

where $\{X_1, \dots, X_d\}$ is the associated Frenet frame for γ. We call a helix γ *holomorphic* when all its complex torsions are constant. One can easily find that every circle on a Kähler manifold is holomorphic. In the study of helices in a Kähler manifold their complex torsions play an important role. In their paper [**MOh**], Ohnita and the second-named author proved that a smooth curve on a complete simply connected Kähler manifold $\tilde{M}_n$ of non-zero constant holomorphic sectional curvature (that is, $\tilde{M}_n = \boldsymbol{C}P^n$ or $\boldsymbol{C}H^n$) is a holomorphic helix if and only if it is generated by a holomorphic Killing vector field. They also proved that two holomorphic helices γ_1, γ_2 in $\tilde{M}_n$ are congruent with respect to an isometry of $\tilde{M}_n$ if and only if the following three conditions hold;

1) they have same proper order d,
2) they have same curvatures,
3) either

$$(\text{complex torsion } \tau_{ij} \text{ of } \gamma_1) = (\text{complex torsion } \tau_{ij} \text{ of } \gamma_2)$$

for every $1 \leq i < j \leq d$, or

$$(\text{complex torsion } \tau_{ij} \text{ of } \gamma_1) = -(\text{complex torsion } \tau_{ij} \text{ of } \gamma_2)$$

for every $1 \leq i < j \leq d$.

Here, in the condition 3), the former holds if γ_1, γ_2 are congruent with respect to a holomorphic isomery, and the latter holds if they are congruent with respect to an anti-holomorphic isometry.

Now let M denote a geodesic sphere of radius r $(0 < r < \pi/2)$ and ι an isometric embedding of M into $\boldsymbol{C}P^n$. We shall show that for every geodesic γ on a geodesic sphere M the curve $\iota \circ \gamma$ in $\boldsymbol{C}P^n$ is a helix of order 4. By using (1.3) and the equality $A\phi = \phi A$ we observe that $\langle \dot{\gamma}(t), \xi\rangle$ is constant along γ:

$$\nabla_{\dot{\gamma}}\langle \dot{\gamma}(t), \xi\rangle = \langle \dot{\gamma}(t), \phi A\dot{\gamma}\rangle = \langle \dot{\gamma}, A\phi\dot{\gamma}\rangle = -\langle \phi A\dot{\gamma}, \dot{\gamma}\rangle = 0. \tag{2.2}$$

We shall call this constant the *structure torsion* of γ and denote by $\sin\theta$ with $0 \leqq |\theta| \leqq \pi/2$.

PROPOSITION 2.1. *Let M be a geodesic sphere of radius r $(0 < r < \pi/2)$ in $\boldsymbol{C}P^n$ of holomorphic sectional curvature* 4. *We denote by ι an isometric embedding of M into $\boldsymbol{C}P^n$. Then the extrinsic shape $\iota \circ \gamma$ of a geodesic γ on M is as follows*:

(1) *Suppose the radius r satisfies $\pi/4 \leq r < \pi/2$. If the structure torsion of γ is $\pm\cot r$, then the curve $\iota \circ \gamma$ is a geodesic.*
(2) *When $r \neq \pi/4$, if the structure torsion of γ is ± 1 (i.e. $\dot{\gamma} = \pm\xi$), then the curve $\iota \circ \gamma$ is a circle of curvature $2|\cot 2r|$ and of complex torsion*

$\mp\mathrm{sgn}(\cot 2r)\cdot 1$ *in* $\boldsymbol{CP}^n$, *where* $\mathrm{sgn}(a)$ *denotes the signature of a real number* a. *This circle lies on a totally geodesic* $\boldsymbol{CP}^1$.

(3) *If* γ *has null structure torsion* (*i.e.* $\dot\gamma$ *is orthogonal to* ξ), *then the curve* $\iota\circ\gamma$ *is circle of curvature* $\cot r$ *and of null complex torsion in* $\boldsymbol{CP}^n$. *This circle lies on a totally geodesic* $\boldsymbol{RP}^2$.

(4) *Generally, if the structure torsion of* γ *is of the form* $\sin\theta$ $(0<|\theta|<\pi/2$, $\sin\theta\neq\pm\cot r)$, *then the curve* $\iota\circ\gamma$ *is a holomorphic helix of proper order* 4 *whose curvatures are described as*

$$\kappa_1 = |\cot r - \tan r\cdot\sin^2\theta|,\quad \kappa_2 = \tan r\cdot|\sin\theta|\cos\theta,\quad \kappa_3=\cot r.$$

Its complex torsions are described as

$$\tau_{12} = \begin{cases} -\sin\theta, & \text{if } \cot r - \tan r\cdot\sin^2\theta > 0 \\ \sin\theta, & \text{if } \cot r - \tan r\cdot\sin^2\theta < 0, \end{cases}$$

$$\tau_{14} = \begin{cases} -\mathrm{sgn}(\sin\theta)\cos\theta, & \text{if } \cot r - \tan r\cdot\sin^2\theta > 0 \\ \mathrm{sgn}(\sin\theta)\cos\theta, & \text{if } \cot r - \tan r\cdot\sin^2\theta < 0, \end{cases}$$

$$\tau_{23} = \mathrm{sgn}(\sin\theta)\cos\theta,\quad \tau_{34}=\sin\theta,\quad \tau_{13}=\tau_{24}=0.$$

This helix $\iota\circ\gamma$ *lies on a totally geodesic* $\boldsymbol{CP}^2$.

We call a smooth curve simple if it does not have self-intersection points. More precisely, an open curve σ is called simple if $\sigma(t_1)\neq\sigma(t_2)$ for every t_1, t_2 $(t_1\neq t_2)$, and a closed curve σ is called simple if $\sigma(t_1)\neq\sigma(t_2)$ for every t_1,t_2 $(0\leqq t_1<t_2<\mathrm{length}(\sigma))$. This proposition guarantees that every geodesic on a geodesic sphere in $\boldsymbol{CP}^n$ is generated by some Killing vector field on $\boldsymbol{CP}^n$ as a curve in $\boldsymbol{CP}^n$. Thus we have

Corollary 2.2. *Every geodesic on a geodesic sphere in a complex projective space is a simple curve.*

Proof of Proposition 2.1. Let $\tilde\nabla$ and ∇ denote the Riemannian connections of $\boldsymbol{CP}^n$ and M, respectively. Let γ be a geodesic with structure torsion $\sin\theta$ on M. For simplicity we also denote the curve $\iota\circ\gamma$ by γ. For the first step, by use of (1.1) we have

$$\tilde\nabla_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + \langle A\dot\gamma,\dot\gamma\rangle\mathcal{N}_M = \langle A\dot\gamma,\dot\gamma\rangle\mathcal{N}_M.$$

Therefore when $\langle A\dot\gamma,\dot\gamma\rangle = \cot r - \tan r\cdot\sin^2\theta = 0$, which is the case of the first assertion, the curve $\iota\circ\gamma$ is a geodesic in $\boldsymbol{CP}^n$. If it is not this case, we set as

$$\kappa_1 = |\langle A\dot\gamma,\dot\gamma\rangle| = |\cot r - \tan r\cdot\sin^2\theta|,$$

$$X_2 = \begin{cases} \mathcal{N}_M & \text{if } \langle A\dot\gamma,\dot\gamma\rangle>0, \\ -\mathcal{N}_M & \text{if } \langle A\dot\gamma,\dot\gamma\rangle<0. \end{cases}$$

Since we have

$$\tilde{\nabla}_{\dot\gamma}\mathcal{N}_M = -A\dot\gamma = -\langle A\dot\gamma,\dot\gamma\rangle\dot\gamma + (\langle A\dot\gamma,\dot\gamma\rangle\dot\gamma - A\dot\gamma),$$

and $\|A\dot\gamma\|^2 = \cot^2 r + (\tan^2 r - 2)\sin^2\theta$ is constant along γ, we obtain

$$\tilde{\nabla}_{\dot\gamma}X_2 = -\kappa_1\dot\gamma + \kappa_2 X_3,$$

where

$$\kappa_2 = \sqrt{\|A\dot\gamma\|^2 - \langle A\dot\gamma,\dot\gamma\rangle^2} = \tan r|\sin\theta|\cos\theta,$$

$$X_3 = \begin{cases} (1/\kappa_2)(\langle A\dot\gamma,\dot\gamma\rangle\dot\gamma - A\dot\gamma) & \text{if } \langle A\dot\gamma,\dot\gamma\rangle > 0, \\ -(1/\kappa_2)(\langle A\dot\gamma,\dot\gamma\rangle\dot\gamma - A\dot\gamma) & \text{if } \langle A\dot\gamma,\dot\gamma\rangle < 0. \end{cases}$$

Thus, when $\theta = 0$ or $\theta = \pm\pi/2$, we have $\kappa_2 = 0$ and observe that the curve $\iota\circ\gamma$ is a circle of curvature $\cot r$ or $2|\cot 2r|$, respectively.

We continue the calculations for general case. It follows from the formula (1.4) on the covariant derivative of the shape operator that

$$\tilde{\nabla}_{\dot\gamma}(\langle A\dot\gamma,\dot\gamma\rangle\dot\gamma - A\dot\gamma) = -\kappa_2^2\mathcal{N}_M + \langle\dot\gamma,\xi\rangle\phi(\dot\gamma).$$

Since we see that $\|\phi\dot\gamma\|^2 = 1 - \langle\dot\gamma,\xi\rangle^2 = 1 - \sin^2\theta$ is constant along γ, we obtain

$$\tilde{\nabla}_{\dot\gamma}X_3 = -\kappa_2 X_2 + \kappa_3 X_4,$$

with

$$\kappa_3 = \frac{1}{\kappa_2}|\langle\dot\gamma,\xi\rangle|\sqrt{1 - \langle\dot\gamma,\xi\rangle^2} = \cot r,$$

$$X_4 = \begin{cases} \left(1/\sqrt{1 - \langle\dot\gamma,\xi\rangle^2}\right)\phi(\dot\gamma) = (1/\cos\theta)\phi(\dot\gamma), & \text{if } \langle A\dot\gamma,\dot\gamma\rangle\cdot\langle\dot\gamma,\xi\rangle > 0, \\ \left(-1/\sqrt{1 - \langle\dot\gamma,\xi\rangle^2}\right)\phi(\dot\gamma) = (-1/\cos\theta)\phi(\dot\gamma), & \text{if } \langle A\dot\gamma,\dot\gamma\rangle\cdot\langle\dot\gamma,\xi\rangle < 0. \end{cases}$$

Finally we have

$$\tilde{\nabla}_{\dot\gamma}\phi(\dot\gamma) = \langle\dot\gamma,\xi\rangle A\dot\gamma - \langle A\dot\gamma,\dot\gamma\rangle\xi.$$

As we find

$$|\langle\langle\dot\gamma,\xi\rangle A\dot\gamma - \langle A\dot\gamma,\dot\gamma\rangle\xi, X_3\rangle| = \kappa_3\cos\theta,$$

$$\|\langle\dot\gamma,\xi\rangle A\dot\gamma - \langle A\dot\gamma,\dot\gamma\rangle\xi\|^2 = \cot^2 r\cdot\cos^2\theta = \kappa_3^2\cos^2\theta,$$

we get

$$\tilde{\nabla}_{\dot\gamma}X_4 = -\kappa_3 X_3,$$

and know that $\iota\circ\gamma$ is a helix of proper order 4 when $\sin\theta \neq 0, \pm1, \pm\cot r$. By direct computation one can easily obtain the assertion on complex torsions. $\square$

Proposition 2.1 also tells congruency of two geodesics on a geodesic sphere.

PROPOSITION 2.3. *On a geodesic sphere M in a complex projective space, two geodesics are congruent if and only if the absolute values of their structure torsion coincide.*

PROOF. Let γ_1 and γ_2 be geodesics on M. By Proposition 2.1 and Maeda-Ohnita's result on congruency of helices on a complex projective space, we find that two holomorphic helices $\iota \circ \gamma_1$ and $\iota \circ \gamma_2$ are not congruent as curves in $\boldsymbol{C}P^n$ if the absolute values of their structure torsion do not coincide; $|\langle \dot{\gamma}_1, \xi \rangle| \neq |\langle \dot{\gamma}_2, \xi \rangle|$. We here note that the isometric immersion ι is equivariant, so that for each isometry φ on M there is an isometry $\tilde{\varphi}$ on $\boldsymbol{C}P^n$ with $\tilde{\varphi} \circ \iota = \iota \circ \varphi$. Therefore in this case γ_1 and γ_2 are not congruent.

