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Convergence of the normalized solution of the Maurer-Cartan equation in the Barannikov-Kontsevich construction

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Abstract. We give a detailed proof of convergence of a normalized solution of the Maurer-Cartan equation in the Barannikov-Kontsevich construction.

1. Introduction.

The purpose of this paper is to show that the potential of the formal Frobenius manifold constructed by Barannikov and Kontsevich in [**1**], converges. Now we recall the definition of Frobenius manifolds, which were introduced and investigated by B. Dubrovin: cf. [**3**].

According to [**3**] and [**8**], a *Frobenius manifold* is a quadruple $(M, \mathscr{T}_M^f, g, A)$. Here M is a supermanifold in one of the standard categories (C^∞, analytic, algebraic, formal, etc.), $\mathscr{T}_M^f$ is the sheaf of flat vector fields tangent to an affine structure, g is a flat Riemannian metric (non-degenerate even symmetric quadratic tensor) such that $\mathscr{T}_M^f$ consists of g-flat tangent fields. Finally, A is an even symmetric tensor $A : S^3(\mathscr{T}_M) \to \mathscr{O}_M$, where $\mathscr{O}_M$ is the sheaf of germs of functions on M in the sense of supermanifold. All the data must satisfy the following conditions:

(a) *Potentiality of A*. Everywhere locally there exists a function Φ such that $A(X, Y, Z) = XYZ\Phi$ for any flat vector fields X, Y, and Z. Φ is called *potential*.

(b) *Associativity*. A and g together define a unique symmetric multiplication $\circ : S^2(\mathscr{T}_M) \to \mathscr{T}_M$ such that $A(X, Y, Z) = g(X \circ Y, Z) = g(X, Y \circ Z)$. Then this multiplication must be associative.

Thus given $(M, \mathscr{T}_M^f, g)$, Frobenius manifold structure on it is determined by a potential satisfying the associativity condition. In a formal Frobenius manifold, the potential Φ is a formal power series. If Φ converges, then we can consider that the Frobenius manifold is in the holomorphic category.

The Barannikov-Kontsevich construction is one of large classes of formal Frobenius manifolds. We explain it in §2. On the other hand, quantum co-

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homology which was discoverd by physicists is also a large class of formal Frobenius manifolds (cf. [**6**]). Its potential is called Gromov-Witten potential. In general, it is difficult to prove the convergence of Gromov-Witten potential.

In this paper, we give a detailed proof of convergence of the normalized solution of the Maurer-Cartan equation and the potential in the Barannikov-Kontsevich construction. Consequently, the Barannikov-Kontsevich construction gives a large class of holomorphic Frobenius manifolds. We state the precise statement in §3, Theorem 3.1 and Corollary 3.2. We remark that Cao-Zhou [**2**] mentioned the convergence of the Barannikov-Kontsevich construction without proof.

2. Barannikov-Kontsevich constructions.

In this section, we briefly recall the construction of Barannikov-Kontsevich [**1**]. We use the notation in Manin [**7**].

Let M be a compact connected Kähler manifold of dimension n whose canonical bundle K_M is holomorphically trivial. It follows from the condition $K_M = 0$ that there exists a nowhere vanishing holomorphic volume form $\Omega \in H^0(M, \Omega_M^n)$. It is defined up to a multiplication by a constant. Let us fix a choice of Ω.

Put

$$\mathbf{t}^{p,q} := \Gamma\left(M, \bigwedge^p \bar{T}_M^* \otimes \bigwedge^q T_M\right),$$

$$\mathbf{t}^n := \bigoplus_{p+q=n} \mathbf{t}^{p,q}, \quad \mathbf{t} := \bigoplus_n \mathbf{t}^n.$$

We define $\boldsymbol{Z}$- and $\boldsymbol{Z}_2$-grading on $\mathbf{t}$, as follows:

$$\boldsymbol{Z}\text{-grading: } |\gamma| := p + q \tag{1}$$

$$\boldsymbol{Z}_2\text{-grading: } \tilde{\gamma} := p + q \mod 2 \quad \text{for } \gamma \in \mathbf{t}^{p,q}.$$

$\mathbf{t}$ is endowed with differential $\bar{\partial}$ and wedge product $\wedge$. Then $(\mathbf{t}, \wedge, \bar{\partial})$ is a supercommutative differential graded algebra with respect to the grading above.

Moreover $\mathbf{t}$ is endowed with the standard Schouten-Nijenhuis bracket. Explicitly, for $X = X_1 \wedge \cdots \wedge X_p$, $Y = Y_1 \wedge \cdots \wedge Y_q$ (where X_i, Y_j are vector fields of type $(1,0)$) and $f \in C^\infty(M)$, define

$$\begin{cases} [X \bullet Y] = (-1)^p \sum_{s,t} (-1)^{s+t} \widehat{X_s} \wedge [X_s, Y_t] \wedge \widehat{Y}_t \\ [X \bullet f] = (-1)^p \sum_{s=1}^{p} (-1)^s X_s(f) \widehat{X_s}, \end{cases} \tag{2}$$

where $\widehat{X_s} := X_1 \wedge \cdots \wedge X_{s-1} \wedge X_{s+1} \wedge \cdots \wedge X_p$. For $\varphi = d\bar{z}_I \otimes X$, $\psi = d\bar{z}_J \otimes Y$, define

$$[\varphi \bullet \psi] = (-1)^{j(p+1)}\, d\bar{z}_I \wedge d\bar{z}_J \otimes [X \bullet Y].$$

Then one can see that this bracket satisfies the following formulas:

$$\begin{cases} [a \bullet b] = -(-1)^{(\tilde{a}+1)(\tilde{b}+1)}[b \bullet a] \\ [a \bullet [b \bullet c]] = [[a \bullet b] \bullet c] + (-1)^{(\tilde{a}+1)(\tilde{b}+1)}[b \bullet [a \bullet c]] \\ [a \bullet bc] = [a \bullet b]c + (-1)^{(\tilde{a}+1)\tilde{b}}b[a \bullet c], \end{cases} \tag{3}$$

and $\bar{\partial}$ is the derivation with respect to both $\wedge$ and $[\bullet]$.

Now using Ω, we define another differential Δ on $\mathbf{t}$. Let $A^{p,q}(M) := \{\text{smooth } (p,q)\text{-forms on } M\}$. We consider

$$I : \mathbf{t}^{p,q} \to A^{n-q,p}(M)$$

defined by

$$\begin{cases} I(d\bar{z}_I \otimes X_1 \wedge \cdots \wedge X_p) := d\bar{z}_I \wedge i_{X_1} \cdots i_{X_p}\Omega & \text{for } X_i\text{: vector fields} \\ I(d\bar{z}_I \otimes f) := d\bar{z}_I \wedge f\Omega & \text{for } f\text{: functions,} \end{cases} \tag{4}$$

where i_X denotes interior product. Clearly,

$$\bar{\partial} I = I\bar{\partial}. \tag{5}$$

Now define another differential $\Delta : \mathbf{t}^{p,q} \to \mathbf{t}^{p,q-1}$ by the formula:

$$\Delta I = I\partial. \tag{6}$$

The operators $\bar{\partial}$ and Δ satisfy the following properties:

$$\begin{cases} \bar{\partial}^2 = \bar{\partial}\Delta + \Delta\bar{\partial} = \Delta^2 \\ \Delta(1) = 0 \\ [\alpha \bullet \beta] = (-1)^{\tilde{\alpha}}\{\Delta(\alpha \wedge \beta) - (\Delta\alpha) \wedge \beta - (-1)^{\tilde{\alpha}}\alpha \wedge (\Delta\beta)\}. \end{cases} \tag{7}$$

The last formula in (7) is known as the Tian-Todorov lemma. A super-commutative algebra satisfying the properties (3) and (7) is called *differential Gerstenhaber-Batalin-Vilkovisky algebra* (see Manin [**7**], §5).

Formulas (4), (5) and (6) imply that I induces isomorphisms: $H(\mathbf{t}, \bar{\partial}) \cong H^*_{\bar{\partial}}(M)$ and $H(\mathbf{t}, \Delta) \cong H^*_{\partial}(M)$. Consequently, $\mathbf{H} := H(\mathbf{t}, \Delta)$ is finite dimensional.

