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Condition for global existence of holomorphic solutions of a certain differential equation on a Stein domain of $\boldsymbol{C}^{n+1}$ and its applications

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Abstract. We give a necessary and sufficient condition for the existence of global solutions of some partial differential equation which is locally solvable and give some applications in complex analysis of several variables.

§0. Introduction.

In this paper, we deal with the problem on the existence of global holomorphic solutions of some partial differential equation on a Stein domain of $\boldsymbol{C}^{n+1}$ which is locally solvable. About this problem Wakabayashi [**9**] in 1968 pointed out that equation $\partial u/\partial x_1 = f$ has no global solution even in a simply connected Stein domain or a Runge domain in $\boldsymbol{C}^n$ in general. In 1972, Suzuki [**8**] gave a necessary and sufficient condition for the existence of global solutions of the same equation for an arbitrary f. In 1981, Wakabayashi gave a necessary and sufficient condition for the existence of global solutions of equation $Du = f$ for an arbitrary f, where D is an arbitrary nonsingular holomorphic vector field on a Stein manifold of dimension 2. The study was unpublished (see [**10**]).

Now we deal with an equation $\partial(f_1, \dots, f_n, u)/\partial(x_1, \dots, x_{n+1}) = g$ which is more general than $\partial u/\partial x_1 = f$ but less general than $Du = f$. The integral curves of this equation are prime sets of $f = (f_1, \dots, f_n)$ (see Definition 1.2) and this equation is regarded as a family of holomorphic 1-forms on the prime sets of f. And we give a necessary and sufficient condition for the existence of global solutions of such equation for an arbitrary g (Theorem 2.1). This result include that of Suzuki as a special case. Finally, we give some applications to $\mathrm{Aut}(\boldsymbol{C}^{n+1})$ in algebraic category (Theorem 3.2 and 3.3) and the existence of an immersion of some Stein holomorphic family of open Riemann surfaces (Theorem 3.5).

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§1. Preliminaries.

Let X be a connected complex manifold of dimension $n+1$ and $f_1, \dots, f_n$ be holomorphic functions on X. We set $D = \{(y) = (y_1, \dots, y_n) \in \boldsymbol{C}^n; y = f(p), p \in X \text{ where } f = (f_1, \dots, f_n)\}$.

Definition 1.1. We say that the triple (X, f, D) satisfies condition (α) if $f^{-1}(y)$ is a pure 1 dimensional analytic subset of X for every $y \in D$.

In this paper, we consider only the triple (X, f, D) which satisfies condition (α).

Definition 1.2. An irreducible component S of $f^{-1}(y)$ will be called a prime set (of f).

Let $\{S_\nu\}_{\nu=1,2,\dots}$ be a sequence of mutually distinct prime sets, that is $S_\nu \cap S_\mu \neq \emptyset$ $(\nu \neq \mu)$.

Definition 1.3. The following set E will be called a limit set of $\{S_\nu\}$. $E = \{p \in X;$ for every neighborhood $U(p)$ of p in X, $U(p) \cap S_\nu \neq \emptyset$ for infinitely many $\nu\}$.

H. Shiga showed the following

Lemma 1.4 (Proposition 1 in [**7**]). *If the limit set E of $\{S_\nu\}$ contains a point p_0 of some prime set S_0 such that there is no other prime set through p_0, then $S_0 \subset E$.*

Remark 1.5. In case $n = 1$, above lemma is true even for a point p_0 which is an intersection point with other prime sets by Lemma 1 of [**5**]. But if $n \geq 2$, it is not true any more. For example (see Example 1 in [**7**]), let $X = \boldsymbol{C}(x_1, x_2, x_3)$, $f_1 = x_1 x_2$ and $f_2 = x_3$. Then $(X, f, \boldsymbol{C}^2)$ satisfies condition (α). Now let $S_\nu = \{x_1 = 0, x_3 = 1/\nu\}$, $S_0 = \{x_2 = x_3 = 0\}$, then the limit set E of $\{S_\nu\}_{\nu=1,2,\dots}$ contains $(0,0,0) \in S_0$ and $E \not\subset S_0$.

