J. Math. Soc. Japan
Vol. 53, No. 2, 2001

Blow-up solutions for ordinary differential equations associated to harmonic maps and their applications

Dedicated to Proferssor Norio Shimakura on his sixtieth birthday

By Takeyuki NAGASAWA and Atsushi TACHIKAWA

(Received Apr. 27, 1999)
(Revised Nov. 29, 1999)

Abstract. In this paper, the blow-up of solutions of ordinary differential equations, which are deduced from the equation of equivariant harmonic maps, is studied. Its direct consequence is the non-existence or existence result of equivariant harmonic maps between warped product manifolds. As another application we prove the non-existence of a harmonic map from an Euclidean space to a Hadamard manifold with a certain nondegeneracy condition at infinity, provided sectional curvatures of the Hadamard manifold are bounded from above by a slowly decaying negative function of the distance from a fixed point.

1. Introduction.

Let $\boldsymbol{R}_+ \times_g S^{m-1}$ be a warped product manifold $(\boldsymbol{R}_+, dt^2) \times (S^{m-1}, d\theta^2)$ equipped with the metric $dt^2 + g(t)^2\, d\theta^2$. $\boldsymbol{R}^m$ and $\boldsymbol{H}^m$ are typical examples of such manifolds:

$$\boldsymbol{R}^m = \boldsymbol{R}_+ \times_f S^{m-1} \quad \text{where } f(t) = t,$$

$$\boldsymbol{H}^m = \boldsymbol{R}_+ \times_h S^{m-1} \quad \text{where } h(r) = \sinh r.$$

We call a map $U : M = [0, T) \times_f S^{m-1} \to N = \boldsymbol{R}_+ \times_h S^{n-1}$ *equivariant* if it can be written as

$$U(t, \theta) = (r(t), \varphi(\theta)) \in \boldsymbol{R}_+ \times_h S^{n-1}$$

for some function $r : [0, T) \to \boldsymbol{R}_+$ and a map $\varphi : S^{m-1} \to S^{n-1}$. An equivariant map $U = (r, \varphi)$ is harmonic if and only if φ is a harmonic map with the constant energy density $e(\varphi)$ as a map from S^{m-1} to S^{n-1} and r is a solution to

---

2000 *Mathematics Subject Classification*. Primary 58E20; Secondary 34A34.
Key Words and Phrases. Harmonic maps, warped product manifolds, equivariant hamonic maps, blow-up solutions, rotationally nondegeneracy conditions.
This research was partly supported by Grant-in-Aid for Scientific Research (Nos. 10740080 and 09640177), Ministry of Education, Culture, Sports, Science and Technology, Japan.

$$\ddot{r} + \frac{(m-1)\dot{f}(t)}{f(t)}\dot{r} - \frac{\mu^2 h(r) h'(r)}{f^2(t)} = 0, \tag{1.1}$$

$$r(0) = 0,$$

with $\mu = \sqrt{2e(\varphi)}$. Here $\dot{}$ and $'$ mean d/dt and d/dr respectively. $h(r)$ is a nonnegative function behaving like r near $r = 0$ (for precise statement see §2). The existence of such φ, called the *eigenmap*, is shown by Ueno [**15**] for some pairs of m and n. The equations like (1.1) were treated in [**7**], [**12**], [**10**] in the context of rotationally symmetric harmonic maps. Moreover, as mentioned in [**17**], in order to consider equivariant harmonic maps between complex manifolds, it is necessary to treat a more general equation

$$\ddot{r}(t) + \left(p\frac{\dot{f}_1(t)}{f_1(t)} + q\frac{\dot{f}_2(t)}{f_2(t)}\right)\dot{r}(t) - \left(\mu^2 \frac{h_1(r(t))h_1'(r(t))}{f_1^2(t)} + \nu^2 \frac{h_2(r(t))h_2'(r(t))}{f_2^2(t)}\right) = 0 \tag{1.2}$$

for $t > 0$ with the initial condition

$$r(0) = 0. \tag{1.3}$$

The above problem was considered by several authors in [**17**], [**16**], [**9**]. In particular Nagasawa-Ueno [**9**] investigated the equation (1.2) under the condition

$$\int^{\infty} \left(\frac{1}{f_1(t)} + \frac{1}{f_2(t)}\right) dt < \infty. \tag{1.4}$$

In this paper we supply the case

$$\int^{\infty} \sqrt{\frac{\mu^2}{f_1(t)^2} + \frac{\nu^2}{f_2(t)^2}}\, dt = \infty, \tag{1.5}$$

which has not been considered in [**9**] yet. Such a case includes the one that the source manifold is $\boldsymbol{R}^m$. Let

$$h(r) = \begin{cases} \max\{h_1(r), h_2(r)\} & \text{if } \nu \neq 0, \\ h_1(r) & \text{if } \nu = 0. \end{cases}$$

Then, under the condition (1.5), we shall show that if h satisfies

$$\int^{\infty} \frac{dr}{h(r)} < \infty,$$

then the solutions $r(t)$ blows up in finite $t = T$ except zero solution. Moreover for given $T > 0$, there uniquely exists the solution $r(t)$ which blows up at $t = T$ (Theorem 3.1). In contrast with the above case, if h satisfies

$$\int^{\infty} \frac{dr}{h(r)} = \infty,$$

then all solutions are global solutions which tend to infinity as $t \to \infty$ except zero solution (Theorem 3.2). As a direct application of these theorems we can show the non-existence or existence of equivariant harmonic maps between warped product manifolds (Corollaries 3.1, 3.2).