Introduce a linear functional on $\mathbf{t}$ by

$$\int \gamma := \int_M I(\gamma) \wedge \Omega \quad \text{for } \gamma \in \mathbf{t}. \tag{8}$$

Then $\int$ satisfies the following identities:

$$\begin{aligned} \forall \omega, \eta \in \mathbf{t}, \quad & \int (\bar{\partial}\omega) \wedge \eta = (-1)^{\tilde{\omega}+1} \int \omega \wedge (\bar{\partial}\eta) \\ & \int (\Delta\omega) \wedge \eta = (-1)^{\tilde{\omega}} \int \omega \wedge (\Delta\eta). \end{aligned} \tag{9}$$

We define a symmetric pairing g on $\mathbf{H}$ by

$$g([\omega],[\eta]) := \int \omega \wedge \eta.$$

Then g is well-defined and nondegenerate.

Choose a homogeneous basis $\{[\gamma_a]\}_a$ ($\gamma_a \in \ker \Delta$) of $\mathbf{H}$. The $\partial\bar{\partial}$-lemma on Kähler manifolds (cf. Griffiths-Harris [**4**]) implies that we can choose γ_a in $\ker \Delta \cap \ker \bar{\partial}$. We assume that γ_0 equals 1. Let $\{t^a\}$ be the dual basis of $\{[\gamma_a]\}$. Define

$$|t^a| := 2 - |\gamma_a|. \tag{10}$$

Then $\boldsymbol{C}[[t_{\mathbf{H}}]] \hat{\otimes} \mathbf{t}$ inherits a natural grading. Here $\boldsymbol{C}[[t_{\mathbf{H}}]]$ is a formal power series ring in the superalgebra sense. See (13).

In [**1**], Barannikov and Kontsevich showed the following:

Theorem 2.1 (Barannikov-Kontsevich [**1**]). *There exists a solution to the Maurer-Cartan equation*

$$\bar{\partial}\Gamma(t) + \frac{1}{2}[\Gamma(t) \bullet \Gamma(t)] = 0 \tag{11}$$

in formal power series with value in $\mathbf{t}$

$$\Gamma(t) = \sum_a \gamma_a t^a + \sum_{\substack{a_1 < \cdots < a_k \\ k \geq 2}} \gamma_{a_1 \cdots a_k} t^{a_1} \cdots t^{a_k} \in (\boldsymbol{C}[[t_{\mathbf{H}}]] \hat{\otimes} \mathbf{t})^2$$

such that

(i) γ_a *are chosen as above,*
(ii) $\gamma_{a_1 \cdots a_k} \in \operatorname{Im} \Delta$ *for* $k \geq 2$,
(iii) $\partial_0 \Gamma(t) = 1$, *where* ∂_0 *is the coordinate vecter field corresponding to* $[1] \in \mathbf{H}$.

We call such a solution $\Gamma(t)$ *normalized*, and denote $\Gamma(t) = \Gamma_1(t) + \Delta B(t)$, where $\Gamma_1(t) := \sum_a \gamma_a t^a$.

Theorem 2.2 (Barannikov-Kontsevich [**1**]). *Put*

$$\Phi(t) = \int\left(\frac{1}{6}\Gamma(t)^3 - \frac{1}{2}\bar{\partial}B(t) \wedge \Delta B(t)\right). \tag{12}$$

Then Φ determines a formal Frobenius manifold structure on $(\mathbf{H}, g)$.

Using Γ, we can define another differential $\bar{\partial}_\Gamma$ on $\boldsymbol{C}[[t_\mathbf{H}]] \hat{\otimes} \mathbf{t}$ by $\bar{\partial}_\Gamma\varphi(t) := \bar{\partial}\varphi(t) + [\Gamma(t) \bullet \varphi(t)]$. Then we can easily show that inclusions induce the following isomorphisms (see Manin [7], §5):

$$H(\boldsymbol{C}[[t_\mathbf{H}]] \hat{\otimes} \mathbf{t}, \bar{\partial}_\Gamma) \cong \frac{\ker \Delta \cap \ker \bar{\partial}_\Gamma}{\operatorname{Im} \Delta\bar{\partial}_\Gamma} \cong H(\boldsymbol{C}[[t_\mathbf{H}]] \hat{\otimes} \mathbf{t}, \Delta) \cong \boldsymbol{C}[[t_\mathbf{H}]] \otimes \mathbf{H}.$$

We note that the homology $H(\bar{\partial}_\Gamma)$ inherits a natural multiplication from $\boldsymbol{C}[[t_\mathbf{H}]] \hat{\otimes} \mathbf{t}$, because $\bar{\partial}_\Gamma$ is a derivation with respect to the wedge product.

We identify $\boldsymbol{C}[[t_\mathbf{H}]] \otimes \mathbf{H}$ with the space of vector fields on $\mathbf{H}$ by the formula $[\gamma_a] = \partial/\partial t^a$. This space acts on $\boldsymbol{C}[[t_\mathbf{H}]] \hat{\otimes} \mathbf{t}$ as derivation. Define a map $\psi : \boldsymbol{C}[[t_\mathbf{H}]] \otimes \mathbf{H} \to H(\bar{\partial}_\Gamma)$ by $\psi(X) := X\Gamma \mod \operatorname{Im} \bar{\partial}_\Gamma$. Then ψ is algebra isomorphism, if we define a multiplication on $\boldsymbol{C}[[t_\mathbf{H}]] \otimes \mathbf{H}$ by the potential Φ in Theorem 2.1. Namely, for $X, Y \in \boldsymbol{C}[[t_\mathbf{H}]] \otimes \mathbf{H}$, their product $X \circ Y$ is a unique element satisfying $(X \circ Y)\Gamma \equiv X\Gamma \wedge Y\Gamma \mod \operatorname{Im} \bar{\partial}_\Gamma$.

3. Convergence of $\Gamma(t)$ in $C^{k+\theta}$.

In this section, we keep the same notation as in the previous section. $\{[\gamma_a]\}_{a=1}^N$ is a basis of $\mathbf{H}$. We assume that γ_a is even for $1 \le a \le m$, and odd for $m+1 \le a \le N$. $\{t^a\}$ is the dual basis. In order to distinguish the odd basis from the even one, we denote t^{m+i} by τ^i for $1 \le i \le l$ $(= N - m)$. Then

$$\boldsymbol{C}[[t_\mathbf{H}]] = \boldsymbol{C}[[t^1, \ldots, t^m]] \otimes \bigwedge(\tau^1, \ldots, \tau^l) \tag{13}$$

by definition. Later, when we need to distinguish even and odd, we use (τ^i), when not, (t^{m+i}).

Let $\Gamma \in \boldsymbol{C}[[t_\mathbf{H}]] \hat{\otimes} \mathbf{t}$. We can represent it as $\Gamma = \sum_\alpha \Gamma_\alpha(t)\tau^\alpha$ where $\Gamma_\alpha(t) \in \boldsymbol{C}[[t^1, \ldots, t^m]] \hat{\otimes} \mathbf{t}$, and τ^α denotes $\tau^{\alpha_1} \cdots \tau^{\alpha_l}$. Then it makes sense to ask whether Γ_α is smooth in the coordinate $(z^1, \ldots, z^n, t^1, \ldots, t^m)$. Here $(z^1, \ldots, z^n)$ is a local coordinate of M. We split $\mathbf{H}$ into even and odd parts: $\mathbf{H} = \mathbf{H}^{ev} \oplus \mathbf{H}^{odd}$. Let U be an open set in $\mathbf{H}^{ev}$. We say that Γ is smooth on U if Γ_α is smooth on $U \times M$ for each α. Our goal is the following.

Theorem 3.1. *There exists a normalized solution of the Maurer-Cartan equation* (11)

$$\Gamma = \Gamma_1 + \Delta B \in (\boldsymbol{C}[[t_\mathbf{H}]] \hat{\otimes} \mathbf{t})^2$$

such that Γ and B are smooth on a sufficiently small neighbourhood of the origin in $\mathbf{H}^{ev}$.