Definition 1.6. Let $\{S_\nu\}_{\nu=1,2,\dots}$ be a sequence of mutually distinct prime sets and S_0 a prime set. We say that the sequence S_ν converges to $S_0 (S_\nu \to S_0)$ if there is a point $p_0 \in S_0$, which is not an intersection point with other prime sets, such that $\operatorname{dist}(S_\nu, p_0) \to 0$ $(\nu \to \infty)$.

It is easy to see that $S_\nu \to S_0$ independently of the choice of such a point p_0 by Lemma 1.4.

Definition 1.7. A prime set S_0 is regular if for every $\{S_\nu\}$ such that $S_\nu \to S_0$ $(\nu \to \infty)$ the limit set E of $\{S_\nu\}$ is equal to S_0.

Definition 1.8. If the matrix $(\partial f_i/\partial x_j)_{(i=1,\dots,n;j=1,\dots,n+1)}$ is of rank n for every point p of X, where $(x_1,\dots,x_n)$ is a local coordinate of p (we call it rank condition in short) and every prime set of f is regular, we say that (X,f,D) satisfies condition (β).

We notice if (X,f,D) satisfies rank condition, (X,f,D) satisfies condition (α). We regard a prime set S as a point q and we denote by V the set of all such points. We shall define a neighborhood of q as follows: Let S_q be the prime set corresponding to q. From rank condition S_q does not intersect with other prime sets. Let the tube Σ_W be the all prime sets passing through a neighborhood W of an arbitrary point on S_q in X and $U(q)$ be the points of V corresponding to the prime sets passing through Σ_W. It is easy to see that V is a topological space with the neighborhood system $\{U(q); q \in V\}$.

Proposition 1.9. *If (X,f,D) satisfies condition (β), V is regarded as an unramified Riemann domain over D and (X,f,V) is regarded as a fiber space whose fibers are irreducible.*

Proof. First we show V is a Hausdorff space. Let q and q' be points in V such that $q \neq q'$. We take a sufficiently small neighborhood W_1 of some point on S_q in X such as $S_{q'} \cap \overline{W_1} = \emptyset$. Let $U(q)$ be the points of V corresponding to the prime sets passing through W_1. Now we can take a neighborhood W_2 of some point on $S_{q'}$ in X sufficiently small such that each prime set passing through W_2 does not pass through W_1. Because if not, there is a sequence of mutually distinct prime sets $\{S_\nu\}$ such that the limit set E of $\{S_\nu\}$ contains $S_{q'}$ (by Lemma 1.4) and $E \cap \overline{W_1} \neq \emptyset$. It is a contradiction because $E = S_{q'}$ since $S_{q'}$ is regular and $S_{q'} \cap \overline{W_1} \neq \emptyset$. When we set $U(q')$ to be the points of V corresponding to the prime sets passing through W_2, we conclude $U(q) \cap U(q') = \emptyset$.

Secondly we define a projection map of V to D, where D is a domain in $\boldsymbol{C}^n$ (because f is an open map from condition (α)). From rank condition there is a neighborhood W in X at every point $p \in X$, an integer j and a biholomorphic map Φ of W into $D \times \boldsymbol{C}$ such as $y_1 = f_1, \dots, y_n = f_n, y_{n+1} = x_j$. For a point $q' \in U(q)$ which is defined from Σ_W we correspond a point $y = f(S_{q'})$. Such map $\pi : V \to D$ is well defined and locally homeomorphic. □

Example 1.10 (see Example 3 in Fujita [**3**]). Let $X = \{(x_1,x_2,x_3,x_4) \in \boldsymbol{C}^4;$ $x_1x_3 + x_2x_4 - 1 = 0\}$, $y_1 = f_1 = x_1$ and $y_2 = f_2 = x_2$. Then X is a Stein manifold and f_1 and f_2 are holomorphic functions on X. Then $D = \boldsymbol{C}^2 - (0,0)$ and for every point $y \in D$ $f^{-1}(y)$ is an irreducible 1 dimensional analytic subset of

X. It is easy to see that (X, f, D) satisfies condition (β). We note that D is not pseudoconvex.

The following proposition follows from Definition 1.7.

PROPOSITION 1.11. *If* (X, f, D) *satisfies rank condition and* V *is a Hausdorff space, then* (X, f, D) *satisfies condition* (β).

§2. Main theorem.

