J. Math. Soc. Japan
Vol. 53, No. 4, 2001

Non-linearizability of n-subhyperbolic polynomials at irrationally indifferent fixed points

By Yûsuke OKUYAMA

(Received May 12, 2000)

Abstract. We study the non-linearlizability conjecture (NLC) for polynomials at non-Brjuno irrationally indifferent fixed points. A polynomial is n-subhyperbolic if it has exactly n recurrent critical points corresponding to irrationally indifferent cycles, other ones in the Julia set are preperiodic and no critical orbit in the Fatou set accumulates to the Julia set. In this article, we show that NLC and, more generally, the cycle-version of NLC are true in a subclass of n-subhyperbolic polynomials. As a corollary, we prove the cycle-version of the Yoccoz Theorem for quadratic polynomials.

We also study several specific examples of n-subhyperbolic polynomials. Here we also show the scaling invariance of the Brjuno condition: if an irrational number α satisfies the Brjuno condition, then so do $m\alpha$ for every positive integer m.

1. Introduction.

In this paper, we always assume $\lambda = e^{2\pi i\alpha}$ ($\alpha \in \boldsymbol{R}\backslash\boldsymbol{Q}$). We consider a holomorphic germ f at $z_0 \in \boldsymbol{C}$ fixing z_0 with *multiplier* $f'(z_0) = \lambda$. Then z_0 is called an *irrationally indifferent* fixed point of f and α is called the *rotation number* of f at z_0. This name is derived from a rigorous relationship between holomorphic dynamics near an irrationally indifferent fixed point and that of analytic circle homeomorphisms. For example, rotation numbers of holomorphic germs are topologically invariant. For more details, see [**21**].

We say f to be *linearizable* at z_0 if there exists a neighborhood D of z_0 and a conformal map $w = h(z)$ from D to the unit disk $\boldsymbol{D}$ with $h(z_0) = 0$ such that $h \circ f \circ h^{-1}$ is a rotation $w \mapsto \lambda w$ on $\boldsymbol{D}$.

Brjuno showed in [**2**] that if α satisfies the *Brjuno condition*, then such f is always linearizable at z_0. The Brjuno condition is defined in terms of the continued fraction expansion. For the rigorous definition, see [**23**] or [**16**].

In fact the Brjuno condition is the best possible. Yoccoz proved in [**23**] that *if a quadratic polynomial is linearizable at an irrationally indifferent fixed point*

2000 *Mathematics Subject Classification*. Primary 37F50; Secondary 37F10, 37F25, 37F30, 30D05.

Key Words and Phrases. n-subhyperbolic polynomial, quadratic polynomial, irrationally indifferent cycle, Brjuno condition, non-linearizability, renormalization, Teichmüller space.

Partially supported by JSPS Research Fellowships for Young Scientists and British Council Grants for JSPS Fellows.

whose multiplier is λ, then α satisfies the Brjuno condition. It is not known whether we can extend this result for polynomials of degree more than two.

The *non-linearizability conjecture* is the following (cf. [**5**] and [**19**]).

NLC. If a polynomial of degree $d \geq 2$ is linearizable at an irrationally indifferent fixed point whose multiplier is λ, then α satisfies the Brjuno condition.

It follows from the Yoccoz theorem that NLC is true in the class of quadratic polynomials. In this paper, we shall prove NLC in the class of *piecewise* 1-*subhyperbolic* polynomials, which is a subclass of *n-subhyperbolic* polynomials defined below.

Let P be a polynomial of degree $d \geq 2$. For details and proofs of the basic results of complex dynamics, see [**16**] and [**17**].

Notation. The Fatou set $F(P)$ is the largest open set such that the iterates $\{P^n|F; n \in \boldsymbol{N}\}$ form a normal family. The Julia set $J(P)$ is the complement of the Fatou set. The filled-in Julia set $K(P)$ consists of those $z \in \boldsymbol{C}$ such that its orbit $\{P^n(z)\}_{n\geq 0}$ is bounded.

The Julia set $J(P)$ is equal to the boundary of $K(P)$.

Notation. For a point $z \in \boldsymbol{C}$, we call $\{P^n(z)\}_{n\geq 0}$ the *orbit*. z is *periodic* and $\{P^i(z)\}_{i=0}^{m-1}$ is a *cycle* if $z, P(z), \dots, P^{m-1}(z)$ are distinct and $P^m(z) = z$ for some $m \in \boldsymbol{N}$.

Let $\mathscr{Z} = \{z_v\}_{v=1}^m$ be a cycle. The *multiplier* of $\mathscr{Z}$ is defined by $(P^m)'(z_v)$, and $\mathscr{Z}$ is *irrationally indifferent* if each z_v is an irrationally indifferent fixed point of P^m. Moreover, if P^m is linearizable at each z_v, or equivalently if $\mathscr{Z} \subset F(P)$, then $\mathscr{Z}$ is called a *Siegel cycle* and the Fatou component containing z_v is called the *Siegel disk of P at* z_v. Otherwise $\mathscr{Z}$ is called a *Cremer cycle*. We call each point of a Siegel cycle a *Siegel* (*periodic*) *point*, and call each one of a Cremer cycle a *Cremer* (*periodic*) *point*.

Definition. For an irrationally indifferent cycle $\mathscr{Z} = \{z_v\}_{v=1}^m$ of P, the *singular set* $\mathscr{S} = \mathscr{S}(\mathscr{Z})$ is defined by $\bigcup_{v=1}^m \bar{S}_v$ (S_v is the Siegel disk at z_v) if $\mathscr{Z}$ is a Siegel cycle, and by $\mathscr{Z}$ itself if $\mathscr{Z}$ is a Cremer cycle.

Theorem 1.1 (Mañé [**12**]). *For each singular set $\mathscr{S}$ of P, there exists a recurrent critical point c such that $\omega(c) \supset \partial\mathscr{S}$.*

Here $\omega(c) = \{z \in \boldsymbol{C};$ there exists $n_k \to \infty$ such that $z = \lim P^{n_k}(c)\}$ is the *omega limit set* of c, and c is *recurrent* if $c \in \omega(c)$.

Definition. A recurrent critical point c *corresponds to* an irrationally indifferent cycle $\mathscr{Z}$ if $\omega(c) \supset \partial\mathscr{S}(\mathscr{Z})$.

In this paper, we always count the number of critical points *with multiplicity*.

Definition (n-subhyperbolicity). For a non-negative integer n, a polynomial P is *n-subhyperbolic* if

(a) there exist exactly n recurrent critical points corresponding to irrationally indifferent cycles,

(b) every critical point in $J(P)$ other than the ones in (a) is preperiodic, and

(c) no critical orbit in $F(P)$ accumulates to $J(P)$.

An n-subhyperbolic polynomial is *n-hyperbolic* if there is no preperiodic critical point in $J(P)$.

By definition, a quadratic polynomial with an irrationally indifferent cycle is 1-hyperbolic. A 0-subhyperbolic polynomial is subhyperbolic in a classical sense. For $n \geq 1$, an n-subhyperbolic polynomial is obtained by "blowing-up" preperiodic critical points of a 0-subhyperbolic polynomial in its Julia set.

We shall precisely state Main Theorem of this paper in Section 6, which says that NLC is true in the class of *piecewise* 1-*subhyperbolic* polynomials. For simplicity, we first treat the class of 1-hyperbolic and 1-subhyperbolic polynomials.

Theorem 1 (NLC). *If a* 1-*subhyperbolic polynomial is linearizable at an irrationally indifferent fixed point whose multiplier is λ, then α satisfies the Brjuno condition.*

In particular, we also have the following earlier result.

Corollary 1 ([**18**]). *Suppose that a cubic polynomial P is linearizable at an irrationally indifferent fixed point and let its rotation number be α. If there exists a critical point of P iterated into a cyclic Fatou component which is a superattractive or attractive basin or a Siegel disk, then α satisfies the Brjuno condition.*

More generally, we also have a positive answer for the *cycle-version* of NLC.

Theorem 2 (Cycle version of NLC). *If a* 1-*subhyperbolic polynomial has a Siegel cycle whose multiplier is λ, then α satisfies the Brjuno condition.*

As a corollary, we have the *cycle-version* of the Yoccoz Theorem.

Corollary 2. *If a quadratic polynomial has a Siegel cycle whose multiplier is λ, then α satisfies the Brjuno condition.*

By studying specific examples of n-subhyperbolic polynomials, we also have:

Theorem 3 (Scaling invariance of the Brjuno condition). *If α satisfies the Brjuno condition, then $m\alpha$ $(m \in \boldsymbol{N})$ also satisfies the Brjuno condition.*

In the rest of this paper, we shall prove Main Theorem. We first prove Theorem 1 in 1-*hyperbolic* case. In Section 2, we consider the *linearizability-*

preserving perturbations for an arbitrary polynomial P. Preserving the linearizability of P at every irrationally indifferent periodic point, this perturbation increases the number of the foliated equivalence classes of acyclic critical points in $F(P)$.

In Section 3, we shall survey the structure theorem of Teichmüller spaces of polynomials and their uniformization in parameter spaces, and consider a local lifting of this uniformization into the representation space of polynomials.

In Section 4, we shall apply linearizability-preserving perturbations to a 1-hyperbolic polynomial in order to increase the dimension of its Teichmüller spaces. We shall complete the proof of Theorem 1 in 1-hyperbolic case.

In Section 5, we shall define *weak renormalizations* of polynomials by *strong separation* and show that they are in fact *strongly* renormalizable under a certain condition.

In Section 6, we shall define a subclass of n-subhyperbolic polynomials, that is, *piecewise* 1*-subhyperbolic polynomials* in terms of strong separation, and state Main Theorem in this paper. Applying strong renormalizations to piecewise 1-subhyperbolic polynomials, we complete the proof of Main Theorem and Theorem 2.

In Section 7, we conclude with several examples of n-subhyperbolic polynomials. We shall prove Theorem 3 here.

Acknowledgement. The author would like to express his gratitude to Prof. Masahiko Taniguchi and Prof. Toshiyuki Sugawa for many valuable discussions and advices, and also to Prof. Anthony Manning, Prof. Oleg Kozlovski and Prof. Sebastian van Strien for helpful comments. He would like to also thank Prof. Caroline Series for inviting him as a long term visitor to Warwick university and the referee for reading the draft carefully and giving him many useful comments.

2. Linearizability-preserving perturbation.

Definition. Let P be a polynomial. A point is said to be *acyclic* if it is neither periodic nor preperiodic point of P.

The *grand orbit* of $x \in \boldsymbol{C}$ is

$$\mathrm{GO}(x, P) := \{y \in \boldsymbol{C}; P^n(x) = P^m(y) \text{ for some } n, m \geq 0\}.$$

Two points $x, y \in \boldsymbol{C}$ are in the *foliated equivalence class* of P if $\overline{\mathrm{GO}(x, P)} = \overline{\mathrm{GO}(y, P)}$.

N_{AC} is the number of the foliated equivalence classes of acyclic critical points of P in $F(P)$.

Main result in this section is the following:

THEOREM 2.1. *Let P be an n-hyperbolic polynomial of degree $d \geq 2$. Then there exists an n-hyperbolic polynomial $\hat{P}$ with the same degree as P such that*

(i) *If P is linearizable at an irrationally indifferent fixed point whose multiplier is λ, then $\hat{P}$ is also so, and*

(ii) $N_{AC}(\hat{P}) = d - n - 1$.

NOTATION. For $r > 0$ and $x \in \boldsymbol{C}$, we put $\boldsymbol{D}(x, r) := \{z \in \boldsymbol{C}; |z - x| < r\}$ and $\boldsymbol{D}_r := \boldsymbol{D}(0, r)$. For $C \in \boldsymbol{C}$ and $U \subset \boldsymbol{C}$, we set $C \cdot U := \{Cz; z \in U\}$.

