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Vietoris continuous selections on scattered spaces

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Abstract. We prove that a countable regular space has a continuous selection if and only if it is scattered. Further we prove that a paracompact scattered space admits a continuous selection if each of its points has a countable pseudo-base. We also provide two examples to show that: (1) paracompactness can not be replaced by countable compactness even together with (collectionwise) normality, and (2) having countable pseudo-base at each of its points can not be omitted even in the class of regular Lindelöf linearly ordered spaces.

1. Introduction.

Let X be a topological space, and let $\mathscr{F}(X)$ be the set of all non-empty closed subsets of X. Let us recall the definition of the *Vietoris topology* τ_V on $\mathscr{F}(X)$. The base for τ_V is defined by the collection of sets

$$\langle \mathscr{V} \rangle = \{F \in \mathscr{F}(X) : F \subset \bigcup \mathscr{V} \text{ and } F \cap V \neq \varnothing \text{ for all } V \in \mathscr{V}\}$$

where $\mathscr{V}$ runs over all finite families of non-empty open subsets of X. If $\mathscr{V} = \{V_0, V_1, \dots, V_n\}$ is a finite family of open subsets of X, then in some cases, we shall write $\langle V_0, V_1, \dots, V_n \rangle$ instead of $\langle \mathscr{V} \rangle$. Let $\mathscr{F} \subset \mathscr{F}(X)$. A map $\sigma : \mathscr{F} \to X$ is a *selection* for $\mathscr{F}$ if $\sigma(F) \in F$ for every $F \in \mathscr{F}$. A selection $\sigma : \mathscr{F} \to X$ is a *continuous selection* for $\mathscr{F}$ if it is continuous with respect to the relative topology of the Vietoris topology τ_V on $\mathscr{F}(X)$. We say X *has* (or *does not have*) *a continuous selection* if there is (no) continuous selection for $\mathscr{F}(X)$.

Ernest Michael has discovered a simple sufficient condition for the existence of a continuous selection on a Hausdorff space X: if there exists a linear order $<$ on X such that the induced order topology is weaker than the original topology and every non-empty closed subspace of X has $<$-minimal element, then the space X has a continuous selection [**9**]. The selection in this case is constructed by assigning to each non-empty closed subset of X its $<$-minimal element. In fact Michael has proved that this condition is not only sufficient but also necessary for connected Hausdorff spaces. Later on, van Mill and Wattel have proved the

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same for compact Hausdorff spaces [**11**]. It is still unknown if the condition is necessary for all regular spaces, that is all presently known regular spaces with continuous selections satisfy it as well. While this shows that the existence of a special linear order on a space with continuous selection plays an important role, mere existence of some linear order does not suffice to imply the existence of a continuous selection: the real line $\boldsymbol{R}$ is a linearly ordered (metric) space without any continuous selection [**4**].

In the next section we completely characterize countable regular spaces which admit a continuous selection by proving that *a countable regular space has a continuous selection if and only if it is scattered,* see Theorem 2.4. A space is *scattered* if and only if every its non-empty closed subset has an isolated point. We also give an example of a countable Hausdorff scattered space without any continuous selection (see Example 2.5), thereby demonstrating that the assumption of regularity is essential in the above characterization. Unfortunately, scatteredness is no longer a sufficient condition for the existence of a continuous selection outside of the class of countable spaces. Indeed, in Section 3 we construct an example of a scattered (collectionwise) normal, countably compact, first countable space which does not have a continuous selection, see Example 3.1. The first countability is a novel feature of our example, since without it the one point compactification of an uncountable discrete set provides an example of a Hausdorff compact scattered space without any continuous selection. Further, in Section 4, we show that scatteredness and linear orderability even combined together do not guarantee the existence of a continuous selection. This is accomplished by constructing an example of a Lindelöf scattered linearly ordered space without a continuous selection, see Example 4.1. It should be pointed out that both our examples have size ω_1, which is the smallest possible cardinality. Finally, we prove that a paracompact scattered space admits a continuous selection provided that every point has a countable pseudo-base, see Theorem 2.3. This is a possible approach to investigate relations between order-like conditions on topological bases and continuous selections for Vietoris hyperspaces (for similar results in the metric case, see [**5**]). Concerning the condition of paracompactness, our Example 3.1 shows that, in Theorem 2.3, paracompactness cannot be weakened to collectionwise normality.