Let $\mathcal{O}_U$ be the sheaf of the germs of holomorphic functions on U. Put $\mathcal{O} := \mathcal{O}_U \otimes \bigwedge(\tau^1, \dots, \tau^l)$. We remark that if $\Gamma(t)$ is smooth, then we can consider $\bar{\partial}_\Gamma$ in the smooth category, and obtain an algebra homomorphism $\mathbf{H} \otimes \mathcal{O} \to H(\bar{\partial}_\Gamma) : X \mapsto X\Gamma$.

For $X = [a], Y = [b] \in \mathbf{H}$, define $g(X, Y) := \int ab$. When we regard $(U, \mathcal{O})$ as a supermanifold, its tangent sheaf is identified with $\mathcal{O} \otimes \mathbf{H}$. Then we can regard g as a Riemannian metric on U. From (8), (12) and the result above, we obtain the following immediately.

COROLLARY 3.2. *Let Γ be as in Theorem* 3.1, *and Φ be the potential which takes the form of* (12). *Then Φ is holomorphic on U. Consequently*, $(U, \mathcal{O}, g, \Phi)$ *is a Frobenius manifold in the sense of Manin* [**8**].

This is straightforward.

We will prove Theorem 3.1 by modifying the arguments in the Kodaira-Spencer deformation theory (cf. Kodaira [**5**]). The proof is divided into two parts: Proposition 1 and Proposition 2. In the first part, we shall prove the $C^{k+\theta}$-convergence, and in the second part, the regularity of the resulting solution.

We introduce the Hölder norms on the space $\mathbf{t}$. Let U be an open set in a Euclidean space $\boldsymbol{R}^n$, k be a nonnegative integer, and $0 < \theta < 1$. For $f \in C^k(U)$, define

$$|f|_{k+\theta}^U := \sum_{|\alpha| \le k} \sup_{x \in U} |D^\alpha f(x)| + \sum_{|\alpha| = k} \sup_{\substack{x, y \in U \\ |x-y| \le 1}} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x - y|^\theta},$$

where α is multi-index. Next, we fix a finite covering $\{V_j\}_{j \in I}$ of M such that (z_j) are coordinate on V_j. For $\gamma \in \mathbf{t}$,

$$\gamma = \sum_{p,q=0}^{n} \sum_{\substack{\alpha_1 < \cdots < \alpha_p \\ \beta_1 < \cdots < \beta_q}} \gamma_{j\bar{\alpha}_1 \cdots \bar{\alpha}_p}^{\beta_1 \cdots \beta_q}(z_j)\, d\bar{z}_j^{\alpha_1} \wedge \cdots \wedge d\bar{z}_j^{\alpha_p} \otimes \frac{\partial}{\partial z_j^{\beta_1}} \wedge \cdots \wedge \frac{\partial}{\partial z_j^{\beta_q}},$$

the Hölder norm $|\gamma|_{k+\theta}$ is defined as follows:

$$|\gamma|_{k+\theta} := \sup |\gamma_{j\bar{\alpha}_1 \cdots \bar{\alpha}_p}^{\beta_1 \cdots \beta_q}(z_j)|_{k+\theta}^{V_j},$$

where the sup is over all $j \in I$; $p, q = 1, \dots, n$; $\alpha_1 < \cdots < \alpha_p$; $\beta_1 < \cdots < \beta_q$. We also introduce the Hölder norms on $A^{*,*}(M)$, that is, the space of all the (p,q)-forms on M.

Let $\Gamma(t) = \sum \gamma_\alpha t^\alpha \in \boldsymbol{C}[[t_\mathbf{H}]] \hat{\otimes} \mathbf{t}$, $\gamma_\alpha \in \mathbf{t}$. Here $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index, and t^α denotes $(t^1)^{\alpha_1} \cdots (t^N)^{\alpha_N}$. Then we define

$$|\Gamma|_{k+\theta}(t) := \sum_\alpha |\gamma_\alpha|_{k+\theta} t^\alpha \in \boldsymbol{C}[[t^1, \dots, t^N]]. \tag{14}$$

In (14), we forget the grading of (t^i). So, $t^i t^j = t^j t^i$ for all i, j in $\boldsymbol{C}[[t^1, \dots, t^N]]$, though $\boldsymbol{C}[[t_{\mathbf{H}}]]$ is graded commutative.

Clearly, if $|\Gamma|_{k+\theta}(t)$ converges on a domain U, then $\Gamma(t)$ is $C^{k+\theta}$ class on U. Indeed,

$$|\Gamma_\alpha|_{k+\theta}(t) = \left.\frac{\partial^{|\alpha|}}{(\partial t^{m+1})^{\alpha_1} \cdots (\partial t^{m+l})^{\alpha_l}}\right|_{t^{m+1}=\cdots=t^{m+l}=0} |\Gamma|_{k+\theta}(t)$$

converges. So we shall prove the convergence of $|\Gamma|_{k+\theta}(t)$ for a certain specific choice of $\Gamma(t)$.

Fix a Kähler metric on M. Let ω be its Kähler form; $\bar{\partial}$ be $\bar{\partial}$-operator acting on differential forms on M; $\bar{\partial}^*$ be the adjoint of $\bar{\partial}$ with respect to the L^2-inner product induced by the Kähler metric on M; $\Delta_{\bar{\partial}} = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$ be the Laplacian; $G_{\bar{\partial}}$ be its Green operator. Similarly, we consider $\partial^*, \Delta_\partial$, and G_∂. Let

$$L : A^{*,*}(M) \to A^{*,*}(M)$$

be the map defined by $L(\eta) := \eta \wedge \omega$, and Λ be its adjoint. Then the following is well-known (cf. Griffiths-Harris [**4**]):

$$\begin{gathered} \Delta_{\bar{\partial}} = \Delta_\partial \quad G_{\bar{\partial}} = G_\partial \\ [\Lambda, \partial] = \sqrt{-1}\bar{\partial}^* \quad [\Lambda, \bar{\partial}] = -\sqrt{-1}\partial^*. \end{gathered} \tag{15}$$

We choose $\Gamma(t)$ as follows. Let $\{[\gamma_a]\}$ be a basis of $\mathbf{H}$. We can assume that $I\gamma_a$ are harmonic forms, that is,

$$\Delta_{\bar{\partial}}(I\gamma_a) = 0. \tag{16}$$

Then the condition $\bar{\partial}\gamma_a = \Delta\gamma_a = 0$ is satisfied. Define

$$\Gamma_0 := 0, \quad \Gamma_1 := \sum_a \gamma_a t^a.$$

For $n \geq 2$, we define Γ_n inductively, as follows:

$$\psi_n := -\frac{1}{2} \sum_{i+j=n} [\Gamma_i \bullet \Gamma_j]$$

$$\Gamma_n := I\bar{\partial}^* G I \psi_n,$$

where $G := G_{\bar{\partial}} = G_\partial$, and I is defined by (4). Here, Γ_n is homogeneous of degree n in t^a.

Lemma 3.3. *Let* $\Gamma = \sum_{n\geq 1} \Gamma_n$ *be as above. Then* Γ *is a normalized solution. More precisely, if we define*

$$B_n := \sqrt{-1} I \Lambda G I \psi_n, \quad B = \sum_{n \ge 2} B_n, \tag{17}$$

then $\Gamma = \Gamma_1 + \Delta B$, *and* Γ *is homogeneous of degree* 2 *with respect to the grading on* $\boldsymbol{C}[[t_{\mathbf{H}}]] \hat{\otimes} \mathbf{t}$ *induced by* (1) *and* (10).