For a C^1-function f, we set $\mu[f] := \bar{\partial} f / \partial f$. For an open set $V \subset \boldsymbol{C}$, we identify a Beltrami coefficient on V with a function $\mu \in L^\infty(V)$ such that $\|\mu\|_\infty < 1$, and for a C^1-function $f : V \to W$ and a Beltrami coefficient μ on W, we define the pullback $f^*\mu$ of μ on V by

$$(f^*\mu)(z) = \frac{\overline{\partial f(z)}\mu(f(z)) + \bar{\partial} f(z)}{\overline{\bar{\partial} f(z)}\mu(f(z)) + \partial f(z)}.$$

Theorem 2.1 follows from the three Lemmas below. We fix a polynomial P of degree $d \geq 2$ arbitrarily.

LEMMA 2.1. *Let c be a non-periodic critical point in $F(P)$ with multiplicity $k \geq 2$. Then there exist an analytic Jordan neighborhood U of c in $F(P)$, a quasi-conformal automorphism Φ of $\boldsymbol{C}$ and a polynomial $\hat{P}$ with the same degree as P such that*

- *$\overline{U}$ contains neither critical point other than c nor periodic point,*
- *$\hat{P}$ has exactly k distinct critical points in $\Phi(U)$, which are simple, and*
- $P = \Phi^{-1} \circ \hat{P} \circ \Phi$ *on* $\boldsymbol{C} \setminus U$.

PROOF. Let Ω be the component of $F(P)$ containing c. Then it is not a Siegel disk of P. By assumption, c is not a superattracting periodic point. There exists an analytic Jordan neighborhood U of c in Ω such that

- $\overline{U}$ contains neither critical point other than c nor periodic point of P,
- for all $n \in \boldsymbol{N}$, $P^n(U) \cap U = \emptyset$, and
- $P|U$ is a proper map onto a Jordan domain V.

We choose a quasiregular extension $Q : U \to V$ of $P|\partial U : \partial U \to \partial V$ so that

- $Q|\partial U = P|\partial U$, and
- Q has exactly k distinct branch points in U, which are simple.

We set a quasiregular endomorphism of $\boldsymbol{C}$:

$$\tilde{P} := \begin{cases} P & (\text{on } \boldsymbol{C} \setminus U) \\ Q & (\text{on } U). \end{cases}$$

In the above, we have chosen the neighborhood U so that

$$\mu(z) := \begin{cases} (P^*)^i \mu[Q](z) & \text{if } P^i(z) \in U \text{ for some } i \in \boldsymbol{N} \cup \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

becomes a $\tilde{P}$-invariant Beltrami differential on $\boldsymbol{C}$.

Let Φ be a quasiconformal automorphism of $\boldsymbol{C}$ with $\mu[\Phi] = \mu$. Then we have $\mu[\Phi \circ \tilde{P}] = \mu[\Phi]$, so $\hat{P} := \Phi \circ \tilde{P} \circ \Phi^{-1}$ is holomorphic. By construction, we have $\deg \hat{P} = \deg P$. □

The first and second lemma says that we can *decompose* a critical point c with multiplicity k in $F(P)$ to k distinct simple and non-periodic critical points near c.

Lemma 2.2. *Let c be a periodic critical point in $F(P)$ (so it is a superattracting periodic point) with multiplicity $k \geq 1$. Then there exist an analytic Jordan neighborhood U of c in $F(P)$, a quasiconformal automorphism Φ of $\boldsymbol{C}$ and a polynomial $\hat{P}$ with the same degree as P such that*

- *$\overline{U}$ contains no critical point other than c,*
- *$\hat{P}$ has exactly k distinct critical points in $\Phi(U)$, which are simple,*
- *$\Phi(c)$ is not a critical point of $\hat{P}$, and*
- *$P = \Phi^{-1} \circ \hat{P} \circ \Phi$ on $(\boldsymbol{C} \setminus U) \cup \{c\}$.*

The inverse of this perturbation is well-known (cf. [**3**] VI. 5). For readers' convenience, we write the proof.

Proof. Let p be the period of c and Ω the component of $F(P)$ containing c. Then Ω is the superattractive fixed basin of P^p. There exists an analytic Jordan neighborhood U of c in Ω such that

- $\overline{U}$ contains no critical point other than c,
- $P^p(U) \Subset U$, and
- $P|U$ is a proper map onto a Jordan domain V.

We choose a quasiregular extension $Q : U \to V$ of $P|\partial U : \partial U \to \partial V$ so that

- $Q = P$ on $\partial U \cup \{c\}$,
- Q is holomorphic on $\overline{P^p(U)}$,
- Q has exactly distinct k critical points, which are simple, in $P^p(U) \setminus \{c\}$, and
- c is not a critical point of Q.

By the same way as that in the previous proof, we have such a polynomial $\hat{P}$ in this Lemma. □

The third lemma says that we can *move* slightly each critical orbit of P in $F(P)$.

LEMMA 2.3. *Let c be a non-periodic and simple critical point in $F(P)$. Then there exist an analytic Jordan neighborhood U of c in $F(P)$, a quasiconformal automorphism Φ of $\boldsymbol{C}$ and a polynomial $\hat{P}$ with the same degree as P such that*

- *$\overline{U}$ contains neither critical point other than c nor periodic point,*
- *$\hat{P}$ has one and only one critical point $\Phi(c)$ in $\Phi(U)$, which is simple,*
- *$P = \Phi^{-1} \circ \hat{P} \circ \Phi$ on $\boldsymbol{C} \setminus U$, and*
- *$\Phi(c)$ is an acyclic critical point of $\hat{P}$ and not foliated orbit equivalent to any other critical point of $\hat{P}$.*

PROOF. Let Ω be the component of $F(P)$ containing c. By assumption, c is not a superattracting periodic point of P and Ω is not a Siegel disk of P. Thus there exists an analytic Jordan neighborhood U of c in Ω such that

- $\overline{U}$ contains no critical point other than c,
- for all $n \in \boldsymbol{N}$, $P^n(U) \cap U = \emptyset$, and
- $P|U$ is a proper map onto a Jordan domain V.

There exists a small neighborhood V_0 of the critical value $P(c)$ such that for every $v \in V_0$, there exists a quasiregular extension $Q : U \to V$ of $P|\partial U : \partial U \to \partial V$ satisfying:

- $Q|\partial U = P|\partial U$,
- c is one and only one branch point of Q in U, which is simple, and
- $Q(c) = v$.

Let $\hat{P}$ be a polynomial obtained by the same way as that in the proof of Lemma 2.1. We choose $v \in V_0 \setminus \{P(c)\}$ so that for $\hat{P}$, $\Phi(c)$ is acyclic and not foliated orbit equivalent to any other critical point. □

REMARK. In constructing the quasiregular extensions Q of $P|\partial U$ appearing in the above proofs, we can use, for example, the following lemma:

LEMMA 2.4 ([**9**]). *Let Σ_k $(k \geq 1)$ be the quotient of $\boldsymbol{D}^k$ by the action of the symmetric group S_k and we put the set of normalized Blaschke products* (*or proper holomorphic maps of $\boldsymbol{D}$ onto itself fixing* 0, 1) *of degree* $k+1$:

$$\mathscr{B}_k := \left\{ z \prod_{j=1}^{k} \left(\frac{1-\bar{a}_j}{1-a_j} \right) \left(\frac{z-a_j}{1-\bar{a}_j z} \right); \ |a_j| < 1 \ \textit{for} \ 1 \leq j \leq k \right\}.$$

The map $\Sigma_k \to \Sigma_k$ which maps the set of zeros of $B \in \mathscr{B}_d$ to the critical set of B is a homeomorphism.

PROOF OF THEOREM 2.1. Let P be an n-hyperbolic polynomial. We recall that there exists $d-n-1$ critical points in $F(P)$.

The perturbations in the above Lemmas preserve the n-hyperbolicity. Thus by applying Lemma 2.1 and 2.2 to every critical point in $F(P)$ either periodic or with multiplicity $k \geq 2$ and using Lemma 2.3 for simple critical points in $F(P)$

finite times, we have a polynomial $\hat{P}$ with the same degree as P, a quasiconformal automorphism Φ of $\boldsymbol{C}$ and an open set $U \subset \boldsymbol{C}$ which is relatively compact in $(F(P)\backslash(\text{Siegel disks of } P))$ such that

- $P = \Phi^{-1} \circ \hat{P} \circ \Phi$ on $\boldsymbol{C}\backslash U$, and
- $\hat{P}$ is n-hyperbolic and $N_{AC}(\hat{P}) = d - n - 1$.

Consequently, $\hat{P}$ has desired properties. □

3. Teichmüller spaces of polynomials.

Let P be a polynomial of degree $d \geq 2$ and have an irrationally indifferent fixed point z_0 whose multiplier is λ. Then there exists an affine transformation A with $A(z_0) = 0$ such that $A \circ P \circ A^{-1} =: P_A$ is a monic polynomial of degree d. If P is 1-subhyperbolic and linearizable at z_0, P_A is also 1-subhyperbolic and linearizable at the origin. Thus we assume that $P \in \mathscr{P}_d$ and $z_0 = 0$ without any loss of generality. Here for $d \geq 2$, we set

$$\mathscr{P}_d := \{P(z) = \lambda z + a_2 z^2 + \cdots + a_{d-1} z^{d-1} + z^d\} \cong \boldsymbol{C}^{d-2}, \quad \text{and}$$

$$\tilde{\mathscr{P}}_d := \{P(z) = \lambda z + a_2 z^2 + \cdots + a_{d-1} z^{d-1} + a_d z^d \ (a_d \neq 0)\} \cong \boldsymbol{C}^{d-2} \times \boldsymbol{C}^*.$$

We fix $P \in \mathscr{P}_d$ arbitrarily.

Definition. The *deformation space* of P *rel* the origin is

$$\mathrm{Def}(\boldsymbol{C}, 0, P) := \left\{ \phi \,\middle|\, \begin{array}{l} \phi \text{ is a quasiconformal automorphism of } \boldsymbol{C} \\ \text{fixing } 0 \text{ and } \phi \circ P \circ \phi^{-1} =: P_\phi \text{ is a polynomial.} \end{array} \right\} \Big/ \sim,$$

where $\phi_1 \sim \phi_2$ if there exists an affine transformation $h_c(z) := cz$ such that $h_c \circ \phi_1 = \phi_2$. The equivalence class of ϕ is also written by ϕ so long as the discussion is independent of the choice of representative.

We set $\mathscr{H} := \{h_c; c \in \boldsymbol{C}^*\}$. Since rotation numbers of holomorphic germs are topologically invariant, as we have stated in Introduction, we have $P_\phi \in \tilde{\mathscr{P}}_d$ for $\phi \in \mathrm{Def}(\boldsymbol{C}, 0, P)$.

By the Ahlfors-Bers measurable Riemann mapping theorem [**1**], the map from $\mathrm{Def}(\boldsymbol{C}, 0, P)$ to the set $M_1(\boldsymbol{C}, P)$ of P-invariant Beltrami differentials on $\boldsymbol{C}$:

$$\mathrm{Def}(\boldsymbol{C}, 0, P) \ni \phi \mapsto \mu[\phi] \in M_1(\boldsymbol{C}, P)$$

is bijective. Hence we identify $\mathrm{Def}(\boldsymbol{C}, 0, P)$ with $M_1(\boldsymbol{C}, P)$, which has a structure of a complex manifold.

Definition. The *quasiconformal automorphism group* of P *rel* the origin is

$$\mathrm{QC}(\boldsymbol{C}, 0, P) := \left\{ \omega \,\middle|\, \begin{array}{l} \omega \text{ is a quasiconformal automorphism of } \boldsymbol{C} \\ \text{fixing } 0 \text{ and } P_\omega = P. \end{array} \right\}.$$

It acts on $\mathrm{Def}(\boldsymbol{C},0,P)$ by $\omega(\phi)=\phi\circ\omega^{-1}$.