Throughout this paper, all spaces are assumed to be Hausdorff.

2. Countable spaces.

We start with the following easy lemma.

Lemma 2.1. *Let X be a space which has a disjoint cover $\{U_\alpha : \alpha \in \kappa\}$ of clopen subsets such that each U_α has a continuous selection. Then X has a continuous selection.*

A family $\mathscr{U}$ of open subsets of X is a *pseudo-base* at $x \in X$ if $\bigcap \mathscr{U} = \{x\}$.

Lemma 2.2. *Let X be a space and $x \in X$. Let $\{U_n : n \in \omega\}$ be a decreasing pseudo-base at x consisting of clopen subsets of X. Suppose $U_0 = X$. If there exists a continuous selection σ_n for $\mathscr{F}(U_n - U_{n+1})$ for $n \in \omega$, then X has a continuous selection.*

Proof. Let $E_n = U_n - U_{n+1}$. For $F \in \mathscr{F}(X) - \{\{x\}\}$ let $n(F) = \min\{n : F \cap E_n \neq \emptyset\}$. Define a selection $\sigma : \mathscr{F}(X) \to X$ as follows: $\sigma(F) = \sigma_{n(F)}(F \cap E_{n(F)})$ provided that $F \neq \{x\}$, and $\sigma(\{x\}) = x$. Now it is easy to check that σ is continuous. □

For every ordinal number α, we define by transfinite induction the α-derivative of a space $X : X^{(0)} = X; X^{(\alpha+1)} = (X^{(\alpha)})' = \{x \in X^{(\alpha)} : x$ is not an isolated point of $X^{(\alpha)}\}$. $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$ if α is limit. Notice that X is scattered if and only if $X^{(\alpha)} = \emptyset$ for some ordinal α. For a scattered space X, the *height* $h(X)$ of X is the smallest ordinal α such that $X^{(\alpha)} = \emptyset$. The set $X^{(\alpha)} - X^{(\alpha+1)}$ is called the α-th level of X. For every α, each $x \in X^{(\alpha)} - X^{(\alpha+1)}$ is an isolated point of $X^{(\alpha)}$, thus there exists a neighborhood V_x of x such that $V_x \cap X^{(\alpha)} = \{x\}$.

Theorem 2.3. *Let X be a paracompact scattered space such that every point $x \in X$ has a countable pseudo-base. Then X has a continuous selection.*

Proof. We prove our theorem using transfinite induction for the height of a space. If $h(X) = 1$ then X is a discrete space, and so X has a continuous selection by Lemma 2.1.

Suppose $h(X) < \gamma$ implies that X has a continuous selection. Let $h(X) = \gamma$.

Case 1. $\gamma = \alpha + 1$ is a successor ordinal. Then $X^{(\alpha)}$ is a closed discrete subset of X. Since a scattered paracompact space is strongly zero-dimensional, for every $x \in X^{(\alpha)}$ there exists a clopen neighborhood V_x of x such that the collection $\{V_x : x \in X^{(\alpha)}\}$ is discrete in X and $V_x \cap X^{(\alpha)} = \{x\}$. Let $\{U_n : n \in \omega\}$ be a decreasing countable pseudo-base at x consisting of clopen sets of X with $U_0 = V_x$. Since $h(U_n) < \alpha$, by inductive hypothesis each U_n has a continuous selection, and so does its closed subspace $U_n - U_{n+1}$. Therefore $V_x = U_0 = \bigcup_{n\in\omega}(U_n - U_{n+1})$ has a continuous selection by Lemma 2.2. Let $V = X - \bigcup\{V_x : x \in X^{(\alpha)}\}$. Then V is clopen and $h(V) < \gamma$. Hence V has a continuous selection by the inductive assumption. Since $\{V_x : x \in X^{(\alpha)}\} \cup \{V\}$ is a discrete cover of X, X has a continuous selection by Lemma 2.1.

Case 2. γ is a limit ordinal. Then X has a discrete clopen cover $\mathscr{V}$ with $h(V) < \gamma$ for each $V \in \mathscr{V}$. Now X has a continuous selection by Lemma 2.1. □

In the above theorem paracompactness can not be weakened to collectionwise normality, and countable pseudo-base can not weakened to either having a nested base of size ω_1 or linear orderability. The necessary counterexamples will be provided in later sections. In the case that X is first countable instead of countable pseudo-base the above theorem was proved in [**1**].