Proof. By definition,

$$\bar{\partial}\Gamma + \frac{1}{2}[\Gamma \bullet \Gamma] = 0 \Leftrightarrow \begin{cases} \bar{\partial}\Gamma_1 = 0 \quad \text{and} \\ \bar{\partial}\Gamma_n = -(1/2) \sum_{i+j=n} [\Gamma_i \bullet \Gamma_j] = \psi_n \quad \forall n \ge 2. \end{cases}$$

From (16), we have $\bar{\partial}\Gamma_1 = 0$. Therefore it is sufficient to prove inductively the following:

$$\begin{cases} \bar{\partial}\Gamma_n = \psi_n \\ \Gamma_n = \Delta B_n \\ |\Gamma_n| = 2. \end{cases} \tag{$(*)_n$}$$

We assume that $(*)_1, \dots, (*)_{n-1}$ hold. Then

$$\begin{aligned} \bar{\partial}\psi_n &= -\frac{1}{2} \sum_{i+j=n} ([\bar{\partial}\Gamma_i \bullet \Gamma_j] - [\Gamma_i \bullet \bar{\partial}\Gamma_j]) \\ &= -\frac{1}{2} \sum_{i+j+k=n} [[\Gamma_i \bullet \Gamma_j] \bullet \Gamma_k]. \end{aligned}$$

The right hand side vanishes because the Jacobi identity reads:

$$[[\Gamma_i \bullet \Gamma_j] \bullet \Gamma_k] + [[\Gamma_j \bullet \Gamma_k] \bullet \Gamma_i] + [[\Gamma_k \bullet \Gamma_i] \bullet \Gamma_j] = 0.$$

On the other hand, because of the Tian-Todorov lemma (7), we have $\psi_n \in \operatorname{Im} \Delta$. Namely, $I\psi_n \in \ker \bar{\partial} \cap \operatorname{Im} \partial$. We have

$$\begin{aligned} I\psi_n &= \partial\partial^* G I \psi_n \quad \text{because } I\psi_n \in \operatorname{Im} \partial \\ &= \sqrt{-1}\partial(\Lambda\bar{\partial} - \bar{\partial}\Lambda) G I \psi_n \quad \text{from (15)} \\ &= \bar{\partial}(\sqrt{-1}\partial\Lambda G I \psi_n) \\ &= \bar{\partial}\bar{\partial}^* G I \psi_n. \end{aligned}$$

Therefore we obtain $\psi_n = \bar{\partial}\Gamma_n$ and $\Gamma_n = \Delta B_n$. Finally, because

$$I \Lambda I, IGI : \mathbf{t} \to \mathbf{t}$$

preserve $\boldsymbol{Z}$-grading, we have

$$|\Gamma_n| = |B_n| - 1 = |\psi_n| - 1 = 2. \qquad \square$$

For $f = \sum_\alpha a_\alpha t^\alpha$, $g = \sum_\beta b_\beta t^\beta \in \boldsymbol{C}[[t^1, \dots, t^N]]$, we define:

$$f \ll g \overset{\text{def}}{\Longleftrightarrow} |a_\alpha| \le |b_\alpha| \quad \text{for all } \alpha.$$

If $f \ll g$ and g converges, then f also converges. For $b, c \in \boldsymbol{R}_{>0}$, define

$$A(t) = A(b, c; t) := \frac{b}{16c} \sum_{\mu=1}^{\infty} \frac{c^\mu}{\mu^2} (t^1 + \cdots + t^N)^\mu.$$

Then $A(t)$ converges on $\{t \in \boldsymbol{C}^N \mid |t^i| < 1/Nc\}$, and satisfies

$$A(t)^2 \ll \frac{b}{c} A(t). \tag{18}$$

Proposition 1. *Let $\Gamma(t) = \Gamma_1(t) + \Delta B(t)$ be chosen as in Lemma* 3.3. *Then, for fixed integer $k \ge 2$ and real number $0 < \theta < 1$, there exist sufficiently large numbers b, c which satisfy*

$$|\Gamma|_{k+\theta}(t) \ll A(t) \quad \textit{and} \quad |B|_{k+1+\theta}(t) \ll A(t).$$

To prove this, we need the following two lemmas.

Lemma 3.4. *For all $\varphi \in A^{*,*}(M)$ and $\gamma \in \mathbf{t}$, we have*

(i) $|G\varphi|_{k+\theta} \le C_1 |\varphi|_{k-2+\theta}$,
(ii) $|\Lambda\varphi|_{k+\theta} \le C_2 |\varphi|_{k+\theta}$,
(iii) $C_3^{-1} |I\varphi|_{k+\theta} \le |\varphi|_{k+\theta} \le C_3 |I\varphi|_{k+\theta}$,
(iv) $|\Delta\gamma|_{k+\theta} \le C_4 |\gamma|_{k+1+\theta}$,

where C_1, C_2, C_3 and C_4 are some positive constants depending on k, θ, not on φ, γ.

Proof. The first inequality is well-known in the theory of elliptic operators (cf. Kodaira [**5**], Appendix, etc.). $\Lambda : A^{*,*} \to A^{*,*}$ and $I : \mathbf{t} \to A^{*,*}$ are operators of order 0, and $\Delta : \mathbf{t} \to \mathbf{t}$ is of order 1. Hence we obtain the remaining inequalities. $\square$

Lemma 3.5. *There exists a positive constant C_5 depending on k, θ such that*

$$|[\varphi \bullet \psi]|_{k-1+\theta} \le C_5 |\varphi|_{k+\theta} |\psi|_{k+\theta}$$

for all $\varphi, \psi \in \mathbf{t}$.

Proof. In general, if $U \subset \boldsymbol{R}^l$ is an open set, and $f, g \in C^{k+\theta}(U)$, we have

$$|fg|_{k-1+\theta} \le C |f|_{k-1+\theta} |g|_{k-1+\theta}$$

for some constant C.

By ∂_i, we denote $\partial/\partial z_i$. Let $\varphi = f\,d\bar{z}_I \otimes \partial_{i_1} \wedge \cdots \wedge \partial_{i_p}$, $\psi = g\,d\bar{z}_J \otimes \partial_{j_1} \wedge \cdots \wedge \partial_{j_q} \in \mathbf{t}$. Then, from (2), we have

$$[\varphi \bullet \psi] = \pm d\bar{z}_I \wedge d\bar{z}_J \otimes \left\{ \sum_{a=1}^{p} \pm f(\partial_{i_a} g)\partial_{i_1} \wedge \cdots \wedge \widehat{\partial_{i_a}} \wedge \cdots \wedge \partial_{i_p} \wedge \partial_{j_1} \wedge \cdots \wedge \partial_{j_q} \right.$$
$$\left. + \sum_{b=1}^{q} \pm g(\partial_{j_b} f)\partial_{i_1} \wedge \cdots \wedge \partial_{i_p} \wedge \partial_{j_1} \wedge \cdots \wedge \widehat{\partial_{j_b}} \wedge \cdots \wedge \partial_{j_q} \right\}.$$

Hence

$$|[\varphi \bullet \psi]|_{k-1+\theta} \leq \sum_{a=1}^{p} |f \partial_{i_a} g|_{k-1+\theta} + \sum_{b=1}^{q} |g \partial_{j_b} f|_{k-1+\theta}$$
$$\leq 2nC|\varphi|_{k+\theta}|\psi|_{k+\theta}.$$

Since general elements in $\mathbf{t}$ are represented as sum of at most 4^n such elements, we obtain

$$|[\varphi \bullet \psi]|_{k-1+\theta} \leq 2n4^n C|\varphi|_{k+\theta}|\psi|_{k+\theta}. \qquad \square$$

Proof of Proposition 1. Recall that $\Gamma_1 = \sum \gamma_a t^a$ and $A(t) = (b/16) \cdot (t^1 + \cdots + t^N) + \text{higher terms}$. Therefore, if

$$b \geq 16 \max_a |\gamma_a|_{k+\theta}, \tag{19}$$

then it follows that $|\Gamma_1|_{k+\theta}(t) \ll A(t)$.