We set a normal subgroup of it:

$$\mathrm{QC}_0(\boldsymbol{C},0,P):=\left\{\omega\in\mathrm{QC}(\boldsymbol{C},0,P)\;\middle|\;\begin{array}{l}\text{There exists a uniformly quasiconformal}\\ \text{isotopy }\{\omega_t\in\mathrm{QC}(\boldsymbol{C},0,P);\,0\le t\le 1\}\\ \text{with }\omega_0=\omega\text{ and }\omega_1=\mathrm{id}_{\boldsymbol{C}}.\end{array}\right\}.$$

Now we define the Teichmüller space, and state its structure theorem and the discreteness of its modular group. For the full account and proof, see [**15**].

Definition. The *Teichmüller space* is

$$\mathrm{Teich}(\boldsymbol{C},P):=\mathrm{Def}(\boldsymbol{C},0,P)/\mathrm{QC}_0(\boldsymbol{C},0,P).$$

The equivalence class of ϕ is written by $[\phi]$.

In McMullen-Sullivan [**15**], we define the deformation space and so on without "*rel* the origin" and write them by $\mathrm{Def}(\boldsymbol{C},P)$, $\mathrm{QC}(\boldsymbol{C},P)$ and $\mathrm{QC}_0(\boldsymbol{C},P)$. By definition, we have

$$\mathrm{Def}(\boldsymbol{C},0,P)\cong\mathrm{Def}(\boldsymbol{C},P)\;\;(\cong M_1(\boldsymbol{C},P))\quad\text{and}$$

$$\mathrm{QC}_0(\boldsymbol{C},0,P)=\mathrm{QC}_0(\boldsymbol{C},P),$$

so this $\mathrm{Teich}(\boldsymbol{C},P)$ agrees with the Teichmüller space of P *defined by the usual way*. In particular,

$$\mathrm{Teich}(\boldsymbol{C},P)=\mathrm{Def}(\boldsymbol{C},0,P)/\mathrm{QC}_0(\boldsymbol{C},P).$$

Theorem 3.1 (The structure theorem [**15**]). *$\mathrm{Teich}(\boldsymbol{C},P)$ is a finite dimensional connected and simply connected complex manifold whose complex dimension is equal to*

$$N_{AC}+N_{LF}-N_P,$$

where

- *N_{AC} is the number of the foliated equivalence classes of acyclic critical points in $F(P)$,*
- *N_{LF} is the number of invariant line fields on the Julia set of P, and*
- *N_P is the number of parabolic cycles.*

Moreover, the canonical projection $\pi:\mathrm{Def}(\boldsymbol{C},0,P)\to\mathrm{Teich}(\boldsymbol{C},P)$ is a holomorphic submersion.

In $\mathrm{Teich}(\boldsymbol{C},P)$, the *Teichmüller metric d* is defined by

$$d([\phi_1],[\phi_2]):=\frac{1}{2}\inf\left\{\log\frac{1+\|\mu[\phi'\circ\phi''^{-1}]\|_\infty}{1-\|\mu[\phi'\circ\phi''^{-1}]\|_\infty};\;\phi'\sim\phi_1\text{ and }\phi''\sim\phi_2\right\}.$$

Theorem 3.2 (Discreteness of modular group [**15**]). *The modular group rel the origin* $\mathrm{Mod}(\boldsymbol{C},0,P):=\mathrm{QC}(\boldsymbol{C},0,P)/\mathrm{QC}_0(\boldsymbol{C},P)$, *which is a subgroup of the Teichmüller modular group* $\mathrm{Mod}(\boldsymbol{C},P):=\mathrm{QC}(\boldsymbol{C},P)/\mathrm{QC}_0(\boldsymbol{C},P)$, *acts on* $(\mathrm{Teich}(\boldsymbol{C},P),d)$ *isometrically, biholomorphically and properly discontinuously.*

We put $\mathscr{M}:=\tilde{\mathscr{P}}_d/(\mathscr{H}$-conjugation$)$. By Theorem 3.2, we have:

Theorem 3.3 (Uniformization in the parameter space). *The map*

$$\eta:\mathrm{Teich}(\boldsymbol{C},P)\ni[\phi]\mapsto[P_\phi]\in\mathscr{M},$$

is holomorphic, and every fiber $\eta^{-1}([P_\phi])$ *is discrete for* $[P_\phi]\in\eta(\mathrm{Teich}(\boldsymbol{C},P))$. *Here* $[P]\in\mathscr{M}$ *is the equivalence class of* $P\in\tilde{\mathscr{P}}_d$.

Proof. $\mathrm{Mod}(\boldsymbol{C},0,P)$ is a covering transformation group of $\eta:\mathrm{Teich}(\boldsymbol{C},P)\to\eta(\mathrm{Teich}(\boldsymbol{C},P))$ and its action on $\mathrm{Teich}(\boldsymbol{C},P)$ is properly discontinuous. □

In the rest of this section, we prepare a lemma needed later. Let $p:\tilde{\mathscr{P}}_d\to\mathscr{M}$ be the canonical projection.

Lemma 3.1. *If* $P(z)=\lambda z+a_2z^2+\cdots+a_{d-1}z^{d-1}+z^d\in\mathscr{P}_d$ $(d\geq 3)$ *satisfies that*:

(*) *for any* $c\in\boldsymbol{C}\backslash\{1\}$ *with* $c^{d-1}=1$, *there exists* $j\in\{2,3,\dots,d-1\}$ *such that* $a_j\neq 0$ *and* $c^{j-1}\neq 1$,

then there exists a local holomorphic section s of p from a neighborhood of $[P]$ *into* $\mathscr{P}_d$ *which maps* $[P]$ *to* P.

Proof. We fix $P\in\mathscr{P}_d$ satisfying (*). We write the j-th coefficient of $P_1\in\mathscr{P}_d$ as $a_j(P_1)$ $(j=2,3,\dots,d-1)$. For each $c\in\boldsymbol{C}\backslash\{1\}$ with $c^{d-1}=1$, we fix such $j=j_c\in\{2,3,\dots,d-1\}$ that $a_j(P)\neq 0$ and $c^{j-1}\neq 1$ and choose such a neighborhood V_j of $a_j(P)$ in $\boldsymbol{C}^*$ that $V_j\cap(c^{j-1}\cdot V_j)=\emptyset$.

For $c\in\boldsymbol{C}\backslash\{1\}$ with $c^{d-1}=1$, we set $V(c):=\{P_1\in\mathscr{P}_d;\,a_{j_c}(P_1)\in V_{j_c}\}\subset\mathscr{P}_d$ and set $V:=\bigcap_{c^{d-1}=1,\,c\neq 1}V(c)$. If two distinct elements P_1 and P_2 of V satisfy $p(P_1)=p(P_2)$, or equivalently $P_1(z)=P_2(cz)/c$ for some $c\in\boldsymbol{C}\backslash\{0,1\}$, then this c satisfies that $c^{d-1}=1$ and $a_j(P_1)=a_j(P_2)c^{j-1}$ for all $j=2,\dots,d-2$. Thus we have $a_{j_c}(P_2)c^{j_c-1}\in(c^{j_c-1}\cdot V_{j_c})\cap V_{j_c}$ but it contradicts the definition of V_{j_c}. Therefore p is injective on V. We choose $s:=(p|V)^{-1}$. □

4. Proof of Theorem 1 in 1-hyperbolic case.

Lemma 4.1. *If a* 1*-hyperbolic polynomial* $P\in\mathscr{P}_d$ $(d\geq 2)$ *satisfies* $N_{AC}=d-2$, *then we have* $\dim\mathrm{Teich}(\boldsymbol{C},P)=\dim\mathscr{M}=d-2$.

Proof. Since no critical point in $F(P)$ accumulates to $J(P)$, P has no parabolic periodic point. Thus we have $N_P=0$ and $\dim\mathrm{Teich}(\boldsymbol{C},P)=N_{IL}+d-2$.

By Theorem 3.3, we have $\dim \mathrm{Teich}(\boldsymbol{C}, P) \le \dim \mathscr{M} = d-2$. Therefore we have $\dim \mathrm{Teich}(\boldsymbol{C}, P) = d-2$ and $N_{IL} = 0$. □

LEMMA 4.2. *If $P \in \mathscr{P}_d$ $(d \ge 3)$ satisfies the condition $(*)$ and* $\dim \mathrm{Teich}(\boldsymbol{C}, P) = d-2$, *then P is quasiconformally stable in $\mathscr{P}_d$, i.e., there exists a neighborhood $U \subset \mathscr{P}_d$ of P such that every element of U is quasiconformally conjugate to P.*

PROOF. By Lemma 3.1, we have a local section s of p from a neighborhood $V \subset \mathscr{M}$ of $[P]$ into $\mathscr{P}_d$ which maps $[P]$ to P. From $\dim \mathrm{Teich}(\boldsymbol{C}, P) = \dim \mathscr{M}$, it follows that $\eta(\mathrm{Teich}(\boldsymbol{C}, P))$ is an open neighborhood of $[P]$. We set $U := s(\eta(\mathrm{Teich}(\boldsymbol{C}, P)) \cap V)$. It is an open neighborhood of P in $\mathscr{P}_d$, and every element of U is quasiconformally conjugate to P by definition of the Teichmüller space. □

LEMMA 4.3. *$P \in \mathscr{P}_d$ $(d \ge 3)$ not satisfying $(*)$ is not 1-hyperbolic.*

PROOF. From the assumption, it follows that $P(cz)/c = P(z)$ for some $c \in \boldsymbol{C} \setminus \{0, 1\}$, and so $P'(cz) = P'(z)$. Thus if z_0 is a critical point of P, then z_0/c $(\ne z_0)$ is another one. Furthermore, if z_0 is contained in $J(P)$, then z_0/c is also so.

Suppose that P is 1-hyperbolic. Then P has one and only one critical point in $J(P)$. It is a contradiction. □

Combining the above lemmas and Theorem 4.1, we prove Theorem 1 in the case where $N_{AC} = d-2$.

THEOREM 4.1 ([**20**], Théorème IV.2.1). *If a quasiconformally stable element of $\mathscr{P}_d$ $(d \ge 2)$ is linearizable at the origin, then the rotation number α satisfies the Brjuno condition.*

For readers' convenience, we give a proof of Theorem 4.1 a little simpler than Pérez-Marco's original one. In this proof, we only use the *J-stability* of quasiconformally stable elements of $\mathscr{P}_d$.

PROOF. In the case $d = 2$, it trivially follows from the Yoccoz theorem. We set $d \ge 3$. We fix a quasiconformally stable element $P \in \mathscr{P}_d$. Then Julia set depends continuously at P in the Hausdorff topology (cf. [**13**]).

We assume that P is linearizable at the origin. Then $0 \notin J(P)$. By the continuity of Julia sets, we choose $r > 0$ and a neighborhood V of P in $\mathscr{P}_d$ so that every element of V has a Siegel disk at the origin including $\boldsymbol{D}_r$. Thus there exists $B > 0$ such that $P[b](z) := P(z) + z^2/b \in V$ for $|b| > B$. We put, for $b \in \boldsymbol{C}$,

$$Q_b(z) := \frac{1}{b} P[b](bz) = \lambda z + z^2 + O(bz^2) \quad \text{as } z \to 0 \quad (Q_0(z) = \lambda z + z^2).$$

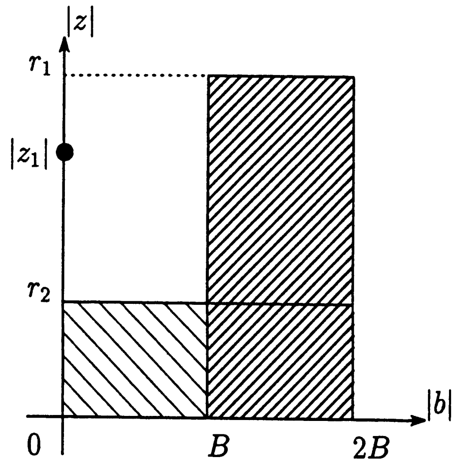

Figure 1. The Reinhardt domain where H is holomorphic.

Any element of $\{Q_b; B < |b| < 2B\}$ has a Siegel disk at the origin which includes $\boldsymbol{D}_{r_1}$ $(r_1 := r/2B)$.