Theorem 2.4. *A countable regular space X has a continuous selection if and only if it is scattered.*

Proof. The "if" part of our theorem follows from Theorem 2.3 since regular countable spaces are paracompact and have countable pseudobase at all points.

To prove the "only if" part, assume that X is a countable regular space that is not scattered. Then X has a closed subspace without the Baire property. (A space has the *Baire property* if the intersection of countably many open dense subsets of it is dense.) Theorem 1.2 of [**5**] now implies that X does not have a continuous selection. □

The following example shows that Hausdorffness is not enough in Theorem 2.4.

Example 2.5. There exists a countable, first countable scattered Hausdorff space without a continuous selection.

Let $X = \boldsymbol{Q} \times \{0,1\}$, and let $\boldsymbol{Q}_i = \boldsymbol{Q} \times \{i\}$ for $i \in \{0,1\}$ where $\boldsymbol{Q}$ denotes the rational numbers. For $x \in \boldsymbol{Q}$ we write $x^0 = \{x\} \times \{0\}$ and $x^1 = \{x\} \times \{1\}$. Let the topology τ on X be generated by the singletons of $\boldsymbol{Q}_0$ together with all sets of the form $V_\varepsilon(x^1) = \{x^1\} \cup \{y^0 \in \boldsymbol{Q}_0 : x - \varepsilon < y < x + \varepsilon\} - \{x^0\}$, where $<$ is the usual order of the real line, $\varepsilon > 0$ and $x \in \boldsymbol{Q}$. Clearly (X, τ) is a first countable, scattered, Hausdorff space. Since a point $y \in \boldsymbol{Q}_1$ and the closed set $\boldsymbol{Q}_1 - \{y\}$ can not be separated by disjoint open sets, X is not regular. We show that X has no continuous selection. Suppose that there exists a continuous selection $\sigma : \mathscr{F}(X) \to X$. Let $\sigma(\boldsymbol{Q}_1) = y$, and $\sigma(S) = y'$, where $S = \boldsymbol{Q}_1 - \{y\}$. Choose disjoint neighborhoods V of y and W of y' such that $V \cap \boldsymbol{Q}_1 = \{y\}$. By the continuity of σ we can choose Vietoris open neighborhoods $\langle V_0, V_1, \dots, V_n\rangle$ of $\boldsymbol{Q}_1$ and $\langle W_0, W_1, \dots, W_m\rangle$ of S such that $\sigma(\langle V_0, V_1, \dots, V_n\rangle) \subset V$ and $\sigma(\langle W_0, W_1, \dots, W_m\rangle) \subset W$ respectively. Without loss of generality we may assume that $V_0 \subset V$. This implies $V_0 \cap \boldsymbol{Q}_1 = \{y\}$. We may also assume that $y \notin \bigcup_{i=0}^m W_i$. Since y and S can not be separated by disjoint open sets in X, there exists $z \in (\bigcup_{i=0}^m W_i) \cap \bigcap\{V_i : y \in V_i\} \cap \boldsymbol{Q}_0$. Since $S \cup \{z\} \in \langle V_0, V_1, \dots, V_n\rangle \cap \langle W_0, W_1, \dots, W_m\rangle$, we must have $\sigma(S \cup \{z\}) \in V \cap W$, a contradiction.

3. A scattered (collectionwise) normal, countably compact, first countable example.

According to Theorem 2.3 a compact scattered space with a countable pseudo-base at each of its points has a continuous selection (hence it must be linearly orderable by the theorem of van Mill and Wattel [**11**]). The following example shows that compactness can not be weakened to countable compactness and that paracompactness in Theorem 2.3 can not be weakened to collectionwise normality.

Recall that a subset $A \subset \omega_1$ is *stationary* if A has non-empty intersection with any closed unbounded subset of ω_1.

Example 3.1. Let $X = \omega_1 \times (\omega + 1)$ be the product of the space ω_1 of countable ordinals and the convergence sequence $\omega + 1$. Then X is a scattered, (collectionwise) normal, countably compact, first countable space that does not have a continuous selection even for $\mathscr{F}_2(X)$, where $\mathscr{F}_2(X) = \{F \in \mathscr{F}(X) : |F| \leq 2\}$.