Conversely, we suppose that they have the same absolute value of structure torsion. When $\langle \dot{\gamma}_1(0), \xi \rangle = \langle \dot{\gamma}_2(0), \xi \rangle$, we can set $\dot{\gamma}_1(0) = (\cos\theta)u + (\sin\theta)\xi_{\gamma_1(0)}$ and $\dot{\gamma}_2(0) = (\cos\theta)v + (\sin\theta)\xi_{\gamma_2(0)}$ with unit tangent vectors u, v orthogonal to ξ. Then we can choose by Lemma 1.1 an isometry φ of M with $d\varphi_{\gamma_1(0)}(u) = v$ and $d\varphi_{\gamma_1(0)}(\xi_{\gamma_1(0)}) = \xi_{\gamma_2(0)}$. Hence, $d\varphi_{\gamma_1(0)}(\dot{\gamma}_1(0)) = \dot{\gamma}_2(0)$, so that $\varphi \circ \gamma_1(t) = \gamma_2(t)$ for any t. When $\langle \dot{\gamma}_1(0), \xi \rangle = -\langle \dot{\gamma}_2(0), \xi \rangle$, we also obtain the desired isometry by Lemma 1.1 and get the conclusion. □

We are now in a position to study length of closed geodesics on a geodesic sphere in a complex projective space. Let $\Pi : S^{2n+1} \to \boldsymbol{C}P^n$ denote the Hopf fibration of a unit sphere. For a smooth curve γ on $\boldsymbol{C}P^n$ a smooth curve $\tilde{\gamma}$ is called a horizontal lift of γ if $\dot{\tilde{\gamma}}(t)$ is a horizontal vector and $d\Pi(\dot{\tilde{\gamma}}(t)) = \dot{\gamma}(t)$ for all t. Our idea lies on considering a horizontal lift of a holomorphic helix $\iota \circ \gamma$ for every geodesic γ on a geodesic sphere. The following elementary lemma is useful in our argument.

LEMMA 2.4. *Let σ be a smooth simple curve on $\boldsymbol{C}P^n$. Suppose a horizontal lift $\tilde{\sigma}$ of σ on a standard unit sphere S^{2n+1} is represented as*

$$\tilde{\sigma}(t) = Ae^{\sqrt{-1}at} + Be^{\sqrt{-1}bt} + Ce^{\sqrt{-1}ct} + De^{\sqrt{-1}dt},$$

which is a curve on $\boldsymbol{C}^{n+1}$ with non-zero vectors $A, B, C, D \in \boldsymbol{C}^{n+1}$ and mutually distinct real numbers a, b, c, d which satisfy $a + b + c + d = 0$ and $a \neq 0$. Then σ is closed if and only if all the ratios $b/a, c/a, d/a$ are rational. In this case, its length is

$$\mathrm{length}(\sigma) = 2\pi \times \mathrm{L.C.M.}\left\{\frac{1}{|b-a|}, \frac{1}{|c-a|}, \frac{1}{|d-a|}\right\}.$$

Here for positive numbers $\alpha_1, \alpha_2, \alpha_3$, we denote by L.C.M.$\{\alpha_1, \alpha_2, \alpha_3\}$ *the minimum value of the set* $\{j\alpha_1 \mid j = 1, 2, \ldots\} \cap \{j\alpha_2 \mid j = 1, 2, \ldots\} \cap \{j\alpha_3 \mid j = 1, 2, \ldots\}$.

PROOF. Suppose σ is a closed curve with length t_0. There is a number ψ with $\tilde{\sigma}(t_0) = e^{\sqrt{-1}\psi}\tilde{\sigma}(0)$. Without loss of generality, we may suppose $at_0 = \psi$.

Since σ is simple we find that $\tilde{\sigma}^{(k)}(t_0) = e^{\sqrt{-1}\psi}\tilde{\sigma}^{(k)}(0)$ for every $k = 0, 1, 2, \dots$. Generally, if non-zero vectors $v_1, v_2, v_3 \in \boldsymbol{C}^{n+1}$ and complex numbers $\lambda_1, \lambda_2, \lambda_3$ satisfy

$$\begin{cases} \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0, \\ \lambda_1 a_1 v_1 + \lambda_2 a_2 v_2 + \lambda_3 a_3 v_3 = 0, \\ \lambda_1 a_1^2 v_1 + \lambda_2 a_2^2 v_2 + \lambda_3 a_3^2 v_3 = 0, \end{cases}$$

for some relatively distinct real numbers a_1, a_2, a_3, then $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Therefore we find that $(b-a)t_0/(2\pi), (c-a)t_0/(2\pi)$ and $(d-a)t_0/(2\pi)$ are integers, hence $(c-a)/(b-a)$ and $(d-a)/(b-a)$ are rational. Since $a+b+c+d=0$, this tells us that b/a is rational. By the same argument as above we can see that c/a and d/a are rational.

Conversely, we suppose that $b/a, c/a$ and d/a are rational. Then one can easily find that σ is closed by considering the reverse of this argument. Its length t_0 is the smallest number with satisfying that all $(b-a)t_0/(2\pi), (c-a)t_0/(2\pi)$ and $(d-a)t_0/(2\pi)$ are integers. Thus we get the conclusion. □

THEOREM 2.5. *Let γ be a geodesic on a geodesic sphere M of radius r $(0 < r < \pi/2)$ in $\boldsymbol{CP}^n$ of holomorphic sectional curvature* 4.

(1) *If the structure torsion of γ is ± 1, then γ is closed and its length is $\pi \sin 2r$.*

(2) *If γ has null structure torsion, then γ is also closed and its length is $2\pi \sin r$.*

(3) *When the structure torsion of γ is of the form $\sin\theta$ $(0 < |\theta| < \pi/2)$, it is closed if and only if*

$$\sin\theta = \frac{\pm q}{\sin r \sqrt{p^2 \tan^2 r + q^2}}$$

with some relatively prime positive integers p and q with $q < p \tan^2 r$. In this case, its length is

$$\text{length}(\gamma) = \begin{cases} 2\pi\sqrt{p^2 \sin^2 r + q^2 \cos^2 r}, & \text{if } pq \text{ is even} \\ \pi\sqrt{p^2 \sin^2 r + q^2 \cos^2 r}, & \text{if } pq \text{ is odd.} \end{cases}$$

PROOF. The assertions of (1) and (2) are direct consequences of Proposition 1 and Theorem of [**AMU**, p. 718]. We here show (3). If we denote the Riemannian connection on a standard unit sphere S^{2n+1} by $\hat{\nabla}$, then it satisfies $\hat{\nabla}_U V = \tilde{\nabla}_U V + \langle U, JV \rangle J\mathcal{N}_{S^{2n+1}}$ for each vector field U, V on $\boldsymbol{CP}^n$. Here $\mathcal{N}_{S^{2n+1}}$ denotes the outward unit normal vector field on S^{2n+1} in $\boldsymbol{C}^{n+1}$ and vector fields on $\boldsymbol{CP}^n$ are regarded as horizontal vector fields on S^{2n+1} with respect to the Hopf fibration. By use of this relation and Proposition 2.1, we can easily check that every horizontal lift $\hat{\gamma}$ of $\imath \circ \gamma$ onto S^{2n+1} is a helix of proper order 3:

$$\begin{cases} \hat{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = \ell_1 \hat{X}_2, \\ \hat{\nabla}_{\dot{\hat{\gamma}}}\hat{X}_2 = -\ell_1\dot{\hat{\gamma}} + \ell_2\hat{X}_3, \\ \hat{\nabla}_{\dot{\hat{\gamma}}}\hat{X}_3 = -\ell_2\hat{X}_2, \end{cases}$$

where

$$\ell_1 = |\cot r - \tan r \cdot \sin^2\theta|, \quad \ell_2 = |\sin\theta|\sqrt{\tan^2 r\cos^2\theta + 1},$$

$$\hat{X}_2 = \begin{cases} \mathcal{N}_M, & \text{if } \cot r - \tan r\cdot\sin^2\theta > 0, \\ -\mathcal{N}_M, & \text{if } \cot r - \tan r\cdot\sin^2\theta < 0, \end{cases}$$

$$\hat{X}_3 = \begin{cases} (1/\ell_2)(\langle A\dot{\gamma}, \dot{\gamma}\rangle\dot{\gamma} - A\dot{\gamma} - \sin\theta J\mathcal{N}_{S^{2n+1}}), & \text{if } \cot r - \tan r\cdot\sin^2\theta > 0, \\ (-1/\ell_2)(\langle A\dot{\gamma}, \dot{\gamma}\rangle\dot{\gamma} - A\dot{\gamma} - \sin\theta J\mathcal{N}_{S^{2n+1}}), & \text{if } \cot r - \tan r\cdot\sin^2\theta < 0, \end{cases}$$

with the shape operator A of M in $\boldsymbol{CP}^n$ and the complex structure J on $\boldsymbol{C}^{n+1}$. Regarding $\hat{\gamma}$ as a curve in $\boldsymbol{C}^{n+1}$, we find that it satisfies the following ordinary differential equation:

$$\hat{\gamma}^{(4)} + (\cot^2 r + \cos^2\theta + \tan^2 r\sin^2\theta)\hat{\gamma}'' + \sin^2\theta(\tan^2 r\cos^2\theta + 1)\hat{\gamma} = 0.$$

Thus we can see that $\hat{\gamma}$ is of the form

$$\begin{aligned} \hat{\gamma}(t) &= A\exp(\sqrt{-1}t\tan r\sin\theta) + B\exp(-\sqrt{-1}t\tan r\sin\theta) \\ &\quad + C\exp(\sqrt{-1}t\sqrt{\cot^2 r + \cos^2\theta}) + D\exp(-\sqrt{-1}t\sqrt{\cot^2 r + \cos^2\theta}) \end{aligned}$$

with some non-zero vectors $A, B, C, D \in \boldsymbol{C}^{n+1}$ which span the complex 3-space $\boldsymbol{C}^3$ which is also spanned by $\hat{\gamma}(0)$ and horizontal lifts of $\dot{\gamma}(0)$ and $\xi_{\gamma(0)}$.

Put $\alpha_r(\theta) = \sqrt{(\cot^2 r + \cos^2\theta)\cot^2 r\sin^{-2}\theta}$. Since γ is simple, we know by Lemma 2.4 that γ is closed if and only if $\alpha_r(\theta)$ is rational. In this case, its length is

$$\begin{aligned} \text{length}(\gamma) = 2\pi \times \text{L.C.M.}\biggl\{ &\frac{1}{2\tan r|\sin\theta|}, \frac{1}{|\sqrt{\cot^2 r + \cos^2\theta} - \tan r|\sin\theta||}, \\ &\frac{1}{\sqrt{\cot^2 r + \cos^2\theta} + \tan r|\sin\theta|} \biggr\}. \end{aligned}$$

When $\alpha_r(\theta)$ is rational, denoting $\alpha_r(\theta) = p/q$ by relatively prime positive integers p and q, we see that $\sin\theta = \pm q(\sin r\sqrt{p^2\tan^2 r + q^2})^{-1}$. As $\sin^2\theta < 1$, we get $q < p\tan^2 r$. By direct calculation we find

$$\mathrm{length}(\gamma) = 2\pi\sqrt{p^2\sin^2 r + q^2\cos^2 r} \times \mathrm{L.C.M.}\left\{\frac{1}{2q}, \frac{1}{|p-q|}, \frac{1}{p+q}\right\}.$$

For relatively prime positive integers p, q we see that $2q, |p-q|, p+q$ are relatively prime if pq is even, and that $q, |p-q|/2, (p+q)/2$ are relatively prime if pq is odd. We hence obtain the expression of $\mathrm{length}(\gamma)$. Conversely, if $\sin\theta$ is of that form then $\alpha_r(\theta)$ is rational. Therefore we get the conclusion. $\square$

Remark. When the structure torsion is not equal to ± 1, every horizontal lift $\hat{\gamma}$ of $\imath\circ\gamma$ for a geodesic γ on M is closed if and only if γ is closed.

When we study the length spectrum of geodesics on a Riemannian manifold N, in order to avoid the influence of the action of the isometry group of N, we consider the moduli space of geodesics under the action of isometries. The moduli space $\mathrm{Geod}(N)$ of geodesics on N is the quotient space of the set of all geodesics on N under the congruency relation. We call a smooth curve σ *open* if it is not closed. For convenience we set $\mathrm{length}(\sigma) = \infty$ for an open curve σ. We define the *length spectrum* $\mathscr{L}_N : \mathrm{Geod}(N) \to \boldsymbol{R}\cup\{\infty\}$ of N by $\mathscr{L}_N([\gamma]) = \mathrm{length}(\gamma)$, where $[\gamma]$ denotes the congruency class containing a geodesic γ. We also call the image $\mathrm{Lspec}(N) = \mathscr{L}_N(\mathrm{Geod}(N))\cap\boldsymbol{R}$ the length spectrum of N. For example, the length spectrum of a standard unit sphere is $\mathrm{Lspec}(S^m) = \{2\pi\}$.

As a direct consequence of Theorem 2.5, for a geodesic sphere M of radius r in $\boldsymbol{CP}^n$, we can see that

$$\begin{aligned}\mathrm{Lspec}(M) = {}& \{\pi\sin 2r\}\cup\{2\pi\sin r\}\\ &\cup\left\{2\pi\sqrt{p^2\sin^2 r + q^2\cos^2 r}\;\middle|\; \begin{array}{c}p \text{ and } q \text{ are relatively prime}\\ \text{positive integers which satisfy}\\ pq \text{ is even and } q < p\tan^2 r\end{array}\right\}\\ &\cup\left\{\pi\sqrt{p^2\sin^2 r + q^2\cos^2 r}\;\middle|\; \begin{array}{c}p \text{ and } q \text{ are relatively prime}\\ \text{positive integers which satisfy}\\ pq \text{ is odd and } q < p\tan^2 r\end{array}\right\}.\end{aligned}$$

Therefore we obtain the following.