As a further application of the analysis on the equation (1.2), using the blow-up (super-)solutions of (1.2), we can also deduce non-existence results for entire harmonic maps between Hadamard manifolds under some conditions on the curvatures and a kind of nondegeneracy condition. For example, in [**13**], Tachikawa proved that if N is a Hadamard n-manifold whose sectional curvatures are bounded from above by a negative constant, then there exists no entire harmonic map $U : \boldsymbol{R}^m \to N$ whose expression u with respect to a normal coordinate system centered at $U(0)$ satisfies

$$\sum_{i=1}^{n}\sum_{\alpha=1}^{m}\left(D_\alpha \frac{u^i}{|u|}\right)^2 \geq \frac{\varepsilon}{|x|^2} \quad \text{for all } x \in \boldsymbol{R}^m \tag{1.6}$$

for some constant $\varepsilon > 0$. Here $|\cdot|$ denotes the standard Euclidean norms. Let us call such a condition the *rotationally nondegeneracy condition* (for its etymology see Remark 4.1). Note that equivariant harmonic maps satisfy (1.6). In [**14**], the above result was extended for the case that the source manifold is a simple manifold M with a pole p_0, under the condition

$$-r^2 \min\{k_M(p), 0\} \leq \text{const.} \quad \text{as } r = \operatorname{dist}(p_0, p) \to \infty, \tag{1.7}$$

where $k_M(p)$ is the minimum of the sectional curvature of M at p. Here, a Riemannian manifold is said to be *simple* if it is diffeomorphic to the Euclidean m-space $\boldsymbol{R}^m$ and furnished with a metric for which associated Laplace-Beltrami operator is uniformly elliptic, and $p_0 \in M$ is said to be a *pole* of M if the exponential map at $p_0 \in M$ gives a diffeomorphism between M and the Euclidean space. The non-existence results of this type can be found in [**10**] also.

Moreover, in [**2**], Akutagawa-Tachikawa showed non-existence of harmonic maps satisfying the rotationally nondegeneracy condition (1.6) at infinity.

Note that in the results mentioned above it is assumed that the sectional curvature of the target manifold N bounded from above by a negative constant. Theorem 3.1 enables us to treat more general cases of target manifold N (Theorem 4.1).

2. A comparison theorem.

In the following we are always assuming the condition (1.5) on f_1 and f_2. From geometrical point of view, we impose the following condition. The constants p and q are related to the dimension of source manifold. Hence they are originally positive integers, but we do not necessarily assume that. They are always assumed

$$\begin{cases} p \geq 1 \quad \text{and} \quad q \geq 1, \\ \text{or} \\ p + q \geq 1 \ \text{provided} \ f_1(t) \equiv c f_2(t) \ \text{for some constant} \ c > 0. \end{cases} \tag{2.1}$$

The functions f_i and h_j are warping functions, which are smooth functions defined on $[0, \infty)$ satisfying

$$f_i(t) > 0 \quad \text{for} \ t > 0, \tag{2.2}$$

$$\dot{f}_i(t) \geq 0 \quad \text{for} \ t \geq 0, \tag{2.3}$$

$$f_i(t) = a_i t + O(t^3) \quad \text{as} \ t \downarrow 0 \quad \text{for some} \ a_i > 0, \tag{2.4}$$

$$1 \leq pt\frac{\dot{f}_1(t)}{f_1(t)} + qt\frac{\dot{f}_2(t)}{f_2(t)} \quad \text{for} \ t \geq 0, \tag{2.5}$$

$$(h_i h_i')'(r) \geq 0 \quad \text{for} \ r \geq 0, \tag{2.6}$$

$$h_j(r) = b_j r + O(r^3) \quad \text{as} \ r \downarrow 0 \quad \text{for some} \ b_j > 0. \tag{2.7}$$

If $\mu + \nu = 0$, then $r(t) \equiv 0$ is unique solution to (1.2), (1.3) under the above conditions. Therefore without loss of generality we may assume

$$\mu > 0. \tag{2.8}$$

In this section we show a comparison theorem for two solutions to (1.2).

Theorem 2.1. *We assume* (2.1)–(2.8). *Let* r_1 *and* r_2 *be two solutions to* (1.2) *on some interval* $[a, b)$. *If*

$$\lim_{t \downarrow a} f_1(t)^p f_2(t)^q (r_1(t) - r_2(t))(\dot{r}_1(t) - \dot{r}_2(t)) \geq 0,$$

then it holds that

$$(r_1(t) - r_2(t))^2 \geq (r_1(a) - r_2(a))^2 \quad \text{for} \ t \in [a, b).$$

Proof. We denote $r_1 - r_2$ by R, which satisfies

$$\ddot{R} + \left(p\frac{\dot{f_1}}{f_1} + q\frac{\dot{f_2}}{f_2}\right)\dot{R}$$
$$= \frac{\mu^2}{f_1^2}(h_1(r_1)h_1'(r_1) - h_1(r_2)h_1'(r_2)) + \frac{\nu^2}{f_2^2}(h_2(r_1)h_2'(r_1) - h_2(r_2)h_2'(r_2)).$$

