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Invariants and semi-direct products for finite group actions on tensor categories

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Abstract. Suppose a group G acts on a tensor category $\mathscr{C}$ over a field k. Then we have the tensor category $\mathscr{C}^G$ of G-invariant objects in $\mathscr{C}$, and the semi-direct product tensor category $\mathscr{C}[G]$. We show that if G is finite and $k[G]$ is semi-simple, there exists a one-to-one correspondence between categories with action of $\mathscr{C}^G$ and categories with action of $\mathscr{C}[G]$.

Introduction.

If a group G acts on a ring S, we have the ring of G-invariants S^G and the skew group ring $S[G]$. This paper deals with analogous constructions for a tensor category in place of a ring. Suppose that G acts on a tensor category $\mathscr{C}$ over a field k. This means that for each $\sigma \in G$, a tensor functor $\sigma_* : \mathscr{C} \to \mathscr{C}$ is given and for each $\sigma, \tau \in G$, a tensor isomorphism $\sigma_* \circ \tau_* \cong (\sigma\tau)_*$ is given in a coherent way. The tensor category $\mathscr{C}^G$ consists of objects C of $\mathscr{C}$ equipped with isomorphisms $\sigma_* C \cong C$ satisfying certain coherence conditions. The tensor category $\mathscr{C}[G]$ is just the product $\bigoplus_{\sigma \in G} \mathscr{C}$ as a category, whose objects are expressed as $\bigoplus_{\sigma \in G}(C_\sigma, \sigma)$ with $C_\sigma \in \mathscr{C}$, and the tensor product in $\mathscr{C}[G]$ is defined by $(C, \sigma) \otimes (D, \tau) = (C \otimes \sigma_* D, \sigma\tau)$.

For a tensor category $\mathscr{A}$, an $\mathscr{A}$-module means a category with associative action of $\mathscr{A}$. We assume here categories have direct sums and direct summands.

Our result is that if G is finite and $k[G]$ is semi-simple, then $\mathscr{C}^G$-modules and $\mathscr{C}[G]$-modules are in one-to-one correspondence. It is given by assigning to a $\mathscr{C}[G]$-module $\mathscr{X}$ the $\mathscr{C}^G$-module $\mathscr{X}^G$ of G-invariant objects of $\mathscr{X}$.

We notice that the rings S^G and $S[G]$ are not generally Morita equivalent. They are equivalent through the functor assigning to an $S[G]$-module X the S^G-module X^G only when S is a G-Galois extension over S^G.

The above result is a simple consequence of the one-to-one correspondence between modules over the tensor category of $k[G]$-modules and modules over the tensor category of $k[G]^*$-modules given in [**T**], where $k[G]^*$ is the dual of the

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group algebra. This correspondence may be thought of as a version of the duality for cross products in [**BM**] and [**NT**].

As an application, we describe $\mathscr{C}^G$-modules whose underlying categories are the categories of k^n-modules for $n \geq 0$ when $\mathscr{C}$ is the tensor category of $k[A]^*$-modules with twisted associativity given by a 3-cocycle of a group A. In this case $\mathscr{C}[G]$-modules have a simple description and so we know about $\mathscr{C}^G$-modules through our correspondence.

In Section 1 some definitions and constructions for modules over tensor categories are reviewed. In Section 2 group actions on tensor categories are considered and the definitions of $\mathscr{C}^G$ and $\mathscr{C}[G]$ are given. In Section 3 we restate the duality theorem of [**T**] in the case of a group algebra. In Section 4 we deduce the correspondence between $\mathscr{C}^G$-modules and $\mathscr{C}[G]$-modules. In Section 5 we show that the semi-simplicity of categories are preserved under the correspondence. In Section 6 we consider a tensor category of $k[A]^*$-modules with 3-cocycle twist, called a group tensor category, and describe modules over it. In Section 7 we apply our correspondence to a group tensor category with a group action. In Section 8 the verification of the pentagon axiom for $\mathscr{C}[G]$ is given.

1. Modules over tensor categories.

We reproduce here some of definitions about tensor category modules from [**T**]. See [**T**] for details. We make a little use of terms in 2-category theory, whose meanings are explained in our context. See [**B**] for generalities on 2-categories.

We fix a field k throughout. A k-linear category is a category in which the hom-sets are k-vector spaces, the compositions are k-bilinear and finite direct sums exist. A k-linear functor between k-linear categories is a functor which is linear on all hom-spaces. Let $\mathrm{Hom}(\mathscr{X},\mathscr{Y})$ denote the category of k-linear functors $\mathscr{X} \to \mathscr{Y}$. A tensor category is a k-linear monoidal category. Notations for monoidal structures of a general tensor category are as follows: $(A,B) \mapsto A.B$ denotes the tensor product operation, I the unit object, $\alpha_{A,B,C} : (A.B).C \to A.(B.C)$ the associativity isomorphism, $\lambda_A : I.A \to A$ the left unit isomorphism, $\rho_A : A.I \to A$ the right unit isomorphism.

For a tensor category $\mathscr{A}$, $\mathscr{A}^{\otimes \mathrm{op}}$ stands for the tensor category whose underlying category is the same as $\mathscr{A}$ and tensor product operation is opposite to $\mathscr{A}$.

For a tensor category $\mathscr{A}$, a left $\mathscr{A}$-module is a k-category $\mathscr{X}$ equipped with a bilinear functor $\mathscr{A} \times \mathscr{X} \to \mathscr{X} : (A,X) \mapsto A.X$ and isomorphisms of associativity $\alpha_{A,B,X} : (A.B).X \to A.(B.X)$ and unitality $\lambda_X : I.X \to X$ for $A,B \in \mathscr{C}$, $X \in \mathscr{X}$ satisfying the conditions of naturality and coherence similar to the ones for a monoidal category.

For tensor categories $\mathscr{A}$ and $\mathscr{B}$, an $(\mathscr{A},\mathscr{B})$-bimodule is a k-category $\mathscr{X}$ equipped with bilinear functors $\mathscr{A}\times\mathscr{X}\to\mathscr{X}$ and $\mathscr{X}\times\mathscr{B}\to\mathscr{X}$ and isomorphisms $\alpha_{A,A',X}:(A.A').X\to A.(A'.X)$, $\alpha_{X,B,B'}:(X.B).B'\to X.(B.B')$, $\alpha_{A,X,B}:(A.X).B\to A.(X.B)$, $\lambda_X:I.X\to X$, $\rho_X:X.I\to X$ for $A,A'\in\mathscr{A}$, $B,B'\in\mathscr{B}$, $X\in\mathscr{X}$ satisfying the conditions of naturality and coherence.

For left $\mathscr{A}$-modules $\mathscr{X}$ and $\mathscr{Y}$, an $\mathscr{A}$-linear functor $\mathscr{X}\to\mathscr{Y}$ is a pair (L,β) of a k-linear functor $L:\mathscr{X}\to\mathscr{Y}$ and a family β of isomorphisms $\beta_{A,X}:L(A.X)\to A.L(X)$ for $A\in\mathscr{A}$, $X\in\mathscr{X}$ which are natural in A, X and commute with the isomorphisms of associativity and unitality for $\mathscr{X}$ and $\mathscr{Y}$.

For $\mathscr{A}$-linear functors $(L,\beta),(L',\beta'):\mathscr{X}\to\mathscr{Y}$, an $\mathscr{A}$-linear transformation $(L,\beta)\to(L',\beta')$ is a natural transformation $\sigma:L\to L'$ commuting with β and β'.