Theorem 2.6. *On a geodesic sphere M in $\boldsymbol{CP}^n$, there exist infinitely many congruency classes of closed geodesics. Moreover the length spectrum* $\mathrm{Lspec}(M)$ *of M is a discrete unbounded subset in the real line $\boldsymbol{R}$.*

For a length spectrum $\lambda\in\mathrm{Lspec}(N)$ we call the cardinality $m_N(\lambda)$ of the set $\mathscr{L}_N^{-1}(\lambda)$ the *multiplicity* of λ. When the multiplicity of a length spectrum is 1 we say it is *simple*. Clearly for a geodesic sphere M in a complex projective space, we see by the expression of $\mathrm{Lspec}(M)$ that $m_M(\lambda) < \infty$ at each λ. We here study the first, the second and the third length spectrum, that is, the minimum, the second minimum and the third minimum of the length spectrum.

PROPOSITION 2.7. *Let M be a geodesic sphere of radius r $(0 < r < \pi/2)$ in $\boldsymbol{C}P^n$ of holomorphic sectional curvature* 4.

(1) *The first length spectrum of M is $\pi \sin 2r$, which is the length of geodesics with structure torsion ± 1. It is simple.*

(2) *The second length spectrum of M is also simple. When $0 < r \le \pi/4$, it is $2\pi \sin r$, which is the length of geodesics with null structure torsion. When $\pi/4 < r < \pi/2$, it is π, which is the length of geodesics with structure torsion $\pm\cot r$.*

(3) *The third length spectrum is also simple. When $\pi/4 < r < \pi/2$, it is $2\pi \sin r$, which is the length of geodesics with null structure torsion. When $\sqrt{2m-1} \le \cot r < \sqrt{2m+1}$ $(m = 1, 2, \dots)$, in particular, $0 < r \le \pi/4$, it is $\pi\sqrt{4m(m+1)\sin^2 r + 1}$, which is the length of geodesics with structure torsion $\pm 1/\left(\sin r\sqrt{(2m+1)^2\tan^2 r + 1}\right)$.*

PROOF. (1) We compare $\pi \sin 2r$, $2\pi\sqrt{p^2 \sin^2 r + q^2 \cos^2 r}$ for relatively prime positive integers p, q with even pq and $p \tan^2 r > q$, and $\pi\sqrt{p^2 \sin^2 r + q^2 \cos^2 r}$ for relatively prime positive integers p, q with odd pq and $p \tan^2 r > q$. When $r \le \pi/4$, as $\tan r \le 1$, we have

$$\begin{cases} \sqrt{p^2 + q^2 \cot^2 r} \ge \sqrt{4 + \cot^2 r} > \sqrt{5} > \cos r & \text{if } pq \text{ is even,} \\ \sqrt{p^2 + q^2 \cot^2 r} \ge \sqrt{9 + \cot^2 r} > \sqrt{10} > 2\cos r & \text{if } pq \text{ is odd,} \end{cases}$$

and when $r > \pi/4$, as $\tan r > 1$, we have

$$\begin{cases} \sqrt{p^2 \tan^2 r + q^2} > 1 > \sin r & \text{if } pq \text{ is even,} \\ \sqrt{p^2 \tan^2 r + q^2} \ge \sqrt{\tan^2 r + 1} > 2\sin r & \text{if } pq \text{ is odd.} \end{cases}$$

In any case we obtain $\pi \sin 2r = 2\pi \sin r \cos r$ is the minimum length spectrum.

(2) It is trivial that $2\pi \sin r < 2\pi \sin r\sqrt{p^2 + q^2 \cot^2 r}$ for positive integers p, q. When $r > \pi/4$, we have $2\pi \sin r > \sqrt{2}\pi > \pi$ and $\pi\sqrt{p^2 \sin^2 r + q^2 \cos^2 r} \ge \pi$, where equality holds if and only if $(p, q) = (1, 1)$. When $r \le \pi/4$, for relatively prime positive integers p, q with odd pq and $p \tan^2 r > q$, we have $\sqrt{p^2 + q^2 \cot^2 r} \ge \sqrt{9 + \cot^2 r} \ge \sqrt{10} > 2$.

(3) We can obtain similarly this assertion. $\square$

Since the sectional curvature of a geodesic sphere M of radius r in $\boldsymbol{CP}^n$ lies in the interval $[\cot^2 r, 4+\cot^2 r]$, the first length spectrum of M is smaller than $2\pi/\sqrt{4+\cot^2 r}$ if $\tan^2 r > 2$. Hence M is an example of a Berger sphere, as was pointed out in [**W**]. But for other length spectrum we find the following lemma of Klingenberg's type holds, which is well-known for some geometers.

COROLLARY 2.8. *Let M be a geodesic sphere of radius r in $\boldsymbol{CP}^n$ of holomorphic sectional curvature* 4. *Except geodesics with structure torsion* ± 1, *every geodesic γ on M satisfies* $\mathrm{length}(\gamma) > 2\pi/\sqrt{4+\cot^2 r}$.

Length spectrum is of course not necessarily simple. For example when M is a geodesic sphere of radius $\pi/4$ in $\boldsymbol{CP}^n$, we have

$$\begin{aligned}\mathrm{Lspec}(M) = \{&\pi, \sqrt{2}\pi, \sqrt{5}\pi, \sqrt{10}\pi, \sqrt{13}\pi, \sqrt{17}\pi, 5\pi, \sqrt{26}\pi, \sqrt{29}\pi,\\ &\sqrt{34}\pi, \sqrt{37}\pi, \sqrt{41}\pi, \sqrt{50}\pi, \sqrt{53}\pi, \sqrt{58}\pi, \sqrt{61}\pi, \sqrt{65}\pi, \sqrt{73}\pi, \dots\}\end{aligned}$$

and the multiplicity of $\sqrt{65}\pi$ is two; it is the common length of geodesics of structure torsions $3/\sqrt{65}$ and $7/\sqrt{65}$. Every spectrum which is smaller than $\sqrt{65}\pi$ is simple.

THEOREM 2.9. *Let M be a geodesic sphere of radius r $(0 < r < \pi/2)$ in $\boldsymbol{CP}^n$ of holomorphic sectional curvature* 4.

(1) *If $\tan^2 r$ is irrational, then every length spectrum of M is simple.*
(2) *If $\tan^2 r$ is rational, then the multiplicity of each length spectrum of M is finite. But it is not uniformly bounded*; $\limsup_{\lambda\to\infty} m_M(\lambda) = \infty$. *In this case, the growth order of m_M is not so rapid. It satisfies* $\lim_{\lambda\to\infty} \lambda^{-\delta} m_M(\lambda) = 0$ *for arbitrary positive δ.*

This theorem guarantees that on a geodesic sphere of radius r with irrational $\tan^2 r$ in a complex projective space, two closed geodesics are congruent if and only if they have the same length. On the other hand, if $\tan^2 r$ is rational, this theorem shows that we can not classify congruency classes of geodesics only by their length.

PROOF OF THEOREM 2.9. (1) If two distinct pairs of positive integers (p,q) and $(\tilde{p},\tilde{q})$ satisfy $p^2\sin^2 r + q^2\cos^2 r = \tilde{p}^2\sin^2 r + \tilde{q}^2\cos^2 r$, we see that $\tan^2 r = (\tilde{q}^2 - q^2)/(p^2 - \tilde{p}^2)$ is rational. Since $\pi\sin 2r$ and $2\pi\sin r$ are simple by Proposition 2.7, we get the assertion.

(2) The assertion is a direct consequence of the following Lemmas 2.10 and 2.11. We apply these lemmas by putting $\tan^2 r = a/b$ with relatively prime positive integers a, b and $\varepsilon_1 = 0$, $\varepsilon_2 = \tan^2 r$. $\square$

In order to complete the proof of Theorem 2.9, we need to make a study of integers. For relatively prime positive integers a, b and a positive integer λ, we define a set $\mathscr{S}(a, b; \lambda)$ of pairs of integers as

$$\mathscr{S}(a, b; \lambda) = \left\{ (p, q) \in \boldsymbol{Z} \times \boldsymbol{Z} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime integers which} \\ \text{satisfy } pq \text{ is even and } ap^2 + bq^2 = \lambda \end{array} \right\}.$$

For relatively prime positive integers λ, v we introduce a map

$$\mathscr{S}(a, b; \lambda) \times \mathscr{S}(1, ab; v) \to \mathscr{S}(a, b; \lambda v)$$

by

$$((p, q), (\tilde{p}, \tilde{q})) \mapsto (p, q) \star (\tilde{p}, \tilde{q}) = (p\tilde{p} - bq\tilde{q}, \tilde{p}q + ap\tilde{q}).$$

Since λ and v are relatively prime, it is well-defined; $p\tilde{p} - bq\tilde{q} \neq 0, \tilde{p}q + ap\tilde{q} \neq 0$. It satisfies the following properties.

P1) If we denote as $(P, Q) = (p, q) \star (\tilde{p}, \tilde{q})$ we have

$$\sqrt{a}P + \sqrt{-b}Q = (\sqrt{a}p + \sqrt{-b}q)(\tilde{p} + \sqrt{-ab}\tilde{q}).$$

In particular,

$$\arg(\sqrt{a}P + \sqrt{-b}Q) = \arg(\sqrt{a}p + \sqrt{-b}q) + \arg(\tilde{p} + \sqrt{-ab}\tilde{q}),$$

where $\arg(z)$ denotes the argument of a complex number z with $|\arg(z)| \leq \pi/2$.

P2) If we define the conjugate $\overline{(p, q)}$ of $(p, q) \in \mathscr{S}(a, b; \lambda)$ by $\overline{(p, q)} = (p, -q)$, we have

$$\overline{(p, q) \star (\tilde{p}, \tilde{q})} = \overline{(p, q)} \star \overline{(\tilde{p}, \tilde{q})}.$$

P3) For fixed (p, q) the correspondence $(\tilde{p}, \tilde{q}) \mapsto (p, q) \star (\tilde{p}, \tilde{q})$ is injective, and the correspondence $(p, q) \mapsto (p, q) \star (\tilde{p}, \tilde{q})$ is also injective for fixed $(\tilde{p}, \tilde{q})$.

P4) When $a = 1$, the operation $\star$ is commutative; $(p, q) \star (\tilde{p}, \tilde{q}) = (\tilde{p}, \tilde{q}) \star (p, q)$ for every $(p, q) \in \mathscr{S}(1, b; \lambda)$ and $(\tilde{p}, \tilde{q}) \in \mathscr{S}(1, b; v)$.

We here note that $(p, q) \neq \overline{(p, q)}$ for every $(p, q) \in \mathscr{S}(a, b; \lambda)$, because $q \neq 0$.

Lemma 2.10. *For arbitrary real numbers $\varepsilon_1, \varepsilon_2$ with $\varepsilon_2 > \varepsilon_1 \geq 0$, the cardinality $m_{(a,b;\varepsilon_1,\varepsilon_2)}(\lambda)$ of the set $\{(p, q) \in \mathscr{S}(a, b; \lambda) \mid \varepsilon_1 p < q < \varepsilon_2 p\}$ satisfies*

$$\limsup_{\lambda \to \infty} m_{(a,b;\varepsilon_1,\varepsilon_2)}(\lambda) = \infty.$$

Proof. Take an arbitrary pair of relatively prime positive integers (p_0, q_0) which satisfies $\varepsilon_1 p_0 < q_0 < \varepsilon_2 p_0$ and $p_0 q_0$ is even. We put $\lambda_0 = ap_0^2 + bq_0^2$. For a small positive real number δ we inductively choose odd integers p_i $(i = 1, 2, \ldots)$ as

i) $2\sqrt{ab} \cot \delta < p_1 < p_2 < p_3 < \cdots$,

ii) $p_i \not\equiv \pm p_j \pmod{\ell}$ for all $j < i$ and for all prime divisors ℓ of $p_j^2 + 4ab$.

We set $\lambda_i = p_i^2 + 4ab$. It is easy to see that any two of $\lambda_1, \lambda_2, \lambda_3, \dots$ are relatively prime. Thus, omitting finite number of p_i's if necessary, we obtain $(p_i, 2) \in \mathscr{S}(1, ab; \lambda_i)$ $(i \geq 1)$ such that any two of $\lambda_0, \lambda_1, \lambda_2, \dots$ are relatively prime and that $|\arg(p_i + 2\sqrt{-ab})| \leq \delta$ $(i \geq 1)$.