Suppose that $J(Q_0)$ intersects $\boldsymbol{D}_{r_1}$. Then there exists $z_1 \in \boldsymbol{D}_{r_1}^*$ and $q > 0$ such that $Q_0^q(z_1) = z_1$ since $J(Q_0)$ is the closure of the set of all repelling periodic points of Q_0. We set:

$$H(b,z) := \frac{z}{Q_b^q(z) - z} \quad (b \in \boldsymbol{D}_{2B},\ z \in \boldsymbol{D}_{r_1}),$$

which meromorphically depends on each variables and is uniformly continuous on $(\overline{\boldsymbol{D}_{2B}} \times \overline{\boldsymbol{D}_{r_1}}) \backslash$(a small neighborhood of poles).

Since $\boldsymbol{D}_{r_1}$ is included in the Siegel disk of Q_b at 0 for $B < |b| < 2B$, Q_b has no periodic point in $\boldsymbol{D}_{r_1}$ for $B < |b| < 2B$. Thus $H(b,z)$ is holomorphic on $\{b; B < |b| < 2B\} \times \boldsymbol{D}_{r_1}$. On the other hand, since $H(b,0) = 1/(\lambda^q - 1)$ is a bounded constant for every $b \in \overline{\boldsymbol{D}_{2B}}$, there exists $0 < r_2 < r_1$ such that $H(b,z)$ is also holomorphic on $\boldsymbol{D}_{2B} \times \boldsymbol{D}_{r_2}$. See Figure 1.

By the Hartogs continuation theorem, $H(b,z)$ is actually holomorphic on $\boldsymbol{D}_{2B} \times \boldsymbol{D}_{r_1}$. It contradicts the assumption $Q_0^q(z_1) = z_1$ and $z_1 \in \boldsymbol{D}_{r_1}^*$.

Thus $Q_0(z) = \lambda z + z^2$ also has the Siegel disk at 0 including $\boldsymbol{D}_{r_1}$. From the Yoccoz theorem, it follows that α satisfies the Brjuno condition. □

Let us complete the proof of Theorem 1 in 1-hyperbolic case.

Let P be a 1-hyperbolic polynomial of degree $d \geq 2$ and have an irrationally indifferent fixed point z_0 whose multiplier is λ.

Suppose that P is linearizable at z_0. Let $\hat{P}$ be the polynomial in Theorem 2.1 derived from P. By considering an affine conjugation of $\hat{P}$, we assume that $\hat{P} \in \mathscr{P}_d$ and $\hat{P}$ is linearizable at the origin. By applying Lemma 4.1–4.3 and Theorem 4.1 to $\hat{P}$, we conclude that α satisfies the Brjuno condition since $N_{AC}(\hat{P}) = d - 2$. □

5. Renormalization of polynomials.

Throughout this section, we will assume the following.

Standing Hypothesis. P is a monic polynomial of degree $d \geq 2$ and its filled-in Julia set $K(P)$ is connected.

Then there exists the unique conformal map $\phi : \boldsymbol{C} \backslash \bar{\boldsymbol{D}} \to \boldsymbol{C} \backslash K(P)$ such that $\phi(z)/z \to 1$ as $z \to \infty$. We have $\phi(z^d) = P(\phi(z))$ for $z \in \boldsymbol{C} \backslash \bar{\boldsymbol{D}}$, and set $G := \log|\phi^{-1}|$ which is a Green function of $\boldsymbol{C} \backslash K(P)$ with the pole ∞.

For an angle $t \in \boldsymbol{R}$, an *external ray* R_t is defined by

$$R_t := \{\phi(\exp(r + 2\pi i t)); 0 < r < \infty\} \subset \boldsymbol{C}.$$

An external ray R_t *lands* at a point $x \in \partial K(P) = J(P)$ if

$$\lim_{r \to +0} \phi(\exp(r + 2\pi i t)) = x.$$

We call x the *landing point* of R_t and t an *external angle* at x.

For any external ray R_t, its image $P(R_t) = R_{dt}$ is again an external ray. An external ray is *periodic* if $P^n(R_t) = R_t$, or equivalently $d^n t \equiv t \bmod \boldsymbol{Z}$ for some $n \in \boldsymbol{N}$. Such n is the *period* of R_t and the least such n is the *fundamental period.*

The following result is assembled from contributions of Douady, Hubbard, Sullivan and Yoccoz. For the proof, see, for example, [**16**].

Landing theorem. Every periodic external ray lands on a repelling or parabolic periodic point of P. Conversely, let x be a repelling or parabolic periodic point of P. Then x is a landing point, and every ray landing at x is periodic with the same fundamental period.

An external ray is *preperiodic* if $P^k(R_t)$ is periodic for some $k \in \boldsymbol{N}$. Any external ray with rational angle is preperiodic. An external ray R_t lands at a point x if and only if $P(R_t) = R_{dt}$ lands at $P(x)$. By these facts, we have:

Corollary 5.1. *Every external ray with rational angle lands on such a point as is eventually mapped to either repelling or parabolic periodic point by P.*

For $n \in \boldsymbol{N}$ and $k \in \boldsymbol{N} \cup \{0\}$, the *regular (n,k)-partition* $\mathscr{R}_n^{(k)}$ is defined by

$$\mathscr{R}_n^{(k)} = \bigcup_{t=0}^{d^k(d^n-1)-1} \bar{R}_{t/(d^k(d^n-1))}.$$

Definition (Strong separation). Let $\mathscr{C}$ be a closed subset of $\boldsymbol{C} \backslash \operatorname{int} K(P)$. Then $\boldsymbol{C} \backslash \mathscr{C}$ is a *strong separation* (of $(P, K(P))$) if

(i) $P(\mathscr{C}) \subset \mathscr{C}$,

(ii) Each component of $\boldsymbol{C} \backslash \mathscr{C}$ contains at most one cyclic Fatou component or Cremer periodic point,

(iii) Each component of $\boldsymbol{C}\setminus\mathscr{C}$ contains no preperiodic critical point eventually mapped to a repelling or parabolic periodic point,

(iv) Let C be a cyclic Fatou component or Cremer periodic point and c be a critical point. If C and c are contained in a same component of $\boldsymbol{C}\setminus\mathscr{C}$, then for every $n\in\boldsymbol{N}$, $P^n(C)$ and $P^n(c)$ are also contained in a same component of $\boldsymbol{C}\setminus\mathscr{C}$, and

(v) Let p be the period of the above C and U_i be the component of $\boldsymbol{C}\setminus\mathscr{C}$ containing $P^i(C)$ for $i=1,2,\dots,p$. Then the union $\bigcup_{i=1}^p U_i$ contains at least one critical point.

Let $n_0(P)<+\infty$ be the least common multiple of periods of:

- cyclic Fatou components,
- Cremer points, and
- repelling periodic points to which critical points are eventually mapped.

Lemma 5.1 (Kiwi-Geyer). *Suppose that* $n=n_0(P)$. *There exists a positive integer* $\mathscr{K}$ *such that for every* $k\geq\mathscr{K}$, $\mathscr{R}_n^{(k)}$ *gives a strong separation.*

We fix such a $\mathscr{K}$ as in Lemma 5.1. An integer k is said to be *admissible* if $k\geq\mathscr{K}$.

Proof. We follow Kiwi's argument in [**10**] and Geyer's one in [**7**].

Let $\mathscr{S}$ be the set of all cyclic Fatou components of P and Cremer periodic points of P. We note that $\mathscr{R}_n^{(0)}$ is the union of the closures of the *fixed* rays of P^n, and that all elements of $\mathscr{S}$ are P^n-*invariant*. Therefore from Goldberg-Milnor theorem ([**8**], Theorem 3.3), it follows that every component of $\boldsymbol{C}\setminus\mathscr{R}_n^{(0)}$ contains at most one element of $\mathscr{S}$.

Let $U_k(z)$ be the component of $\boldsymbol{C}\setminus\mathscr{R}_n^{(k)}$ containing z ($k\in\boldsymbol{N}\cup\{0\}, z\in\boldsymbol{C}\setminus\mathscr{R}_n^{(k)}$). Since $P(\mathscr{R}_n^{(0)})=\mathscr{R}_n^{(0)}$ and $\mathscr{R}_n^{(k)}=P^{-k}(\mathscr{R}_n^{(0)})$, we have: for $k\in\boldsymbol{N}\cup\{0\}$,

(a) $\mathscr{R}_n^{(k)}\subset\mathscr{R}_n^{(k+1)}$,

(b) $P(\mathscr{R}_n^{(k)})\subset\mathscr{R}_n^{(k)}$,

(c) $U_{k+1}(z)\subset U_k(z)$ $(z\in\boldsymbol{C}\setminus\mathscr{R}_n^{(k+1)})$, and

(d) $P^i(U_{k+i}(z))=U_k(P^i(z))$ $(i\in\boldsymbol{N}\cup\{0\}, z\in\boldsymbol{C}\setminus\mathscr{R}_n^{(k+i)})$.

Claim. *For a critical point c and an element of $C\in\mathscr{S}$, there exists $k(c,C)\in\boldsymbol{N}\cup\{0\}$ such that for every $l\geq k(c,C)$,*

(*) $U_l(c)=U_l(C)\Rightarrow U_l(P^i(c))=U_l(P^i(C))$ $(i\in\boldsymbol{N})$.

Proof. If (*) holds for every $l\geq 0$, then we set $k(c,C)=0$.

Suppose that for some $l_0\geq 0$, (*) does not hold. Then there exists $i\geq 1$ such that $U_{l_0}(P^i(c))\neq U_{l_0}(P^i(C))$, so we have $U_{l_0+i}(c)\neq U_{l_0+i}(C)$ by (d). Thus for every $l\geq l_0+i$, we have $U_l(c)\neq U_l(C)$ by (c), and set $k(c,C)=l_0+i$. Then (*) trivially holds for every $l\geq k(c,C)$. □

We take a critical point c arbitrarily. If c satisfies that $P^k(c)$ is a repelling or parabolic periodic point for some $k \in \boldsymbol{N}$, we write the least such k by $k(c)$. Otherwise, we put $k(c) = 0$.

We set

$$\mathscr{K} := \max\{k(c), k(c, C); C \in \mathscr{S} \text{ and } c \text{ is a critical point.}\} < +\infty.$$

For $k \geq \mathscr{K}$, $\mathscr{R}_n^{(k)}$ satisfies (i) and (ii) by (b) and (a) respectively, and (iv) by Claim. By definition of $n = n_0$, $\mathscr{R}_n^{(0)}$ contains every repelling or parabolic periodic point to which a critical point is eventually mapped. Thus by definition of $\mathscr{K}$, $\mathscr{R}_n^{(k)}$ satisfies (iii) for $k \geq \mathscr{K}$.

We fix $C \in \mathscr{S}$ and $k \geq \mathscr{K}$ arbitrarily. If C is neither Siegel disk nor Cremer point, then (v) follows from the well-known facts.

Suppose that C is a Siegel disk or Cremer point. Let p be the period of it and z_0 be the Siegel point in C if C is a Siegel disk, and C itself otherwise. If $\bigcup_{i=0}^{p-1} U_k(P^i(C))$ contains no critical point, $P^p | U_{k+p}(C) : U_{k+p}(C) \to U_k(C)$ is a conformal isomorphism between simply connected domains fixing z_0. Since $U_{k+p}(C) \subset U_k(C)$ and $|(P^p)'(z_0)| = 1$, it follows that $U_{k+p}(C) = U_k(C)$. However it contradicts $U_k(C) \not\subset K(P)$.

Consequently, (v) holds in every case. □

For an angle $t \in \boldsymbol{R}$ and an opening $\Theta \geq 0$, an *external sector* $S_{t,\Theta}$ is defined by

$$S_{t,\Theta} := \{\phi(\exp(r + 2\pi i(t + \theta r))); 0 < r < \infty, |\theta| \leq \Theta\} \subset \boldsymbol{C}.$$

Let E be a bounded subset of $\boldsymbol{C}$. Then we call $l(E) := \max_{z \in \partial E} G(z)$ the *level* of E, and call a point $z_0 \in \partial E$ with $G(z_0) = l(E)$ a *top* of E.