Assume that for all $i = 1, \dots n$,

$$|\Gamma_i|_{k+\theta}(t) \ll A(t), \tag{20}$$

for some $b, c > 0$. Using Lemma 3.4, 3.5, and (17), we have

$$|B_{n+1}|_{k+1+\theta}(t) = |I\Lambda GI\psi_{n+1}|_{k+1+\theta}(t) \ll C_1 C_2 C_3^2 |\psi_{n+1}|_{k-1+\theta}(t). \tag{21}$$

We denote $\Gamma_1 + \cdots + \Gamma_n$ by Γ^n. Then

$$|\psi_{n+1}|_{k-1+\theta}(t) = \frac{1}{2} \left| \sum_{\substack{i+j=n+1 \\ i,j \geq 1}} [\Gamma_i \bullet \Gamma_j] \right|_{k-1+\theta} (t) \ll \frac{1}{2} |[\Gamma^n \bullet \Gamma^n]|_{k-1+\theta}(t).$$

For $\varphi = \sum \varphi_\alpha t^\alpha \in \boldsymbol{C}[[t_{\mathbf{H}}]] \hat{\otimes}\, \mathbf{t}$, we have

$$
\begin{aligned}
|[\varphi \bullet \varphi]|_{k-1+\theta}(t) &\ll \sum_{\alpha,\beta} t^\alpha t^\beta |[\varphi_\alpha \bullet \varphi_\beta]|_{k-1+\theta} \\
&\ll \sum_{\alpha,\beta} t^\alpha t^\beta C_5 |\varphi_\alpha|_{k+\theta} |\varphi_\beta|_{k+\theta} \\
&= C_5 |\varphi|_{k+\theta}(t) |\varphi|_{k+\theta}(t),
\end{aligned}
$$

by Lemma 3.5. Since $|\Gamma^n|_{k+\theta} \ll A(t)$ by the assumption (20), we obtain

$$
\begin{aligned}
|\psi_{n+1}|_{k-1+\theta} &\ll \frac{1}{2} |[\Gamma^n \bullet \Gamma^n]|_{k-1+\theta} \\
&\ll \frac{C_5}{2} |\Gamma^n|_{k+\theta} |\Gamma^n|_{k+\theta} \\
&\ll \frac{C_5}{2} A(t)^2 \\
&\ll \frac{C_5 b}{2c} A(t) \quad \text{from (18).} \qquad (22)
\end{aligned}
$$

From (21) and (22), we have

$$
|B_{n+1}|_{k+1+\theta} \ll \frac{C_1 C_2 C_3^2 C_5 b}{2c} A(t).
$$

Choose b satisfying (19). Next, choose c so that c satisfies

$$
c \geq \frac{1}{2} C_1 C_2 C_3^2 C_4 C_5 b. \qquad (23)
$$

Then

$$
\begin{aligned}
|\Gamma_{n+1}|_{k+\theta} &= |\Delta B_{n+1}|_{k+\theta} \\
&\ll C_4 |B_{n+1}|_{k+1+\theta} \\
&\ll A(t).
\end{aligned}
$$

The conditions (19) and (23) are independent of n. Therefore once we choose b and c satisfying (19) and (23), we can apply this argument for all n. Hence for such b and c

$$
|\Gamma|_{k+\theta}(t) \ll A(t) \quad \text{and} \quad |B|_{k+1+\theta} \ll C_4^{-1} A(t) \ll A(t). \qquad \square
$$

Remark. Since b and c depend on k and θ, the convergence radius of $\Gamma(t)$ also depends on k and θ.

From Proposition 1, we can deduce Corollary 3.2. However, because in order to observe the multiplicative structure of the resulting Frobenius manifold, it seems suitable to use $\Gamma(t)$, we prove the regularity of $\Gamma(t)$ in the next section.

4. Regularity of $\Gamma(t)$.

In the previous section, we proved that $\Gamma(t)$ is $C^{k+\theta}$. In this section we shall prove that $\Gamma(t)$ is C^{∞} on a sufficiently small neighbourhood of the origin in $\mathbf{H}^{ev}$. Since $B_n = (\sqrt{-1}/2) I\Lambda GI(\sum_{i+j=n}[\Gamma_i \bullet \Gamma_j])$, we have

$$B = \frac{\sqrt{-1}}{2} I\Lambda GI([\Gamma \bullet \Gamma]).$$

Therefore if $\Gamma(t)$ is C^{∞}, then B is also C^{∞}. See Kodaira [**5**], Theorem 7.10.

In this section, we separate even and odd, again. $(t^1 \dots, t^m)$ denotes even parameter, and $(\tau^1, \dots, \tau^l)$ denotes odd. Put

$$\varphi(t) := I\Gamma(t), \quad \varphi_n := I\Gamma_n \quad \text{and} \tag{24}$$

$$S := \{(t^1, \dots, t^m) \in \boldsymbol{C}^m \mid |t^i| < r \text{ for } \forall i\} \quad \text{for small } r > 0.$$

We assume that $\Gamma(t)$ is $C^{k+\theta}$ on S. Let π be a projection $M \times S \to M$. Then we can regard φ as a $C^{k+\theta}$ section of

$$V = \pi^* \left(\overset{*}{\bigwedge} T_M^* \otimes \overset{*}{\bigwedge} \overline{T}_M^* \right) \otimes \bigwedge (\tau^1, \dots, \tau^l) \to M \times S.$$

In order to prove that $\Gamma(t)$ is C^{∞}, it is sufficient to prove that φ is so. We consider the equation that $\varphi(t)$ satisfies.

Since $\Gamma(t)$ satisfies

$$\bar{\partial}\Gamma + \frac{1}{2}[\Gamma \bullet \Gamma] = 0,$$

$\varphi(t)$ satisfies

$$\bar{\partial}\varphi + \frac{1}{2} I[I\varphi \bullet I\varphi] = 0.$$

Here, we have $\varphi_1(t) = \sum (I\gamma_a) t^a$ by definition. Because we choose γ_a so that they satisfy (16), we have $\bar{\partial}(I\gamma_a) = \bar{\partial}^*(I\gamma_a) = 0$. Therefore $\bar{\partial}^* \varphi_1(t) = 0$. If $n \geq 2$, then $\bar{\partial}^* \varphi_n = 0$ because $\varphi_n = \bar{\partial}^* GI\psi_n$. Hence $\varphi(t)$ satisfies the following:

$$\Delta_{\bar{\partial}}\varphi + \frac{1}{2}\bar{\partial}^* I[I\varphi \bullet I\varphi] = 0.$$

Using (15) and $\Delta\Gamma = 0$, we can rewrite this equation as follows:

$$\Delta_{\bar\partial}\varphi + \frac{\sqrt{-1}}{2}\partial\Lambda I[I\varphi \bullet I\varphi] = 0.$$

On the other hand, since φ is holomorphic in $(t^1, \dots, t^m)$, we obtain the following:

$$\left(-\sum_{i=1}^{m}\frac{\partial^2}{\partial t^i \partial \bar{t}^i} + \Delta_{\bar\partial}\right)\varphi + \frac{\sqrt{-1}}{2}\partial\Lambda I[I\varphi \bullet I\varphi] = 0. \tag{25}$$

Unlike the Kodaira-Spencer theory, our $\varphi(t)$ is possibly nonzero even if $t = 0$. Perhaps the quasi-linear equation (25) is not elliptic. However, we will prove the regularity, modifying the argument in Kodaira [**5**], appendix, §8.