Now consider the subset

$$\mathscr{I}_k = \{(p_0, q_0) \star (p_1, q_1) \star \cdots \star (p_k, q_k) \mid q_i = \pm 2, i = 1, 2, \dots, k\}$$

of $\mathscr{S}(a, b; \lambda_0\lambda_1\lambda_2 \cdots \lambda_k)$. We find the cardinality of this set is 2^k by the following argument. Suppose

$$(p_0, q_0) \star (p_1, q_1) \star \cdots \star (p_k, q_k) = (p_0, q_0) \star (p_1, q_1') \star \cdots \star (p_k, q_k').$$

If $q_i \neq q_i'$ we have $(p_i, q_i') = \overline{(p_i, q_i)}$. As any element of $\mathscr{S}(1, ab; \lambda_1 \cdots \lambda_k)$ is not equal to its conjugate, properties P2) and P3) guarantee that $q_{i_0} = q_{i_0}'$ for some i_0 $(i_0 \geq 1)$. We can then cancel (p_{i_0}, q_{i_0}) by properties P4) and P3), and conclude by induction that $q_i = q_i'$ for all i. Since every $(p, q) \in \mathscr{I}_k$ satisfies

$$\begin{aligned}&|\arg(\sqrt{a}p + \sqrt{-b}q) - \arg(\sqrt{a}p_0 + \sqrt{-b}q_0)|\\ &\qquad \leq |\arg(p_1 + 2\sqrt{-ab})| + \cdots + |\arg(p_k + 2\sqrt{-ab})| \leq k\delta,\end{aligned}$$

by taking δ so that it satisfies

$$\begin{aligned}\delta\ (= \delta_k) < \frac{1}{k} \min\Biggl\{ &\tan^{-1}\left(\sqrt{\frac{b}{a}}\frac{q_0}{p_0}\right) - \tan^{-1}\left(\sqrt{\frac{b}{a}}\varepsilon_1\right),\\ &\tan^{-1}\left(\sqrt{\frac{b}{a}}\varepsilon_2\right) - \tan^{-1}\left(\sqrt{\frac{b}{a}}\frac{q_0}{p_0}\right)\Biggr\}\end{aligned}$$

we find that $\mathscr{I}_k$ is contained in $\{(p, q) \in \mathscr{S}(a, b; \lambda) \mid \varepsilon_1 p < q < \varepsilon_2 p\}$ and get the conclusion. $\square$

For positive integers a, b and λ we denote by $\tilde{m}_{a,b}(\lambda)$ the cardinality of the set $\{(p, q) \in \boldsymbol{Z} \times \boldsymbol{Z} \mid ap^2 + bq^2 = \lambda\}$, and by $d(\lambda)$ the number of positive divisors of λ. Trivially we have $m_{(a,b;\varepsilon_1,\varepsilon_2)}(\lambda) \leq \tilde{m}_{a,b}(\lambda)$ for every ε_1 and ε_2.

LEMMA 2.11. (1) *We have* $\tilde{m}_{a,b}(\lambda) \leq w_{ab} \min\{d(a), d(b)\}\, d(\lambda)$, *where* w_{ab} *is the number of roots of unity in the quadratic field* $\boldsymbol{Q}(\sqrt{-ab})$ (*i.e.* $w_{ab} = 4$ *if* $ab = 1$, $w_{ab} = 6$ *if* $ab = 3$, *and* $w_{ab} = 2$, *otherwise*).

(2) *In particular, we have* $\limsup_{\lambda\to\infty} \lambda^{-\delta}\tilde{m}_{a,b}(\lambda) = 0$ *for all positive* δ.

PROOF. The second assertion follows from the first assertion and the well-known result $\lim_{\lambda\to\infty} \lambda^{-\delta} d(\lambda) = 0$ (see [**HW**, Theorem 315]). We show the first assertion (cf. [**HW**, §18.7] for the case $a = b = 1$).

First we treat the case $a = 1$. Let $\lambda = \prod_{\ell} \ell^{n(\ell;\lambda)}$ be the factorization of λ. Here ℓ runs over all primes and $n(\ell;\lambda)$ is a non-negative integer with $n(\ell;\lambda) = 0$ for almost all ℓ. We denote by $\mathcal{O}$ the ring of integers of the imaginary quadratic field $\boldsymbol{Q}(\sqrt{-b})$, which is either $\boldsymbol{Z}[\sqrt{-b}]$ or $\boldsymbol{Z}[(1+\sqrt{-b})/2]$. (Concerning basic facts on quadratic fields, see for example [**IR**, §13.1]). For a prime ℓ there are three possibilities for the prime ideal decomposition of ℓ in $\mathcal{O}$:

i) ℓ ramifies: $\ell\mathcal{O} = \mathfrak{p}_\ell^2$,

ii) ℓ decomposes: $\ell\mathcal{O} = \mathfrak{p}_\ell \bar{\mathfrak{p}}_\ell$ $(\mathfrak{p}_\ell \neq \bar{\mathfrak{p}}_\ell)$,

iii) ℓ remains prime: $\ell\mathcal{O} = \mathfrak{p}_\ell$,

where $\mathfrak{p}_\ell$ is a prime ideal of $\mathcal{O}$ and $^- : \mathcal{O} \to \mathcal{O}$ denotes the automorphism of $\mathcal{O}$ induced by $\sqrt{-b} \mapsto -\sqrt{-b}$. We denote by $\mathscr{P}_r, \mathscr{P}_d, \mathscr{P}_p$ the set of primes ℓ which ramify, decompose, remain prime in $\mathcal{O}$, respectively. When there is a pair (p,q) of integers with $p^2 + bq^2 = \lambda$, we put $\mathfrak{a} = (p + \sqrt{-b}q)\mathcal{O}$. It follows from $\lambda\mathcal{O} = \mathfrak{a}\bar{\mathfrak{a}}$ that the prime ideal decomposition of $\mathfrak{a}$ must be of the form

$$\mathfrak{a} = \prod_{\ell \in \mathscr{P}_r} \mathfrak{p}_\ell^{n(\ell;\lambda)} \prod_{\ell \in \mathscr{P}_d} \mathfrak{p}_\ell^{i(\ell)} \bar{\mathfrak{p}}_\ell^{n(\ell;\lambda)-i(\ell)} \prod_{\ell \in \mathscr{P}_p} \mathfrak{p}_\ell^{n(\ell;\lambda)/2},$$

with some integer $i(\ell)$ with $0 \leq i(\ell) \leq n(\ell;\lambda)$ for $\ell \in \mathscr{P}_d$. Hence for this λ, the possibility for $\mathfrak{a}$ is at most $\prod_{\ell \in \mathscr{P}_d}(n(\ell;\lambda)+1)$. If two pairs (p,q) and $(\tilde{p},\tilde{q})$ give the same $\mathfrak{a}$, which means that $(p+\sqrt{-b}q)\mathcal{O} = (\tilde{p}+\sqrt{-b}\tilde{q})\mathcal{O}$, then $(p+\sqrt{-b}q)/(\tilde{p}+\sqrt{-b}\tilde{q})$ is a unit of $\mathcal{O}$. As $\mathcal{O}$ has no unit other than roots of unity, we have $\tilde{m}_{1,b}(\lambda) \leq w_b \prod_{\ell \in \mathscr{P}_d}(n(\ell;\lambda)+1)$. Since $d(\lambda) = \prod_\ell (n(\ell;\lambda)+1)$, we complete the proof for the case $a = 1$.

In general case, if a pair (p,q) of integers satisfies $ap^2 + bq^2 = \lambda$, then the pair $(P,Q) = (ap,q)$ of integers satisfies $P^2 + abQ^2 = a\lambda$. Therefore we have $\tilde{m}_{a,b}(\lambda) \leq w_{ab}\, d(a\lambda) \leq w_{ab}\, d(a) d(\lambda)$. Changing the role of a and b, we also have $\tilde{m}_{a,b}(\lambda) \leq w_{ab}\, d(b) d(\lambda)$, so that we get the conclusion. □

Remark. In Theorem 2.9 (2), even if we give a restriction on structure torsion of geodesics on M, we obtain the same assertion; the multiplicity of length spectrum of geodesics with restricted structure torsion is not uniformly bounded. For arbitrary α, β with $0 \leq \alpha < \beta \leq 1$, we denote by $\mathrm{Geod}_{(\alpha,\beta)}(M)$ the set of congruency classes of geodesics on M whose structure torsion lies in the interval (α,β). Restrict the length spectrum $\mathscr{L}_M$ onto this set and denote by $\mathscr{L}_{(\alpha,\beta)}$. Then the multiplicity $m_{(\alpha,\beta)}(\lambda)$ of the restricted length spectrum $\mathscr{L}_{(\alpha,\beta)}$ also satisfies $\limsup_{\lambda\to\infty} m_{(\alpha,\beta)}(\lambda) = \infty$.

Finally we make mention of the growth of the number of congruency classes of geodesics with respect to their length spectrum for a geodesic sphere in a complex projective space. For a Riemannian manifold N we denote by $n_N(\lambda)$ the cardinality of the set $\{[\gamma] \in \mathrm{Geod}(N) \mid \mathscr{L}_N([\gamma]) \leq \lambda\}$.

THEOREM 2.12. *For a geodesic sphere M of radius r $(0 < r < \pi/2)$ in $\boldsymbol{CP}^n$ of holomorphic sectional curvature* 4 *we have*

$$\lim_{\lambda\to\infty} \frac{n_M(\lambda)}{\lambda^2} = \frac{3r}{\pi^4 \sin 2r}.$$

PROOF. For positive real numbers α, β, λ and a positive integer d, we put $n_{\alpha,\beta}(\lambda)$ and $k_{\alpha,\beta}(\lambda; d)$ the cardinality of the sets

$$\left\{ (p,q) \in \boldsymbol{Z} \times \boldsymbol{Z} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime integers with} \\ \alpha p^2 + \beta q^2 \le \lambda^2 \text{ and } \alpha p > \beta q > 0 \end{array} \right\},$$

$$K(\lambda; d) = \{(p,q) \in d\boldsymbol{Z} \times d\boldsymbol{Z} \mid \alpha p^2 + \beta q^2 \le \lambda^2, \alpha p > \beta q > 0\},$$

respectively. Here $d\boldsymbol{Z}$ denotes the set $\{dj \mid j \in \boldsymbol{Z}\}$. Since the correspondence $(p,q) \mapsto (dp, dq)$ of $K(\lambda/d; 1)$ to $K(\lambda; d)$ is bijective, we find the following relation between $n_{\alpha,\beta}(\lambda)$ and $k_{\alpha,\beta}(\lambda; 1)$ by using the Möbius function μ;

$$n_{\alpha,\beta}(\lambda) = \sum_{d\ge 1} \mu(d) k_{\alpha,\beta}(\lambda; d) = \sum_{1\le d\le [\lambda/\sqrt{\alpha+\beta}]} \mu(d) k_{\alpha,\beta}(\lambda/d; 1), \tag{2.3}$$

where $[\delta]$ denotes the integer part of a real number δ.

We first study the growth of $n_{\alpha,\beta}(\lambda)$ by estimating $k_{\alpha,\beta}(\lambda; 1)$. Consider the following three sets in a plane $\boldsymbol{R}^2$;

$$\begin{aligned}
E_\lambda &= \{(x,y) \in \boldsymbol{R}^2 \mid \alpha x^2 + \beta y^2 \le \lambda^2, \alpha x \ge \beta y \ge 0\}, \\
E_\lambda^+ &= \{(x,y) \in \boldsymbol{R}^2 \mid x \ge 0, y \ge 0, \alpha(x-1)^2 + \beta(y-1)^2 \le \lambda^2, \alpha x \ge \beta(y-1)\}, \\
E_\lambda^- &= \{(x,y) \in \boldsymbol{R}^2 \mid x \ge 0, y \ge 1, \alpha x^2 + \beta y^2 \le \lambda^2, \alpha(x-1) \ge \beta y\}.
\end{aligned}$$

Since $k_{\alpha,\beta}(\lambda; 1)$ coincides with the area of the union of quadrates

$$R_\lambda = \bigcup_{(p,q)\in K(\lambda;1)} \{(x,y) \in \boldsymbol{R}^2 \mid p \le x \le p+1, q \le y \le q+1\},$$

and $E_\lambda^- \subset R_\lambda \subset E_\lambda^+$, we have $\mathrm{Area}(E_\lambda^-) < k_{\alpha,\beta}(\lambda; 1) < \mathrm{Area}(E_\lambda^+)$. Put

$$C_{\alpha,\beta} = \frac{1}{2\sqrt{\alpha\beta}} \tan^{-1} \sqrt{\frac{\alpha}{\beta}}.$$

As $E_\lambda^- \subset E_\lambda \subset E_\lambda^+$ and $\mathrm{Area}(E_\lambda) = C_{\alpha,\beta}\lambda^2$, there exist positive constants C_1, C_2 with

$$|k_{\alpha,\beta}(\lambda; 1) - C_{\alpha,\beta}\lambda^2| < \mathrm{Area}(E_\lambda^+) - \mathrm{Area}(E_\lambda^-) < C_1\lambda + C_2, \tag{2.4}$$

for every λ. By using (2.3) and (2.4) we get

$$\left|\frac{n_{\alpha,\beta}(\lambda)}{\lambda^2} - C_{\alpha,\beta} \sum_{1 \le d \le [\lambda/\sqrt{\alpha+\beta}]} \frac{\mu(d)}{d^2}\right| < \frac{1}{\lambda^2} \sum_{1 \le d \le [\lambda/\sqrt{\alpha+\beta}]} \left(C_1 \frac{\lambda}{d} + C_2\right)$$

$$< \frac{C_1}{\lambda}(1 + \log(\lambda/\sqrt{\alpha+\beta})) + \frac{C_2}{\lambda\sqrt{\alpha+\beta}}$$

$$\to 0 \ (\lambda \to \infty).$$

We hence obtain

$$\lim_{\lambda\to\infty} \frac{n_{\alpha,\beta}(\lambda)}{\lambda^2} = C_{\alpha,\beta} \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = C_{\alpha,\beta} \prod_{\ell} \left(1 - \frac{1}{\ell^2}\right) = \frac{C_{\alpha,\beta}}{\zeta(2)} = \frac{6C_{\alpha,\beta}}{\pi^2},$$

where in the product ℓ runs over all prime positive integers and ζ denotes the Riemann zeta function.