It is easy to check the following (see Figure 2):

- $l(\boldsymbol{C} \backslash S_{t,\Theta}) = 1/(2\Theta)$ and the top of $\boldsymbol{C} \backslash S_{t,\Theta}$ is $\phi(-\exp(1/(2\Theta) + 2\pi i t))$,
- $\overline{S_{t,\Theta}} \backslash S_{t,\Theta} = \overline{R_t} \backslash R_t$. In particular, for a rational angle, $\overline{S_{t,\Theta}} \backslash S_{t,\Theta}$ agrees with the landing point of R_t. It is called the *landing point* of $S_{t,\Theta}$, and
- $P(S_{t,\Theta}) = S_{dt,\Theta}$.

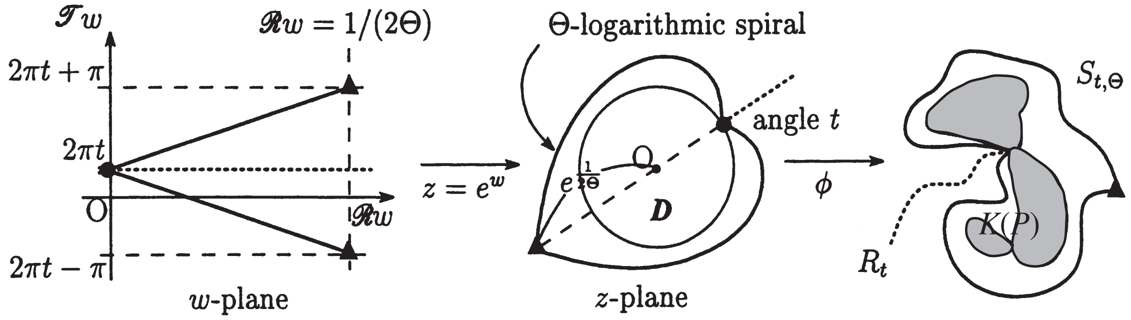

Figure 2. External sector $S_{t,\Theta}$ (t: rational)

An external sector $S_{t,\Theta}$ is *periodic* if $d^n t \equiv t \bmod \boldsymbol{Z}$ for some $n \in \boldsymbol{N}$. Such n is the *period* of it and the least such n is the *fundamental period.*

For $n \in \boldsymbol{N}$ and $k \in \boldsymbol{N} \cup \{0\}$, the *regular wedge* (n,k)*-partition* $\mathscr{S}_n^{(k)}(\Theta)$ is defined by

$$\mathscr{S}_n^{(k)}(\Theta) = \bigcup_{t=0}^{d^k(d^n-1)-1} \overline{S}_{t/(d^k(d^n-1)),\Theta}.$$

Definition. A component of $\boldsymbol{C} \setminus \mathscr{S}_n^{(k)}(\Theta)$ is called a *puzzle piece* of $\mathscr{S}_n^{(k)}(\Theta)$.

It is easy to check the following:

- $P(\mathscr{S}_n^{(k)}(\Theta)) \subset \mathscr{S}_n^{(k)}(\Theta)$, and
- Let V be a puzzle piece of $\mathscr{S}_n^{(k)}(\Theta)$. Then $P^{-1}(V)$ is a finite union $\{U_i\}_{i=1}^m$ of puzzle pieces of $\mathscr{S}_n^{(k+1)}(\Theta)$ and $P|U_i : U_i \to V$ is proper for $i = 1, \dots, m$.

Definition. Let U be a puzzle piece. A point $x \in \partial U$ is a *vertex* of U if $x \in J(P)$ or ∂U is not analytic at x. Such x is said *landing* if $x \in J(P)$, and said *crossing* otherwise. Each component of $\partial U \setminus \{\text{vertices}\}$ is called an *edge* of U.

For an angle $t \in \boldsymbol{R}$ and a curvature $\theta \in \boldsymbol{R}$, an external θ*-logarithmic spiral* $R_{t,\theta}$ is defined by

$$R_{t,\theta} := \{\phi(\exp(r + 2\pi i(t + \theta r)));\ 0 < r < +\infty\} \subset \boldsymbol{C}.$$

It follows that $P(R_{t,\theta}) = R_{dt,\theta}$.

By construction of $\mathscr{S}_n^{(k)}(\Theta)$, we have (see Figure 3):

Lemma 5.2 (Structure of puzzle pieces). *Let U be a puzzle piece of $\mathscr{S}_n^{(k)}(\Theta)$ $(\Theta > 0)$. Then*

(i) *U is simply connected or equivalently, $\boldsymbol{C} \setminus U$ is connected,*

(ii) *$\sharp\{$crossing vertices$\} = \sharp\{$landing vertices$\} \geq 1$,*

(iii) *U has an angle $2\arctan(2\pi\Theta) \in (0, \pi)$ at each crossing vertex,*

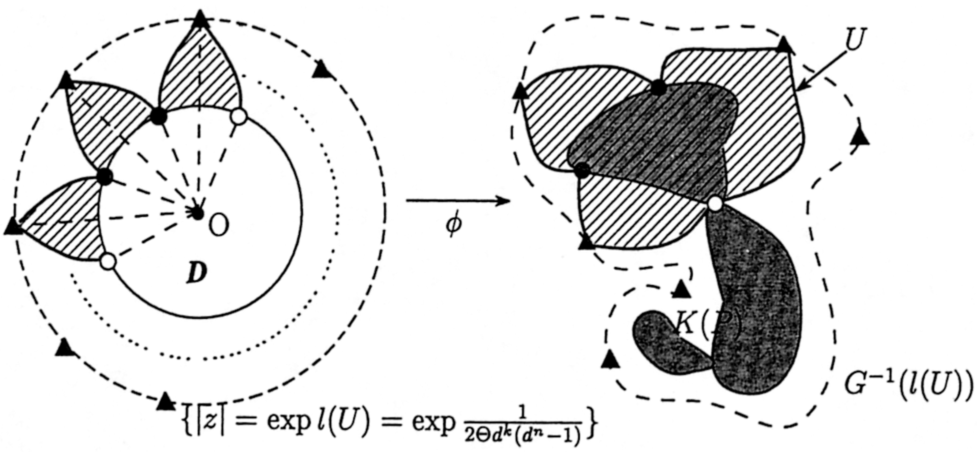

Figure 3. A puzzle piece U of a regular wedge (n,k)-partition $\mathscr{S}_n^{(k)}(\Theta)$

(iv) $l(U) = 1/(2\Theta d^k(d^n - 1))$, *and a point on* ∂U *is a top of* U *if and only if it is a crossing vertex of* U, *and*

(v) *every edge of* U *is a subarc of an external* θ*-logarithmic spiral* ($\theta = \Theta$ *or* $-\Theta$) *between a landing vertex and a crossing vertex.*

We also have:

Lemma 5.3 (Separation is independent of its opening.). *Let* U *be a puzzle piece of* $\mathscr{S}_n^{(k)}(\Theta)$ *and* U' *that of* $\mathscr{S}_n^{(k)}(\Theta')$. *If* $0 \leq \Theta < \Theta'$, *then one and only one of the following holds*:

- $U \cap U' = \emptyset$, *and*
- $U' \subsetneq U$ *and* $U \cap K(P) = U' \cap K(P)$.

Corollary 5.2. *Suppose that* $n = n_0(P)$. *If* k *is admissible*, *then for all* $\Theta \geq 0$, $\mathscr{S}_n^{(k)}(\Theta)$ *gives a strong separation.*

From now on, we always assume that $n = n_0(P)$. This regular wedge partition induces a *weak renormalization* around each periodic point which is neither repelling nor parabolic:

Definition. Let x be a periodic point of P which is neither repelling nor parabolic and p the period of x. Suppose that k is admissible and $\Theta > 0$. Then $(P^p|U, U, V)$ is a *weak renormalization* around x (induced by $\mathscr{S}_n^{(k)}(\Theta)$) if U is the puzzle piece of $\mathscr{S}_n^{(k+p)}(\Theta)$ containing x and if V is that of $\mathscr{S}_n^{(k)}(\Theta)$ containing x.

Proposition 5.1. *A weak renormalization* $(P^p|U, U, V)$ *around* x *satisfies*:

(i) x *is the only non-repelling periodic point in* U,

(ii) U *contains no preperiodic critical point eventually mapped to a repelling or parabolic cycle*,

(iii) $U \subset V$ *and* $P^p|U : U \to V$ *is proper*, *and*

(iv) *The degree of* $P^p|U$ *is more than one.*

Proof. Suppose that this weak renormalization is induced by $\mathscr{S}_n^{(k)}(\Theta)$. Noting that $\mathscr{S}_n^{(k+p)}(\Theta)$ also gives a strong separation, we have easily (i), (ii) and (iv). Since $\mathscr{S}_n^{(k+p)}(\Theta) = P^{-p}(\mathscr{S}_n^{(k)}(\Theta))$, we have (iii). $\square$

A weak renormalization $(P^p|U, U, V)$ around x is *renormalizable* if it is topologically conjugate to a polynomial on $\overline{U}$. If $U \Subset V$, then $(P^p|U, U, V)$ is renormalizable (cf. [**6**]). In general, we have:

Theorem 5.1 (Strong renormalization). *Let* $(P^p|U, U, V)$ *be a weak renormalization around* x. *If every landing vertex of* V *is eventually mapped to a repelling periodic point*, *then it is strongly renormalizable*:

There exists a polynomial P_0 *without preperiodic critical point eventually mapped to a repelling or parabolic cycle in* $J(P_0)$ *such that* $(P^p|U, U, V)$ *is hybrid* *quasiconformally conjugate to* P_0, *i.e., there exists a quasiconformal automorphism* Φ *of* $\boldsymbol{C}$ *satisfying*:

- $P^p = \Phi^{-1} \circ P_0 \circ \Phi$ *on* $\overline{U}$,
- $\mu[\Phi] \equiv 0$ *on the filled-in Julia set of* $(P^p|U, U, V)$, *which is defined by* $K(P^p|U, U, V) := \bigcap_{n \in N} (P^p)^{-n}(\overline{U})$, *and*
- $\Phi(K(P^p|U, U, V)) = K(P_0)$.

Therefore P_0 *has the unique non-repelling periodic point* $\Phi(x)$ (*thus it is a fixed point of* P_0).

P_0 *is called a strong renormalization of* $(P^p|U, U, V)$.

L. Geyer pointed out this theorem in his thesis [**7**], but at present, it has not been published and his proof seems to have several gaps. For readers' convenience, we will give a proof.

Proof of Theorem 5.1. We prove the following lemma to consider the *quasiconformal opening* of P^p near landing points which are critical points of P^p later.

Lemma 5.4. *Let U be a puzzle piece of* $\mathscr{S}_n^{(k)}(\Theta)$ $(\Theta > 0)$. *If every landing vertex of U is eventually mapped to a repelling periodic point, then U is a quasidisk.*

Proof. We put $L := \{t_j := \exp(2\pi i \cdot j/(d^k(d^n - 1))); j = 0, 1, \ldots, d^k(d^n - 1) - 1\}$. Let ψ be a continuous function on $W := \phi^{-1}(\partial\mathscr{S}_n^{(k)}(\Theta)\setminus\{\text{landing points}\}) \cup L \cong S^1$:

$$\psi(x) := \begin{cases} \phi(x) & \text{if } x \in W\setminus L \\ \text{landing point of } R_{\arg t_j/2\pi} & \text{if } x = t_j \in L. \end{cases}$$

Since ψ induces a continuous and injective map from S^1 onto ∂U, ∂U is a Jordan curve.

Let $\{v_i\}_{i=1}^n$ be the set of all landing vertices of U. By Lemma 5.2 (ii), we have $n \geq 1$.

For $\{x, y\} \subset \partial U$, $C(\{x, y\})$ is the component of $\partial U\setminus\{x, y\}$ with smaller diameter. We put $\tilde{C}(\{x, y\}) := \partial U\setminus\overline{C(\{x, y\})}$.