Proof. Suppose that $\sigma : \mathscr{F}_2(X) \to X$ is a continuous selection on $\mathscr{F}_2(X)$. For $\alpha, \beta \in \omega_1, \alpha < \beta$ and $n, m \in \omega + 1$, we put

$$\varepsilon(F) = \begin{cases} 0, & \text{if } \sigma(F) = \langle \alpha, m \rangle \\ 1, & \text{if } \sigma(F) = \langle \beta, n \rangle, \end{cases}$$

where $F = \{\langle \alpha, m \rangle, \langle \beta, n \rangle\} \in \mathscr{F}_2(X)$, and we define

$$W(\alpha, \beta) = \begin{cases} [0, \alpha] \times (\omega + 1), & \text{if } \varepsilon(\{\langle \alpha, \omega \rangle, \langle \beta, \omega \rangle\}) = 0 \\ (\alpha, \beta] \times (\omega + 1), & \text{if } \varepsilon(\{\langle \alpha, \omega \rangle, \langle \beta, \omega \rangle\}) = 1. \end{cases}$$

Then $W(\alpha, \beta)$ is a neighborhood of $\sigma(\{\langle \alpha, \omega \rangle, \langle \beta, \omega \rangle\})$.

For $\alpha, \beta \in \omega_1$, $\alpha < \beta$ and $n \in \omega$, we use the following notations:

$$V(\alpha, \beta, n) = (\alpha, \beta] \times (\{m : m \geq n\} \cup \{\omega\}),$$

$$V(\alpha, \omega_1, n) = (\alpha, \omega_1) \times (\{m : m \geq n\} \cup \{\omega\}).$$

By the continuity of σ, for every $\alpha, \beta \in \omega_1$ with $\alpha < \beta$ there exist three ordinals $\rho(\alpha, \beta)$, $\lambda(\alpha, \beta)$, $n(\alpha, \beta)$ such that $\rho(\alpha, \beta) < \alpha \leq \lambda(\alpha, \beta) < \beta$, $n(\alpha, \beta) < \omega$ and

$$\sigma(\langle V(\rho(\alpha, \beta), \alpha, n(\alpha, \beta)), V(\lambda(\alpha, \beta), \beta, n(\alpha, \beta)) \rangle) \subset W(\alpha, \beta).$$

Hence, by using the pressing down lemma and a well-known fact that every countable union of non-stationary sets of ω_1 is non-stationary, for every $\alpha < \omega_1$ we can choose four ordinals $\rho(\alpha), \lambda(\alpha), n(\alpha), i(\alpha)$ and a stationary set S_α of ω_1 such that

$$\rho(\alpha) < \alpha \leq \lambda(\alpha) < \omega_1, \quad n(\alpha) < \omega, \quad i(\alpha) < 2$$

and

$$\rho(\alpha,\beta)=\rho(\alpha),\quad \lambda(\alpha,\beta)=\lambda(\alpha),\quad n(\alpha,\beta)=n(\alpha),\quad \varepsilon(\{\langle\alpha,\omega\rangle,\langle\beta,\omega\rangle\})=i(\alpha)$$

for every $\beta\in S_\alpha$.

Therefore we have $\varepsilon(F)=i(\alpha)$ for every $F\in\langle V(\rho(\alpha),\alpha,n(\alpha)),V(\lambda(\alpha),\omega_1,n(\alpha))\rangle$ and every $\alpha<\omega_1$.

Again by the pressing down lemma, we obtain the following fact:

Fact. There exist three ordinals ρ_0, n_0, i_0 and a stationary set S of ω_1 such that

$$\rho_0<\omega_1,\quad n_0<\omega,\quad i_0<2$$

and

$$\rho(\alpha)=\rho_0,\quad n(\alpha)=n_0,\quad i(\alpha)=i_0$$

for all $\alpha\in S$.

This fact implies that

(1) $\varepsilon(F)=i_0$ for every $F\in\langle V(\rho_0,\alpha,n_0),V(\lambda(\alpha),\omega_1,n_0)\rangle$ and $\alpha\in S$.

Let D be the set of accumulating points of S in ω_1, and put

$$D_0=\{\beta\in D:\lambda(\gamma)<\beta \text{ for any } \gamma<\beta\}.$$

Then it is easily seen that D_0 is a closed unbounded set of ω_1, so we pick a $\beta\in D_0$. We choose $s,t\in\omega$ satisfying $n_0\le s$, $n_0\le t$ and $s\ne t$. We now suppose $\sigma(\{\langle\beta,s\rangle,\langle\beta,t\rangle\})=\langle\beta,s\rangle$. Then, by the continuity of σ, there exists an $\alpha_0\in\omega_1$ such that $\rho_0<\alpha_0<\beta$ and

(2) $$\sigma(\langle(\alpha_0,\beta]\times\{s\},(\alpha_0,\beta]\times\{t\}\rangle)\subset(\rho_0,\beta]\times\{s\}.$$

By the definitions of D and D_0, we can find an $\alpha\in S$ so that $\alpha_0<\alpha\le\lambda(\alpha)<\beta$.