We now put $n^{\mathrm{o}}_{\alpha,\beta}(\lambda)$ and $n^{\mathrm{e}}_{\alpha,\beta}(\lambda)$ the cardinalities of the sets

$$\left\{ (p,q) \in \boldsymbol{Z} \times \boldsymbol{Z} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime integers} \\ \text{which satisfy } pq \text{ is odd}, \alpha p > \beta q > 0 \text{ and} \\ \alpha p^2 + \beta q^2 \le \lambda^2 \end{array} \right\},$$

$$\left\{ (p,q) \in \boldsymbol{Z} \times \boldsymbol{Z} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime integers} \\ \text{which satisfy } pq \text{ is even}, \alpha p > \beta q > 0 \text{ and} \\ \alpha p^2 + \beta q^2 \le \lambda^2 \end{array} \right\},$$

respectively. In order to study the growth of $n^{\mathrm{o}}_{\alpha,\beta}(\lambda)$ we estimate the cardinality $k^{\mathrm{o}}_{\alpha,\beta}(\lambda; d)$ of the set

$$K^{\mathrm{o}}(\lambda; d) = \left\{ (2p+1, 2q+1) \in d\boldsymbol{Z} \times d\boldsymbol{Z} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are positive integers with} \\ \alpha(2p+1)^2 + \beta(2q+1)^2 \le \lambda^2, \\ \alpha(2p+1) > \beta(2q+1) > 0 \end{array} \right\}.$$

Clearly we have $k^{\mathrm{o}}_{\alpha,\beta}(\lambda; d) = 0$ for even d, so that

$$n^{\mathrm{o}}_{\alpha,\beta}(\lambda) = \sum_{d \ge 1} \mu(d) k^{\mathrm{o}}_{\alpha,\beta}(\lambda; d) = \sum_{\substack{1 \le d \le [\lambda/\sqrt{\alpha+\beta}] \\ d \text{ is odd}}} \mu(d) k^{\mathrm{o}}_{\alpha,\beta}(\lambda/d; 1).$$

If $(2p+1, 2q+1) \in K^{\mathrm{o}}(\lambda; 1)$, then the pair of integer (p,q) satisfies

$$\alpha p^2 + \beta q^2 \le \lambda^2/4 \quad \text{and} \quad \alpha p + (\alpha - \beta)/2 > \beta q > 0.$$

On the other hand if a pair of positive integers (p,q) satisfies

$$\alpha(p+1)^2 + \beta(q+1)^2 \le \lambda^2/4 \quad \text{and} \quad \alpha p + (\alpha - \beta)/2 > \beta q,$$

then $(2p+1, 2q+1) \in K^{\mathrm{o}}(\lambda; 1)$. Hence we find by similar argument on $k_{\alpha,\beta}(\lambda/d; 1)$ that

$$\left| k^{\mathrm{o}}_{\alpha,\beta}(\lambda; 1) - \frac{C_{\alpha,\beta}}{4}\lambda^2 \right| < C_1'\lambda + C_2',$$

for every λ with some positive constants C_1', C_2'. Thus we obtain

$$\lim_{\lambda\to\infty} \frac{n^{\mathrm{o}}_{\alpha,\beta}(\lambda)}{\lambda^2} = \frac{C_{\alpha,\beta}}{4} \sum_{\substack{1\le d<\infty, \\ d \text{ is odd}}} \frac{\mu(d)}{d^2} = \frac{C_{\alpha,\beta}}{4} \prod_{\ell} \left(1 - \frac{1}{\ell^2}\right) = \frac{4}{3}\frac{C_{\alpha,\beta}}{4\zeta(2)} = \frac{2C_{\alpha,\beta}}{\pi^2},$$

where in the product ℓ runs over all prime positive integers with $\ell \neq 2$. This also leads us to

$$\lim_{\lambda\to\infty} \frac{n^{\mathrm{e}}_{\alpha,\beta}(\lambda)}{\lambda^2} = \lim_{\lambda\to\infty} \left(\frac{n_{\alpha,\beta}(\lambda)}{\lambda^2} - \frac{n^{\mathrm{o}}_{\alpha,\beta}(\lambda)}{\lambda^2} \right) = \frac{4C_{\alpha,\beta}}{\pi^2}.$$

Coming back to our situation, we see by use of Theorem 2.5 that

$$n_M(\lambda) = 2 + n^{\mathrm{e}}_{\sin^2 r, \cos^2 r}\left(\frac{\lambda}{2\pi}\right) + n^{\mathrm{o}}_{\sin^2 r, \cos^2 r}\left(\frac{\lambda}{\pi}\right), \quad \text{for } \lambda > 2\pi \sin r,$$

hence we obtain the conclusion. □

3. Geodesics on geodesic spheres and tubes around complex hyperplanes in a complex hyperbolic space.

In this section we study geodesics on horospheres, geodesic spheres and tubes around complex hyperplanes in a complex hyperbolic space.

Let M be a real hypersurface in $\boldsymbol{C}H^n$ which is congruent to either one of a horosphere, a geodesic sphere of radius r $(0 < r < \infty)$, or a tube of radius r $(0 < r < \infty)$ around a totally geodesic complex hyperplane $\boldsymbol{C}H^{n-1}$, and $\imath$ be an isometric embedding of M into $\boldsymbol{C}H^n$. For every geodesic γ on M we see by (2.2) that $\langle \dot\gamma, \xi \rangle$ is constant along γ. We call this constant the structure torsion of γ. By just the same calculation as in the proof of Proposition 2.1, we find that $\imath \circ \gamma$ is a helix of order 4 in $\boldsymbol{C}H^n$:

$$\tag{3.1} \begin{cases} \tilde\nabla_{\dot\gamma}\dot\gamma = \kappa_1 X_2, \\ \tilde\nabla_{\dot\gamma} X_2 = -\kappa_1 \dot\gamma + \kappa_2 X_3, \\ \tilde\nabla_{\dot\gamma} X_3 = -\kappa_2 X_2 + \kappa_3 X_4, \\ \tilde\nabla_{\dot\gamma} X_4 = -\kappa_3 X_3, \end{cases}$$

where

$$\kappa_1 = \langle A\dot\gamma, \dot\gamma \rangle, \quad \kappa_2 = \sqrt{\|A\dot\gamma\|^2 - \langle A\dot\gamma, \dot\gamma\rangle^2},$$

$$\kappa_3 = \frac{1}{\kappa_2} |\langle \dot\gamma, \xi \rangle| \sqrt{1 - \langle \dot\gamma, \xi\rangle^2},$$

$$X_2 = \mathcal{N}_M, \quad X_3 = \frac{1}{\kappa_2}(\langle A\dot{\gamma}, \dot{\gamma}\rangle\dot{\gamma} - A\dot{\gamma}),$$

$$X_4 = \begin{cases} \left(1/\sqrt{1 - \langle\dot{\gamma}, \xi\rangle^2}\right)\phi(\dot{\gamma}), & \text{if } \langle\dot{\gamma}, \xi\rangle > 0, \\ \left(-1/\sqrt{1 - \langle\dot{\gamma}, \xi\rangle^2}\right)\phi(\dot{\gamma}), & \text{if } \langle\dot{\gamma}, \xi\rangle < 0, \end{cases}$$

with a unit normal vector field $\mathcal{N}_M$ and the shape operator A of M in $\boldsymbol{C}H^n$. Thus we get the following result on the extrinsic shape of geodesics on those real hypersurfaces in $\boldsymbol{C}H^n$.

Proposition 3.1. *Let M be a real hypersurface in $\boldsymbol{C}H^n$ of holomorphic sectional curvature -4 which is congruent to one of a horosphere, a geodesic sphere of radius r $(0 < r < \infty)$, and a tube of radius r $(0 < r < \infty)$ around complex hyperplane $\boldsymbol{C}H^{n-1}$. We denote by $\imath$ an isometric embedding of M into $\boldsymbol{C}H^n$. Then the extrinsic shape $\imath \circ \gamma$ of a geodesic γ on M is as follows*:

(1) *If the structure torsion of γ is ± 1, then the curve $\imath \circ \gamma$ is a circle of complex torsion ∓ 1 in $\boldsymbol{C}H^n$. Its curvature is* 2 *if M is congruent to a horosphere,* $2\coth 2r$ *if M is congruent to a geodesic sphere of radius r or a tube of radius r around a complex hyperplane.*

(2) *If γ has null structure torsion, then the curve $\imath \circ \gamma$ is a circle of null complex torsion in $\boldsymbol{C}H^n$. Its curvature is* 1 *if M is a horosphere,* $\coth r$ *if M is congruent to a geodesic sphere of radius r, and* $\tanh r$ *if M is congruent to a tube of radius r around a complex hyperplane.*

(3) *Generally, if the structure torsion of γ is of the form* $\sin\theta$ $(0 < |\theta| < \pi/2)$, *then the curve $\imath \circ \gamma$ is a holomorphic helix of proper order* 4 *whose complex torsions are described as*;

$$\tau_{12} = -\tau_{34} = -\sin\theta, \quad \tau_{14} = -\tau_{23} = -\mathrm{sgn}(\sin\theta)\cos\theta, \quad \tau_{13} = \tau_{24} = 0.$$

Its curvatures are as the following table:

	horosphere	*geodesic sphere of radius r*	*tube of radius r around* $\boldsymbol{C}H^{n-1}$
κ_1	$1 + \sin^2\theta$	$\coth r + \tanh r \sin^2\theta$	$\tanh r + \coth r \sin^2\theta$
κ_2	$\lvert\sin\theta\rvert\cos\theta$	$\tanh r\lvert\sin\theta\rvert\cos\theta$	$\coth r\lvert\sin\theta\rvert\cos\theta$
κ_3	1	$\coth r$	$\tanh r$

Each of these holomorphic helices lies on some totally geodesic complex plane $\boldsymbol{C}H^2$ in $\boldsymbol{C}H^n$.

This proposition and Lemma 1.2 guarantee the following.

COROLLARY 3.2. *Let M be one of a horosphere, a geodesic sphere, and a tube around a complex hyperplane in a complex hyperbolic space. Then two geodesics on M are congruent if and only if the absolute values of their structure torsion coincide.*

COROLLARY 3.3. *Every geodesic on a horosphere, a geodesic sphere and a tube around complex hyperplane in a complex hyperbolic space is a simple curve.*

Let H_1^{2n+1} denote an anti-de Sitter space in $\boldsymbol{C}^{n+1}$. In order to obtain information on length of closed geodesics on geodesic spheres and tubes around complex hyperplanes, we study their horizontal lifts. Denoting the connection on H_1^{2n+1} by $\hat{\nabla}$, for vector fields U, V on $\boldsymbol{C}H^n$ we have

$$\hat{\nabla}_U V = \tilde{\nabla}_U V - \langle U, JV\rangle J\mathcal{N}_{H_1^{2n+1}}, \tag{3.2}$$

where $\mathcal{N}_{H_1^{2n+1}}$ denotes the time-like outward unit normal vector field on H_1^{2n+1} in $\boldsymbol{C}^{n+1}$ and in the left-hand side U, V are regarded as horizontal vector fields on H_1^{2n+1}.

When M is a geodesic sphere of radius r in $\boldsymbol{C}H^n$, each horizontal lift $\hat{\gamma}$ of $\imath \circ \gamma$ for a geodesic γ on M is a helix of order 3 in general sense: In fact, denoting by $\sin\theta$ $(0 \le |\theta| \le \pi/2)$ the structure torsion of γ, we find

$$\begin{cases} \hat{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = \ell_1 \hat{X}_2, \\ \hat{\nabla}_{\dot{\hat{\gamma}}}\hat{X}_2 = -\ell_1\dot{\hat{\gamma}} + \ell_2\hat{X}_3, \\ \hat{\nabla}_{\dot{\hat{\gamma}}}\hat{X}_3 = \ell_2\hat{X}_2, \end{cases}$$

where

$$\ell_1 = \coth r + \tanh r \cdot \sin^2\theta, \quad \ell_2 = |\sin\theta|\sqrt{1 - \tanh^2 r\cos^2\theta},$$

$$\hat{X}_2 = \mathcal{N}_M, \quad \hat{X}_3 = \frac{1}{\ell_2}(\langle A\dot{\gamma}, \dot{\gamma}\rangle\dot{\gamma} - A\dot{\gamma} + \sin\theta J\mathcal{N}_{H_1^{2n+1}}),$$

with the shape operator A of M in $\boldsymbol{C}H^n$ and the complex structure J on $\boldsymbol{C}^{n+1}$. Regarding $\hat{\gamma}$ as a curve in $\boldsymbol{C}^{n+1}$, we find that it satisfies the following ordinary differential equation;

$$\hat{\gamma}^{(4)} + (\coth^2 r - \cos^2\theta + \tanh^2 r\sin^2\theta)\hat{\gamma}'' + \sin^2\theta(1 - \tanh^2 r\cos^2\theta)\hat{\gamma} = 0.$$

Then $\hat{\gamma}$ is of the form

$$\hat{\gamma}(t) = A\exp(\sqrt{-1}t\tanh r\sin\theta) + B\exp(-\sqrt{-1}t\tanh r\sin\theta)$$
$$+ C\exp(\sqrt{-1}t\sqrt{\coth^2 r - \cos^2\theta}) + D\exp(-\sqrt{-1}t\sqrt{\coth^2 r - \cos^2\theta})$$

with some non-zero vectors $A, B, C, D \in \boldsymbol{C}^{n+1}$ which span the complex 3-space $\boldsymbol{C}^3$ which is also spanned by $\hat{\gamma}(0)$ and horizontal lifts of $\dot{\gamma}(0)$ and $\xi_{\gamma(0)}$. With the aid of Lemma 2.4 we get the following.