A set $\{\gamma_i\}_{i=1}^n$ of open subarcs of ∂U is *admissible* if

- $\gamma_i \cap \{v_i\}_{i=1}^n = v_i$ and $\operatorname{diam}\gamma_i < \operatorname{diam}(\partial U\setminus\gamma_i)$ for $i = 1, \ldots, n$, and
- all elements of $\{\gamma_i\}_{i=1}^n$ are mutually disjoint.

For an admissible $\{\gamma_i\}_{i=1}^n$, it follows that $C(\{x, y\}) \subset \gamma_i$ for every $\{x, y\} \subset \gamma_i$.

Since ∂U is Jordan and $\partial U\setminus\{v_i\}_{i=1}^n$ is a *finite* union of quasiarcs (piecewise analytic arcs without cusp), it follows that:

Lemma 5.5. *If there exists an admissible* $\{\gamma_i\}_{i=1}^n$ *and a positive constant* $M > 0$ *such that*

$$\operatorname{diam} C(\{x, y\}) \leq M|x - y|$$

for every $\{x, y\} \subset \gamma_i$ *for some* $i \in \{1, \ldots, n\}$, *then* ∂U *is a quasicircle.*

Proof. For each $\delta > 0$, there exists $M_1, M_2 > 0$ such that
(a.1) if $|x - y| \geq \delta$, then $\operatorname{diam} \tilde{C}(\{x, y\}) \leq M_1|x - y|$, and
(a.2) if either C or $\tilde{C}$ is disjoint from $\{v_i\}_{i=1}^n$, then $\operatorname{diam} C(\{x, y\}) \leq M_2|x - y|$.

For a moment, we only consider such an $\{x, y\} \subset \partial U$ that $C(\{x, y\}) \cap \{v_i\}_{i=1}^n \neq \emptyset$ but $C(\{x, y\}) \not\subset \gamma_i$ for all $i = 1, \ldots, n$. Such $\{x, y\}$ belongs to either A_1 or A_2, where

$$A_1 := \{\{x, y\}; \tilde{C} \cap \{v_i\}_{i=1}^n \neq \emptyset\},$$

and

$$A_2 := \{\{x, y\}; \tilde{C} \cap \{v_i\}_{i=1}^n = \emptyset\}.$$

If $A_1 \neq \emptyset$, we put $\delta_1 := \inf\{|x - y|; \{x, y\} \in A_1\} > 0$ and set $\delta = \delta_1$. Otherwise we fix $\delta > 0$ arbitrarily. From (a.1) and (a.2), it follows that

$$\operatorname{diam} C(\{x, y\}) \leq \begin{cases} M_1|x - y| & \text{if } \{x, y\} \in A_1, \\ M_2|x - y| & \text{if } \{x, y\} \in A_2 \end{cases}$$

for such an $\{x, y\} \subset \partial U$ that $C(\{x, y\}) \cap \{v_i\}_{i=1}^n \neq \emptyset$ but $C(\{x, y\}) \not\subset \gamma_i$ for all $i = 1, \ldots, n$. Therefore under the assumption, we have

$$\operatorname{diam} C(\{x, y\}) \leq \max\{M_1, M_2, M\}|x - y|$$

for every $\{x, y\} \subset \partial U$. Thus ∂U is a quasicircle (cf. [**22**] or [**11**]). □

We will find below an admissible $\{\gamma_i\}_{i=1}^n$ satisfying the assumption of Lemma 5.5.

Case 1. Let v_i be a landing vertex of U and a repelling periodic point of P. Without any loss of generality, we assume that every external ray landing at v_i is fixed by P. We write $v = v_i$ and put $\rho := P'(v) \in \boldsymbol{C} \setminus \overline{\boldsymbol{D}}$.

We choose a linearizing chart (D, h) at v, i.e., $h : D \to \boldsymbol{D}$ is conformal, $h(v) = 0$ and $h(P(z)) = \rho h(z)$ $(z \in D)$, such that v is the only vertex contained in $\overline{D}$.

Since D is a linearizing coordinate neighborhood of v, we have:
(*) for any $k \in \boldsymbol{N}$, a branch of P^{-k} fixing v is defined and univalent on D and $(P^{-k})'(v) = \rho^{-k}$.

We write this branch as P^{-k} in below since we focus on a *local dynamics* of P around v.

Choose $r > 0$ so that $\boldsymbol{D}(v, 3r) \subset D$. Let $\gamma^{(0)}$ be a component of $\partial U \cap \boldsymbol{D}(v, r)$ containing v and put $\gamma^{(k)} := P^{-k}(\gamma^{(0)})$ for $k \in \boldsymbol{N}$. By taking $r > 0$ small enough, we assume that $\operatorname{diam} \gamma^{(0)} < \operatorname{diam}(\partial U \setminus \gamma^{(0)})$.

Proposition 5.2. *For any $k \in \boldsymbol{N}$, $\gamma^{(k)}$ is an open subarc of ∂U containing v, and $\gamma^{(k)} \subsetneq \gamma^{(k-1)}$. Moreover,* $\operatorname{diam} \gamma^{(k)} \to 0$ *as* $k \to \infty$.

PROOF. $\gamma^{(0)}$ is the union of v and two subarcs of external $\pm\Theta$-logarithmic spirals landing at v. By assumption, these spirals are also fixed by P. On the other hand, we have $h(\gamma^{(k)}) = \rho^{-k} \cdot h(\gamma^{(0)})$ $(k \in \boldsymbol{N})$. Combining P^{-1}-invariance and P^{-1}-contractiveness of $\gamma^{(0)}$, we have $\gamma^{(k)} \subsetneq \gamma^{(k-1)}$.

Since $\operatorname{diam} h(\gamma^{(k)}) \to 0$ as $k \to \infty$, we have $\operatorname{diam} \gamma^{(k)} \to 0$ as $k \to \infty$. □

Now we fix an $\{x, y\} \subset \gamma^{(1)}$ arbitrarily. Choose the least $k \in \boldsymbol{N}$ such that $C(\{x, y\}) \not\subset \gamma^{(k+1)}$. Setting $X := P^k(x)$ and $Y := P^k(y)$, we have:

- $P^k(C(\{x, y\})) \subset \gamma^{(0)}$, and
- $P^k(C(\{x, y\})) = C(\{X, Y\}) \not\subset \gamma^{(1)}$.

Since P^{-k} is conformal on $\boldsymbol{D}(v, 3r)$, we have:

CLAIM.

$$\frac{\operatorname{diam} C(\{x, y\})}{|x - y|} \leq M \frac{\operatorname{diam} C(\{X, Y\})}{|X - Y|} \quad (M = 64). \tag{1}$$

PROOF. By applying the Koebe distortion theorem to $P^{-k}(v + 3rw)$ on $w \in \boldsymbol{D}$, we have

$$|P^{-k}(z) - v| \leq 3r\rho^{-k} \frac{|w|}{(1 - |w|)^2} \quad (z = v + 3rw).$$

Putting $z \in C(\{X, Y\})$ $(\subset \gamma^{(0)}) \subset \boldsymbol{D}(v, r)$, we have $|w| \leq \operatorname{diam} C(\{X, Y\})/(3r)$ and so $C(\{x, y\}) \subset \boldsymbol{D}(v, (9/4)\rho^{-k} \operatorname{diam} C(\{X, Y\}))$. Thus

$$\operatorname{diam} C(\{x, y\}) \leq \frac{9}{2}\rho^{-k} \operatorname{diam} C(\{X, Y\}). \tag{2}$$

Next, since $X, Y \in \boldsymbol{D}(v, r)$, $\boldsymbol{D}(X, |X - Y|) \subset \boldsymbol{D}(X, 2r) \subset \boldsymbol{D}(v, 3r)$. By applying the Koebe distortion theorem to $P^{-k}(X + 2rw)$ on $w \in \boldsymbol{D}$, we have

$$2r|(P^{-k})'(X)| \frac{|w|}{(1 + |w|)^2} \leq |P^{-k}(z) - x| \quad (z = X + 2rw).$$

Putting $z = Y$, we have

$$\frac{1}{4}|(P^{-k})'(X)|\,|X - Y| \leq |x - y|. \tag{3}$$

Finally, by applying Koebe distortion theorem to $P^{-k}(v + 3rw)$ on $w \in \boldsymbol{D}$, we have

$$3r\rho^{-k} \frac{1 - |w|}{(1 + |w|)^3} \leq \left|\frac{d}{dw}(P^{-k}(z))\right| \quad (z = v + 3rw).$$

Putting $w = (X - v)/3r$, we have

$$9\rho^{-k}/2^5 \le |(P^{-k})'(X)|. \tag{4}$$

Summing up (2), (3) and (4), we have the claimed inequality (1). □

We put $\gamma_v := \gamma^{(1)}$ in this case.

Case 2. Let v_i be a landing vertex of U and not periodic. From the assumption, there exists $k \in \boldsymbol{N}$ such that $P^k(v_i)$ is a repelling periodic point. Let k_0 be the least such k. Without loss of generality, we assume that every external ray landing at $P^{k_0}(v_i)$ is fixed by P. We write $v = v_i$ and put $v_0 := P^{k_0}(v_i)$.

Choose two simply connected domains $D \ni v$ and $D_0 \ni v_0$ such that $P^{k_0}|D : D \to D_0$ is proper and $D \backslash \{v\}$ contains no critical point of P^{k_0}. By the same way as that in Case 1, we can choose D_0 so that it is a linearizing coordinate neighborhood of v_0 for P containing no other vertex than v_0 in $\overline{D_0}$.

Let $Q : D \to D$ be a lift of the branch of $(P|D_0)^{-1}$ fixing v_0, which is univalent, by $P^{k_0}|D$. We have:

(**) for any $k \in \boldsymbol{N}$, Q^k is defined and univalent on D with $Q^k(v) = v$ and $(Q^k)'(v) = \tilde{\rho}^k$, where $Q'(v) =: \tilde{\rho} \in \boldsymbol{D}^*$ (by the Schwarz lemma).

Choose $r > 0$ so that $\boldsymbol{D}(v, 3r) \subset D$ and let $\gamma^{(0)}$ be a component of $\partial U \cap \boldsymbol{D}(v, r)$ containing v. We can use Q as a substitute for P^{-1} in Case 1 by choosing Q such that $Q(\gamma^{(0)}) \subset \gamma^{(0)}$. Put $\gamma^{(k)} = Q^k(\gamma^{(0)})$ for $k \in \boldsymbol{N}$ and assume that $\operatorname{diam} \gamma^{(0)} < \operatorname{diam}(\partial U \backslash \gamma^{(0)})$ by taking $r > 0$ small enough.

By the argument similar to that in Case 1, it follows that for any $k \in \boldsymbol{N}$, $\gamma^{(k)}$ is an open subarc of ∂U containing v, $\gamma^{(k)} \subsetneq \gamma^{(k-1)}$ and $\operatorname{diam} \gamma^{(k)} \to 0$ as $k \to \infty$. We set $X := Q^{-k}(x)$ and $Y := Q^{-k}(y)$ for $\{x, y\} \subset \gamma_1$ by the least $k \in \boldsymbol{N}$ with $C(\{x, y\}) \not\subset \gamma^{(k+1)}$. Then $C(\{X, Y\}) \not\subset \gamma^{(1)}$ and Claim in Case 1 also holds in this case.

We put $\gamma_v := \gamma^{(1)}$ in this case, too.

We have now defined the open subarc γ_v for each $v \in \{v_i\}_{i=1}^n$. We write $\gamma_i := \gamma_{v_i}$. If every γ_i is small enough, this $\{\gamma_i\}_{i=1}^n$ is admissible. From the proof of Lemma 5.5, the $\{X, Y\}$ in Case 1 and 2 satisfies that

$$\frac{\operatorname{diam} C(\{X, Y\})}{|X - Y|} \le \max\{M_1, M_2\}.$$

Therefore from Claim, it follows that $\{\gamma_i\}_{i=1}^m$ satisfies the assumption in Lemma 5.5.