Thus we have the category $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},\mathscr{Y})$ whose objects are $\mathscr{A}$-linear functors $\mathscr{X}\to\mathscr{Y}$ and morphisms are $\mathscr{A}$-linear transformations.

For $\mathscr{A}$-linear functors $(L,\beta):\mathscr{X}\to\mathscr{Y}$ and $(M,\gamma):\mathscr{Y}\to\mathscr{Z}$, their composite $(M,\gamma)\circ(L,\beta)$ is defined to be the $\mathscr{A}$-linear functor $(M\circ L,\delta):\mathscr{X}\to\mathscr{Z}$, where

$$\delta_{A,X}=\gamma_{A,L(X)}\circ M(\beta_{A,X}).$$

Thus we have the composition functors

$$\mu_{\mathscr{X},\mathscr{Y},\mathscr{Z}}:\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},\mathscr{Z})\times\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},\mathscr{Y})\to\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},\mathscr{Z}),$$

which are (strictly) associative. Also we have the identity $\mathscr{A}$-linear functors $\mathrm{Id}_{\mathscr{X}}$ in $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},\mathscr{X})$, which are (strictly) unital for composition.

The collection of the categories $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},\mathscr{Y})$ for all $\mathscr{A}$-modules $\mathscr{X},\mathscr{Y}$ together with the compositions $\mu_{\mathscr{X},\mathscr{Y},\mathscr{Z}}$ and the identities $\mathrm{Id}_{\mathscr{X}}$ is referred as the 2-category of $\mathscr{A}$-modules and denoted by $\mathscr{A}$-Mod. $\mathscr{A}$-modules, $\mathscr{A}$-linear functors, $\mathscr{A}$-linear transformations are also called 0-cells, 1-cells, 2-cells of the 2-category $\mathscr{A}$-Mod, respectively. The composition of 1-cells and of 2-cells given by the composition functor $\mu_{\mathscr{X},\mathscr{Y},\mathscr{Z}}$ is called the horizontal composition, while the composition of 2-cells inside the hom-category $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},\mathscr{Y})$ is called the vertical composition.

An $\mathscr{A}$-linear functor $L:\mathscr{X}\to\mathscr{Y}$ is called an equivalence of $\mathscr{A}$-modules if there exist an $\mathscr{A}$-linear functor $L':\mathscr{Y}\to\mathscr{X}$ and invertible $\mathscr{A}$-linear transformations $L\circ L'\to\mathrm{Id}_{\mathscr{X}'}$ and $L'\circ L\to\mathrm{Id}_{\mathscr{X}}$. It can be shown that this amounts to requiring L to be plainly an equivalence of categories.

Let $\mathscr{M}$ be a $(\mathscr{B},\mathscr{A})$-bimodule. If $\mathscr{Y}$ is a left $\mathscr{B}$-module, the category $\mathrm{Hom}_{\mathscr{B}}(\mathscr{M},\mathscr{Y})$ becomes a left $\mathscr{A}$-module. The action is defined by

$$(A.L)(M)=L(M.A)$$

for $A\in\mathscr{A}$, $M\in\mathscr{M}$, $L\in\mathrm{Hom}_{\mathscr{B}}(\mathscr{M},\mathscr{Y})$.

Moreover we have a functor

$$\Phi_{\mathscr{Y},\mathscr{Y}'} : \mathrm{Hom}_{\mathscr{B}}(\mathscr{Y},\mathscr{Y}') \to \mathrm{Hom}_{\mathscr{A}}(\mathrm{Hom}_{\mathscr{B}}(\mathscr{M},\mathscr{Y}),\mathrm{Hom}_{\mathscr{B}}(\mathscr{M},\mathscr{Y}'))$$
$$F \mapsto (L \mapsto F \circ L)$$

for $\mathscr{B}$-modules $\mathscr{Y},\mathscr{Y}'$. The functors $\Phi_{\mathscr{Y},\mathscr{Y}'}$ preserve horizontal compositions and unit 1-cells.

The 2-functor $\mathrm{Hom}_{\mathscr{B}}(\mathscr{M},-) : \mathscr{B}\text{-Mod} \to \mathscr{A}\text{-Mod}$ consists of the assignment

$$\mathscr{B}\text{-module } \mathscr{Y} \mapsto \mathscr{A}\text{-module } \mathrm{Hom}_{\mathscr{B}}(\mathscr{M},\mathscr{Y})$$

and the collection of the functors $\Phi_{\mathscr{Y},\mathscr{Y}'}$ for all $\mathscr{B}$-modules $\mathscr{Y},\mathscr{Y}'$.

For a right $\mathscr{A}$-module $\mathscr{X}$ and a left $\mathscr{A}$-module $\mathscr{Y}$, the tensor product category $\mathscr{X} \otimes_{\mathscr{A}} \mathscr{Y}$ is defined. Its objects are finite direct sums of symbols $[X, Y]$ for $X \in \mathscr{X}$, $Y \in \mathscr{Y}$, and morphisms are sums of compositions of symbols

$$[f,g] : [X,Y] \to [X',Y']$$

for $f : X \to X'$ and $g : Y \to Y'$, and

$$\alpha_{X,A,Y} : [X.A, Y] \to [X, A.Y]$$

for $X \in \mathscr{X}$, $A \in \mathscr{A}$, $Y \in \mathscr{Y}$ and the formal inverse of $\alpha_{X,A,Y}$. These generating morphisms are subject to the relations of functoriality, naturality and coherence. The precise definition is in [**T**], but will not be needed in the sequel.

Let $\mathscr{M}$ be a $(\mathscr{B},\mathscr{A})$-bimodule. If $\mathscr{X}$ is a left $\mathscr{A}$-module, $\mathscr{M} \otimes_{\mathscr{A}} \mathscr{X}$ becomes a left $\mathscr{B}$-module. The action is defined by

$$B.[M,X] = [B.M,X]$$

for $B \in \mathscr{B}$, $M \in \mathscr{M}$, $X \in \mathscr{X}$.

Moreover we have a functor

$$\Psi_{\mathscr{X},\mathscr{X}'} : \mathrm{Hom}_{\mathscr{A}}(\mathscr{X},\mathscr{X}') \to \mathrm{Hom}_{\mathscr{B}}(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{X}, \mathscr{M} \otimes_{\mathscr{A}} \mathscr{X}')$$
$$G \mapsto ([M,X] \mapsto [M,G(X)])$$

for $\mathscr{A}$-modules $\mathscr{X},\mathscr{X}'$. The functors $\Psi_{\mathscr{X},\mathscr{X}'}$ preserve horizontal compositions and unit 1-cells.

The 2-functor $\mathscr{M} \otimes_{\mathscr{A}} - : \mathscr{A}\text{-Mod} \to \mathscr{B}\text{-Mod}$ consists of the assignment

$$\mathscr{A}\text{-module } \mathscr{X} \mapsto \mathscr{B}\text{-module } \mathscr{M} \otimes_{\mathscr{A}} \mathscr{X}$$

and the collection of the functors $\Psi_{\mathscr{X},\mathscr{X}'}$ for all $\mathscr{A}$-modules $\mathscr{X},\mathscr{X}'$.