We multiply both sides by $f_1^p f_2^q R$. It follows from (2.6) that

$$f_1^p f_2^q R \times \text{the right-hand side} \geq 0.$$

Therefore we obtain

$$\frac{d}{dt}(f_1^p f_2^q R\dot{R}) - f_1^p f_2^q \dot{R}^2 \geq 0,$$

in particular

$$\frac{d}{dt}(f_1^p f_2^q R\dot{R}) \geq 0.$$

By the assumption we get

$$f_1^p f_2^q R\dot{R} \geq 0 \quad \text{for } t \in [a, b).$$

We obtain from (2.2) that

$$\frac{d}{dt}(R^2) \geq 0 \quad \text{for } t \in (a, b),$$

and

$$R(t)^2 \geq R(a)^2 \quad \text{for } t \in [a, b).$$ □

3. Structure of solutions.

We define $h(r)$ by

$$h(r) = \begin{cases} \max\{h_1(r), h_2(r)\} & \text{if } \nu \neq 0, \\ h_1(r) & \text{if } \nu = 0. \end{cases}$$

Proposition 3.1. *We assume* (2.1)–(2.8) *and*

(3.1) $$\int^\infty \frac{dr}{h(r)} < \infty.$$

Then there exists a solution to (1.2), (1.3) *blowing up at* $t = T < \infty$.

PROOF. We take $t_0 > 0$ and fixed. Let r be a solution to (1.2), (1.3) with $r(t_0) = r_0 > 0$, which uniquely exists (see [**16**]). Put $r'(t_0) = \beta(t_0, r_0)$, and

$$\phi(r_0, t_0) = \int_{r_0}^{\infty} \{\beta(r_0, t_0)^2 + \gamma_1(t_0)^2(h_1(r)^2 - h_1(r_0)^2) + \gamma_2(t_0)^2(h_2(r)^2 - h_2(r_0)^2)\}^{-1/2}\, dr,$$

where

$$\gamma_1(t_0) = \frac{\mu}{f_1(t_0)}, \quad \gamma_2(t_0) = \frac{\nu}{f_2(t_0)}.$$

We have already shown that $\phi(r_0, t_0)$ is well-defined, and

$$\phi(r_0, t_0) \geq f_1(t_0)^p f_2(t_0)^q \int_{t_0}^{T} \frac{d\tau}{f_1(\tau)^p f_2(\tau)^q} \quad [\mathbf{16},\ \text{Lemma 3.5}], \tag{3.2}$$

$$\lim_{r_0 \to \infty} \phi(r_0, t_0) = 0 \quad [\mathbf{9},\ \text{Lemma 4.1}]. \tag{3.3}$$

Here $[0, T)$ is a life span of r. By (3.3) (r_0, t_0) satisfies

$$\phi(r_0, t_0) < f_1(t_0)^p f_2(t_0)^q \int_{t_0}^{\infty} \frac{d\tau}{f_1(\tau)^p f_2(\tau)^q}$$

for sufficiently large r_0, and the life span must be a finite interval by (3.2). □

THEOREM 3.1. *We assume* (2.1)–(2.8) *and* (3.1). *Then the following facts hold.*

1. *All solutions to* (1.2), (1.3) *blow up in finite time except zero solution.*
2. *For any* $T \in (0, \infty)$ *there exists a solution to* (1.2), (1.3) *which blows up at* $t = T$.

PROOF. For a positive number $\lambda > 0$ put

$$f_i(\lambda^{-1} t) = F_{i,\lambda}(t).$$

We consider the problem

$$\begin{cases} \ddot{\tilde{r}}(t) + \left(p\dfrac{\dot{F}_{1,\lambda}(t)}{F_{1,\lambda}(t)} + q\dfrac{\dot{F}_{2,\lambda}(t)}{F_{2,\lambda}(t)}\right)\dot{\tilde{r}}(t) \\ \qquad - \dfrac{1}{\lambda^2}\left(\mu^2 \dfrac{h_1(\tilde{r}(t))h_1'(\tilde{r}(t))}{F_{1,\lambda}(t)^2} + \nu^2 \dfrac{h_2(\tilde{r}(t))h_2'(\tilde{r}(t))}{F_{2,\lambda}(t)^2}\right) = 0, \\ \tilde{r}(0) = 0. \end{cases} \tag{3.4}$$

Since $F_{i,\lambda}$ satisfies (2.2)–(2.5), there exists a solution $\tilde{r}$ to (3.4) which blows up at finite T by Proposition 3.1. Put $\tilde{r}_\lambda(t) = \tilde{r}(\lambda t)$. Then (3.4) reduces to

$$\begin{cases} \ddot{\tilde{r}}_\lambda(t) + \left(p\dfrac{\dot{f}_1(t)}{f_1(t)} + q\dfrac{\dot{f}_2(t)}{f_2(t)}\right)\dot{\tilde{r}}_\lambda(t) \\ \qquad -\left(\mu^2\dfrac{h_1(\tilde{r}_\lambda(t))h_1'(\tilde{r}_\lambda(t))}{f_1(t)^2} + \nu^2\dfrac{h_2(\tilde{r}_\lambda(t))h_2'(\tilde{r}_\lambda(t))}{f_2(t)^2}\right) = 0, \\ \tilde{r}_\lambda(0) = 0. \end{cases}$$