In this section we assume that X is a Stein (univalent) domain of $\boldsymbol{C}^{n+1}$ of $n+1$ complex variables $x_1, \dots, x_{n+1}$ and $f_1, \dots, f_n$ are holomorphic functions on X. For a given holomorphic functions $g(x)$ on X, we consider the following partial differential equation:

$$\frac{\partial(f_1, \dots, f_n, u)}{\partial(x_1, \dots, x_{n+1})} = g, \tag{1}$$

where u is an unknown function. We show the following

THEOREM 2.1. *The equation* (1) *has a global solution* u *for an arbitrary* g *on* X, *if and only if*

(a) (X, f, D) *satisfies condition* (β);

(b) *every prime set of* f *is simply-connected*;

(c) V *is a Stein manifold.*

LEMMA 2.2. *If conditions* (a), (b), (c) *are satisfied, equation* (1) *has a global solution for an arbitrary* g.

PROOF. Let $p = (x^0)$ be an arbitrary point of X and $y^0 = f(x^0)$. From rank condition there is an integer j such that

$$\Delta_j = \det \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_{j-1}} & \dfrac{\partial f_1}{\partial x_{j+1}} & \cdots & \dfrac{\partial f_1}{\partial x_{n+1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_{j-1}} & \dfrac{\partial f_n}{\partial x_{j+1}} & \cdots & \dfrac{\partial f_n}{\partial x_{n+1}} \end{bmatrix} \neq 0$$

at p. Then there is a sufficiently small neighborhood W of p such that $\Phi : y_1 = f_1, \dots, y_n = f_n, y_{n+1} = x_j$ is a biholomorphic map of W to a neighborhood of (y^0, x_j^0). We transform (1) by Φ as follows:

$$\frac{\partial(y_1, \dots, y_n, u)}{\partial(y_1, \dots, y_n, y_{n+1})} \frac{\partial(y_1, \dots, y_n, y_{n+1})}{\partial(x_1, \dots, x_{n+1})} = g(x_1, \dots, x_{n+1}),$$

that is,

$$\det\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \dfrac{\partial u}{\partial y_1} & \dfrac{\partial u}{\partial y_2} & \dfrac{\partial u}{\partial y_3} & \dots & \dfrac{\partial u}{\partial y_n} & \dfrac{\partial u}{\partial y_{n+1}} \end{bmatrix} \det\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dots & \dfrac{\partial f_1}{\partial x_j} & \dots & \dfrac{\partial f_1}{\partial x_{n+1}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dots & \dfrac{\partial f_n}{\partial x_j} & \dots & \dfrac{\partial f_n}{\partial x_{n+1}} \\ 0 & \dots & 1 & \dots & 0 \end{bmatrix} = g.$$

Thus,

$$(-1)^{j+n+1}\Delta_j \frac{\partial u}{\partial y_{n+1}} = g.$$

Therefore, the restriction of equation (1) to a prime set S passing through p defines a holomorphic 1-form on S with local coordinate x_j $(j = 1, \dots, n+1)$ of the form

$$du = \frac{g}{(-1)^{n+2}\Delta_1}\,dx_1 = \frac{g}{(-1)^{n+3}\Delta_2}\,dx_2 = \dots = \frac{g}{(-1)^{2n+2}\Delta_{n+1}}\,dx_{n+1}, \tag{2}$$

because S is a characteristic curve which satisfies $dx_1/\Delta_1 = -dx_2/\Delta_2 = \dots = (-1)^n(dx_{n+1}/\Delta_{n+1})$. So we can regard equation (1) as an analytic family of holomorphic 1-forms on V.