We have also an $\mathscr{A}$-linear functor

$$\mathscr{X} \to \mathrm{Hom}_{\mathscr{B}}(\mathscr{M}, \mathscr{M} \otimes_{\mathscr{A}} \mathscr{X})$$
$$X \mapsto (M \mapsto [M,X])$$

for an $\mathscr{A}$-module $\mathscr{X}$, and a $\mathscr{B}$-linear functor

$$\mathscr{M} \otimes_{\mathscr{A}} \mathrm{Hom}_{\mathscr{B}}(\mathscr{M}, \mathscr{Y}) \to \mathscr{Y}$$
$$[M, L] \mapsto L(M)$$

for a $\mathscr{B}$-module $\mathscr{Y}$. These are natural in $\mathscr{X}$ and $\mathscr{Y}$, respectively.

Furthermore we have an equivalence

$$\mathrm{Hom}_{\mathscr{B}}(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{X}, \mathscr{Y}) \to \mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, \mathrm{Hom}_{\mathscr{B}}(\mathscr{M}, \mathscr{Y}))$$
$$F \mapsto (X \mapsto (M \mapsto F([M, X]))).$$

The category of finite dimensional k-vector spaces is denoted by $\mathscr{V}$. This is a tensor category with the usual tensor product. We regard the natural isomorphisms $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ and $k \otimes X \cong X \cong X \otimes k$ for vector spaces X, Y, Z as the identities.

Every k-category $\mathscr{X}$ may be viewed as a $\mathscr{V}$-module. The action of the n-dimensional space k^n on $\mathscr{X}$ is given by the functor $X \mapsto X^n$. We use the symbol $\otimes$ rather than 'dot' for this action. Thus $k^n \otimes X = X^n$ for $X \in \mathscr{X}$.

A k-category $\mathscr{X}$ is said to have direct summands if every idempotent endomorphism $e : X \to X$ in $\mathscr{X}$ is the projection to a direct summand of X. Given a k-category $\mathscr{X}$ we can embed $\mathscr{X}$ to a k-category $\overline{\mathscr{X}}$ having direct summands such that every object of $\overline{\mathscr{X}}$ is a direct summand of an object of $\mathscr{X}$. For any k-category $\mathscr{Y}$ with direct summands, the induced functor

$$\mathrm{Hom}(\overline{\mathscr{X}}, \mathscr{Y}) \to \mathrm{Hom}(\mathscr{X}, \mathscr{Y})$$

is an equivalence. For a construction of $\overline{\mathscr{X}}$ see [**GV**, p. 413].

If $\mathscr{X}$ is an $\mathscr{A}$-module, then $\overline{\mathscr{X}}$ becomes naturally an $\mathscr{A}$-module. For a right $\mathscr{A}$-module $\mathscr{X}$ and a left $\mathscr{A}$-module $\mathscr{Y}$, $\overline{\mathscr{X} \otimes_{\mathscr{A}} \mathscr{Y}}$ is written as $\mathscr{X} \bar{\otimes}_{\mathscr{A}} \mathscr{Y}$. $\mathscr{A}$-Modk denotes the 2-category consisting of the categories $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, \mathscr{Y})$ for left $\mathscr{A}$-modules $\mathscr{X}$, $\mathscr{Y}$ with direct summands. The 2-functors

$$\mathscr{A}\text{-Mod} \underset{\mathrm{Hom}_{\mathscr{B}}(\mathscr{M}, -)}{\overset{\mathscr{M} \otimes_{\mathscr{A}} -}{\rightleftarrows}} \mathscr{B}\text{-Mod}$$

defined above yield the 2-functors

$$\mathscr{A}\text{-Modk} \underset{\mathrm{Hom}_{\mathscr{B}}(\mathscr{M}, -)}{\overset{\mathscr{M} \bar{\otimes}_{\mathscr{A}} -}{\rightleftarrows}} \mathscr{B}\text{-Modk}.$$

2. Group actions on tensor categories.

An action of a group G on a k-category $\mathscr{X}$ consists of data

– functors $\sigma_* : \mathscr{X} \to \mathscr{X}$ for all $\sigma \in G$
– isomorphisms $\phi(\sigma,\tau) : (\sigma\tau)_* \to \sigma_* \circ \tau_*$ for all $\sigma, \tau \in G$
– an isomorphism $\nu : 1_* \to \mathrm{Id}_{\mathscr{X}}$

which make the following diagrams commutative for all $\sigma, \tau, \rho \in G$ and $X \in \mathscr{X}$.

$$\begin{array}{ccc} (\sigma\tau\rho)_* X & \xrightarrow{\phi(\sigma\tau,\rho)_X} & (\sigma\tau)_*\rho_* X \\ {\scriptstyle \phi(\sigma,\tau\rho)_X}\downarrow & & \downarrow{\scriptstyle \phi(\sigma,\tau)_{\rho_* X}} \\ \sigma_*(\tau\rho)_* X & \xrightarrow[\sigma_*(\phi(\tau,\rho)_X)]{} & \sigma_*\tau_*\rho_* X \end{array} \tag{1}$$

$$1_* X \underset{1_*(\nu_X)}{\overset{\phi(1,1)_X}{\rightleftarrows}} 1_* 1_* X \tag{2}$$

$$1_* X \underset{\nu_{1_* X}}{\overset{\phi(1,1)_X}{\rightleftarrows}} 1_* 1_* X \tag{3}$$

Here commutativity of the last two diagrams means that the opposite arrows are inverse to each other.

Let $\mathscr{X}$, $\mathscr{Y}$ be categories with G-action. A G-linear functor $\mathscr{X} \to \mathscr{Y}$ consists of

– a k-linear functor $L : \mathscr{X} \to \mathscr{Y}$
– isomorphisms $\eta(\sigma) : L \circ \sigma_* \to \sigma_* \circ L$ for all $\sigma \in G$

making the following diagram commutative for all $\sigma, \tau \in G$ and $X \in \mathscr{X}$.

$$\begin{array}{ccccc} L((\sigma\tau)_* X) & & \xrightarrow{\eta(\sigma\tau)_X} & & (\sigma\tau)_* L(X) \\ {\scriptstyle L(\phi(\sigma,\tau)_X)}\downarrow & & & & \downarrow{\scriptstyle \phi(\sigma,\tau)_{L(X)}} \\ L(\sigma_*\tau_* X) & \xrightarrow[\eta(\sigma)_{\tau_* X}]{} & \sigma_* L(\tau_* X) & \xrightarrow[\sigma_*\eta(\tau)_X]{} & \sigma_*\tau_* L(X) \end{array} \tag{4}$$

Let $\mathscr{X}$ be a category with G-action. The category of G-invariants in $\mathscr{X}$, denoted by $\mathscr{X}^G$, is a k-category defined as follows. An object of $\mathscr{X}^G$ is a pair (X, f), where X is an object of $\mathscr{X}$ and f is a family of isomorphisms $f(\sigma) : \sigma_* X \to X$ for all $\sigma \in G$ making the following diagram commutative for all $\sigma, \tau \in G$.

$$\begin{array}{ccc} (\sigma\tau)_* X & \xrightarrow{f_{\sigma\tau}} & X \\ {\scriptstyle \phi(\sigma,\tau)_X}\downarrow & & \uparrow{\scriptstyle f_\sigma} \\ \sigma_*\tau_* X & \xrightarrow[\sigma_*(f_\tau)]{} & \sigma_* X \end{array} \tag{5}$$

A morphism $(X, f) \to (X', f')$ in $\mathscr{X}^G$ is a morphism $u : X \to X'$ in $\mathscr{X}$ such that

$$f'(\sigma) \circ \sigma_* u = u \circ f(\sigma)$$

for all $\sigma \in G$.