The life span of $\tilde{r}_\lambda$ is $[0, \lambda^{-1}T)$. Let r be a non-trivial solution to (1.2), (1.3) with $r(t_0) = r_0$. If $\lambda > 0$ is sufficiently small, then

$$t_0 < \lambda^{-1}T, \quad r(t_0) = r_0 > \tilde{r}(\lambda t_0) = \tilde{r}_\lambda(t_0),$$

$$\dot{r}(t_0) > \lambda \frac{d\tilde{r}}{d\tau}\bigg|_{\tau=\lambda t_0} = \dot{\tilde{r}}_\lambda(t_0).$$

Here we use $\tilde{r}(0) = 0$ and the fact

$$\dot{\tilde{r}}(\lambda t_0) = o\left(\frac{1}{\lambda t_0}\right)$$

which is reduced from [**16**, Theorem 2.1]. Theorem 2.1 yields

$$(r(t) - \tilde{r}_\lambda(t))^2 \geq (r(t_0) - \tilde{r}_\lambda(t_0))^2 > 0$$

for $t > t_0$. Since both r and $\tilde{r}_\lambda$ are continuous, we have

$$r(t) > \tilde{r}_\lambda(t) \quad \text{for } t > t_0.$$

Therefore the life span of r is shorter than that of $\tilde{r}_\lambda$.

Since we have already obtained the assertion (1), the proof of (2) is in the same way as the argument in [**9**, §4]. □

Corollary 3.1. *Let $m \geq 2$, $n \geq 2$, and $M = \boldsymbol{R}_+ \times_f S^{m-1}$, $N = \boldsymbol{R}_+ \times_h S^{n-1}$ be warped product manifolds with warping functions f and h. Assume that f and h satisfy* (2.2)–(2.7) *and*

$$\int^\infty \frac{dt}{f(t)} = \infty, \quad \int^\infty \frac{dr}{h(r)} < \infty.$$

Then the following facts hold.

1. *There do not exist equivariant harmonic maps from M to N except constant maps.*

2. *We assume that the eigenmap* $\varphi : S^{m-1} \to S^{n-1}$ *exists for the pair* (m,n). *Then the domain of equivariant harmonic maps is controllable in the following sense. Let* $B_T = [0,T) \times_f S^{m-1} \subset M$. *There exists an equivariant harmonic map from* B_T *onto* N *which is unique among equivariant harmonic maps up to the eigenmap.*

$(f(t),h(r)) = (t, \sinh r)$ is a typical example of Corollary 3.1, which corresponds to $M = \boldsymbol{R}^m$ and $N = \boldsymbol{H}^n$.

Next we assume $\int^{\infty}(dr/h(r)) = \infty$.

PROPOSITION 3.2. *We assume* (2.1)–(2.8) *and*

$$\int^{\infty} \frac{dr}{h(r)} = \infty. \tag{3.5}$$

Then there exist no solution to (1.2), (1.3) *which blows up at* $t = T < \infty$.

PROOF. We know the inequality

$$\int_{r_0}^{r(t)} \frac{dr}{h(r)} \le \int_{t_0}^{t} \sqrt{\frac{\mu^2}{f_1(\tau)^2} + \frac{\nu^2}{f_2(\tau)^2}}\, d\tau \quad \text{on } t \ge t_0$$

for a solution r to (1.2), (1.3) with $r(t_0) = r_0 > 0$ ([**9**, Lemma 2.3ff]). If r blows up at $T < \infty$, then letting $t \uparrow T$ in the inequality, we have

$$\infty = \int_{r_0}^{\infty} \frac{dr}{h(r)} \le \int_{t_0}^{T} \sqrt{\frac{\mu^2}{f_1(\tau)^2} + \frac{\nu^2}{f_2(\tau)^2}}\, d\tau < \infty.$$

This is contradiction. □

THEOREM 3.2. *We assume* (2.1)–(2.8) *and* (3.5). *Then all solutions to* (1.2), (1.3) *are global solutions, and their limit value as* $t \to \infty$ *are infinite except zero solution.*

PROOF. It follows from [**16**, Lemma 3.1] that all solutions are nondecreasing. By Proposition 3.2 it is enough to show that there exists no global solution whose limit value as $t \to \infty$ is finite except zero solution. Let r be a global solution satisfying

$$r(t_0) = r_0 > 0, \quad \lim_{t\to\infty} r(t) = \ell \in (0, \infty).$$

Put smooth functions $\tilde{h}_i(r)$ satisfying (2.6) and

$$\tilde{h}_i(r) = h_i(r) \quad \text{for } r \le 3\ell,$$
$$\tilde{h}_i(r) > h_i(r) \quad \text{for } r > 3\ell,$$
$$\int^{\infty} \frac{dr}{\tilde{h}(r)} < \infty.$$

Here $\tilde{h}(r)$ is

$$\tilde{h}(r) = \begin{cases} \max\{\tilde{h}_1(r), \tilde{h}_2(r)\} & \text{if } \nu \neq 0, \\ \tilde{h}_1(r) & \text{if } \nu = 0. \end{cases}$$