Now, (V, π, D) is an unramified Riemann domain by Proposition 1.9. By virtue of the Cauchy-Kovalevskaya theorem, for every point $p \in X$ there is a neighborhood W and a local holomorphic solution u_W in W of (1). We take W small if necessary such that a neighborhood $U(q)$ of $q \in V$ corresponding to the prime set S_q passing through W is an univalent domain over D. We continue u_W analytically along each prime set passing through W. As equation (1) defines a holomorphic 1-form along each prime set and each prime set is simply-connected, u_W can be continued single-valued and analytically on Σ_W by Hartogs' theorem, that is, if a function $h(x_1, \dots, x_{n+1})$ is holomorphic on $|x_{n+1}| < R$ when we fix $(x_1, \dots, x_n)$ in $|x_i| < r$ $(i = 1, \dots, n)$ and holomorphic on $|x_i| < r$ $(i = 1, \dots, n+1)$ $(r < R)$, then h is holomorphic on $|x_1| < r, \dots, |x_n| < r$, $|x_{n+1}| < R$. We set u for u_W. We consider $U' = U(q')$ similar as $U = U(q)$ such that $U(q) \cap U(q') \neq \emptyset$ and a single-valued holomorphic solution u' on $\Sigma_{W'}$. Since $\partial(f_1, \dots, f_n, u - u')/\partial(x_1, \dots, x_{n+1}) = 0$ on $\Sigma_W \cap \Sigma_{W'}$, there is a holomorphic function φ on $\pi(U) \cap \pi(U')$ such as $u - u' = \varphi(f_1, \dots, f_n)$ on $\Sigma_W \cap \Sigma'_W$. Let

$\{U_i\}_{i=1,2,\dots}$ be a countable open covering of V and u_i be the solution on Σ_{W_i} as above. When $U_i \cap U_j \neq \emptyset$ and $u_i - u_j = \varphi_{ij}(f_1, \dots, f_n), \{\varphi_{ij} \circ \pi : U_i \cap U_j\}$ is a cocycle. Since V is a Stein manifold, such a cocycle is a coboundary, that is, there are holomorphic functions φ_i on U_i such that $\varphi_i - \varphi_j = \varphi_{ij} \circ \pi$ on $U_i \cap U_j \neq \emptyset$. If we set $u = u_i - \varphi_i \circ \pi^{-1}(f_1, \dots, f_n)$ on Σ_{W_i}, u is a global solution of (1) in X. □

Lemma 2.3. *If equation* (1) *has a global solution for an arbitrary* g, *condition* (b) *is satisfied.*

Proof. Since equation (1) has a global solution for $g \equiv 1$ by the assumption, (X, f, D) satisfies rank condition and condition (α) consequently. Therefore we can consider (1) as a family of holomorphic 1-forms on prime sets as (2). If there is a prime set S_0 which is not simply-connected, there is a holomorphic 1-form $a_j(x_j)\,dx_j$ whose integral on S_0 is multi-valued by the Behnke-Stein theorem in [**1**]. If we set

$$a_1(x_1)(-1)^{n+2}\Delta_1 = a_2(x_2)(-1)^{n+3}\Delta_2 = \cdots = a_{n+1}(x_{n+1})(-1)^{2n+2}\Delta_{n+1},$$

it represents a holomorphic function g_0 on S_0. As a consequence of Cartan's Theorem B, there is a holomorphic function g on X such that $g|_{S_0} = g_0$ because S_0 is a nonsingular analytic subset of a Stein domain of X. It is easy to see that equation (1) for such g has not any single-valued holomorphic solution on X. □

We denote by $\mathcal{O}$ the sheaf of holomorphic functions.

Lemma 2.4 (cf. Suzuki [**8**]). *If* (X, f, D) *satisfies condition* (β) *and equation* (1) *has a global solution for an arbitrary* g, $H^1(V, \mathcal{O}) = 0$ *and* V *is a Cousin-I domain consequently.*

Proof. Let L be a linear differntial operator on X such that $Lu = g$ means equation (1). We denote by $\mathcal{O}^L$ the sheaf of local solutions of $Lu = 0$ on X. Since $Lu = g$ is locally solvable, the sequence of sheaves on X

$$0 \to \mathcal{O}^L \xrightarrow{i} \mathcal{O} \xrightarrow{L} \mathcal{O} \to 0$$

is exact. Since X is a Stein domain of $\boldsymbol{C}^{n+1}$, we have $H^p(X, \mathcal{O}) = 0$ for $p \geq 1$ by Cartan's Theorem B. Then we have the exact sequence as follows:

$$0 \to \Gamma(X, \mathcal{O}^L) \xrightarrow{\tilde{i}} \Gamma(X, \mathcal{O}) \xrightarrow{\tilde{L}} \Gamma(X, \mathcal{O}) \to H^1(X, \mathcal{O}^L) \to 0.$$