Theorem 3.4. *Let γ be a geodesic on a geodesic sphere M of radius r $(0 < r < \infty)$ in $\boldsymbol{C}H^n$ of holomorphic sectional curvature -4.*

(1) *If the structure torsion of γ is ± 1, then it is closed with length $\pi \sinh 2r$.*
(2) *If γ has null structure torsion, then it is also closed and its length is $2\pi \sinh r$.*
(3) *When the structure torsion of γ is of the form $\sin\theta$ $(0 < |\theta| < \pi/2)$, it is closed if and only if*

$$\sin\theta = \frac{\pm q}{\sinh r\sqrt{p^2 \tanh^2 r - q^2}}$$

with some relatively prime positive integers p and q with $q < p\tanh^2 r$. In this case, its length is

$$\operatorname{length}(\gamma) = \begin{cases} 2\pi\sqrt{p^2\sinh^2 r - q^2\cosh^2 r}, & \text{if } pq \text{ is even} \\ \pi\sqrt{p^2\sinh^2 r - q^2\cosh^2 r}, & \text{if } pq \text{ is odd}. \end{cases}$$

Remark. When the structure torsion is not equal to ± 1, every horizontal lift $\hat{\gamma}$ of $\imath \circ \gamma$ for a geodesic γ on M is closed if and only if γ is closed.

Next we study geodesics on a tube M of radius r around a complex hyperplane $\boldsymbol{C}H^{n-1}$. Let γ be a geodesic on M of structure torsion $\sin\theta$ $(0 \le |\theta| \le \pi/2)$. When $\cos\theta > \tanh r$, by use of (3.1) and (3.2) we find that each horizontal lift $\hat{\gamma}$ of $\imath \circ \gamma$ on H_1^{2n+1} is a helix of proper order 3:

$$\begin{cases} \hat{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = \ell_1 \hat{X}_2, \\ \hat{\nabla}_{\dot{\hat{\gamma}}}\hat{X}_2 = -\ell_1\dot{\hat{\gamma}} + \ell_2\hat{X}_3, \\ \hat{\nabla}_{\dot{\hat{\gamma}}}\hat{X}_3 = -\ell_2\hat{X}_2, \end{cases}$$

where

$$\ell_1 = |\tanh r + \coth r \sin^2\theta|, \quad \ell_2 = |\sin\theta|\sqrt{\coth^2 r\cos^2\theta - 1},$$

$$\hat{X}_2 = \mathcal{N}_M, \quad \hat{X}_3 = \frac{1}{\ell_2}(\langle A\dot{\gamma}, \dot{\gamma}\rangle\dot{\gamma} - A\dot{\gamma} + \sin\theta J\mathcal{N}_{H_1^{2n+1}}),$$

with the shape operator A of M in $\boldsymbol{C}H^n$ and the complex structure J on $\boldsymbol{C}^{n+1}$.

When $\cos\theta \le \tanh r$, we see that each horizontal lift $\hat{\gamma}$ of $\iota\circ\gamma$ onto H_1^{2n+1} is a helix of proper order 3 in general sense: When $\cos\theta < \tanh r$, it satisfies

$$\begin{cases} \hat{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = \quad \ell_1 \hat{X}_2, \\ \hat{\nabla}_{\dot{\hat{\gamma}}}\hat{X}_2 = -\ell_1\dot{\hat{\gamma}} + \ell_2\hat{X}_3, \\ \hat{\nabla}_{\dot{\hat{\gamma}}}\hat{X}_3 = \quad \ell_2\hat{X}_2, \end{cases}$$

and when $\cos\theta = \tanh r$, it satisfies

$$\begin{cases} \hat{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = \ell_1\hat{X}_2, \\ \hat{\nabla}_{\dot{\hat{\gamma}}}\hat{X}_2 = -\ell_1\dot{\hat{\gamma}} + \{\langle A\dot{\gamma}, \dot{\gamma}\rangle\dot{\gamma} - A\dot{\gamma} + \sin\theta J\mathcal{N}_{H_1^{2n+1}}\}, \\ \hat{\nabla}_{\dot{\hat{\gamma}}}\{\langle A\dot{\gamma}, \dot{\gamma}\rangle\dot{\gamma} - A\dot{\gamma} + \sin\theta J\mathcal{N}_{H_1^{2n+1}}\} = 0. \end{cases}$$

Here $\ell_1, \hat{X}_2$ and $\hat{X}_3$ are the same as in the case $\cos\theta > \tanh r$, and

$$\ell_2 = |\sin\theta|\sqrt{1 - \coth^2 r\cos^2\theta}.$$

In any case, regarding $\hat{\gamma}$ as a curve in $\boldsymbol{C}^{n+1}$, we find that it satisfies the following ordinary differential equation;

$$\hat{\gamma}^{(4)} + (\tanh^2 r - \cos^2\theta + \coth^2 r\sin^2\theta)\hat{\gamma}'' + \sin^2\theta(1 - \coth^2 r\cos^2\theta)\hat{\gamma} = 0.$$

Solving this equation, we have

$$\hat{\gamma}(t) = \begin{cases} A\exp(\sqrt{-1}t\coth r\sin\theta) + B\exp(-\sqrt{-1}t\coth r\sin\theta) \\ \qquad + C\exp(t\sqrt{\cos^2\theta - \tanh^2 r}) + D\exp(-t\sqrt{\cos^2\theta - \tanh^2 r}), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \cos\theta > \tanh r, \\ A + Bt + C\exp(\sqrt{-1}t\tan\theta), \qquad\qquad\qquad\quad \text{if } \cos\theta = \tanh r, \\ A\exp(\sqrt{-1}t\coth r\sin\theta) + B\exp(-\sqrt{-1}t\coth r\sin\theta) \\ \qquad + C\exp(\sqrt{-1}t\sqrt{\tanh^2 r - \cos^2\theta}) \\ \qquad + D\exp(-\sqrt{-1}t\sqrt{\tanh^2 r - \cos^2\theta}), \qquad\quad \text{if } \cos\theta < \tanh r, \end{cases}$$

with some non-zero vectors $A, B, C, D \in \boldsymbol{C}^{n+1}$ which span the complex 3-space $\boldsymbol{C}^3$ which is also spanned by $\hat{\gamma}(0)$ and horizontal lifts of $\dot{\gamma}(0)$ and $\xi_{\gamma(0)}$. For example, when $\cos\theta = \tanh r$ we have

$$A = \frac{1}{\sin^2\theta}\hat{x} + \frac{\cos\theta}{\sin^2\theta}\sqrt{-1}\hat{\xi}, \quad B = \cos\theta\hat{u} - \frac{\cos^2\theta}{\sin\theta}\hat{\xi} + \sqrt{-1}\cot\theta\hat{x},$$

$$C = -\cot^2\theta\hat{x} - \frac{\cos\theta}{\sin^2\theta}\sqrt{-1}\hat{\xi},$$

where $\hat{x} = \hat{\gamma}(0) \in \boldsymbol{C}^{n+1}$ and $\hat{u}, \hat{\xi} \in \boldsymbol{C}^{n+1} \cong T_{\hat{x}}\boldsymbol{C}^{n+1}$ are horizontal lifts of $u, \xi_{\gamma(0)} \in T_{\gamma(0)}M \subset T_{\gamma(0)}\boldsymbol{CH}^n$ with $\dot{\gamma}(0) = (\cos\theta)u + (\sin\theta)\xi_{\gamma(0)}$ and $u \in \langle \xi_{\gamma(0)} \rangle^{\perp}$.

We call a smooth curve σ on a complex hyperbolic space $\boldsymbol{CH}^n$ *unbounded in both directions* if both $\sigma([0,\infty))$ and $\sigma((-\infty,0])$ are unbounded sets. Considering the ideal boundary $\partial\boldsymbol{CH}^n$ of $\boldsymbol{CH}^n$ as a Hadamard manifold, we can define its limit points

$$\sigma(\infty) = \lim_{t\to\infty} \sigma(t), \quad \sigma(-\infty) = \lim_{t\to-\infty} \sigma(t) \in \partial\boldsymbol{CH}^n,$$

at infinity if they exist. We shall call a smooth curve σ on $\boldsymbol{CH}^n$ *horocyclic* if the following conditions hold.

i) It has single point at infinity; $\sigma(\infty) = \sigma(-\infty)$.

ii) If a geodesic ρ on $\boldsymbol{CH}^n$ with $\rho(\infty) = \sigma(\infty)$ crosses σ, then they cross orthogonally at their crossing point.

These conditions are equivalent to the condition that σ is unbounded in both directions and lies on a horosphere. As a consequence of the expressions of horizontal lifts of geodesics on a tube around a complex hyperplane, we obtain the following.

Theorem 3.5. *Let γ be a geodesic of structure torsion $\sin\theta$ $(0 \le |\theta| \le \pi/2)$ on a tube M of radius r $(0 < r < \infty)$ around a complex hyperplane $\boldsymbol{CH}^{n-1}$ in $\boldsymbol{CH}^n$ of holomorphic sectional curvature -4.*

(1) *If the structure torsion of γ is ± 1, then it is closed and its length is $\pi\sinh 2r$.*

(2) *If $\cos\theta > \tanh r$, then it is unbounded in both directions, and has two distinct points at infinity as a curve on $\boldsymbol{CH}^n$.*

(3) *If $\cos\theta = \tanh r$, then it is horocyclic as a curve on $\boldsymbol{CH}^n$.*

(4) *If $0 < \cos\theta < \tanh r$, then it is bounded. Under this situation, it is closed if and only if*

$$\sin\theta = \frac{\pm p}{\cosh r\sqrt{p^2 - q^2\coth^2 r}}$$

with some relatively prime positive integers p and q with $p\tanh^2 r > q$. In this case, its length is

$$\operatorname{length}(\gamma) = \begin{cases} 2\pi\sqrt{p^2\sinh^2 r - q^2\cosh^2 r}, & \text{if } pq \text{ is even} \\ \pi\sqrt{p^2\sinh^2 r - q^2\cosh^2 r}, & \text{if } pq \text{ is odd}. \end{cases}$$

Remark. When the structure torsion is not equal to ± 1, every horizontal lift $\hat{\gamma}$ of $\iota\circ\gamma$ for a geodesic γ on M is closed if and only if γ is closed.

Theorem 3.4 assures that the length spectrum of a geodesic sphere M of radius r in $\boldsymbol{C}H^n$ is of the following form;

$$\begin{aligned}\mathrm{Lspec}(M) &= \{2\pi\sinh r\} \cup \{\pi\sinh 2r\} \\ &\quad \cup \left\{ 2\pi\sqrt{p^2\sinh^2 r - q^2\cosh^2 r} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime} \\ \text{positive integers which satisfy} \\ pq \text{ is even and } q < p\tanh^2 r \end{array} \right\} \\ &\quad \cup \left\{ \pi\sqrt{p^2\sinh^2 r - q^2\cosh^2 r} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime} \\ \text{positive integers which satisfy} \\ pq \text{ is odd and } q < p\tanh^2 r \end{array} \right\},\end{aligned}$$

and Theorem 3.5 assures that the length spectrum of a tube M of radius r around a complex hyperplane $\boldsymbol{C}H^{n-1}$ in $\boldsymbol{C}H^n$ is of the following form;

$$\begin{aligned}\mathrm{Lspec}(M) &= \{\pi\sinh 2r\} \\ &\quad \cup \left\{ 2\pi\sqrt{p^2\sinh^2 r - q^2\cosh^2 r} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime} \\ \text{positive integers which satisfy} \\ pq \text{ is even and } q < p\tanh^2 r \end{array} \right\} \\ &\quad \cup \left\{ \pi\sqrt{p^2\sinh^2 r - q^2\cosh^2 r} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime} \\ \text{positive integers which satisfy} \\ pq \text{ is odd and } q < p\tanh^2 r \end{array} \right\}.\end{aligned}$$

We obtain the following on the multiplicity of the length spectrum.