Consider the problem

$$\begin{cases} \ddot{\tilde{r}}(t) + \left(p\dfrac{\dot{f}_1(t)}{f_1(t)} + q\dfrac{\dot{f}_2(t)}{f_2(t)} \right) \dot{\tilde{r}}(t) \\ \qquad - \left(\mu^2 \dfrac{\tilde{h}_1(\tilde{r}(t))\tilde{h}_1'(\tilde{r}(t))}{f_1(t)^2} + \nu^2 \dfrac{\tilde{h}_2(\tilde{r}(t))\tilde{h}_2'(\tilde{r}(t))}{f_2(t)^2} \right) = 0, \\ \tilde{r}(0) = 0, \quad \tilde{r}(t_0) = r_0 > 0. \end{cases}$$

By Theorem 2.1 it holds that $r(t) = \tilde{r}(t)$ as long as $\tilde{r}(t) \le 3\ell$. Theorem 3.1 yields the existence of t_1 such that

$$\tilde{r}(t_1) = 2\ell.$$

Since r is strictly increasing [**16**, Lemma 3.1], we get contradiction

$$\ell = \lim_{t \to \infty} r(t) > r(t_1) = \tilde{r}(t_1) = 2\ell. \qquad \square$$

Corollary 3.2. *Let $m \ge 2$, $n \ge 2$, and $M = \boldsymbol{R}_+ \times_f S^{m-1}$, $N = \boldsymbol{R}_+ \times_h S^{n-1}$ be warped product manifolds with warping functions f and h. Assume that f and h satisfy* (2.2)–(2.7) *and*

$$\int^{\infty} \frac{dt}{f(t)} = \infty, \qquad \int^{\infty} \frac{dr}{h(r)} = \infty.$$

We also assume that the eigenmap $\varphi : S^{m-1} \to S^{n-1}$ exists for the pair (m, n). Then there exist equivariant harmonic maps from M to N. Moreover if φ is an onto map, then the map $U = (r, \varphi)$ is also an onto map unless r is constant.

When $f_i(t) = t$, the following structure holds regardless of finiteness or infiniteness of $\int^{\infty} (dr/h(r))$.

Theorem 3.3. *The set of all solutions to* (1.2), (1.3) *is a one parameter family* $\{r_\lambda(t) = r(\lambda t)\}_{\lambda \geq 0}$.

Proof. It is easy to see $r_\lambda(t)$ is a solution if so is $r(t)$. The assertion yields from the uniqueness theorem by Ueno [**16**, Corollary 3.4]. □

4. Harmonic maps from $\boldsymbol{R}^m$ to Hadamard manifolds.

In this section we generalize Corollary 3.1 in some aspect. That is, we prove non-existence of a harmonic map with a rotational nondegeneracy at infinity, from $\boldsymbol{R}^m$ to an Hadamard manifold N whose sectional curvature $K(p)$ at $p \in N$ possibly tends to 0 as $(\mathrm{dist}(p_0, p))^{-2}$ for some fixed point $p_0 \in N$. A Riemannian manifold N is said to be an *Hadamard manifold* if it is a complete simply connected Riemannian manifold with nonpositive sectional curvature. Recently, existence and non-existence of harmonic maps from complete non-compact manifolds to Hadamard manifolds are studied by several authors. About "existence" see, for example, [**1**], [**3**], [**8**].

In the following $(\cdot\,,\cdot)$ and $|\cdot|$ stand for the standard Euclidean inner product and norm respectively. For a Riemannian manifold $N = (N^n, g)$, $(\cdot\,,\cdot)_{g(p)}$ stands for the inner product on the tangent space T_pN with respect to the metric g and $\|X\|_{g(p)} = \sqrt{(X, X)_{g(p)}}$. If N has a pole p_0, let $\sigma(p_0, p)(t)$ be the geodesic curve such that $\sigma(p_0, p)(0) = p_0$ and $\sigma(p_0, p)(1) = p$. Let $k_N(p; \pi)$ be the sectional curvature of N at p with respect to the plane section π and $K_{\mathrm{rad}, N}(p; p_0)$ the maximum of the radial curvature of N at p, i.e.

$$K_{\mathrm{rad}, N}(p; p_0) = \max\{k_N(p; \pi) \,|\, \pi \ni \sigma'(p_0, p)(1)\}. \tag{4.1}$$

Moreover, using a normal coordinate system defined by the exponential map at the pole p_0, we define $\lambda_N(p; p_0)$ by

$$\lambda_N(p; p_0) = \inf\left\{ \frac{\|\xi\|_{g(p)}^2}{|\xi|^2} \,\middle|\, \xi \in \boldsymbol{R}^n \text{ with } (\xi, \sigma'(p_0, p)(1))_{g(p)} = 0 \right\}. \tag{4.2}$$

Now, we can state our main result of this section.

Theorem 4.1. *Let $N = (N^n, g)$ be a Hadamard n-manifold. For some fixed point $p_0 \in N$, assume that*

$$\liminf_{\mathrm{dist}(p_0, p) \to \infty} \{\mathrm{dist}(p_0, p)\}^2 |K_{\mathrm{rad}, N}(p; p_0)| > 0. \tag{4.3}$$

Then there exists no harmonic map $U : \boldsymbol{R}^m \to N$ which satisfies the following condition.