Example 2.1. Let G act on the category $\mathscr{V}$ of vector spaces trivially. This means that all σ_*, $\phi(\sigma, \tau)$, v are the identities. Then $\mathscr{V}^G$ is the category of $k[G]$-modules.

Let $\mathscr{C}$ be a tensor category with tensor product $(A, B) \mapsto A.B$, unit object I, associativity isomorphisms $\alpha_{A,B,C} : (A.B).C \to A.(B.C)$, and unit isomorphisms $\lambda_A : I.A \to A$, $\rho_A : A.I \to A$.

An action of G on the tensor category $\mathscr{C}$ means an action of G on the k-category $\mathscr{C}$ preserving the tensor structure. Namely it consists of data

- tensor functors $\sigma_* : \mathscr{C} \to \mathscr{C}$ for all $\sigma \in G$
- isomorphisms $\phi(\sigma, \tau) : (\sigma\tau)_* \to \sigma_* \circ \tau_*$ of tensor functors for all $\sigma, \tau \in G$
- an isomorphism $v : 1_* \to \mathrm{Id}_{\mathscr{C}}$ of tensor functors

making the diagrams (1), (2), (3) commutative with obvious change of letters. We also use the word G-tensor category for tensor category with G-action.

By the definition of a tensor functor, the above σ_* consists of

- a functor $\sigma_* : \mathscr{C} \to \mathscr{C}$
- natural isomorphisms $\psi(\sigma)_{A,B} : \sigma_* A.\sigma_* B \to \sigma_*(A.B)$ for all $A, B \in \mathscr{C}$
- an isomorphism $\iota(\sigma) : I \to \sigma_* I$

making the following diagrams commutative for all $A, B, C \in \mathscr{C}$.

$$\begin{tikzcd}
(\sigma_* A.\sigma_* B).\sigma_* C \arrow[r, "\alpha_{\sigma_* A, \sigma_* B, \sigma_* C}"] \arrow[d, "\psi(\sigma)_{A,B}.\sigma_* C"'] & \sigma_* A.(\sigma_* B.\sigma_* C) \arrow[d, "\sigma_* A.\psi(\sigma)_{B,C}"] \\
\sigma_*(A.B).\sigma_* C \arrow[d, "\psi(\sigma)_{A.B,C}"'] & \sigma_* A.\sigma_*(B.C) \arrow[d, "\psi(\sigma)_{A,B.C}"] \\
\sigma_*((A.B).C) \arrow[r, "\sigma_*(\alpha_{A,B,C})"'] & \sigma_*(A.(B.C))
\end{tikzcd} \tag{6}$$

$$\begin{tikzcd}
I.I \arrow[rr, "\lambda_I"] \arrow[d, "\iota(\sigma).\iota(\sigma)"'] & & I \arrow[d, "\iota(\sigma)"] \\
\sigma_* I.\sigma_* I \arrow[r, "\psi(\sigma)_{I,I}"'] & \sigma_*(I.I) \arrow[r, "\sigma_*(\lambda_I)"'] & \sigma_* I
\end{tikzcd} \tag{7}$$

The requirement that $\phi(\sigma,\tau)$ is a morphism of tensor functors means that the following diagram is commutative for all $A, B \in \mathscr{C}$.

$$\begin{array}{ccc}
(\sigma\tau)_* A.(\sigma\tau)_* B & \xrightarrow{\phi(\sigma,\tau)_A.\phi(\sigma,\tau)_B} & \sigma_*\tau_* A.\sigma_*\tau_* B \\
 & & \Big\downarrow {\scriptstyle \psi(\sigma)_{\tau_* A,\tau_* B}} \\
{\scriptstyle \psi(\sigma\tau)_{A,B}} \Big\downarrow & & \sigma_*(\tau_* A.\tau_* B) \\
 & & \Big\downarrow {\scriptstyle \sigma_*(\psi(\tau)_{A,B})} \\
(\sigma\tau)_*(A.B) & \xrightarrow[\phi(\sigma,\tau)_{A.B}]{} & \sigma_*\tau_*(A.B)
\end{array} \tag{8}$$

In the presence of the commutativity of (3) and (8), $\nu : 1_* \to \mathrm{Id}_{\mathscr{C}}$ is automatically a morphism of tensor functors. Thus we could say that a G-action on the tensor category $\mathscr{C}$ consists of the data σ_*, $\phi(\sigma,\tau)$, ν, $\psi(\sigma)$, $\iota(\sigma)$ making the diagrams of (1), (2), (3), (6), (7), (8) commutative.

Let $\mathscr{C}$ be a G-tensor category. The category $\mathscr{C}^G$ becomes a tensor category as follows. The tensor product is defined by

$$(A,f).(B,g) = (A.B,h) \tag{9}$$

where

$$h(\sigma) = f(\sigma).g(\sigma) \circ \psi(\sigma)_{A,B}^{-1}. \tag{10}$$

The unit object is (I,ι^{-1}). The associativity and unit isomorphisms are inherited from $\mathscr{C}$.

We now construct another tensor category $\mathscr{C}[G]$ from a G-tensor category $\mathscr{C}$. We set $\mathscr{C}[G] = \bigoplus_{\sigma\in G}\mathscr{C}$ as categories. So an object of $\mathscr{C}[G]$ is expressed as $\bigoplus_{\sigma\in G}(A_\sigma,\sigma)$ with $A_\sigma \in \mathscr{C}$, and a morphism from $\bigoplus_{\sigma\in G}(A_\sigma,\sigma)$ to $\bigoplus_{\sigma\in G}(B_\sigma,\sigma)$ is expressed as $\bigoplus_{\sigma\in G}(f_\sigma,\sigma)$ with $f_\sigma : A_\sigma \to B_\sigma$ a morphism in $\mathscr{C}$. The tensor product operation in $\mathscr{C}[G]$ is defined by

$$(A,\sigma).(B,\tau) = (A.\sigma_* B,\sigma\tau) \quad \text{for objects,}$$

$$(f,\sigma).(g,\tau) = (f.\sigma_* g,\sigma\tau) \quad \text{for morphisms.}$$

The unit object is $(I,1)$. The associativity is given by

$$\begin{array}{ccc}
((A,\sigma).(B,\tau)).(C,\rho) = (A.\sigma_* B,\sigma\tau).(C,\rho) = & & ((A.\sigma_* B).(\sigma\tau)_* C,\sigma\tau\rho) \\
{\scriptstyle \alpha_{(A,\sigma),(B,\tau),(C,\rho)}} \Big\downarrow & & \Big\downarrow {\scriptstyle (\alpha(A,\sigma,B,\tau,C),\sigma\tau\rho)} \\
(A,\sigma).((B,\tau).(C,\rho)) = (A,\sigma).(B.\tau_* C,\tau\rho) = & & (A.\sigma_*(B.\tau_* C)),\sigma\tau\rho)
\end{array} \tag{11}$$

where $\alpha(A,\sigma,B,\tau,C)$ is the composite

$$\begin{array}{c} (A.\sigma_*B).(\sigma\tau)_*C \\ \Big\downarrow {\scriptstyle (A.\sigma_*B).\phi(\sigma,\tau)_C} \\ (A.\sigma_*B).\sigma_*\tau_*C \\ \Big\downarrow {\scriptstyle \alpha_{A,\sigma_*B,\sigma_*\tau_*C}} \\ A.(\sigma_*B.\sigma_*\tau_*C) \\ \Big\downarrow {\scriptstyle A.\psi(\sigma)_{B,\tau_*C}} \\ A.\sigma_*(B.\tau_*C). \end{array} \tag{12}$$

The left unitality

$$\lambda_{(A,\sigma)} : (I,1).(A,\sigma) = (I.1_*A,\sigma) \to (A,\sigma)$$

is given by

$$I.1_*A \xrightarrow{I.v_A} I.A \xrightarrow{\lambda_A} A. \tag{13}$$

The right unitality

$$\rho_{(A,\sigma)} : (A,\sigma).(I,1) = (A.\sigma_*I,\sigma) \to (A,\sigma)$$

is given by

$$A.\sigma_*I \xrightarrow{A.\iota(\sigma)^{-1}} A.I \xrightarrow{\rho_A} A. \tag{14}$$

These data satisfy the axiom of a tensor category. In [**M**] Magid introduced the double cross product of tensor categories, of which $\mathscr{C}[G]$ is regarded as a special case. [**M**] does not contain the proof of the axiom of a tensor category for the double cross product. We will verify the pentagon axiom for $\mathscr{C}[G]$ in Section 8.