THEOREM 3.6. *Let M be either a geodesic sphere of radius r or a tube of radius r around a complex hyperplane $\boldsymbol{C}H^{n-1}$ in $\boldsymbol{C}H^n$ of holomorphic sectional curvature -4.*

(1) *The length spectrum* $\mathrm{Lspec}(M)$ *is a discrete unbounded subset of the real line $\boldsymbol{R}$. In particular, there exist infinitely many congruency classes of closed geodesics.*

(2) *If* $\coth^2 r$ *is irrational of the form either*

$$\frac{1}{2q^2}\left\{(p^2+q^2-1) - \sqrt{\{p^2-(q-1)^2\}\{p^2-(q+1)^2\}}\right\}$$

for some relatively prime positive integers p, q with even pq and $p \geq q+3$, or

$$\frac{1}{2q^2}\left\{(p^2+q^2-4) - \sqrt{\{p^2-(q-2)^2\}\{p^2-(q+2)^2\}}\right\}$$

for some relatively prime positive integers p, q with odd pq and $p \geq q+4$, then the multiplicity of the spectrum $\pi\sinh 2r$ is two and other length spectrum of M is simple.

(3) *If* $\coth^2 r$ *is irrational and is not of the form in* (2), *then every length spectrum of* M *is simple.*

(4) *If* $\coth^2 r$ *is rational, the multiplicity of each length spectrum is finite. But it is not uniformly bounded*; $\limsup_{\lambda\to\infty} m_M(\lambda) = \infty$. *The growth order of* $m_M(\lambda)$ *is not so rapid. It satisfies* $\lim_{\lambda\to\infty} \lambda^{-\delta} m_M(\lambda) = 0$ *for arbitrary positive* δ.

Proof. (1) Let Λ be an arbitrary positive number. If positive integers p and q satisfy $p^2\tanh^2 r - q^2 < \Lambda$ and $p\tanh^2 r > q$, then $p^2(1-\tanh^2 r)\tanh^2 r < \Lambda$, so that the number of such pairs of integers is finite. Thus we know that $\mathrm{Lspec}(M)$ is unbounded and discrete.

(2), (3) By the same argument as in the proof of Theorem 2.9 (1) we only need to check the multiplicity of $\pi\sinh 2r$. If relatively prime positive integers p, q with even pq satisfy $\sinh 2r = 2\sqrt{p^2\sinh^2 r - q^2\cosh^2 r}$, then one finds that $\coth^2 r$ is of the first form in the assertion by noticing $p > q\coth^2 r$ and $\coth^2 r$ is irrational. Similarly, the equality $\sinh 2r = \sqrt{p^2\sinh^2 r - q^2\cosh^2 r}$ for relatively prime positive integers p, q with odd pq leads us to the second form of $\coth^2 r$ in the assertion. Conversely, if $\coth^2 r$ is one of these forms then the multiplicity of $\pi\sinh 2r$ is two. We get the assertions (2) and (3).

(4) This assertion follows from the following Lemmas 3.7 and 3.8. We apply these lemmas by putting $\tanh^2 r = a/b$ with relatively prime positive integers a, b $(a < b)$ and $\varepsilon_1 = 0, \varepsilon_2 = \varepsilon = \tanh^2 r$. □

Remark. In (4), the multiplicity of length spectrum of geodesics with restricted structure torsion is not uniformly bounded; $\limsup_{\lambda\to\infty} m_{(\alpha,\beta)}(\lambda) = \infty$, for α, β with $0 < \alpha < \beta < 1$ in the case M is a geodesic sphere and for α, β with $1/\cosh r < \alpha < \beta < 1$ in the case M is a tube of radius r around a complex hyperplane.

Lemma 3.7. *For relatively prime positive integers* a, b *and a positive integer* λ, *we define a set* $\mathscr{T}(a,b;\lambda)$ *of pairs of integers as*

$$\mathscr{T}(a,b;\lambda) = \left\{ (p,q) \in \boldsymbol{Z}\times\boldsymbol{Z} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime integers which} \\ \text{satisfy } pq \text{ is even and } ap^2 - bq^2 = \lambda \end{array} \right\}.$$

For arbitrary real numbers $\varepsilon_1, \varepsilon_2$ *with* $\varepsilon_2 > \varepsilon_1 \geq 0$ *and* $\varepsilon_2 < \sqrt{a/b}$, *the cardinality* $m_{(a,b;\varepsilon_1,\varepsilon_2)}(\lambda)$ *of the set* $\{(p,q) \in \mathscr{T}(a,b;\lambda) \mid \varepsilon_1 p < q < \varepsilon_2 p\}$ *satisfies*

$$\limsup_{\lambda\to\infty} m_{(a,b;\varepsilon_1,\varepsilon_2)}(\lambda) = \infty.$$

Proof. For relatively prime positive integers λ, v we define a map

$$\mathscr{T}(a,b;\lambda) \times \mathscr{T}(1,ab;v) \to \mathscr{T}(a,b;\lambda v)$$

by

$$((p,q),(\tilde{p},\tilde{q})) \mapsto (p,q) \star (\tilde{p},\tilde{q}) = (p\tilde{p} + bq\tilde{q}, \tilde{p}q + ap\tilde{q}).$$

Since λ and ν are relatively prime, it is well-defined; $p\tilde{p} + bq\tilde{q} \neq 0$, $\tilde{p}q + ap\tilde{q} \neq 0$. It satisfies the following properties.

1) $\sqrt{a}P + \sqrt{b}Q = (\sqrt{a}p + \sqrt{b}q)(\tilde{p} + \sqrt{ab}\tilde{q})$ for $(P,Q) = (p,q) \star (\tilde{p},\tilde{q})$.
2) $\overline{(p,q) \star (\tilde{p},\tilde{q})} = \overline{(p,q)} \star \overline{(\tilde{p},\tilde{q})}$, where $\overline{(p,q)} = (p,-q)$.
3) For fixed (p,q) the correspondence $(\tilde{p},\tilde{q}) \mapsto (p,q) \star (\tilde{p},\tilde{q})$ is injective, and the correspondence $(p,q) \mapsto (p,q) \star (\tilde{p},\tilde{q})$ is also injective for fixed $(\tilde{p},\tilde{q})$.
4) When $a = 1$, the operation $\star$ is commutative; $(p,q) \star (\tilde{p},\tilde{q}) = (\tilde{p},\tilde{q}) \star (p,q)$ for every $(p,q) \in \mathscr{T}(1,b;\lambda)$ and $(\tilde{p},\tilde{q}) \in \mathscr{T}(1,b;\nu)$.

The property 1) guarantees that

$$\tanh^{-1}\left(\frac{\sqrt{b}Q}{\sqrt{a}P}\right) = \tanh^{-1}\left(\frac{\sqrt{b}q}{\sqrt{a}p}\right) + \tanh^{-1}\left(\frac{\sqrt{ab}\tilde{q}}{\tilde{p}}\right).$$

Here one should note that $(\sqrt{b}q)/(\sqrt{a}p) < 1$ for $(p,q) \in \mathscr{T}(a,b;\lambda)$. We can prove the assertion by replacing the role of $\arg(\sqrt{a}p + \sqrt{-b}q)$ by $\tanh^{-1}((\sqrt{b}q)/(\sqrt{a}p))$ in the proof of Lemma 2.10. □

For positive integers a, b, λ and a positive real number ε with $\varepsilon < \sqrt{a/b}$, we denote by $\tilde{m}_{a,b;\varepsilon}(\lambda)$ the cardinality of the set $\{(p,q) \in \boldsymbol{Z} \times \boldsymbol{Z} \mid ap^2 - bq^2 = \lambda, \varepsilon p > q > 0\}$. It is clear that $m_{(a,b;\varepsilon_1,\varepsilon_2)}(\lambda) \leq \tilde{m}_{a,b;\varepsilon}(\lambda)$ for every ε_1 and ε_2 with $\varepsilon_1 < \varepsilon_2 \leq \varepsilon$.

Lemma 3.8. (1) *There exists a constant $C_{a,b;\varepsilon}$ depending only on a, b and ε such that $\tilde{m}_{a,b;\varepsilon}(\lambda) \leq C_{a,b;\varepsilon} d(\lambda)$ holds for arbitrary positive λ.*

(2) *In particular, we have $\lim_{\lambda\to\infty} \lambda^{-\delta}\tilde{m}_{a,b;\varepsilon}(\lambda) = 0$ for arbitrary positive δ.*

Proof. If a pair (p,q) of integers satisfies $ap^2 - bq^2 = \lambda$ and $0 < q < \varepsilon p$, then the pair $(P,Q) = (ap,q)$ of integers satisfies $P^2 - abQ^2 = a\lambda$ and $0 < Q < (\varepsilon/a)P$. Since $\varepsilon/a < 1/\sqrt{ab}$ if $\varepsilon < \sqrt{a/b}$, we have $\tilde{m}_{a,b;\varepsilon}(\lambda) \leq \tilde{m}_{1,ab;\varepsilon/a}(a\lambda)$. Hence we may assume $a = 1$ because $d(a\lambda) \leq d(a)d(\lambda)$.

When b is a square, we can associate a pair of positive integers (p,q) satisfying $p^2 - bq^2 = \lambda$ with a positive divisor $p + \sqrt{b}q$ of λ which is greater than $\sqrt{\lambda}$. This shows that $\tilde{m}_{1,b;\varepsilon}(\lambda) \leq d(\lambda)/2$. When b is not a square, our proof goes through almost similarly as that of Lemma 2.11, except the following modification. We denote by $\mathcal{O}$ the ring of integers of the real quadratic field $\boldsymbol{Q}(\sqrt{b})$, and define the "hyperbolic argument" of $p + \sqrt{b}q \in \mathcal{O}$ by

$$\operatorname{argh}(p + \sqrt{b}q) = \tanh^{-1}\frac{\sqrt{b}q}{p}.$$

This is well-defined because b is not a square. If there are distinct two pairs (p,q) and $(\tilde{p},\tilde{q})$ of integers with $0<q<\varepsilon p$, $0<\tilde{q}<\varepsilon\tilde{q}$ and $(p+\sqrt{b}q)\mathcal{O}=(\tilde{p}+\sqrt{b}\tilde{q})\mathcal{O}$, we find that $u=(p+\sqrt{b}q)/(\tilde{p}+\sqrt{b}\tilde{q})$ is a unit of $\mathcal{O}$ such that

i) $0<|\mathrm{argh}(u)|<\tanh^{-1}(\varepsilon\sqrt{b})$,

ii) $u,\bar{u}>0$ (i.e. u is totally positive).

We here note that $\varepsilon\sqrt{b}<1$ and that the unit $u^{-1}=(\tilde{p}+\sqrt{b}\tilde{q})/(p+\sqrt{b}q)$ $(=\bar{u})$ satisfies $\mathrm{argh}(u^{-1})=-\mathrm{argh}(u)$. As the multiplicative group of totally positive units of $\mathcal{O}$ is an infinite cyclic group, we choose its generator u_0 with minimum positive hyperbolic argument. We then obtain the number of totally positive units u of $\mathcal{O}$ with $0<\mathrm{argh}(u)<\tanh^{-1}(\varepsilon\sqrt{b})$ is not greater than $C_{1,b;\varepsilon}=\tanh^{-1}(\varepsilon\sqrt{b})\times\{\mathrm{argh}(u_0)\}^{-1}$, because $\mathrm{argh}(u_0^k)=k\,\mathrm{argh}(u_0)$. This completes the proof with the aid of the argument in the proof of Lemma 2.11. □

Remark. When $a<b$, we can put $\varepsilon=a/b$ in Lemma 3.8. Hence in this case, the cardinality $\tilde{m}_{a,b}(\lambda)$ of the set $\{(p,q)\in\boldsymbol{Z}\times\boldsymbol{Z}\mid ap^2-bq^2=\lambda, ap>bq>0\}$ satisfies $\lim_{\lambda\to\infty}\lambda^{-\delta}\tilde{m}_{a,b}(\lambda)=0$ for arbitrary positive δ.

We here study the first and the second length spectrum of geodesic spheres and tubes around complex hyperplanes in a complex hyperbolic space.

Proposition 3.9. *Let M be a geodesic sphere of radius r in $\boldsymbol{C}H^n$ of holomorphic sectional curvature -4.*

(1) *The first length spectrum is $2\pi\sinh r$, which is the length of geodesics with null structure torsion. It is simple.*

(2) *The second length spectrum is also simple. When $r>(1/2)\log(2+\sqrt{3})$ (i.e. $\coth^2 r<3$), it is $\pi\sqrt{8\sinh^2 r-1}$, which is the length of geodesics with structure torsion $\pm 1/(\sinh r\sqrt{9\tanh^2 r-1})$. When $r\le(1/2)\log(2+\sqrt{3})$, it is $\pi\sinh 2r$, which is the length of geodesics with structure torsion ± 1.*

Proof. We prove this proposition by reductive absurdity.