(4.4) $$\liminf_{|x|\to\infty}\left\{|x|^2\left(\frac{1}{\rho^2\lambda_N(U(x);p_0)}\right)(e(U)(x)-e(\rho)(x))\right\}>0,$$

where

$$\rho(x)=\mathrm{dist}(U(x),p_0),$$

and

$$e(U)(x)=\frac{1}{2}\sum_{\alpha=1}^{m}\|D_\alpha U(x)\|^2_{g(U(x))},\quad e(\rho)(x)=\frac{1}{2}\sum_{\alpha=1}^{m}|D_\alpha\rho(x)|^2.$$

Remark 4.1. Let $u(x)$ be an expression of U with respect to a normal coordinate system on N centered at p_0. Then the condition (4.4) can be written as

(4.5) $$\liminf_{|x|\to\infty}|x|^2\sum_{i=1}^{n}\sum_{\alpha=1}^{m}\left(D_\alpha\frac{u^i(x)}{|u(x)|}\right)^2>0.$$

Compare (4.5) with (1.6). Moreover, for the case that $M=\boldsymbol{R}_+\times_f S^{m-1}$ and $N=\boldsymbol{R}_+\times_h S^{n-1}$, the condition (4.4) can be replaced by a simpler condition: If we write a map $U:M=\boldsymbol{R}_+\times_f S^{m-1}\to N=\boldsymbol{R}_+\times_h S^{n-1}$ as

$$U(t,\theta)=(r(t,\theta),\varphi(t,\theta))\in\boldsymbol{R}_+\times_h S^{n-1},$$

then the condition

$$\liminf_{t\to\infty}\|D_\theta\varphi(t,\theta)\|>0$$

implies (4.4).

Remark 4.2. As mentioned in Section 1, in the former non-existence results of [**2**], [**13**], [**14**], it is assumed that the sectional curvature of the target manifold N bounded from above by a negative constant. In Theorem 4.1, we prove a similar non-existence result as in [**2**] for the case that the target manifold N has sectional curvatures possibly decaying at infinity, using Theorem 3.1.

Before we give the proof of Theorem 4.1, we prepare some differential geometric estimates which are based on [**6**, Lemma 6].

Lemma 4.1. *Let N be a Riemannian n-manifold with a pole p_0, $(y^1,\dots,y^n)$ a normal coordinate system centered at p_0, $(g_{ij}(y))$ the metric tensor with respect to the normal coordinate system. Let ρ be a function of class $C^2(\boldsymbol{R}_+,\boldsymbol{R}_+)$ which satisfies*

(4.6) $$\lim_{t\to 0}\frac{\rho(t)}{t}=1,\quad \rho(t)>0\quad \textit{for any}\ \ t\in(0,\infty).$$

Assume that

$$K_{\mathrm{rad},N}(y;0) \le -\frac{\rho''(t)}{\rho(t)}, \tag{4.7}$$

where $t = |y|$. *Then we have the following estimates*

$$g_{ij}(y)(X^iX^j + y^k\Gamma^i_{k\ell}(y)X^jX^\ell) \ge |\zeta|^2 + t\frac{\rho'(t)}{\rho(t)}g_{ij}(y)\xi^i\xi^j, \tag{4.8}$$

$$g_{ij}(y)X^iX^j \ge |\zeta|^2 + \frac{\rho^2(t)}{t^2}|\xi|^2, \tag{4.9}$$

for all y, $X \in \boldsymbol{R}^n$, *where* $\zeta = t^{-2}(X,y)y$ *and* $\xi = X - \zeta$.

PROOF. We can proceed as in the proof of [**13**, Lemma 1], [**14**, Lemmas 2.1, 2.2] or [**2**, Lemma 2.1], because we can apply Rauch's comparison theorem under the assumption (4.7) on the radial curvatures only. □

LEMMA 4.2. *Let* N *be as in Theorem* 4.1. *Then there exists a positive constant* k, c_0, $c_1 > 0$ *and* $a > 1$ *such that for*

$$\rho_k(t) = \begin{cases} \dfrac{1}{k}\sinh kt & \text{for } 0 < t \le 1 \\ c_0 + c_1 t^a & \text{for } t \ge 1 \end{cases}$$

the following estimates hold:

$$g_{ij}(y)(X^iX^j + y^k\Gamma^j_{k\ell}X^\ell X^j) \ge |\zeta|^2 + |y|\frac{\rho_k'(|y|)}{\rho_k(|y|)}g_{ij}(y)\xi^i\xi^j, \tag{4.10}$$

$$g_{ij}(y)X^iX^j \ge |\zeta|^2 + \frac{\rho_k^2(|y|)}{|y|^2}|\xi|^2 \tag{4.11}$$

for all y *and* $X \in \boldsymbol{R}^n$, *where* $t = |y|$, $\zeta = t^{-2}(X,y)y$ *and* $\xi = X - \zeta$.