Therefore $\tilde{L}$ is onto if and only if $H^1(X, \mathcal{O}^L) = 0$. Since $\tilde{L}$ is onto if and only if equation (1) has a global solution for an arbitrary g, $H^1(X, \mathcal{O}^L) = 0$. Since we can consider naturally such as $H^1(V, \mathcal{O}) \subset H^1(X, \mathcal{O}^L)$, $H^1(V, \mathcal{O}) = 0$. □

Assume that equation (1) has a global solution u_0 for $g \equiv 1$. Let Φ be a locally biholomorphic map defined by $y_1 = f_1, \dots, y_n = f_n, y_{n+1} = u_0$. Let Y be an unramified Riemann domain over $\boldsymbol{C}^{n+1}$ defined by Φ. The domain Y is biholomorphic to X and a Stein manifold. Now we transform (1) by Φ, then we have

$$\frac{\partial u}{\partial y_{n+1}} = g. \tag{3}$$

We can consider (3) as a family of holomorphic 1-forms on each prime set S with coordinate y_{n+1} with parameter t, where S is an irreducible component of $\Phi^{-1}(t)$ and $t = (y_1, \dots, y_n) \in D = \{(y) \in \boldsymbol{C}^n; y_1 = f_1, \dots, y_n = f_n\}$, so that D is an unramified domain over $\boldsymbol{C}^n$. Let $y_1 = \bar{f}_1 = y_1, \dots, y_n = \bar{f}_n = y_n$ and consider $(Y, y, D) = (Y, \bar{f}, D)$.

It is easy to see the following

Lemma 2.5. *To prove that conditions* (a), (b), (c) *are satisfied when equation* (1) *has a global solution for an arbitrary g, it is enough to prove that conditions* (a), (b), (c) *for $(Y, \bar{f}, D)$ and V are satisfied when equation* (3) *has a global solution for an arbitrary g.*

Lemma 2.6. *If equation* (3) *has a global solution for an arbitrary g, V is a Stein manifold.*

Proof. We shall show the lemma by the following four steps 1), 2), 3) and 4).

1) We show $(Y, \bar{f}, D)$ satisfies condition (β) in case $n = 1$. Assume that there are prime sets S_1 and S_2 and a sequence of mutually distinct prime sets $\{S_\nu\}_{\nu=1,2,\dots}$ such that S_ν converges to S_1 and S_2 simultaneously. Let p_i be a point of S_i $(i = 1, 2)$. We take a sufficiently small neighborhood W_1 of p_1 in Y such that Σ_{W_1} corresponds to an univalent domain of V over D. Let $(y_1^0, y_2^0) = \Phi(p_1)$ and $\Gamma_1 = \Phi^{-1}|_{y_2 = y_2^0} \cap W_1$. The line Γ_1 is transversal to prime sets passing through W_1. We give initial values on Γ_1 with a function $1/(y_1 - y_1^0)$ where we regard y_1 as a coordinate of Γ_1 and continue the initial value constantly to each prime set passing through Γ_1. We denote such function by h_0. The function h_0 is meromorphic on Σ_{W_1} and its pole divisor is S_1. We give a Cousin-I data such as h_0 on Σ_{W_1} and 0 on $Y - S_1$. Since Y is a Stein manifold, there is a solution h of the Cousin-I problem. When we set $g = \partial h / \partial y_2$, g is holomorphic on Y because $h = h_0 + k$ on Σ_{W_1} where k is a holomorphic function and $\partial h_0 / \partial y_2 = 0$. Then equation $\partial u / \partial y_2 = g$ has no global solution on Y. Because, if u is a global solution, $\partial (u - h) / \partial y_2 = 0$ and $(u - h)|_{S_\nu}$ is constant. It is a contradiction that $\lim_{S_\nu \ni p_\nu \to p_1} |u - h| = \infty$ and $\lim_{S_\nu \ni p'_\nu \to p_2} |u - h| < \infty$. It is absurd because $\partial u / \partial y_2 = g$ has a global solution for arbitrary g from the assumption. So $(Y, \bar{f}, D)$

satisfies the condition (β) and V is regarded as an unramified Riemann domain over $D \subset \boldsymbol{C}$ by Proposition 1.9. It is well known that V is a Stein manifold.