(1) If we suppose that relatively prime integers p,q with odd pq and $p>q\coth^2 r$ satisfy

$$2\pi\sinh r\ge\pi\sqrt{p^2\sinh^2 r-q^2\cosh^2 r},$$

then $3\le p<2\cosh r$, so that $\coth^2 r<p^2/(p^2-4)$. Since $p-2$ is an odd integer and satisfies $p>(p-2)\coth^2 r$, we have $q\le p-2$, hence

$$4\sinh^2 r\ge p^2\sinh^2 r-q^2\cosh^2 r\ge 4(p-1)\sinh^2 r-(p-2)^2.$$

Thus we obtain $p^2-4<4\sinh^2 r\le p-2$, which is a contradiction.

Next, if we suppose that relatively prime integers p,q with even pq and $p>q\coth^2 r$ satisfy

$$2\pi\sinh r\ge 2\pi\sqrt{p^2\sinh^2 r-q^2\cosh^2 r},$$

then $2 \le p < \cosh r$, so that $\coth^2 r < p^2/(p^2-1)$. Since $p > (p-1)\coth^2 r$, we have $q \le p-1$, hence

$$\sinh^2 r \ge p^2 \sinh^2 r - q^2 \cosh^2 r \ge (2p-1)\sinh^2 r - (p-1)^2.$$

Thus we obtain $2(p^2-1) < 2\sinh^2 r \le p-2$, which is also a contradiction. We therefore obtain the assertion as $2\pi \sinh r < \pi \sinh 2r$.

(2) Repeating the same argument as above, one can easily get the assertion. $\square$

REMARK. The sectional curvature of a geodesic sphere M of radius r in $\boldsymbol{CH}^n$ lies in the interval $[-(4-\coth^2 r), \coth^2 r]$, so that it has positive sectional curvature if $\coth r > 2$. However for this geodesic sphere M, every geodesic γ on M satisfies length$(\gamma) > 2\pi/\sqrt{\coth^2 r}$. Hence it is not an example of a Berger sphere.

PROPOSITION 3.10. *Let M be a tube of radius r around a complex hyperplane $\boldsymbol{CH}^{n-1}$ in $\boldsymbol{CH}^n$ of holomorphic sectional curvature -4.*

(1) *The first length spectrum is simple. It is*

$$\begin{cases} \pi\sqrt{8\sinh^2 r - 1}, & \textit{when } \coth r < \sqrt{3} \ (\textit{i.e. } r > (1/2)\log(2+\sqrt{3})), \\ \pi \sinh 2r, & \textit{when } \coth r \ge \sqrt{3} \ (\textit{i.e. } r \le (1/2)\log(2+\sqrt{3})). \end{cases}$$

It is the length of geodesics with structure torsion $\pm 3/(\cosh r\sqrt{9-\coth^2 r})$ and ± 1, respectively.

(2) *The second length spectrum is also simple. It is*

$$\begin{cases} 2\pi\sqrt{3\sinh^2 r - 1}, & \textit{when } \coth r < \sqrt{2} \ (\textit{i.e. } r > \log(1+\sqrt{2})), \\ \pi \sinh 2r, & \textit{when } \sqrt{2} \le \coth r < \sqrt{3} \\ & \quad (\textit{i.e. } (1/2)\log(2+\sqrt{3}) < r \le \log(1+\sqrt{2})), \\ \pi\sqrt{4m(m+1)\sinh^2 r - 1}, & \\ & \textit{when } \sqrt{2m-1} \le \coth r < \sqrt{2m+1}, \ (m = 2, 3, \dots) \\ & \quad (\textit{i.e. } (1/2)\{\log(\sqrt{2m+1}+1) - \log(\sqrt{2m+1}-1)\} < r \\ & \quad \leqq (1/2)\{\log(\sqrt{2m-1}+1) - \log(\sqrt{2m-1}-1)\}). \end{cases}$$

It is the length of geodesics of structure torsion $\pm 2/(\cosh r\sqrt{4-\coth^2 r})$, ± 1 and $\pm(2m+1)/\left(\cosh r\sqrt{(2m+1)^2 - \coth^2 r}\right)$, respectively.

We now make mention of the growth of the number of congruency classes of geodesics with respect to their length. Since for positive real numbers α, β with $\beta > \alpha$, the area $\mathrm{Area}(E_\lambda)$ of the set

$$E_\lambda = \{(x, y) \in \boldsymbol{R}^2 \mid \alpha x^2 - \beta y^2 \leq \lambda^2, \alpha x \geq \beta y \geq 0\}$$

is $(\lambda^2/2\sqrt{\alpha\beta})\{\log(\sqrt{\alpha} + \sqrt{\beta}) - (1/2)\log(\beta - \alpha)\}$, we obtain the following by just the same way as in the proof of Theorem 2.12.

THEOREM 3.11. *Let M be either a geodesic sphere of radius r or a tube of radius r around a complex hyperplane $\boldsymbol{CH}^{n-1}$ in $\boldsymbol{CH}^n$ of holomorphic sectional curvature -4. Then the number $n_M(\lambda)$ of congruency classes of geodesics with length not greater than λ satisfies*

$$\lim_{\lambda\to\infty} \frac{n_M(\lambda)}{\lambda^2} = \frac{3r}{\pi^4 \sinh 2r}.$$

Finally we study geodesics on a horosphere M. We shall show that every geodesic on a horosphere is an unbounded open curve. Every horizontal lift $\hat{\gamma}$ of $\iota \circ \gamma$ for a geodesic γ on M is a helix of order 3 in general sense: If we denote by $\sin\theta$ $(0 \leq |\theta| \leq \pi/2)$ the structure torsion of γ, we have

$$\begin{cases} \hat{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = \quad \ell_1 \hat{X}_2, \\ \hat{\nabla}_{\dot{\hat{\gamma}}}\hat{X}_2 = -\ell_1\dot{\hat{\gamma}} + \ell_2\hat{X}_3, \\ \hat{\nabla}_{\dot{\hat{\gamma}}}\hat{X}_3 = \quad \ell_2\hat{X}_2, \end{cases}$$

where

$$\ell_1 = 1 + \sin^2\theta, \quad \ell_2 = \sin^2\theta,$$

$$\hat{X}_2 = \mathcal{N}_M, \quad \hat{X}_3 = \frac{1}{\ell_2}(\langle A\dot{\gamma}, \dot{\gamma}\rangle\dot{\gamma} - A\dot{\gamma} + \sin\theta J\mathcal{N}_{H_1^{2n+1}}),$$

with the shape operator A of M in $\boldsymbol{CH}^n$ and the complex structure J on $\boldsymbol{C}^{n+1}$. Regarding $\hat{\gamma}$ as a curve in $\boldsymbol{C}^{n+1}$, we find that it satisfies the ordinary differential equation

$$\hat{\gamma}^{(4)} + 2\sin^2\theta\hat{\gamma}'' + \sin^4\theta\hat{\gamma} = 0.$$

Then we know that $\hat{\gamma}$ is of the form

$$\hat{\gamma}(t) = (A + Bt)\exp(\sqrt{-1}t\sin\theta) + (C + Dt)\exp(-\sqrt{-1}t\sin\theta)$$

with some non-zero vectors $A, B, C, D \in \boldsymbol{C}^{n+1}$ which span the complex 3-space $\boldsymbol{C}^3$ which is also spanned by $\hat{\gamma}(0)$ and horizontal lifts of $\dot{\gamma}(0)$ and $\xi_{\gamma(0)}$. Therefore we get the following.

PROPOSITION 3.12. *Every geodesic on a horosphere in a complex hyperbolic space is unbounded in both directions, hence it is horocyclic as a curve on a complex hyperbolic space.*

4. Some characterizations.

It is well-known that a hypersurface M^n in a Euclidean space $\boldsymbol{R}^{n+1}$ is locally congruent to a standard sphere if and only if all geodesics on M are circles of positive curvature in $\boldsymbol{R}^{n+1}$. On the contrary, there do not exist real hypersurfaces in a non-flat complex space form $\tilde{M}_n$ all of whose geodesics are circles in $\tilde{M}_n$. This fact shows that the condition "all geodesics on M are circles in the ambient space $\tilde{M}_n$" is too strong for real hypersurfaces M in a non-flat complex space form. So we shall consider a weaker condition.

In this section we give some characterizations of geodesic spheres, tubes around complex hyperplanes and horospheres in a non-flat complex space form in connection with Propositions 2.1 and 3.1.

PROPOSITION 4.1. *A real hypersurface M^{2n-1} in a non-flat complex space form $\tilde{M}_n$, $n \geq 3$ (that is, $\tilde{M}_n = \boldsymbol{CP}^n$ or $\boldsymbol{CH}^n$) is locally congruent to one of a geodesic sphere, a tube around complex hyperplane, and a horosphere if and only if at each point $x \in M$ there exist orthonormal tangent vectors $v_1, v_2, \dots, v_{2n-2} \in T_xM$ orthogonal to the structure vector ξ_x such that all geodesics on M emanating x in the direction $v_i + v_j$ $(1 \leq i \leq j \leq 2n-2)$ are circles of positive curvatures in $\tilde{M}_n$.*

PROOF. The "if" part is a consequence of Propositions 2.1 and 3.1. So we show the "only if" part. Let γ_i $(1 \leq i \leq 2n-2)$ be geodesics on M with $\gamma_i(0) = x$ and $\dot{\gamma}_i(0) = v_i$ which satisfy $\tilde{\nabla}_{\dot{\gamma}_i}(\tilde{\nabla}_{\dot{\gamma}_i}\dot{\gamma}_i) = -\kappa_i^2\dot{\gamma}_i$ with some positive constants κ_i. It follows from (1.1) that

$$\tilde{\nabla}_{\dot{\gamma}_i}(\tilde{\nabla}_{\dot{\gamma}_i}\dot{\gamma}_i) = \langle(\nabla_{\dot{\gamma}_i}A)\dot{\gamma}_i, \dot{\gamma}_i\rangle\mathcal{N}_M - \langle A\dot{\gamma}_i, \dot{\gamma}_i\rangle A\dot{\gamma}_i.$$

Comparing the tangential components of these equations on $\tilde{\nabla}_{\dot{\gamma}_i}(\tilde{\nabla}_{\dot{\gamma}_i}\dot{\gamma}_i)$, we find $\langle A\dot{\gamma}_i, \dot{\gamma}_i\rangle A\dot{\gamma}_i = \kappa_i^2\dot{\gamma}_i$, so that at $t = 0$

$$\langle Av_i, v_i\rangle Av_i = \kappa_i^2 v_i. \tag{4.1}$$

Since $\kappa_i \neq 0$ $(1 \leqq i \leq 2n-2)$, this implies

$$Av_i = \kappa_i v_i \ \text{ or } \ Av_i = -\kappa_i v_i \quad \text{for } i = 1, \dots, 2n-2. \tag{4.2}$$

This guarantees that ξ is principal and that

$$\langle Av_i, v_j\rangle = 0 \quad \text{for } 1 \leq i < j \leq 2n-2.$$

Let γ_{ij} $(1 \leq i < j \leq 2n-2)$ be geodesics on M with $\gamma_{ij}(0) = x$ and $\dot{\gamma}_{ij}(0) = (v_i + v_j)/\sqrt{2}$. We similarly find from (4.1) that

$$\langle A(v_i + v_j), (v_i + v_j)\rangle A(v_i + v_j) = 2\kappa_{ij}^2(v_i + v_j)$$

for some positive κ_{ij}. Thus we have $\langle A(v_i+v_j), v_i-v_j\rangle = 0$ for $1 \le i < j \le 2n-2$, so that

$$\langle Av_i, v_i\rangle = \langle Av_j, v_j\rangle \quad \text{for } 1 \le i, j \le 2n-2. \tag{4.3}$$

It follows from (4.2) and (4.3) that $Av = \kappa v$ holds for each tangent vector $v \in T_xM$ orthogonal to ξ_x with some κ. This implies that M is η-umbilic at x. Since x is arbitrary, we get that M is totally η-umbilic in $\tilde{M}_n$, which leads us to the conclusion (see, [**M**], [**T2**]). □

Finally we give a characterization of a horosphere in a complex hyperbolic space.

PROPOSITION 4.2. *A real hypersurface M^{2n-1} in $\boldsymbol{C}H^n$ $n \ge 2$ of holomorphic sectional curvature -4 is locally congruent to a horosphere if and only if at each point $x \in M$ there exist orthonormal tangent vectors $v_1, v_2, \dots, v_{2n-2} \in T_xM$ orthogonal to the structure vector ξ_x such that all geodesics on M emanating x and with the initial vector v_i $(1 \le i \le 2n-2)$ are circles of curvature 1 on $\boldsymbol{C}H^n$.*

PROOF. The "if" part follows from Proposition 3.1. We show the "only if" part. By the discussion in the proof of Proposition 4.1 we find that $Av_i = \pm v_i$ and so that ξ is principal. Therefore M has constant principal curvatures α and ± 1, where $A\xi = \alpha\xi$. From the well-known classification theorem on Hopf hypersurfaces with constant principal curvatures in $\boldsymbol{C}H^n, n \ge 2$ we can see that M is locally congruent to a horosphere (for details, see [**B**]). □

Added in proof. After this paper having been accepted for publication the authors ([**AM2**], [**A**]) obtained corresponding results on length spectrum of geodesic spheres in other symmetric spaces of rank one.

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