Now the proof of Lemma 5.4 is completed. □

Let $(P^p|U, U, V)$ be a weak renormalization around x such that every landing vertex of V is eventually mapped to a repelling cycle. Since $P^p|U : U \to V$ is a proper map and every landing vertex of U is mapped to that of V, every one of U is also eventually mapped to a repelling cycle. Thus U and V are quasidiscs by Lemma 5.4 so have quasiconformal reflections λ_U of ∂U and λ_V of ∂V respectively.

If $\partial U \cap \partial V = \emptyset$, then $(P^p|U, U, V)$ is a polynomial-like map of degree more than one and $K(P^p|U, U, V) \subset U$. Since U contains no preperiodic critical point eventually mapped to a repelling or parabolic periodic point, Theorem 5.1 trivially holds. Thus from now on, we assume that $\partial U \cap \partial V \neq \emptyset$.

For each landing vertex v of U (resp. V), let $\mathscr{V}_v(U)$ (resp. $\mathscr{V}_v(V)$) be the union of v and two edges of U (resp. V) from v. Then it follows that $P^p(\mathscr{V}_v(U)) = \mathscr{V}_{P^p(v)}(V)$.

By Lemma 5.2 (iv), we choose a set $\{N_v;\ v \text{ is a landing vertex of } U\}$, where each N_v is a neighborhood of

$$\mathscr{V}_v(U) \cap G^{-1}\left(\left[0, \frac{l(U)}{d^p}\right]\right)$$

in $\boldsymbol{C} \setminus U$ such that all elements of $\{N_v\}$ are mutually disjoint.

We fix such $\{N_v\}$. We consider the *quasiconformal opening* of P around such landing vertices of U as are critical points of P^p: We put

$$f_0 := \begin{cases} P^p & \text{on } U, \\ \lambda_V \circ P^p \circ \lambda_U & \text{on } N_v \text{ if } v \text{ is a critical point of } P^p, \\ P^p & \text{on } N_v \text{ if } v \text{ is not so.} \end{cases}$$

Then every vertex of U is not a branch point of f_0 which is a quasiregular map on $U \cup (\bigcup N_v)$, and $f_0(N_v)$ is a neighborhood of $\mathscr{V}_{P^p(v)}(U) \cap G^{-1}([0, \ell(U)])$ in $\boldsymbol{C} \setminus V$.

The number of components of $\partial U \cap \partial V$ is finite, and we write the set of them as $\{W_i\}_{i=1}^m$. Each W_i contains one and only one landing vertex v_i of U and is written as $W_i = \mathscr{V}_{v_i}(U)$, and all elements of $\{\overline{W_i}\}_{i=1}^m$ are mutually disjoint.

Proposition 5.3 (Straightening of opened polynomials). *There exist* $\{U_i\}_{i=1}^m$ *and* $\{V_i\}_{i=1}^m$ *such that*

(i) $U_i \subset \bigcup N_v\ (\subset \boldsymbol{C} \setminus U)$, $V_i \subset f_0(\bigcup N_v)\ (\subset \boldsymbol{C} \setminus V)$, $U_i \Subset V_i\ (i = 1, \dots, m)$, *and all elements of* $\{V_i\}_{i=1}^m$ *are mutually disjoint,*

(ii) $U_0 := U \cup (\bigcup_{i=1}^m U_i) \cup (\bigcup_{i=1}^m (f_0)^{-1}(V_i))$ *and* $V_0 := V \cup (\bigcup_{i=1}^m V_i) \cup (\bigcup_{i=1}^m f_0(U_i))$ *are simply connected domains and* $U_0 \Subset V_0$, *and*

(iii) $f_0|U_0 : U_0 \to V_0$ *satisfies*

  (a) f_0 *is a proper map and has the same degree as* $(P^p|U, U, V)$,

  (b) *the filled-in Julia set* $K(f_0|U_0, U_0, V_0) := \bigcap_{n \geq 0} (f_0)^{-n}(\overline{U_0})$ *agrees with* $K(P^p|U, U, V)$, *and*

  (c) $(f_0|U_0, U_0, V_0)$ *is renormalizable, and more strongly, hybrid quasiconformally conjugate to a polynomial* P_0.

PROOF. First we find $\{U_i\}$ and $\{V_i\}$ satisfying (i), (ii) and (iii)-(a).

CASE 1. Suppose that *all* elements of $\{v_i\}_{i=1}^m$ are periodic points of P^p. By assumption, they are repelling periodic points. Put $f := P^p$. Without any loss of generality, we assume that $f(v_{i-1}) = v_i$ $(i = 1, \dots, m+1)$, where $v_0 := v_m$ and $v_{m+1} := v_1$. Then every v_i is a repelling fixed point of f^m so not a critical point of it. Thus we do not need the quasiconformal opening of P around $\{v_i\}_{i=1}^m$.

LEMMA 5.6. *It follows that*:
(***) *for* $i = 1, 2, \dots, m+1$, $f(W_{i-1}) \supset \overline{W_i}$, *where* $W_0 := W_m$ *and* $W_{m+1} := W_1$.

PROOF. It follows from $f(W_{i-1}) = P^p(\mathscr{V}_{v_{i-1}}(U)) = \mathscr{V}_{v_i}(V) \Supset \mathscr{V}_{v_i}(U) = W_i$. $\square$

Since v_i is a repelling fixed point of f^m and $l((f^m)^{-n}(\overline{W_i})) = d^{-pmn} \cdot l(\overline{W_i})$ $(i = 1, 2, \dots, m+1)$, we have:

COROLLARY 5.3. *There exists a neighborhood* V_m *of* $\overline{W_m}$ *in* $\boldsymbol{C} \setminus V$ *and the branch* $\tilde{G}$ *of* $(f^m)^{-1}$ *on* V_m *fixing* v_m *such that* $\tilde{G}(V_m) \Subset V_m$ *and* $\bigcap_{n \geq 0} \tilde{G}^n(V_m) = \{v_m\}$.

If $m = 1$, we put $U_1 := \tilde{G}(V_1) \Subset V_1$. Taking V_1 small enough, we have $U_0 = U \cup (f_0)^{-1}(V_1) \Subset V_0 = V \cup V_1$ since $(f_0)^{-1}(V_1) \setminus U_1 \Subset V$.

If $m > 1$, we choose $U_i \subset \boldsymbol{C} \setminus U$ and $V_i \subset \boldsymbol{C} \setminus V$ $(i = 1, \dots, m-1)$ by (***) so that

- $U_i := G_{i+1}(V_{i+1})$, where G_{i+1} is the branch of f^{-1} on V_{i+1} mapping v_{i+1} to v_i,
- V_i is a neighborhood of $\overline{W_i}$ in $\boldsymbol{C} \setminus V$, and $U_i \Subset V_i$, and
- $G_1(V_1) \Subset V_m$, where G_1 is the branch of f^{-1} on V_1 mapping v_1 to v_m.

We put $U_m := G_1(V_1)$. If $V_1, \dots, V_m$ are small enough, we have $U_0 = U \cup (\bigcup_{i=1}^m (f_0)^{-1}(V_i)) \Subset V_0 = V \cup (\bigcup_{i=1}^m V_i)$ since $(f_0)^{-1}(V_i) \setminus U_{i-1} \Subset V$ for all i.

CASE 2. Suppose that some element of $\{v_i\}_{i=1}^m$ is *not* a periodic point of P^p. We put $f := P^p$. We write "$v_i \to v_j$" if $f(W_i) \cap W_j \neq \emptyset$. We have the fact similar to (***) in Case 1:
(****) If $v_i \to v_j$, then $f(W_i) \supset \overline{W_j}$.
We recall that $f_0 \equiv f$ on ∂U.

For every cycle $C \subset \{v_i\}_{i=1}^m$ of f, we first define U_j and V_j for each $v_j \in C$ applying the argument in Case 1.

We fix v_i which is not a periodic point of f.

If $v_i \to \emptyset$, or equivalently $f(W_i) \cap W_j = \emptyset$ for all $j = 1, \dots, m$, then $v_i \notin K(f|U, U, V)$. We take a neighborhood V_i of $\overline{W_i}$ in $\boldsymbol{C} \setminus V$ arbitrarily and set $U_i = \emptyset$.

If $v_i \to v_j$ and v_j has already had U_j and V_j, we define U_i by the component of $f_0^{-1}(V_j)$ intersecting W_i, and V_i by a neighborhood of $\overline{W_i}$ in $\boldsymbol{C}\setminus V$ satisfying $U_i \Subset V_i$.

Now every v_i of $\{v_i\}_{i=1}^m$ has U_i and V_i. If every V_i is small enough, $\{U_i\}_{i=1}^m$ and $\{V_i\}_{i=1}^m$ satisfy (i) and (ii). By definition of f_0, U_0 and V_0, we have (iii)-(a).

We put $f := P^p$.

Lemma 5.7. $K(f_0|U_0, U_0, V_0) = K(f|U, U, V)$.

Proof. Suppose that $K(f_0|U_0, U_0, V_0)\setminus K(f|U, U, V) \neq \emptyset$ and take an element x of it. Then there exists $k_1 \in \boldsymbol{N}$ such that $f_0^{k_1}(x) \in U_0\setminus\overline{U}$ since $x \notin K(f|U, U, V)$, and there exists $k_2 \geq k_1$ such that $f_0^{k_2}(x)$ is contained in such an $\operatorname{int} U_i$ that v_i is a repelling periodic point of f. Let p_i be the period of v_i. Then $f_0^{k_2}(x) \in \bigcap_{n\geq 0}((f_0)^{p_i})^{-n}(\operatorname{int} U_i)$. On the other hand, by Corollary 5.3, we have $\bigcap_{n\geq 0}((f_0)^{p_i})^{-n}(\operatorname{int} U_i) = \emptyset$. It is a contradiction. □

By the above lemma, we have (iii)-(b).

Let $\tilde{U}_0$ be the subset of $U_0\setminus U$ where f_0 is not conformal. If v_i is a repelling periodic point, then f_0 is conformal on U_i. Therefore we have $f_0^{-n}(\tilde{U}_0) \subset \overline{U}$ for some $n \in \boldsymbol{N}$. Thus $(f_0|U_0, U_0, V_0)$ is hybrid quasiconformally conjugate to a polynomial P_0. □

By the following facts:

- x is the only non-repelling periodic point in U,
- $K(P^p|U, U, V) = K(f_0|U_0, U_0, V_0) \subset U_0$, and
- every branch point of $f_0|U_0$ is contained in U,

P_0 has desired properties in Theorem 5.1. Now we have completed the proof of Theorem 5.1. □

6. Main theorem and proofs.

Let P be an n-subhyperbolic polynomial of degree $d \geq 2$ whose Julia set is connected and have an irrationally indifferent cycle $\mathscr{Z} = \{z_\nu\}_{\nu=1}^m$.

By using the conformal map $\phi = \phi_P : \boldsymbol{C}\setminus\overline{\boldsymbol{D}} \to \boldsymbol{C}\setminus K(P)$ with $\phi(z)/z \to a_d^{1/(d-1)}$, where a_d is the d-th coefficient of P, we obtain the same results as those in the case where P is monic.

We set $n_0 = n_0(P)$. For an admissible k and $\Theta \geq 0$, we write the puzzle piece of $S_{n_0}^{(k)}(\Theta)$ containing z_ν as $U_\nu^{(k)}$, and set $U^{(k)}(\mathscr{Z}) := \bigcup_{\nu=1}^m U_\nu^{(k)}$ and $K^{(k)}(\mathscr{Z}) := K(P) \cap \overline{U^{(k)}(\mathscr{Z})}$. We note that $K^{(k)}(\mathscr{Z})$ is independent of the opening Θ by Lemma 5.3.

Lemma 6.1. *For every admissible k, $K^{(k)}(\mathscr{Z})$ contains at least one recurrent critical point corresponding to $\mathscr{Z}$.*

This lemma follows from Claim in the proof of Main Theorem.