PROOF. Since N has negative sectional curvature and satisfies (4.3), there exists a constant $\kappa > 0$ such that

(4.12)
$$-\kappa^2 \ge \max\{\sup\{K_{\mathrm{rad},N}(y;0) | y \in N, |y| \le 1\}, \sup\{|y|^2K_{\mathrm{rad},N}(y;0) | y \in N, |y| \ge 1\}\}.$$

For a positive constant k, put $\varphi_k(t) = c_0 + c_1t^a$, and choose the positive constants c_0, c_1 and a so that

$$\varphi_k(1) = \frac{1}{k}\sinh k, \quad \varphi_k'(1) = \cosh k, \quad \varphi_k''(1) = k\sinh k. \tag{4.13}$$

Then we have

$$(4.14)\qquad \begin{cases} a = a_k = \dfrac{k\sinh k + \cosh k}{\cosh k} > 1, \quad \searrow 1 \text{ as } k \downarrow 0, \\ c_1 = \dfrac{\cosh^2 k}{k\sinh k + \cosh k}, \quad c_0 = \dfrac{-k + \sinh k \cosh k}{k(k\sinh k + \cosh k)}. \end{cases}$$

Moreover it is easy to see that

$$(4.15)\qquad \left(-t^2 \frac{\varphi_k''(t)}{\varphi_k(t)}\right)' \leq 0.$$

Now put

$$\rho_k(t) = \begin{cases} \dfrac{1}{k}\sinh kt & \text{for } 0 \leq t \leq 1, \\ \varphi_k(t) & \text{for } t \geq 1. \end{cases}$$

Then, by (4.13) ρ_k is of class C^2. By direct calculations, we can see that $\rho_k(t)$ satisfies (4.6) and that

$$(4.16)\qquad -\frac{\rho_k''(t)}{\rho_k(t)} = -k^2 \quad \text{for } 0 \leq t \leq 1.$$

Moreover, noting (4.15) and (4.14), we obtain

$$(4.17)\qquad \inf_{t>1}\left\{-t^2 \frac{\rho_k''(t)}{\rho_k(t)}\right\} = \lim_{t\to\infty}\left\{-t^2 \frac{\rho_k''(t)}{\rho_k(t)}\right\} = -a(a-1) \nearrow 0 \quad \text{as } k \downarrow 0.$$

Now, by (4.12), (4.16) and (4.17), if we take $k > 0$ sufficiently small, we get

$$K_{\mathrm{rad},N}(y;0) \leq -\frac{\rho_k''(|y|)}{\rho_k(|y|)}.$$

Thus (4.7) is also fulfilled by $\rho = \rho_k$. Now, we can apply Lemma 4.1 with $\rho = \rho_k$ and get (4.10). □

Let $u = (u^1(x), \dots, u^n(x))$ be the expression of a harmonic map $U : \boldsymbol{R}^m \to N$ in terms of a normal coordinate system centered at any fixed point q_0 in N. Then u satisfies the following equation of weak form

$$(4.18)\qquad \int_{\boldsymbol{R}^m} \sum_{\alpha=1}^{m} g_{ij}(D_\alpha u^i D_\alpha \varphi^j + \varphi^k \Gamma_{k\ell}^i D_\alpha u^\ell D_\alpha u^j)\, dx = 0$$

for all $\varphi \in C_0^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$.

Proposition 4.1. *Let N be as in Theorem* 4.1, *ρ_k be the function defined in Lemma* 4.2 *and u be the expression of a harmonic map $U : \boldsymbol{R}^m \to N$ with respect to a normal coordinate system on N centered at an arbitrary fixed point $q_0 \in N$. Then we have the following differential inequality*

$$\Delta|u|(x) - \frac{2\rho_k'(|u|)}{\rho_k(|u|)}\{e(u)(x) - e(|u|)(x)\} \geq 0, \tag{4.19}$$

where $e(u) = (1/2)\sum_{\alpha=1}^m g_{ij}(u)D_\alpha u^i D_\alpha u^j$ and $e(|u|) = (1/2)\sum_{\alpha=1}^m (D_\alpha|u|)^2$.

Moreover, if u satisfies (4.4), *then we get*

$$\Delta|u| - \frac{\varepsilon_0}{|x|^2}\rho_k\rho_k'(|u|) \geq 0 \quad on \ \boldsymbol{R}^m \backslash B_{R_0}(0) \tag{4.20}$$

for some $\varepsilon_0 > 0$ and $R_0 > 0$.

Proof. Replacing $\kappa^{-1}\sinh(\kappa t)$ in the proof of [**13**, Proposition 1], [**14**, Proposition 3.1] or [**2**, Proposition 2.1] by $\rho_\kappa(t)$, and using Lemma 4.2, we get the assertion. □

Now we are in a position to prove Theorem 4.1.

Proof of Theorem 4.1. Let $u(x)$ be the expression of a harmonic map $U : \boldsymbol{R}^m \to N$ with respect a normal coordinate system $y = (y^1, \ldots, y^n)$ on N centered at arbitrary fixed point $q_0 \in N$. Take R_0 as in Proposition 4.1 and put $\xi = \sup_{B_{R_0}(0)}|u|$. Assume that U is not a constant map. Then $|u|$ can not remain bounded because of a Liouville-type theorem due to [**5**]. Thus, there exists a compact set $D_0 \subset \boldsymbol{R}^m \backslash B_{R_0}(0)$ on which $|u| \geq \xi + 1$. Let $f_1(t) = f_2(t) = t$, $h_1(r) = h_2(r) = \rho_k(r)$ of Lemma 4.2 and $\mu^2 + \nu^2 = \varepsilon_0$ in (1.2) then it is easy to see that the conditions (2.6), (2.7) and (3.1) are satisfied. Thus, by Theorem 3.1 and Theorem 3.3, there exist a one-parameter family of solutions $r_\lambda(t)$ to