2) Now let $n > 1$ and assume that V is a Hausdorff Stein manifold for Y of dimension $n-1$ such that for every hyperplane T in $\boldsymbol{C}^n$, any connected component Y' of $\bar{f}^{-1}(D \cap T)$ is a Stein submanifold of Y. When we consider equation $(3)'$ which is defined by restricting (3) to Y', we can consider $(3)'$ as a family of holomorphic 1-forms on the prime sets S_t with coordinate y_{n+1} where $t = (y_1, \dots, y_n) \in D \cap T$. Since $(3)'$ has a global solution for an arbitrary g because (3) has a global solution for an arbitrary g and Cartan's Theorem B holds good for Y and Y', V' is a Hausdorff Stein manifold by the assumption of induction, where V' is a topological space defined from $(Y', \bar{f}|_{Y'}, D \cap T)$.

3) Now, we show that V is a Hausdorff space. Let q_1 and q_2 be distinct points of V. We must find disjoint neighborhoods $U(q_1)$ and $U(q_2)$. Since π, which is the projection of V to D in Proposition 1.9, is a local homeomorphism, we need only to consider the case $\pi(q_1) = \pi(q_2)$. Choose $U(q_1)$ and $U(q_2)$ sufficiently small such that $\pi|_{U(q_1)}$ and $\pi|_{U(q_2)}$ are homeomorphisms of $U(q_1)$ and $U(q_2)$ onto an open ball B in $\boldsymbol{C}^n$. Suppose that there exists a point $q_3 \in U(q_1) \cap U(q_2)$. Let T be a hyperplane in $\boldsymbol{C}^n$ containing $\pi(q_1) = \pi(q_2)$ and $\pi(q_3)$. Since each connected component of $\pi^{-1}(T)$ (that is V') is a Hausdorff space and $B \cap T$ is connected, we conclude that $U(q_1) \cap V' = U(q_2) \cap V'$ and $q_1 = q_2$. This is a contradiction.

4) Finally we shall prove the following statements.

If V is an unramified Riemann domain over $\boldsymbol{C}^n$ of n complex variables $x_1, \dots, x_n$ with projection π and satisfies the following conditions:

(1) *V is a Cousin-I domain*;

(2) *Each connected component of $\pi^{-1}(T)$ is a Stein manifold where T is an arbitrary hyperplane in $\boldsymbol{C}^n$, then V is a Stein manifold.*

In fact, we shall prove this by use of the idea of Cartan [**2**]. If we assume that V is not a holomorphically convex domain, there is a boundary point of V such that every holomorphic function on V is continued analytically across such a point. We can choose such a boundary point p and a hyperplane T in $\boldsymbol{C}^n$ that there is a connected component D of $\pi^{-1}(T) \cap V$ whose boundary points contains p. We may assume $T = \{x_n = 0\}$ without loss of generality. Since D is an unramified Stein Riemann domain over $\boldsymbol{C}(x_1, \dots, x_{n-1})$, there is a holomorphic function f on D which can not be continued analytically across every boundary point of D. The function f can be considered as a holomorphic function on a neighborhood U in V which contains D and sufficiently near to D. We give a Cousin-I data such as f/x_n in U and 0 in $V - D$. Since V is a Cousin-I domain, there is a solution g of the Cousin-I problem. Set $h = x_n g$. Then h is holomorphic on V and $h|_D = f$. It is absurd because h is continued analytically across p by the assumption. So, V is a holomorphically convex domain.

To prove that V is a Stein manifold, it is sufficient to prove that if p_1 and p_2 are different points in V, then $f(p_1) \neq f(p_2)$ for some $f \in \mathcal{O}(V)$. To prove this, it is sufficient to prove for the case $\pi(p_1) \neq \pi(p_2)$. If the connected component D of $\pi^{-1}(T)$ which contains p_1 contains p_2 too, $f(p_1) \neq f(p_2)$ for some $f \in \mathcal{O}(D)$ because D is a Stein manifold. Since V is a Cousin-I domain, there is a holomorphic function h on V such that $h|_D = f$ by the same way above. Then $h(p_1) \neq h(p_2)$. If D_i $(i = 1, 2)$ is the connected component of $\pi^{-1}(T)$ which contains p_i and $D_1 \cap D_2 = \emptyset$, there is a holomorphic function on V such that $h|_{D_i} = i$ by the same way above. Then $h(p_1) \neq h(p_2)$ □

Proof of Theorem 2.1. If conditions (a), (b), (c) are satisfied, equation (1) has a global solution for an arbitrary g by Lemma 2.2. If equation has a global solution for an arbitrary g, condition (b) is satisfied by Lemma 2.3 and conditions (a), (c) are satisfied by Lemma 2.4, Lemma 2.5 and Lemma 2.6. □

§3. Applications.