Example 2.2. With respect to the trivial action of G on $\mathscr{V}$, we have the tensor category $\mathscr{V}[G]$. Objects are of the form $\bigoplus_{\sigma\in G}(V_\sigma,\sigma)$ with $V_\sigma \in \mathscr{V}$. The tensor product is given by

$$(V,\sigma).(W,\tau) = (V\otimes W,\sigma\tau).$$

Thus $\mathscr{V}[G]$ is the category of G-graded vector spaces, or the category of $k[G]^*$-modules when G is finite.

Example 2.3. Suppose G acts on a group A. Then the action of G on the tensor category $\mathscr{V}[A]$ is induced. We have obviously $\mathscr{V}[A][G] = \mathscr{V}[A \rtimes G]$.

Let $\mathscr{C}$ be a G-tensor category. We may view a $\mathscr{C}[G]$-module as a category having actions of $\mathscr{C}$ and G in a compatible way. To be precise, a $\mathscr{C}[G]$-module structure on a k-category $\mathscr{X}$ amounts to the data

- a $\mathscr{C}$-module structure on $\mathscr{X} : (.,\alpha,\lambda)$
- a G-action on $\mathscr{X} : (\sigma_*, \phi(\sigma,\tau), \nu)$
- natural isomorphisms $\psi(\sigma)_{C,X} : \sigma_*C.\sigma_*X \to \sigma_*(C.X)$ for all $C \in \mathscr{C}$, $X \in \mathscr{X}$, $\sigma \in G$

making the following diagrams commutative: (6), (8) with appropriate change of letters, and in place of (7), the diagram

$$\begin{tikzcd} I.\sigma_*X \arrow[r, "\iota(\sigma).\sigma_*X"] \arrow[d, "\lambda_{\sigma_*X}"'] & \sigma_*I.\sigma_*X \arrow[d, "\psi(\sigma)_{I,X}"] \\ \sigma_*X & \sigma_*(I.X). \arrow[l, "\sigma_*(\lambda_X)"] \end{tikzcd} \tag{15}$$

Indeed, given these data, we define the action of $\mathscr{C}[G]$ on $\mathscr{X}$ by

$$(C,\sigma).X = C.\sigma_*X$$

and the associativity and the unitality

$$\alpha_{(C,\sigma),(D,\tau),X} : ((C,\sigma).(D,\tau)).X \to (C,\sigma).((D,\tau).X)$$
$$\lambda_X : (I,1).X \to X$$

by the formulas similar to (11), (12), (13).

Example 2.4. $\mathscr{C}$ itself is a $\mathscr{C}[G]$-module: $(C,\sigma).C' = C.\sigma_*C'$.

Example 2.5. A $\mathscr{V}[G]$-module is nothing but a k-category with G-action: $(k,\sigma).X = \sigma_*X$.

Let $\mathscr{X},\mathscr{Y}$ be $\mathscr{C}[G]$-modules. A structure of a $\mathscr{C}[G]$-linear functor on a k-linear functor $L : \mathscr{X} \to \mathscr{Y}$ amounts to the data

- isomorphisms $\beta_{C,X} : L(C.X) \to C.L(X)$ for $C \in \mathscr{C}$, $X \in \mathscr{X}$
- isomorphisms $\eta(\sigma)_X : L(\sigma_*X) \to \sigma_*L(X)$ for $\sigma \in G$, $X \in \mathscr{X}$

satisfying the following conditions.

- (L,β) is a $\mathscr{C}$-linear functor.
- (L,η) is a G-linear functor.

– the diagram

$$\begin{array}{ccc} L(\sigma_* C.\sigma_* X) & \xrightarrow{L(\psi(\sigma)_{C,X})} & L(\sigma_*(C.X)) \\ {\scriptstyle \beta_{\sigma_* C, \sigma_* X}} \downarrow & & \downarrow {\scriptstyle \eta(\sigma)_{C.X}} \\ \sigma_* C.L(\sigma_* X) & & \sigma_* L(C.X) \\ {\scriptstyle \sigma_* C.\eta(\sigma)_X} \downarrow & & \downarrow {\scriptstyle \sigma_* \beta_{C,X}} \\ \sigma_* C.\sigma_* L(X) & \xrightarrow[\psi(\sigma)_{C,L(X)}]{} & \sigma_*(C.L(X)) \end{array}$$

commutes for all $C \in \mathscr{C}$, $\sigma \in G$, $X \in \mathscr{X}$.

If $\mathscr{X}$ is a $\mathscr{C}[G]$-module, $\mathscr{X}^G$ becomes a $\mathscr{C}^G$-module by a similar action to (9), (10).

3. $\mathscr{V}^G$-modules and $\mathscr{V}[G]$-modules.

Hereafter we assume G is a finite group and the characteristic of k does not divide $|G|$. We denote the category of finite dimensional $k[G]$-modules by $\mathscr{V}^G$, and the category of finite dimensional $k[G]^*$-modules by $\mathscr{V}[G]$ (see Example 2.1, 2.2). In this section we review the one-to-one correspondence between $\mathscr{V}^G$-modules and $\mathscr{V}[G]$-modules from [**T**].

We make $\mathscr{V}$ into a $(\mathscr{V}[G], \mathscr{V}^G)$-bimodule. The action of objects are given by

$$Y.V = Y \otimes V, \quad V.X = V \otimes X$$

for $X \in \mathscr{V}^G$, $Y \in \mathscr{V}[G]$, $V \in \mathscr{V}$. The associativity of actions

$$(Y.Y').V \to Y.(Y'.V), \quad (V.X).X' \to V.(X.X')$$

are the identity maps, while

$$(Y.V).X \to Y.(V.X)$$

is the map

$$(y, \tau) \otimes v \otimes x \mapsto (y, \tau) \otimes v \otimes \tau^{-1} x,$$

where $x \in X$, $v \in V$, $\tau \in G$, and (y, τ) is an element in the τ-component of the G-graded space Y.

The duality theorem of [**T**] in the case of a group algebra is as follows.