Now we choose an $E\in\mathscr{F}_2(X)$ with $E\subset\{\alpha,\beta\}\times\{s,t\}$ which leads us to a contradiction, as follows. Set

$$E=\begin{cases}\{\langle\alpha,t\rangle,\langle\beta,s\rangle\}, & \text{if } i_0=0\\ \{\langle\alpha,s\rangle,\langle\beta,t\rangle\}, & \text{if } i_0=1.\end{cases}$$

Case 0: $i_0=0$. By (1) and $\alpha<\beta$ we get $\sigma(E)=\langle\alpha,t\rangle$. On the other hand, by (2), $\sigma(E)=\langle\beta,s\rangle$, a contradiction.

Case 1: $i_0=1$. By (1) and $\alpha<\beta$ we get $\sigma(E)=\langle\beta,t\rangle$. On the other hand, by (2), $\sigma(E)=\langle\alpha,s\rangle$, a contradiction. □

Remark. If one glues together the first point 0 of two copies of the long line, then the resulting space is a linearly ordered, collectionwise normal, countably compact, first countable and it does not have a continuous selection [**7**]. However, unlike Example 3.1, the space is not scattered.

4. A scattered linearly ordered example.

As we described in the introduction, orderability is the key notion for the existence of continuous selections. Since scattered metrizable spaces are 0-dimensional and complete, such spaces have continuous selections [**2**], [**4**] because of topological well-orderability. However linear orderability is not enough to guarantee the existence of continuous selections even for Lindelöf scattered (hence 0-dimensional) spaces. Example 4.1 shows that Lindelöf scattered linearly ordered spaces need not have continuous selections.

Let M be the quotient space obtained from the product space $\omega_1 \times \{0,1\}$ by identifying the points $\langle \omega_1, 0\rangle$ and $\langle \omega_1, 1\rangle$ to a single point ∞ (we consider the discrete topology on $\{0,1\}$). For convenience of explanation we write $\alpha^0 = \langle \alpha, 0\rangle$ and $\alpha^1 = \langle \alpha, 1\rangle$ for $\alpha \in \omega_1$, and $\infty = \omega_1^0 = \omega_1^1$. We consider a natural order on M which induces the same topology on M, that is: $\alpha^0 < \beta^1$ for $\alpha, \beta \in \omega_1$, and $\alpha^0 < \beta^0$ and $\alpha^1 > \beta^1$ for $\alpha < \beta < \omega_1 + 1$. We define $[\alpha^0, \beta^1] = \{\gamma \in M : \alpha^0 \le \gamma \le \omega_1^0 \text{ or } \omega_1^1 \ge \gamma \ge \beta^1\}$. The other notations are somewhat standard, for instance $[\alpha^0, \beta^0] = \{\gamma \in M : \alpha^0 \le \gamma \le \beta^0\}$, $[\beta^1, \alpha^1] = \{\gamma \in M : \beta^1 \ge \gamma \ge \alpha^1\}$, etc.

A space is a *GO-space* if it is homeomorphic to a subspace of a linearly ordered space.

Example 4.1. Let $S \subset \omega_1$ be a stationary set such that $\omega_1 - S$ is also stationary (such a set exists; see [**8**]). Let $L = \{\alpha^i \in M : \alpha \in S, i = 0,1\} \cup \{\infty\}$ be a subspace of M. Then L is a regular Lindelöf scattered GO space which has no continuous selection.

Proof. Assume that there exists a continuous selection $\sigma : \mathscr{F}(L) \to L$.

Claim 1. For every $\alpha \in S$ there exist $\beta(\alpha) \in S$ and $\gamma(\alpha) \in S$ such that

(i) $\alpha \le \beta(\alpha) < \gamma(\alpha) < \omega_1$, and

(ii) $\sigma([\alpha^0, \delta] \cap L) = \beta(\alpha)^0$ for $\delta \in [\gamma(\alpha)^0, \gamma(\alpha)^1]$.