We introduce a new norm on the space of sections of V. Let ψ be a section of V. Then we can represent ψ uniquely as $\psi = \sum_\beta \psi_\beta \tau^\beta$ where ψ_β is a section of $W = \pi^*(\bigwedge^* T_M^* \otimes \bigwedge^* \bar{T}_M^*)$. Let $\{V_j\}$ be a coordinate neighbourhood of M. Then $\{V_j \times S\}$ is a coordinate neighbourhood of $M \times S$. For $f = \sum f_{jAB}(z_j, t)\, dz_j^A \wedge d\bar{z}_j^B \in \Gamma(M \times S, W)$, define

$$|f|_{k+\theta} := \max_{j,A,B} |f_{jAB}(z_j, t)|_{k+\theta}^{V_j \times S}.$$

In order to distinguish this norm from the one in the previous section, we denote the latter by $|\cdot|_{k+\theta}^M$. Next, for $\psi = \sum \psi_\beta \tau^\beta$ and fixed $\rho \in \boldsymbol{R}$ with $0 < \rho < 1$, define

$$|\psi|_{k+\theta}^\rho := \sum_\beta |\psi_\beta|_{k+\theta}\rho^{|\beta|}.$$

For $\varphi = \sum \varphi_{\alpha\beta} t^\alpha \tau^\beta$ defined by (24), we can assume that φ satisfies

$$\begin{aligned} |\varphi|_{k+\theta}(t,\tau) &= \sum_{\alpha,\beta} |\varphi_{\alpha\beta}|_{k+\theta}^M t^\alpha \tau^\beta \\ &\ll A(t,\tau) = \sum_{\alpha,\beta} A_{\alpha\beta} t^\alpha \tau^\beta \\ &= \frac{b}{16c}\sum_{\mu \ge 1}\frac{c^\mu}{\mu^2}(t^1 + \cdots + t^m + \tau^1 + \cdots + \tau^l)^\mu \end{aligned} \tag{26}$$

i.e. $|\varphi_{\alpha\beta}|_{k+\theta}^M \le A_{\alpha\beta}$.

Lemma 4.1. *Under the assumption* (26) *above, we have*

(i) $|\varphi|_0^\rho \le A(r,\rho)$,

(ii) $|\varphi|_\theta^\rho \le 2A(r,\rho) + 2^{1-\theta}\sum_{|\alpha| \ge 1}|\alpha| A_{\alpha\beta} r^{|\alpha|-\theta}\rho^{|\beta|} =: B(r,\rho)$,

where $A(r,\rho) = A(r, \dots, r, \rho, \dots, \rho)$.

Proof. (i) is obvious. Indeed,

$$|\varphi|_0^\rho = \sum_\beta \left|\sum_\alpha \varphi_{\alpha\beta} t^\alpha\right|_0 \rho^{|\beta|} \le \sum_{\alpha,\beta} |\varphi_{\alpha\beta}|_0^M r^{|\alpha|}\rho^{|\beta|} \le A(r,\rho).$$

To prove (ii), it is sufficient to consider locally. For $(x,t),(y,s) \in V_j \times S$, we estimate

$$\frac{|\varphi_\beta(x,t)-\varphi_\beta(y,s)|}{|(x,t)-(y,s)|^\theta}$$

where $\varphi_\beta = \sum_\alpha \varphi_{\alpha\beta}(x)t^\alpha$. We have

$$\begin{aligned}\frac{|\varphi_\beta(x,t)-\varphi_\beta(y,s)|}{|(x,t)-(y,s)|^\theta} &\le \frac{|\varphi_\beta(x,t)-\varphi_\beta(y,t)|+|\varphi_\beta(y,t)-\varphi_\beta(y,s)|}{\left||x-y|^2+|t-s|^2\right|^{\theta/2}} \\ &\le \frac{|\varphi_\beta(x,t)-\varphi_\beta(y,t)|}{|x-y|^\theta}+\frac{|\varphi_\beta(y,t)-\varphi_\beta(y,s)|}{|t-s|^\theta}\end{aligned}$$

$$\frac{|\varphi_\beta(x,t)-\varphi_\beta(y,t)|}{|x-y|^\theta} \le \sum_\alpha \frac{|\varphi_{\alpha\beta}(x)-\varphi_{\alpha\beta}(y)|}{|x-y|^\theta}|t|^{|\alpha|} \le \sum_\alpha A_{\alpha\beta}r^{|\alpha|}$$

$$\frac{|\varphi_\beta(y,t)-\varphi_\beta(y,s)|}{|t-s|^\theta} \le \sum_\alpha |\varphi_{\alpha\beta}(y)|\frac{|s^\alpha-t^\alpha|}{|s-t|^\theta} \le 2^{1-\theta}\sum_{|\alpha|\ge 1}|\alpha|A_{\alpha\beta}r^{|\alpha|-\theta}.$$

Hence

$$|\varphi|_\theta^\rho = \sum_\alpha |\varphi_\beta|_\theta \rho^{|\beta|} \le 2A(r,\rho) + 2^{1-\theta}\sum_{\substack{|\alpha|\ge 1\\ \beta}}|\alpha|A_{\alpha\beta}r^{|\alpha|-\theta}\rho^{|\beta|}. \qquad \square$$

Remark that if r and ρ are sufficiently small, then $A(r,\rho), B(r,\rho)$ are also small. Of course they converge.

Choose a partition of unity $\{\omega_i\}$ subordinate to the open cover $\{V_i\}$. Next, for each $l = 1,2,\dots,$ we choose a C^∞-function $\eta^l(t)$ on S such that

$$\begin{cases}\eta^l(t) \equiv 1 & \text{if } |t| \le (2^{-1}+2^{-l-1})r \\ \eta^l(t) \equiv 0 & \text{if } |t| \ge (2^{-1}+2^{-l})r \\ 0 \le \eta^l(t) \le 1. \end{cases}$$

Put $\omega_j^l(x,t) := \omega_j(x)\eta^l(t)$. Furthermore, we choose a C^∞-function $\chi_j(x)$ with $\operatorname{supp}\chi_j \subset V_j$ which is identically equal to 1 on some neighbourhood of $\operatorname{supp}\omega_j$. Put $\chi_j^l := \chi_j\eta^l$. Because $\eta^l \equiv 1$ on some neighbourhood of $\operatorname{supp}\eta^{l+2}$, $\chi_j^l \equiv 1$ on some neighbourhood of $\operatorname{supp}\omega_j^{l+2}$. Then we shall prove the following:

Proposition 2. *For some small* $r > 0$, $\eta^{2l+1}\varphi$ *is* $C^{k+l+\theta}$. *In particular*, φ *is* C^∞ *on* $M \times \{t \in \boldsymbol{C}^m \mid |t^i| < r/2\}$.

$\omega_j^l\varphi$ can be considered as a vector-valued function with compact support on a $(2n+2m)$-dimensional torus $\boldsymbol{T}^{2n+2m}$. Since $\eta^{2l+1}\varphi = \sum_j \omega_j^{2l+1}\varphi$, to prove Proposition 2, it is sufficient to prove the regularity of $\omega_j^{2l+1}\varphi$. To prove this, we need some lemmas. Let $C^{k+\theta} = C^{k+\theta}(\boldsymbol{T}^l, \boldsymbol{C})$ be the space of $\boldsymbol{C}$-valued $C^{k+\theta}$ functions on $\boldsymbol{T}^l$.

Lemma 4.2. *Let* $u, v \in C^{K+\theta}(\boldsymbol{T}^l, \boldsymbol{C})$. *Then the product* uv *is* $C^{k+\theta}$. *And there exists a positive constant* B_k *depending only on* k *and* l, *but independent of* u *and* v *such that*

$$|uv|_{k+\theta} \le B_k \sum_{r+s=k} (|u|_{r+\theta}|v|_s + |u|_r|v|_{s+\theta}).$$

Lemma 4.3. *Let* $(x^1, \ldots, x^l)$ *be coordinate functions on* $\boldsymbol{T}^l$. *For* $h \in \boldsymbol{R}$ *with* $h \ne 0$, $a = 1, \ldots, l$ *and* $f \in C^{k+\theta}$, *define*

$$\Delta_a^h f(x^1, \ldots, x^l) := \frac{f(x^1, \ldots, x^a + h, \ldots, x^l) - f(x^1, \ldots, x^l)}{h}.$$

Then we have the following:

(i) *If* $f \in C^{k+\theta}$, *then* $\Delta_a^h f \in C^{k+\theta}$ *for all* $h \ne 0$ *and* $a = 1, \ldots, l$.
(ii) *If* $f \in C^{k+1+\theta}$, *then* $|\Delta_a^h f|_{k+\theta} \le |f|_{k+1+\theta}$ *for all* a *and* h $(0 < |h| < 1)$.
(iii) *If* $f \in C^{k+\theta}$ *and for any* $a = 1, \ldots, l$ *and any* h *with* $0 < |h| < 1$ *there exists a positive constant independent of* h *such that*

$$|\Delta_a^h f|_{k+\theta} \le M,$$

then $f \in C^{k+1+\theta}$.