Theorem 3.1. *The 2-functors*

$$\mathscr{V}^G\text{-Modk} \underset{\mathrm{Hom}_{\mathscr{V}[G]}(\mathscr{V},-)}{\overset{\mathscr{V}\bar{\otimes}_{\mathscr{V}^G}-}{\rightleftarrows}} \mathscr{V}[G]\text{-Modk}$$

are quasi-inverse to each other through the adjunction. Namely, for any $\mathscr{V}^G$-module $\mathscr{X}$ and $\mathscr{V}[G]$-module $\mathscr{Y}$ with direct summands, the canonical functors

$$\mathscr{X} \to \mathrm{Hom}_{\mathscr{V}[G]}(\mathscr{V}, \mathscr{V} \bar{\otimes}_{\mathscr{V}^G} \mathscr{X}),$$

$$\mathscr{V} \bar{\otimes}_{\mathscr{V}^G} \mathrm{Hom}_{\mathscr{V}[G]}(\mathscr{V}, \mathscr{Y}) \to \mathscr{Y}$$

are equivalences of $\mathscr{V}^G$-modules and of $\mathscr{V}[G]$-modules, respectively.

In this situation we also say the pair $(\mathscr{V} \otimes_{\mathscr{V}^G} -, \mathrm{Hom}_{\mathscr{V}[G]}(\mathscr{V}, -))$ is a 2-equivalence. The 2-equivalence implies the following:

(i) For every $\mathscr{V}^G$-module $\mathscr{X}$ with direct summands there exist a $\mathscr{V}[G]$-module $\mathscr{Y}$ with direct summands and an equivalence $\mathscr{X} \to \mathrm{Hom}_{\mathscr{V}[G]}(\mathscr{V}, \mathscr{Y})$ of $\mathscr{V}^G$-modules.

(ii) For $\mathscr{V}[G]$-modules $\mathscr{Y}, \mathscr{Y}'$ with direct summands, the functor

$$\mathrm{Hom}_{\mathscr{V}[G]}(\mathscr{Y}, \mathscr{Y}') \to \mathrm{Hom}_{\mathscr{V}^G}(\mathrm{Hom}_{\mathscr{V}[G]}(\mathscr{V}, \mathscr{Y}), \mathrm{Hom}_{\mathscr{V}[G]}(\mathscr{V}, \mathscr{Y}'))$$

is an equivalence.

Recall that $\mathscr{V}[G]$-modules are just k-categories with G-action (Example 2.5). So we may use the notations σ_*, $\phi(\sigma, \tau)$ of G-action for $\mathscr{V}[G]$-modules. The left $\mathscr{V}[G]$-module structure on the bimodule $\mathscr{V}$ is trivial: $\sigma_* V = V$. The associativity $(Y.V).X \to Y.(V.X)$ for $Y = (k, \sigma)$ takes the form

$$b_\sigma : \sigma_* V \otimes X \to \sigma_*(V \otimes X)$$

$$v \otimes x \mapsto v \otimes \sigma^{-1} x.$$

Proposition 3.2. *For any $\mathscr{V}[G]$-module $\mathscr{X}$, we have an equivalence of $\mathscr{V}^G$-modules*

$$\mathscr{X}^G \simeq \mathrm{Hom}_{\mathscr{V}[G]}(\mathscr{V}, \mathscr{X}).$$

Proof. This is [**T**, Proposition 3.4] adapted to our present situation. But we give here a direct description of the equivalence. Recall that an object of $\mathrm{Hom}_{\mathscr{V}[G]}(\mathscr{V}, \mathscr{X})$, a G-linear functor $\mathscr{V} \to \mathscr{X}$, is represented as a pair (F, η) of a k-linear functor $\mathscr{V} \to \mathscr{X}$ and a family η of isomorphisms

$$\eta(\sigma)_V : F(\sigma_* V) \to \sigma_* F(V)$$

for $\sigma \in G$, $V \in \mathscr{V}$. Let

$$\Phi : \mathscr{X}^G \to \mathrm{Hom}_{\mathscr{V}[G]}(\mathscr{V}, \mathscr{X})$$

be the functor sending (X,f) to (F,η), where F is given by

$$F(V)=V\otimes X$$

(here $\otimes$ denotes the canonical action of $\mathscr{V}$ on $\mathscr{X}$, see Section 1) and η is defined by

$$\eta(\sigma)_V : F(\sigma_* V)=F(V)=V\otimes X \xrightarrow{1\otimes f(\sigma)^{-1}} V\otimes \sigma_* X=\sigma_*(V\otimes X)=\sigma_* F(V).$$

It is easy to see that Φ is an equivalence. Indeed, a quasi-inverse to Φ is the functor sending (F,η) to (X,f) with $X=F(k)$ and $f(\sigma)=\eta(\sigma)_k^{-1}$.

The proof will be completed once we show that Φ has a structure of a $\mathscr{V}^G$-linear functor. Let $M\in\mathscr{V}^G$, $(X,f)\in\mathscr{X}^G$. We wish to define an isomorphism

$$\beta_{M,(X,f)} : \Phi(M.(X,f))\to M.\Phi(X,f)$$

in $\mathrm{Hom}_{\mathscr{V}[G]}(\mathscr{V},\mathscr{X})$. Let $\Phi(X,f)=(F,\eta)$ as above. As the left action of $\mathscr{V}^G$ on $\mathrm{Hom}_{\mathscr{V}[G]}(\mathscr{V},\mathscr{X})$ comes from the right action on $\mathscr{V}$, we have $M.(F,\eta)=(H,\xi)$, where

$$H(V)=F(V.M)=V\otimes M\otimes X,$$

and

$$\xi(\sigma)_V : H(\sigma_* V)\to\sigma_* H(V)$$

is the composite

$$F(\sigma_* V\otimes M)\xrightarrow{F(b_\sigma)} F(\sigma_*(V\otimes M))\xrightarrow{\eta(\sigma)_{V\otimes M}}\sigma_* F(V\otimes M),$$

where b_σ was given preceding to Proposition 3.2. This is equal to

$$V\otimes M\otimes X\xrightarrow{1\otimes\sigma_M^{-1}\otimes 1} V\otimes M\otimes X\xrightarrow{1\otimes 1\otimes f(\sigma)^{-1}} V\otimes M\otimes\sigma_* X,$$

that is,

$$V\otimes M\otimes X\xrightarrow{1\otimes\sigma_M^{-1}\otimes f(\sigma)^{-1}} V\otimes M\otimes\sigma_* X,$$

where $\sigma_M : M\to M$ is the action of σ on M. On the other hand, by the definition of the action of $\mathscr{V}^G$ on $\mathscr{X}^G$ we have $M.(X,f)=(M\otimes X,h)$, where

$$h(\sigma) : \sigma_*(M\otimes X)=M\otimes\sigma_* X\xrightarrow{\sigma_M\otimes f(\sigma)} M\otimes X.$$

So we have $\Phi(M\otimes X,h)=(H',\xi')$, where

$$H'(V)=V\otimes M\otimes X$$

and

$$\xi'(\sigma)_V : H'(\sigma_* V)\to\sigma_* H'(V)$$

is the map

$$V \otimes M \otimes X \xrightarrow{1 \otimes h(\sigma)^{-1}} V \otimes \sigma_*(M \otimes X).$$

Hence $H = H'$ and $\xi = \xi'$. Thus the identity $H \to H'$ gives the required isomorphism $\beta_{M,(X,f)}$.

It is obvious that $\beta_{M,(X,f)}$ is compatible with associativity of actions. Thus $(\Phi, \beta) : \mathscr{X}^G \to \mathrm{Hom}_{\mathscr{V}[G]}(\mathscr{V}, \mathscr{X})$ is a $\mathscr{V}^G$-linear functor. $\square$

4. $\mathscr{C}^G$-modules and $\mathscr{C}[G]$-modules.

Let $\mathscr{C}$ be a tensor category with G-action.