Let $A = (\omega_1 - S) \cap (\alpha, \omega_1)$. Note that for each $\delta \in A$ one has $\sigma([\alpha^0, \delta^0] \cap L) < \delta^0$ because δ^0 is not in L.

Since A is stationary, by the pressing down lemma, there exists $\beta(\alpha)$ such that $B = \{\delta \in A : \beta(\alpha)^0 = \sigma([\alpha^0, \delta^0] \cap L)\}$ is stationary. By the continuity of σ, $\sigma([\alpha^0, \omega_1^0] \cap L) = \beta(\alpha)^0$. Again by the continuity of σ at the point $[\alpha^0, \omega_1^0] \cap L \in \mathscr{F}(L)$, for each $\delta < \beta(\alpha)$ there exists an open set $\mathscr{V}_\delta = \langle V_0^\delta, V_1^\delta, \dots, V_{n_\delta}^\delta \rangle$ of $\mathscr{F}(L)$ such that $\sigma(\mathscr{V}_\delta) \subset (\delta, \beta(\alpha)^0]$. Let $V_\delta = \bigcup_{i \le n_\delta} V_i^\delta$. Since $\omega_1^0 = \omega_1^1 \in V_\delta$, there exists $\gamma_\delta < \omega_1$ such that $[\alpha^0, \gamma] \cap L \in \mathscr{V}_\delta$ for all $\gamma \in [\gamma_\delta^0, \gamma_\delta^1]$. Let $\gamma(\alpha) =$

$\sup\{\gamma_\delta : \delta < \alpha\}$. Then $\sigma([\alpha^0, \gamma] \cap L) = \beta(\alpha)$ for $\gamma \in [\gamma(\alpha)^0, \gamma(\alpha)^1]$. The proof of Claim 1 is complete.

The same claim holds for α^1.

CLAIM 2. For each $\alpha \in S$ there exist $\zeta(\alpha) \in S$ and $\eta(\alpha) \in S$ such that

(i) $\alpha \le \zeta(\alpha) < \eta(\alpha)$, and

(ii) $\sigma([\delta, \alpha^1] \cap L) = \zeta(\alpha)^1$ for $\delta \in [\eta(\alpha)^0, \eta(\alpha)^1]$.

Let $C_\alpha = [\gamma(\alpha), \omega_1)$ and $D_\alpha = [\eta(\alpha), \omega_1)$ for $\alpha \in S$, and $C_\alpha = D_\alpha = \omega_1$ for $\alpha \in \omega_1 - S$. We take the diagonal intersection $E = \{\delta : \delta \in C_\alpha \cap D_\alpha$ for each $\alpha < \delta\}$ of $C_\alpha \cap D_\alpha$'s. Then E is a closed unbounded set in ω_1 (see [**8**]). Choose points ρ and α_n in $S \cap E$ such that $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \cdots$ and $\rho = \lim_{n\to\infty} \alpha_n$. Since $\rho \in E$ and $\alpha_n < \rho$, it follows that $\rho \in C_{\alpha_n}$. Hence $\sigma([\alpha_n^0, \rho^1] \cap L) = \beta(\alpha_n)^0$. Keeping in mind this fact, we can prove $\sigma([\rho^0, \rho^1] \cap L) = \sigma(\lim_{n\to\infty} [\alpha_n^0, \rho^1] \cap L) = \lim_{n\to\infty} \sigma([\alpha_n^0, \rho^1] \cap L) = \lim_{n\to\infty} \beta(\alpha_n)^0 = \rho^0$. On the other hand, by the same argument, we have: $\sigma([\rho^0, \rho^1] \cap L) = \sigma(\lim_{n\to\infty} [\rho^0, \alpha_n^1] \cap L) = \lim_{n\to\infty} \zeta(\alpha)^1 = \rho^1$, a contradiction. □

There is a standard way (see for instance [**12**]) to embed a GO-space X as a closed subspace in a linearly ordered space X^* which, in turn, is a subspace of the linearly ordered space $X \times \boldsymbol{Z}$ equipped with the lexicographical order of X and $\boldsymbol{Z}$, here $\boldsymbol{Z}$ denotes the set of integers. In our case the resulting linearly ordered space X^* is automatically Lindelöf and scattered. Therefore *there exists a Lindelöf scattered linearly ordered space which has no continuous selection*.

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