Lemma 4.4 ($C^{k+\theta}$ a priori estimate). *Let* U *be a domain in* $\boldsymbol{T}^l$. *Suppose that the second-order linear partial differential operator* E *with* C^∞ *coefficients defined on* $\overline{U}$ *is of diagonal type in the principal part and strongly elliptic. Let* $0 < \theta < 1$. *Then for all integer* $k \ge 0$, *there exists a positive constant* C *such that*

$$|f|_{k+2+\theta} \le C(|Ef|_{k+\theta} + |f|_0)$$

for all $f \in C^{k+2+\theta}$ *with* $\operatorname{supp} f \subset U$. *Here* C *is independent of* f.

See Kodaira [**5**], appendix §8, Theorem 2.3, Lemma 8.1 and Lemma 8.2. Put

$$E := -\sum_{i=1}^m \frac{\partial^2}{\partial t^i \partial \bar{t}^i} + \Delta_{\bar\partial}.$$

E is a second-order strongly elliptic operator of diagonal type in the principal part. If we consider that $V_i \times S \subset \boldsymbol{T}^{2n+2m}$, then there exists a positive constant C_0 such that

$$|\psi|_{k+\theta} \le C_0(|E\psi|_{k-2+\theta} + |\psi|_0) \tag{27}$$

for all sections ψ of W with $\operatorname{supp}\psi \subset V_j \times S$. This estimate (27) is true for all sections of W. Let $\psi = \sum \psi_\beta \tau^\beta$ be a section of V. Since $E\psi = \sum (E\psi_\beta)\tau^\beta$ and for all β

$$|\psi_\beta|_{k+\theta} \le C_0(|E\psi_\beta|_{k-2+\theta} + |\psi_\beta|_0),$$

we obtain the following:

$$\begin{aligned} |\psi|^\rho_{k+\theta} &= \sum_\beta |\psi_\beta|_{k+\theta}\rho^{|\beta|} \\ &\le \sum_\beta C_0(|E\psi_\beta|_{k-2+\theta} + |\psi_\beta|_0)\rho^{|\beta|} \\ &= C_0(|E\psi|^\rho_{k-2+\theta} + |\psi|^\rho_0). \end{aligned} \tag{28}$$

Here the constant C_0 in (28) is same as the one in (27). We prove Proposition 2, using this.

PROOF OF PROPOSITION 2. (I) First, we shall prove that $\omega_j^3\varphi \in C^{k+1+\theta}$.

$\omega_j^3\varphi$ can be considered as a function on $\boldsymbol{T}^{2n+2m}$. Therefore we can define $\Delta_a^h(\omega_j^3\varphi)$. By Lemma 4.3, it is sufficient to prove the following: *for each* $a = 1, \dots, 2n+2m$ *and each* β, *there exists a positive constant* K *such that* $|\Delta_a^h\omega_j^3\varphi_\beta|_{k+\theta} \le K$ *for all* $h \in \boldsymbol{R}$ *with* $0 < |h| < 1$.

For simplicity, denote $\omega := \omega_j^3$, $\chi := \chi_j^1$. If $|\Delta_a^h\omega\varphi|^\rho_{k+\theta} \le K$, then we have $|\Delta_a^h\omega\varphi_\beta|_{k+\theta} \le \rho^{-|\beta|}K$ for each β. Therefore, we shall prove that:

$$|\Delta_a^h\omega\varphi|^\rho_{k+\theta} \le K.$$

We have

$$\begin{aligned} E(\omega\varphi) &= E(\omega\chi\varphi) = [E,\omega](\chi\varphi) + \omega E(\chi\varphi) \\ &= -\frac{\sqrt{-1}}{2}\omega\partial\Lambda I[I\chi\varphi \bullet I\chi\varphi] + [E,\omega](\chi\varphi). \end{aligned}$$

Therefore

$$\begin{aligned} E(\Delta_a^h\omega\varphi) &= \Delta_a^h E(\omega\varphi) + [E,\Delta_a^h](\omega\varphi) \\ &= -\frac{\sqrt{-1}}{2}\Delta_a^h(\omega\partial\Lambda I[I\chi\varphi \bullet I\chi\varphi]) + \Delta_a^h([E,\omega](\chi\varphi)) + [\Delta_{\bar\partial},\Delta_a^h](\omega\varphi) \\ &=: F_1. \end{aligned} \tag{29}$$

Here we used the following facts:

$$\left[-\sum_{i=1}^{m}\frac{\partial^2}{\partial t^i\partial\bar{t}^i}, \Delta_a^h\right] = 0 \quad \text{and} \quad \chi \equiv 1 \text{ on a neighbourhood of } \operatorname{supp}\omega.$$

Using (28), we obtain

$$|\Delta_a^h\omega\varphi|_{k+\theta}^{\rho} \leq C_0(|F_1|_{k-2+\theta}^{\rho} + |\Delta_a^h\omega\varphi|_0^{\rho}).$$

Since $\omega\varphi$ is $C^{k+\theta}$, we have

$$|\Delta_a^h\omega\varphi|_0^{\rho} \leq \sum_{\beta}|\omega\varphi_\beta|_{k+\theta}\rho^{|\beta|}, \tag{30}$$

from Lemma 4.3. The right hand side of (30) is independent of h.

Let us estimate $|F_1|_{k-2+\theta}^{\rho}$. First, we have $|\Delta_a^h([E,\omega](\chi\varphi))|_{k-2+\theta}^{\rho} \leq K$. Here K is a positive constant which is independent of h. Indeed, since $[E,\omega]$ is first order operator, $[E,\omega](\chi\varphi)$ is $C^{k-1+\theta}$.

Secondly, we have $|[\Delta_{\bar\partial}, \Delta_a^h](\chi\varphi)|_{k-2+\theta}^{\rho} \leq K$. Indeed Δ_a^h acts only on coefficients of $\Delta_{\bar\partial}$ which is smooth.

Finally we estimate $|\Delta_a^h(\omega\partial\Lambda I[I\chi\varphi \bullet I\chi\varphi])|_{k-2+\theta}^{\rho}$. Since

$$\Delta_a^h(\omega\partial\Lambda I[I\chi\varphi \bullet I\chi\varphi]) = \sum_{\beta,\gamma} \pm\Delta_a^h(\omega\partial\Lambda I[I\chi\varphi_\beta \bullet I\chi\varphi_\gamma])\tau^\beta\tau^\gamma,$$

we have

$$|\Delta_a^h(\omega\partial\Lambda I[I\chi\varphi \bullet I\chi\varphi])|_{k-2+\theta}^{\rho} \leq \sum_{\beta,\gamma}|\Delta_a^h(\omega\partial\Lambda I[I\chi\varphi_\beta \bullet I\chi\varphi_\gamma])|_{k-2+\theta}\rho^{|\beta|+|\gamma|}.$$

Lemma 4.5.

$$|\Delta_a^h(\omega\partial\Lambda I[I\chi\varphi_\beta \bullet I\chi\varphi_\gamma])|_{k-2+\theta} \leq C_1(|\Delta_a^h\omega\varphi_\beta|_{k+\theta}|\varphi_\gamma|_\theta + |\varphi_\beta|_\theta|\Delta_a^h\omega\varphi_\gamma|_{k+\theta}) + K$$

where C_1 is a positive constant which is independent of h, ω and χ.

Postponing the proof of this lemma, we shall finish the proof of (I). If we assume Lemma 4.5, we have

$$\begin{aligned}|\Delta_a^h(\omega\partial\Lambda I[I\chi\varphi \bullet I\chi\varphi])|_{k-2+\theta}^{\rho} &\leq 2C_1\sum_{\beta,\gamma}|\Delta_a^h\omega\varphi_\beta|_{k+\theta}|\varphi_\gamma|_\theta\rho^{|\beta|+|\gamma|} + K \\ &= 2C_1|\Delta_a^h\omega\varphi|_{k+\theta}^{\rho}|\varphi|_\theta^{\rho} + K.\end{aligned}$$

From Lemma 4.1, we have $|\varphi|_\theta^{\rho} \leq B(r,\rho)$. Therefore we obtain

$$|\Delta_a^h(\omega\varphi)|_{k+\theta}^{\rho} \leq 2C_0C_1B(r,\rho)|\Delta_a^h\omega\varphi|_{k+\theta}^{\rho} + K.$$

If we choose r and ρ such that

$$2C_0C_1B(r,\rho) \le 1/2, \tag{31}$$

then it follows that $|\Delta_a^h \omega\varphi|_{k+\theta}^{\rho} \le K$.