Definition 3.1. If $f_1, \dots, f_n$ are polynomials in $\boldsymbol{C}^{n+1}$ of $n+1$ complex variables $x_1, \dots, x_{n+1}$ such that $(\partial f_i/\partial x_j)_{(i=1,\dots,n; j=1,\dots,n+1)}$ is of rank n for every $(x) \in \boldsymbol{C}^{n+1}$ and $f^{-1}(y)$ is a simply-connected irreducible 1 dimensional analytic subset for every $(y) = (y_1, \dots, y_n) \in \boldsymbol{C}^n$, then we say $(f_1, \dots, f_n)$ satisfies condition (γ).

Theorem 3.2. *If $(f_1, \dots, f_n)$ satisfies condition (γ), differential equation*

$$\frac{\partial(f_1, \dots, f_n, u)}{\partial(x_1, \dots, x_{n+1})} = \varphi(f_1, \dots, f_n), \tag{4}$$

where $\varphi(y_1, \dots, y_n)$ is an entire function such as $\varphi \neq 0$ (at any point in $\boldsymbol{C}^n$), has always a global solution u and the map $y_1 = f_1, \dots, y_n = f_n, y_{n+1} = u$ is an automorphism of $\boldsymbol{C}^{n+1}$.

Proof. Now equation (4) is a special case of (1) and $(X, f, D) = (\boldsymbol{C}^{n+1}, f, \boldsymbol{C}^n)$. Since $V = \boldsymbol{C}^n$ is a Stein domain and $(\boldsymbol{C}^{n+1}, f, \boldsymbol{C}^n)$ satisfies rank condition, $(\boldsymbol{C}^{n+1}, f, \boldsymbol{C}^n)$ satisfies condition (β) from Proposition 1.11. Then equation has a global solution u by Lemma 2.2. From the proof in Lemma 2.2, if we restrict equation (4) to the prime set $S = \{f^{-1}(y)\}$, it defines a holomorphic 1-form du which does not take zero on S. Let $\tilde{S}$ be the compactification of S and $\infty = \tilde{S} - S$. Then du can be considered as an Abel's differential on $\tilde{S}$ and it has a pole of order 2 at ∞ since degree of $du = -2$ by the Riemann-Roch theorem. Then $u|_S$ takes every value once by the residue theorem. So the map $y_1 = f_1, \dots, y_n = f_n, y_{n+1} = u$ is an automorphism of $\boldsymbol{C}^{n+1}$. □

Theorem 3.3. *Let* $(f_1,\dots,f_n)$ *satisfy condition* (γ) *and* u *be an entire function. The map* $y_1=f_1,\dots,y_n=f_n,y_{n+1}=u$ *is an automorphism of* $\boldsymbol{C}^{n+1}$ *if and only if* u *satisfies equation*

$$\frac{\partial(f_1,\dots,f_n,u)}{\partial(x_1,\dots,x_{n+1})}=\varphi(f_1,\dots,f_n), \tag{5}$$

where $\varphi(y_1,\dots,y_n)$ *is an entire function such as* $\varphi\neq 0$.

Proof. If $(f_1,\dots,f_n)$ satisfies condition (γ), there is a polynomial f_{n+1} such that the map $y_1=f_1,\dots,y_n=f_n,y_{n+1}=f_{n+1}$ is an automorphism of $\boldsymbol{C}^{n+1}$ by Corollary of Theorem 4 in Fujita [**3**]. Then it is easy to see that if the map $y_1=f_1,\dots,y_n=f_n,y_{n+1}=u$ is an automorphism of $\boldsymbol{C}^{n+1}$, $u=\varphi(f_1,\dots,f_n)f_{n+1}+\psi(f_1,\dots,f_n)$, where ψ is an arbitrary entire function and φ is an entire function such as $\varphi\neq 0$. Now,

$$\frac{\partial(f_1,\dots,f_n,u)}{\partial(x_1,\dots,x_{n+1})}=\varphi(f_1,\dots,f_n)\frac{\partial(f_1,\dots,f_{n+1})}{\partial(x_1,\dots,x_{n+1})}=c\varphi(f_1,\dots,f_n),$$

where c is a constant such as $c\neq 0$.