Definition. Suppose that P is an n-subhyperbolic polynomial with connected Julia set and has an irrationally indifferent cycle $\mathscr{Z}$. Then P is *piecewise* 1-*subhyperbolic* for $\mathscr{Z}$ if there exists an admissible k such that $K^{(k)}(\mathscr{Z})$ contains only one recurrent critical point corresponding to $\mathscr{Z}$.

Now we state Main Theorem in this paper.

Main Theorem (Cycle-version of NLC). *If an n-subhyperbolic polynomial with connected Julia set has a Siegel cycle whose multiplier is λ and for which it is piecewise* 1-*subhyperbolic, then α satisfies the Brjuno condition.*

Proof. Suppose that P satisfies the assumption. Let $\mathscr{Z} = \{z_\nu\}_{\nu=1}^{m}$ be a Siegel cycle whose multiplier is λ and for which P is piecewise 1-subhyperbolic, and let c be the only recurrent critical point corresponding to $\mathscr{Z}$ which is contained in $K^{(k)}(\mathscr{Z})$.

We fix $\Theta > 0$. Without loss of generality, we assume that $c \in U_1^{(k)}$. By definition, $(P^m|U_1^{(k+m)}, U_1^{(k+m)}, U_1^{(k)})$ is a weak renormalization around z_1. Since P is n-subhyperbolic, no critical point in $F(P)$ accumulates to $J(P)$ so P has no parabolic cycle. Therefore every vertex of $U_1^{(k)}$ is eventually mapped to a repelling cycle.

By applying Theorem 5.1 to $(P^m|U_1^{(k+m)}, U_1^{(k+m)}, U_1^{(k)})$, we have a strong renormalization P_0 of it. Let Φ be a quasiconformal automorphism of $\boldsymbol{C}$ giving the hybrid quasiconformal conjugacy between them in Theorem 5.1.

Claim. $\overline{U_1^{(k+m)}}$ *contains a recurrent critical point of P corresponding to $\mathscr{Z}$.*

Proof. By the Mañé Theorem, P_0 has a recurrent critical point c_0 corresponding to the irrationally indifferent fixed point $\Phi(z_1)$. Therefore $\Phi^{-1}(c_0)$ is a recurrent critical point of P^m corresponding to the irrationally indifferent fixed point z_1 of P^m in $\partial K(P^m|U_1^{(k+m)}, U_1^{(k+m)}, U_1^{(k)}) \subset \overline{U_1^{(k+m)}}$. Thus $\{P^i(\Phi^{-1}(c_0))\}_{i=0}^{m-1}$ contains a recurrent critical point of P corresponding to $\mathscr{Z}$. From the uniqueness of c, it follows that $\Phi^{-1}(c_0) = c$. □

Let us continue to prove Main Theorem. Thus $\Phi(c)$ is only one recurrent critical point of P_0 corresponding to $\Phi(z_1)$. Thus P_0 is a 1-subhyperbolic polynomial without critical point eventually mapped to a repelling or parabolic cycle. Furthermore, since P_0 is linearizable at $\Phi(z_1)$, P_0 has no Cremer cycle. Thus P_0 has no preperiodic critical point in $J(P_0)$.

Consequently, P_0 is a 1-hyperbolic polynomial and linearizable at the irrationally indifferent fixed point $\Phi(z_1)$ whose multiplier is λ. From the result in Section 4 (Theorem 1 in 1-hyperbolic case), it follows that α satisfies the Brjuno condition. □

Proof of Theorem 2. Let P be a 1-subhyperbolic polynomial and have an irrationally indifferent cycle $\mathscr{Z} = \{z_\nu\}_{\nu=1}^{m}$ whose multiplier is λ.

Suppose that $K(P)$ is disconnected. Let K_1 be the component of $K(P)$ containing z_1 and $k \in \boldsymbol{N}$ be the least such one that $P^k(K_1) \cap K_1 \neq \emptyset$. Then $\mathscr{Z}^1 := \{(P^k)^j(z_1)\}_{j=0}^{m/k-1}$ is an irrationally indifferent cycle of P^k whose multiplier is λ. Let G be a Green function of $\boldsymbol{C} \backslash K(P)$ with its pole ∞ and U_r be the component of $\boldsymbol{C} \backslash G^{-1}(r)$ containing z_1 for $r > 0$. It is known that for sufficiently small r, $(P^k|U_r, U_r, P^k(U_r))$ is a polynomial-like map of degree more than two whose filled-in Julia set agrees with K_1.

Let P_0 be a polynomial which is hybrid quasiconformal conjugate to $(P^k|U_r, U_r, P^k(U_r))$. P_0 is a polynomial whose Julia set is connected, and have an irrationally indifferent cycle $\Phi(\mathscr{Z}^1)$ whose multiplier is λ, where Φ is a quasiconformal automorphism of $\boldsymbol{C}$ giving the hybrid conjugacy $P^k = \Phi^{-1} \circ P_0 \circ \Phi$ on $\overline{U_r}$.

By the Mañé theorem, there exists a recurrent critical point c_0 of P_0 corresponding to $\Phi(\mathscr{Z}^1)$. Then $c := \Phi^{-1}(c_0)$ is a recurrent critical point of P^k corresponding to $\mathscr{Z}^1$. Thus $\{P^i(c)\}_{i=0}^{k-1}$ contains a recurrent critical point of P corresponding to $\mathscr{Z}$. Since P is 1-subhyperbolic, we assume, without loss of generality, that c is the only recurrent critical point of P that corresponds to an irrationally indifferent cycle of P. Then P_0 is a 1-subhyperbolic polynomial with connected Julia set.

If $\mathscr{Z}$ is a Siegel cycle of P, then $\Phi(\mathscr{Z}^1)$ is a Siegel cycle of P_0. Clearly P_0 is piecewise 1-subhyperbolic for $\Phi(\mathscr{Z}^1)$. Therefore from Main Theorem, it follows that α satisfies the Brjuno condition. □

7. Examples and case studies.

We conclude with several examples of n-subhyperbolic polynomials.

Example 1. $P(z) = \lambda z + z^2$ is a typical example of 1-hyperbolic polynomials. The only critical point of it corresponds to the origin.

Example 2. $P(z) = \lambda z(1+z)^{d-1}$ $(d \geq 3)$ has $d-2$ critical points eventually mapped to the origin which is a fixed point and another one corresponds to the origin. Thus it is 1-hyperbolic.

Example 3. $P_t(z) = \lambda z(1 - (t+1)/(2t)z + 1/(3t)z^2)$ is a 1-hyperbolic polynomial if $|t|$ is sufficiently large (cf. [**4**] §18.2). We note that two critical points are 1 and t and that t is contained in the superattractive basin of ∞ at that time.

Example 4. The family $\mathscr{F} = \{P_t; t \in \boldsymbol{C}\}$ of such polynomials as the above is an *algebraic* family over $t \in \boldsymbol{C}$ with *bifurcations*.

Let M_d $(d \geq 2)$ be the connectedness locus of $\{z^d + c; c \in \boldsymbol{C}\}$.

Theorem 7.1 ([**14**], Theorem 1.3). *Let f_t be a holomorphic family of rational maps with bifurcations. Then there is a $d \geq 2$ such that for any $c \in M_d$ and $m > 0$, the family contains a polynomial-like map $f_t^n : U \to V$ hybrid conjugate to $z^d + c$ with* $\mathrm{mod}(V \setminus U) > m$.

By the above, $\mathscr{F}$ contains actually 1-subhyperbolic polynomial which is not 1-hyperbolic. Furthermore, for every small $\varepsilon > 0$, there exists such a 1-subhyperbolic polynomial of $\mathscr{F}$ that the Hausdorff dimension of its Julia set is more than $2 - \varepsilon$.

Example 5. By applying Theorem 7.1 to $\mathscr{P}_d$ $(d \geq 3)$, we obtain completely general examples of n-subhyperbolic polynomials with irrationally indifferent fixed points.

Example 6. $P(z) = \lambda z + z^d$ $(d \geq 3)$ satisfies $P(cz)/c = P(z)$, where c is a prime $(d-1)$-th root of unity, so it is $(d-1)$-hyperbolic. However P is semi-conjugate to $Q(w) = \lambda^{d-1} w(1+w)^{d-1}$ by $w = z^{d-1}/\lambda$. Thus P is linearizable at the origin if and only if so is Q. From Example 2 $(d \geq 3)$ and the Yoccoz theorem $(d = 2)$, we have:

Theorem 7.2. *If $P(z) = \lambda z + z^d$ $(d \geq 2)$ is linearizable at the origin, then $(d-1)\alpha$ satisfies the Brjuno condition.*

Theorem 3 directly follows from the Brjuno theorem and Theorem 7.2.

References

[1] L. V. Ahlfors and L. Bers, Riemann's mapping theorem for variable metrics, Ann. of Math., **72** (1960), 385–404.
[2] A. D. Brjuno, Analytical form of differential equations, Trans. Moscow Math. Soc., **25** (1971), 199–239.
[3] L. Carleson and T. W. Gamelin, Complex Dynamics, Springer-Verlag (1993).
[4] R. L. Devaney, A first course in chaotic dynamical systems, Addison-Wesley Publishing Company (1992).
[5] A. Douady, Disques de Siegel et anneaux de Herman, Sém. Bourbaki, **39** (1986–87), 151–172.
[6] A. Douady and J. H. Hubbard, On the dynamics of polynomial like mappings, Ann. Sc. E.N.S., 4ème série, **18** (1985), 287–343.
[7] L. Geyer, Linearization of analytic maps at irrationally indifferent fixed points, Thesis (1998).
[8] L. Goldberg and J. Milnor, Fixed points of polynomial maps. II. Fixed point portraits., Ann. Sc. E.N.S., 4ème série, **26** (1993), 51–98.
[9] M. Heins, On a class of conformal metrics, Nagoya Math. J., **21** (1962), 1–50.
[10] J. Kiwi, Non-accessible Critical Points of Cremer Polynomials, Stony Brook IMS Preprint 1995/2 (1995).

[11] O. Lehto, Univalent Functions and Teichmüller Spaces, Springer-Verlag (1987).
[12] R. Mañé, On the theorem of Fatou, Bol. Soc. Bras. Mat., **24** (1993), 1–11.
[13] R. Mañé, P. Sad and D. Sullivan, On the dynamics of rational maps, Ann. Sc. E.N.S., 4ème série, **16** (1983), 193–217.
[14] C. McMullen, The Mandelbrot set is universal, In Tan Lei, editor, The Mandelbrot Set, Theme and Variations, London Mathematical Society Lecture Note Series, **274** (2000), 1–18.
[15] C. McMullen and D. Sullivan, Quasiconformal Homeomorphisms and Dynamics III: The Teichmüller space of a holomorphic dynamical system, Adv. Math., **135** (1998), 351–395.
[16] J. Milnor, Dynamics in one complex variable, Vieweg (1999).
[17] S. Morosawa, Y. Nishimura, M. Taniguchi and T. Ueda, Holomorphic Dynamics, Cambredge studies in advanced mathematics, **66** (1999).
[18] Y. Okuyama, On the non-linearizability of analytic germs and polynomials at irrationally indifferent fixed points, Proceedings of the second ISAAC Congress, Kluwer (2000).
[19] Y. Okuyama, Non-Linearizability of Polynomials at Irrationally Indifferent Fixed Points, Kodai Math. J., **22** (1999), 56–65.
[20] R. Pérez-Marco, Sur les dynamique holomorphes non linéarisables et une conjecture de V. I. Arnold, Ann. Sc. E.N.S., 4^{e} série, **26** (1993), 565–644.
[21] R. Pérez-Marco, Fixed points and circle maps, Acta Math., **179** (1997), 243–294.
[22] C. Pommerenke, Boundary Behaviour of Conformal Maps, Springer-Verlag (1992).
[23] J. C. Yoccoz, Théorèm de Siegel, nombres de Bruno et polynômes quadratiques, Astérisque, **231** (1996), 3–88.

Yûsuke OKUYAMA
Department of Mathematics
Graduate School of Science
Kyoto University
Kyoto 606-8502, Japan
E-mail: okuyama@kusm.kyoto-u.ac.jp