PROOF OF LEMMA 4.5. For simplicity, we denote $f = \varphi_\beta$ and $g = \varphi_\gamma$. Let

$$f = \sum_{A,B} f_{AB}\, dz^A \wedge d\bar{z}^B, \quad g = \sum_{C,D} g_{CD}\, dz^C \wedge d\bar{z}^D$$

$$\Lambda(dz^A \wedge d\bar{z}^B) = \sum_{C,D} \Lambda_{CD}^{AB}\, dz^C \wedge d\bar{z}^D$$

$$\Omega = h\, dz^1 \wedge \cdots \wedge dz^n.$$

Then

$$I(f) = \sum \pm f_{AB}/h\, d\bar{z}^B \otimes \partial_{z^{n-A}},$$

where $n - A$ denotes the compliment of A in $\{1, \dots, n\}$.

$$[If \bullet Ig] = \sum_{\substack{i \in A \\ A,B,C,D}} \pm (f_{AB}/h)\partial_i(g_{CD}/h)\, d\bar{z}^B d\bar{z}^D \partial_{z^{n-A-i}} \partial_{z^{n-C}} + (f \leftrightarrow g),$$

$$I[If \bullet Ig] = \sum \pm f_{AB}\partial_i(g_{CD}/h)\, dz^E d\bar{z}^F + (f \leftrightarrow g),$$

where E and F are defined so that $I(d\bar{z}^B d\bar{z}^D \partial_{z^{n-A-i}} \partial_{z^{n-C}}) = dz^E d\bar{z}^F$

$$\omega\partial\Lambda I[I\chi f \bullet I\chi g] = \sum \pm \omega\partial_j(\Lambda_{EF}^{GH} \chi f_{AB} \partial_i(\chi g_{CD}/h))\, dz^j dz^G d\bar{z}^H + (f \leftrightarrow g).$$

Therefore it is sufficient to estimate

$$|\Delta_a^h(\omega\partial_j(\Lambda_{EF}^{GH} \chi f_{AB} \partial_i(\chi g_{CD}/h)))|_{k-2+\theta}. \tag{32}$$

When we expand (32) by Leibniz rule, all the terms except

$$|\omega\Lambda_{EF}^{GH} \chi f_{AB} h^{-1} \partial_i\partial_j \Delta_a^h(\chi g_{CD})|_{k-2+\theta} \tag{33}$$

can be estimated by positive multiple of $|\omega f|_{k+\theta}|\chi g|_{k+\theta}$ or $|\chi f|_{k+\theta}|\omega g|_{k+\theta}$. Using Lemma 4.2, we can estimate (33) as follows:

$$\begin{aligned}
&|\omega\Lambda_{EF}^{GH} \chi f_{AB} h^{-1} \partial_i\partial_j \Delta_a^h(\chi g_{CD})|_{k-2+\theta} \\
&\quad \le |\Lambda_{EF}^{GH} f_{AB} h^{-1} \partial_i\partial_j \Delta_a^h(\omega g_{CD})|_{k-2+\theta} + K \\
&\quad \le 2B|\Lambda_{EF}^{GH} h^{-1}|_\theta |f_{AB}|_\theta |\partial_i\partial_j \Delta_a^h(\omega g_{CD})|_{k-2+\theta} + K \\
&\quad \le 2BC|f|_\theta |\Delta_a^h(\omega g)|_{k+\theta} + K.
\end{aligned}$$

Here we used $\omega\chi = \omega$. C_1 is represented as a combination of $C^{k+\theta}$ norms of Λ and Ω. Hence C is independent of χ and ω. □

(II) To complete the proof of Proposition 2, we prove, by induction, the following: *for all* $l = 1, 2, \ldots,$ $\omega_j^{2l+1}\varphi$ *is* $C^{k+l+\theta}$. Here, we do not change r and ρ satisfying (31). Under the assumption that $\omega_j^{2l+1}\varphi$ is $C^{k+l+\theta}$, we prove that $\omega_j^{2l+3}\varphi$ is $C^{k+l+1+\theta}$. To prove this, it is sufficient to prove that

$$|\Delta_a^h(D^l\omega_j^{2l+3}\varphi)|_{k+\theta}^{\rho} \le K$$

where D^l denotes an arbitrary l-th order differential. By the same computation as (29), we obtain

$$\begin{aligned} F_{l+1} &:= E(\Delta_a^h(\omega_j^{2l+3}\varphi)) \\ &= -\frac{\sqrt{-1}}{2}\Delta_a^h(\omega_j^{2l+3}\partial\Lambda I[I\chi_j^{2l+1}\varphi \bullet I\chi_j^{2l+1}\varphi]) \\ &\quad + \Delta_a^h([E, \omega_j^{2l+3}](\chi_j^{2l+1}\varphi)) + [\Delta_{\bar\partial}, \Delta_a^h](\omega_j^{2l+3}\varphi). \end{aligned}$$

Therefore

$$E(\Delta_a^h(D^l\omega_j^{2l+3}\varphi)) = D^lF_{l+1} + [\Delta_{\bar\partial}, D^l](\Delta_a^h\omega_j^{2l+3}\varphi).$$

Here we used $[\Delta_a^h, D^l] = 0$. Hence

$$\begin{aligned} &|\Delta_a^h(D^l\omega_j^{2l+3}\varphi)|_{k+\theta}^{\rho} \\ &\quad \le C_0(|D^lF_{l+1}|_{k-2+\theta}^{\rho} + |[\Delta_{\bar\partial}, D^l](\Delta_a^h\omega_j^{2l+3}\varphi)|_{k-2+\theta}^{\rho} + |\Delta_a^h(D^l\omega_j^{2l+3}\varphi)|_0^{\rho}). \end{aligned}$$

By assumption of induction, $\omega_j^{2l+3}\varphi = \eta^{2l+3}\omega_j^{2l+1}\varphi$ is $C^{k+l+\theta}$. Hence $|\Delta_a^h(D^l\omega_j^{2l+3}\varphi)|_0^{\rho} \le K$. Since $[\Delta_{\bar\partial}, D^l]$ is $(l+1)$-th order, we have

$$|[\Delta_{\bar\partial}, D^l](\Delta_a^h\omega_j^{2l+3}\varphi)|_{k-2+\theta}^{\rho} \le C|\Delta_a^h\omega_j^{2l+3}|_{k+l-1+\theta}^{\rho} \le C|\omega_j^{2l+3}|_{k+l+\theta}^{\rho} \le K.$$

Consider $|D^lF_{l+1}|_{k-2+\theta}^{\rho}$. The same argument as (I) is also valid here. Therefore it is sufficient to estimate

$$|D^l\Delta_a^h(\omega_j^{2l+3}\partial\Lambda I[I\chi_j^{2l+1}\varphi \bullet I\chi_j^{2l+1}\varphi])|_{k+\theta}^{\rho}.$$

By the same computation as Lemma 4.5, we obtain the following:

$$|D^l\Delta_a^h(\omega_j^{2l+3}\partial\Lambda I[I\chi_j^{2l+1}\varphi \bullet I\chi_j^{2l+1}\varphi])|_{k+\theta}^{\rho} \le 2C_1|\varphi|_{\theta}^{\rho}|\Delta_a^h(D^l\omega_j^{2l+3}\varphi)|_{k+\theta}^{\rho} + K$$

where C_1 is the same constant as Lemma 4.5. Since r and ρ are chosen so that they satisfies (31), we obtain the following again:

$$|\Delta_a^h(D^l\omega_j^{2l+3}\varphi)|_{k+\theta}^{\rho} \le K.$$

□

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