If $(f_1,\dots,f_n,u)$ satisfies equation (5), the map $y_1=f_1,\dots,y_n=f_n,y_{n+1}=u$ is an automorphism of $\boldsymbol{C}^{n+1}$ by Theorem 3.2. □

Definition 3.4. Let X be a Stein manifold of dimension $n+1$ $(n\geq 1)$, $D=\{|y_1|<r_1,\dots,|y_n|<r_n\}$ $(r_i>0)$ and $f=(f_1,\dots,f_n)$ be a holomorphic map of X onto D such that

(1) for any $(y)\in D, f^{-1}(y)$ is a one dimensional irreducible analytic subset of X,

(2) f satisfies rank condition and

(3) (X,f,D) is homeomorphic preserving fibers to $D\times R$ where R is an open Riemann surface.

Then we call (X,f,D) a Stein holomorphic family of open Riemann surfaces.

Theorem 3.5. *If* (X,f,D) *is a Stein holomorphic family of open Riemann surfaces which is homeomorphic preserving fibers to* $D\times R$ *where* R *is a simply connected open Riemann surface, then there is an immersion* Φ *of* (X,f,D) *to* $D\times\boldsymbol{C}$, *that is, there is a holomorphic function* u *on* X *such that the rank of the transformation matrix of* $\Phi=(f,u)$ *is* $n+1$ *for every point* p *of* X.

Proof. Since X is a Stein manifold in which Cousin II problem is solvable, there is a global holomorphic section g of the canonical line bundle K_X such as $g\neq 0$. We consider differential equation on X such as

$$\frac{\partial(f_1,\dots,f_n,u)}{\partial(x_1,\dots,x_{n+1})}=g$$

where u is an unknown function and $(x_1, \dots, x_{n+1})$ is a local coordinate of every point of X. Then by the same way as in Lemma 2.2, the above equation has a global solution. □

Remark 3.6. If (X, f, D) is a Stein holomorphic family of open Riemann surfaces of type (g, n) where g is the genus of the fiber, n is the number of components of boundary of the fiber and g and n are independent of points of D, then the local immersion for D exists. See Nishimura [**11**].

Corollary 3.7. *Let R be an arbitrary open Riemann surface and (X, f, D) be a Stein holomorphic family of open Riemann surfaces which is homeomorphic preserving fibers to $D \times R$. Then there is an immersion of $\tilde{X}$, which is an universal covering space of X, to $D \times \boldsymbol{C}$.*

Proof. Since $(\tilde{X}, f, D)$ is a Stein holomorphic family of open Riemann surfaces which is homeomorphic to $D \times \tilde{R}$ where $\tilde{R}$ is an universal covering space of R, there is an immersion of $(\tilde{X}, f, D)$ to $D \times \boldsymbol{C}$ from Theorem 3.5. □

Problem 3.8. *Let X be a Stein domain of $\boldsymbol{C}^{n+1}$ and equation* (1) *has a global solution for an arbitrary g. Then, is X biholomorphically equivalent to some domain in $V \times \boldsymbol{C}$?*

Remark 3.9. Nishino [**6**] showed the following theorem. *Let $f(x_1, x_2)$ be an entire function on X; D be a disk $|y_1| < \rho$; (X, f, D) have a global holomorphic section such as a line which is transversal to fibers; $\{f_{x_1} = f_{x_2} = 0\} = \emptyset$; and $y_1 = f(x_1, x_2)$ for each $y_1 \in D$ be irreducible and conformally eqivalent to $\boldsymbol{C}$. Then X is biholomorphic to $D \times \boldsymbol{C}$.* Therefore, in this case equation (1) has a global solution for an arbitrary g. Now if $y_1 = f(x_1, x_2)$ is conformally equivalent to the unit disk for every $y_1 \in D$ and other conditions are same as above ones, is X biholomorphic to a bounded domain in $D \times \boldsymbol{C}$?

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