$$\ddot{r} + \frac{(m-1)}{t}\dot{r} - \frac{\varepsilon_0\rho_k(r)\rho_k'(r)}{t^2} = 0,$$

or equivalently to the equation

$$\Delta r_\lambda(|x|) - \frac{\varepsilon_0}{|x|^2}\rho_k\rho_k'(r_\lambda(|x|)) = 0,$$

which satisfy $r_\lambda(0) = 0$ and blow up at $|x| = T/\lambda$ for some $T > 0$. Since $r_\lambda(0) = 0$, we can take $\lambda_0 > 0$ sufficiently small so that $D_0 \subset B_{T/\lambda_0}(0)$ and

$$r_{\lambda_0}(|x|) < 1 \quad \text{on } D_0. \tag{4.21}$$

Let

$$\psi(x) = r_{\lambda_0}(|x|) + \xi,$$

then $\psi(x)$ satisfies

$$\Delta\psi(x) - \frac{\varepsilon_0}{t^2}\rho_k \rho_k'(\psi(x)) \leq 0 \quad \text{in } \boldsymbol{R}^m.$$

$$\psi(x) \geq \xi \quad \text{on } \partial B_{R_0}(0) \quad \text{and} \quad \lim_{|x| \to T/\lambda} \psi(x) = \infty.$$

Now, using comparison theorem for elliptic equations, we can see that

$$|u(x)| \leq \psi(x) \quad \text{on } B_{T/\lambda}(0) \backslash B_{R_0}(0).$$

On the other hand (4.21) implies that $|u(x)| > \psi(x)$ on D_0. This is a contradiction. □

Acknowledgement. A part of the work was done while the first author visited the Department of Mathematics/JAMI, Johns Hopkins University, Baltimore, Maryland during 1994, whose hospitality is greatly acknowledged.

References

[1] K. Akutagawa, S. Nishikawa and A. Tachikawa, Harmonic maps between unbounded convex polyhedra in hyperbolic spaces, Invent. Math., **115** (1994), 391–404.
[2] K. Akutagawa and T. Tachikawa, Nonexistence results for harmonic maps between noncompact complete Riemannian manifolds, Tokyo J. Math., **16** (1993), 131–145.
[3] H. I. Choi and A. Treibergs, Gauss maps of spacelike constant mean curvature hypersurfaces of Minkowski space, J. Differential Geom., **32** (1990), 775–817.
[4] D. Gromoll, W. Klingenberg and W. Meyer, Rimannsche Geometrie im Großen, Lecture Notes in Math. **55**, Springer-Verlag, Berlin · Heiderberg · New York, 1968.
[5] S. Hildebrandt, J. Jost and K.-O. Widman, Harmonic mappings and minimal submanifolds, Invent. Math., **62** (1980), 269–298.
[6] S. Hildebrandt and H. Kaul, Two-Dimensional variational problems with obstructions and Plateau's problem for *H*-surfaces in a Riemannian manifold, Comm. Pure Appl. Math., **25** (1972), 187–223.
[7] W. Jäger and H. Kaul, Rotationally symmetric harmonic maps from a ball into a sphere and the regularity problem for weak solutions of elliptic systems, J. Reine Angew. Math., **343** (1983), 146–161.
[8] P. Li and L. Tam, Uniqueness and regularity of proper harmonic maps, Ann. of Math., **137** (1993), 167–201.
[9] T. Nagasawa and K. Ueno, Initial-final value problems for ordinary differential equations and applications to equivariant harmonic maps, J. Math. Soc. Japan, **50** (1998), 545–555.
[10] A. Ratto and M. Rigoli, Elliptic differential inequalities with applications to harmonic maps, J. Math. Soc. Japan, **45** (1993), 321–337.
[11] A. Tachikawa, On interior regularity and Liouville's theorem for harmonic mappings, Manuscripta Math., **42** (1983), 11–40.
[12] A. Tachikawa, Rotationally symmetric harmonic maps from a ball into a warped product manifold, Manuscripta Math., **53** (1985), 235–254.
[13] A. Tachikawa, Harmonic mappings from $\boldsymbol{R}^m$ into an Hadamard manifold, J. Math. Soc. Japan, **42** (1990), 147–153.
[14] A. Tachikawa, Harmonic maps from a Riemannian manifold with a pole into an Hadamard manifold with negative sectional curvatures, Manuscripta Math., **74** (1992), 69–81.

[15] K. Ueno, Some new examples of eigenmaps from S^m to S^n, Proc. Japan Acad., **69** (1993), 205–208.
[16] K. Ueno, A study of ordinary differential equations arising from equivariant harmonic maps, Tokyo J. Math., **17** (1994), 395–416.
[17] H. Urakawa and K. Ueno, Equivariant harmonic maps associated to large group actions, Tokyo J. Math., **17** (1994), 417–437.

Takeyuki NAGASAWA
Mathematical Institute
Tôhoku University
980-8578, JAPAN
E-mail: nagasawa@math.tohoku.ac.jp

Atsushi TACHIKAWA
Department of Mathematics
Faculty of Science and Technology
Science University of Tokyo
278-5810, JAPAN
E-mail: atsushi@ma.noda.sut.ac.jp