Theorem 4.1. *The* 2*-functors*

$$\mathscr{C}^G\text{-Modk} \underset{(-)^G}{\overset{\mathscr{C}\bar{\otimes}_{\mathscr{C}^G}-}{\rightleftarrows}} \mathscr{C}[G]\text{-Modk}$$

are quasi-inverse to each other.

Here if $\mathscr{X}$ is a $\mathscr{C}[G]$-module, then $\mathscr{X}^G$ becomes a $\mathscr{C}^G$-module as noted in Section 3. Also in the tensor product $\mathscr{C} \bar{\otimes}_{\mathscr{C}^G} -$, $\mathscr{C}$ is viewed as a $(\mathscr{C}[G], \mathscr{C}^G)$-bimodule in which the left action of $\mathscr{C}[G]$ on $\mathscr{C}$ is the standard one (Example 2.4), the right action of $\mathscr{C}^G$ on $\mathscr{C}$ comes from the forgetful functor $\mathscr{C}^G \to \mathscr{C}$, and the associativity

$$((X, \sigma).Y).(Z, f) \to (X, \sigma).(Y.(Z, f))$$

for $(X, \sigma) \in \mathscr{C}[G]$, $Y \in \mathscr{C}$, $(Z, f) \in \mathscr{C}^G$ is given by

$$(X.\sigma_* Y).Z \xrightarrow{\alpha_{X, \sigma_* Y, Z}} X.(\sigma_* Y.Z) \xrightarrow{X.(\sigma_* Y.f(\sigma)^{-1})} X.(\sigma_* Y.\sigma_* Z)$$
$$\xrightarrow{X.\psi(\sigma)_{Y,Z}} X.\sigma_*(Y.Z).$$

That $\mathscr{C}$ is really a bimodule is verified in the course of the proof of the theorem.

Lemma 4.2. *Let $\mathscr{B}$ be a tensor category and $\mathscr{M}$ a left $\mathscr{B}$-module. Set $\mathscr{A} = (\mathrm{End}_{\mathscr{B}} \mathscr{M})^{\otimes \mathrm{op}}$ so that $\mathscr{M}$ becomes a $(\mathscr{B}, \mathscr{A})$-bimodule. Suppose that the* 2*-functors*

$$\mathscr{A}\text{-Modk} \underset{\mathrm{Hom}_{\mathscr{B}}(\mathscr{M}, -)}{\overset{\mathscr{M}\bar{\otimes}_{\mathscr{A}}-}{\rightleftarrows}} \mathscr{B}\text{-Modk}$$

are quasi-inverse to each other through the adjunction. Let $j : \mathscr{B} \to \mathscr{E}$ be a tensor functor. Set $\mathscr{N} = \mathscr{E} \bar{\otimes}_{\mathscr{B}} \mathscr{M}$, $\mathscr{D} = (\mathrm{End}_{\mathscr{E}} \mathscr{N})^{\otimes \mathrm{op}}$ so that $\mathscr{N}$ becomes an $(\mathscr{E}, \mathscr{D})$-

bimodule and a tensor functor $i : \mathscr{A} \to \mathscr{D}$ is induced. Then the 2-functors

$$\mathscr{D}\text{-Modk} \underset{\operatorname{Hom}_{\mathscr{E}}(\mathscr{N}, -)}{\overset{\mathscr{N} \bar{\otimes}_{\mathscr{D}} -}{\rightleftarrows}} \mathscr{E}\text{-Modk}$$

are quasi-inverse to each other through the adjunction. Also the diagrams

$$\begin{array}{ccc} \mathscr{D}\text{-Modk} & \xrightarrow{\mathscr{N} \bar{\otimes}_{\mathscr{D}} -} & \mathscr{E}\text{-Modk} \\ \downarrow & & \downarrow \\ \mathscr{A}\text{-Modk} & \xrightarrow[\mathscr{M} \bar{\otimes}_{\mathscr{A}} -]{} & \mathscr{B}\text{-Modk}, \end{array} \qquad \begin{array}{ccc} \mathscr{D}\text{-Modk} & \xleftarrow{\operatorname{Hom}_{\mathscr{E}}(\mathscr{N}, -)} & \mathscr{E}\text{-Modk} \\ \downarrow & & \downarrow \\ \mathscr{A}\text{-Modk} & \xleftarrow[\operatorname{Hom}_{\mathscr{B}}(\mathscr{M}, -)]{} & \mathscr{B}\text{-Modk} \end{array}$$

are commutative up to natural isomorphisms, where the vertical arrows are the restrictions through i and j.

Proof. The commutativity of the right diagram follows readily from the hom-tensor adjoint. For the left one, we have firstly

$$\mathscr{D} = \operatorname{Hom}_{\mathscr{E}}(\mathscr{E} \bar{\otimes}_{\mathscr{B}} \mathscr{M}, \mathscr{E} \bar{\otimes}_{\mathscr{B}} \mathscr{M}) \simeq \operatorname{Hom}_{\mathscr{B}}(\mathscr{M}, \mathscr{E} \bar{\otimes}_{\mathscr{B}} \mathscr{M}).$$

Since $\mathscr{M} \bar{\otimes}_{\mathscr{A}} -$ is quasi-inverse to $\operatorname{Hom}_{\mathscr{B}}(\mathscr{M}, -)$, we have an equivalence

$$\mathscr{M} \bar{\otimes}_{\mathscr{A}} \mathscr{D} \simeq \mathscr{E} \bar{\otimes}_{\mathscr{B}} \mathscr{M} \tag{16}$$

of $\mathscr{B}$-modules. Moreover, this is an equivalence of $(\mathscr{B}, \mathscr{D})$-bimodules.

Indeed, this factors as the composite

$$\mathscr{M} \bar{\otimes}_{\mathscr{A}} \mathscr{D} \simeq \mathscr{M} \bar{\otimes}_{\mathscr{A}} \operatorname{Hom}_{\mathscr{B}}(\mathscr{M}, \mathscr{E} \bar{\otimes}_{\mathscr{B}} \mathscr{M}) \simeq \mathscr{E} \bar{\otimes}_{\mathscr{B}} \mathscr{M}. \tag{17}$$

For any $\mathscr{B}$-module $\mathscr{Y}$, the canonical functor

$$\mathscr{M} \bar{\otimes}_{\mathscr{A}} \operatorname{Hom}_{\mathscr{B}}(\mathscr{M}, \mathscr{Y}) \to \mathscr{Y}$$

is $(\mathscr{B}, (\operatorname{End}_{\mathscr{B}} \mathscr{Y})^{\otimes\mathrm{op}})$-linear. In particular, the second equivalence of (17) is $(\mathscr{B}, \mathscr{D})$-linear. Next, the equivalence

$$\mathscr{D} \simeq \operatorname{Hom}_{\mathscr{B}}(\mathscr{M}, \mathscr{E} \bar{\otimes}_{\mathscr{B}} \mathscr{M})$$

is $(\mathscr{A}, \mathscr{D})$-linear as seen from the diagram

$$\begin{array}{ccc} \mathscr{A} & = & \operatorname{Hom}_{\mathscr{B}}(\mathscr{M}, \mathscr{M}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \operatorname{Hom}_{\mathscr{B}}(\mathscr{M}, j \bar{\otimes} \mathscr{M})} \\ \mathscr{D} & \longrightarrow & \operatorname{Hom}_{\mathscr{B}}(\mathscr{M}, \mathscr{E} \bar{\otimes}_{\mathscr{B}} \mathscr{M}). \end{array}$$

Hence the first equivalence of (17) is $(\mathscr{B}, \mathscr{D})$-linear. So (16) is $(\mathscr{B}, \mathscr{D})$-linear as well.

For any $\mathscr{D}$-module $\mathscr{X}$ we now have

$$\mathscr{M} \bar{\otimes}_{\mathscr{A}} \mathscr{X} \simeq (\mathscr{M} \bar{\otimes}_{\mathscr{A}} \mathscr{D}) \bar{\otimes}_{\mathscr{D}} \mathscr{X} \simeq (\mathscr{E} \bar{\otimes}_{\mathscr{B}} \mathscr{M}) \bar{\otimes}_{\mathscr{D}} \mathscr{X} = \mathscr{N} \bar{\otimes}_{\mathscr{D}} \mathscr{X}$$

as $\mathscr{B}$-modules. This proves the commutativity of the left diagram.

We next show that for any $\mathscr{D}$-module $\mathscr{X}$ the canonical functor

$$\mathscr{X} \to \mathrm{Hom}_{\mathscr{E}}(\mathscr{N}, \mathscr{N} \bar{\otimes}_{\mathscr{D}} \mathscr{X})$$

is an equivalence of $\mathscr{D}$-modules. For this it is enough to show that this is just an equivalence of categories. By the commutativity we have shown, the functor restricted to $\mathscr{A}$-Mod is isomorphic to the functor

$$\mathscr{X} \to \mathrm{Hom}_{\mathscr{B}}(\mathscr{M}, \mathscr{M} \bar{\otimes}_{\mathscr{A}} \mathscr{X}).$$

This is an equivalence of $\mathscr{A}$-modules by assumption.

Similarly we have $\mathscr{N} \bar{\otimes}_{\mathscr{D}} \mathrm{Hom}_{\mathscr{E}}(\mathscr{N}, -) \cong \mathrm{Id}$. $\square$

PROOF OF THEOREM 4.1. The trivial tensor functor $\mathscr{V} \to \mathscr{C}$ induces the tensor functor $\mathscr{V}[G] \to \mathscr{C}[G]$. We have an equivalence

$$\mathscr{C}[G] \otimes_{\mathscr{V}[G]} \mathscr{V} = (\mathscr{C} \otimes \mathscr{V}[G]) \otimes_{\mathscr{V}[G]} \mathscr{V} \simeq \mathscr{C},$$

in which an object $X \in \mathscr{C}$ corresponds to the object $[(X,1),k] \in \mathscr{C}[G] \otimes_{\mathscr{V}[G]} \mathscr{V}$. This is an equivalence of $\mathscr{C}[G]$-modules, because

$$\begin{aligned}(X,\sigma).[(X',1),k] &= [(X,\sigma).(X',1),k] = [(X.\sigma_*X',\sigma),k] \\ &= [(X.\sigma_*X',1).(I,\sigma),k] \cong [(X.\sigma_*X',1),(k,\sigma).k] \\ &= [(X.\sigma_*X',1),k].\end{aligned}$$

Let $\mathscr{D} = (\mathrm{End}_{\mathscr{V}[G]} \mathscr{C})^{\otimes \mathrm{op}}$. Applying the lemma to the 2-equivalence of Theorem 3.1 with $\mathscr{A} = \mathscr{V}^G$, $\mathscr{B} = \mathscr{V}[G]$, $\mathscr{M} = \mathscr{V}$ and $\mathscr{E} = \mathscr{C}[G]$, we have a 2-equivalence

$$\mathscr{D}\text{-Modk} \underset{\mathrm{Hom}_{\mathscr{C}[G]}(\mathscr{C},-)}{\overset{\mathscr{C}\bar{\otimes}_{\mathscr{D}}-}{\rightleftarrows}} \mathscr{C}[G]\text{-Modk}. \tag{18}$$

For any $\mathscr{C}[G]$-module $\mathscr{X}$, by the equivalence $\mathscr{C}[G] \otimes_{\mathscr{V}[G]} \mathscr{V} \simeq \mathscr{C}$, the equivalence $\mathscr{X}^G \to \mathrm{Hom}_{\mathscr{V}[G]}(\mathscr{V}, \mathscr{X})$ of Proposition 3.2 yields an equivalence

$$\Phi_{\mathscr{X}} : \mathscr{X}^G \to \mathrm{Hom}_{\mathscr{C}[G]}(\mathscr{C}, \mathscr{X}).$$

We express an object of $\mathrm{Hom}_{\mathscr{C}[G]}(\mathscr{C}, \mathscr{X})$ as a pair (F,η) of a $\mathscr{C}$-linear functor $F : \mathscr{C} \to \mathscr{X}$ and a family η of isomorphisms $\eta(\sigma) : F(\sigma_* C) \to \sigma_* F(C)$. Then $\Phi_{\mathscr{X}}$ maps an object $(X,f) \in \mathscr{X}^G$ to the object $(F,\eta) \in \mathrm{Hom}_{\mathscr{C}[G]}(\mathscr{C}, \mathscr{X})$ defined as follows. $F : \mathscr{C} \to \mathscr{X}$ is given by

$$F(C) = C.X$$

with the obvious $\mathscr{C}$-linear structure, and η is given by

$$\eta(\sigma)_C : F(\sigma_* C) = \sigma_* C.X \xrightarrow{-.f(\sigma)^{-1}} \sigma_* C.\sigma_* X \xrightarrow{\psi(\sigma)} \sigma_*(C.X) = \sigma_* F(C).$$

In particular, taking $\mathscr{X} = \mathscr{C}$, we have an equivalence

$$\Phi_{\mathscr{C}} : \mathscr{C}^G \to \mathrm{Hom}_{\mathscr{C}[G]}(\mathscr{C}, \mathscr{C}) = \mathscr{D}.$$

Recall that if $\mathscr{X}$ is a $\mathscr{C}$-module, then $\mathscr{X}^G$ becomes a $\mathscr{C}^G$-module. And $\mathscr{D} = (\mathrm{End}_{\mathscr{C}[G]} \mathscr{C})^{\otimes \mathrm{op}}$ acts on $\mathrm{Hom}_{\mathscr{C}[G]}(\mathscr{C}, \mathscr{X})$ through $\mathscr{C}$. We claim that $\Phi_{\mathscr{X}}$ and $\Phi_{\mathscr{C}}$ are compatible with respect to these actions. In particular $\Phi_{\mathscr{C}} : \mathscr{C}^G \to \mathscr{D}$ is an equivalence of tensor categories. The theorem then follows from (18).

To prove the claim, let $(C, g) \in \mathscr{C}^G$, $(X, f) \in \mathscr{X}^G$. Let $\Phi(X, f) = (F, \eta)$ as above and $\Phi(C, g) = (D, \delta)$, that is,

$$D(C') = C'.C,$$

$$\delta(\sigma)_{C'} : D(\sigma_* C') = \sigma_* C'.C \xrightarrow{-.g(\sigma)^{-1}} \sigma_* C'.\sigma_* C \xrightarrow{\psi(\sigma)} \sigma_*(C'.C) = \sigma_* D(C').$$

Then, relative to the action of $\mathrm{End}_{\mathscr{C}[G]} \mathscr{C}$ on $\mathrm{Hom}_{\mathscr{C}[G]}(\mathscr{C}, \mathscr{X})$ we have $(D, \delta).(F, \eta) = (F, \eta) \circ (D, \delta) = (H, \xi)$, where $H = F \circ D$, so

$$H(C') = F(D(C')) = (C'.C).X,$$

and

$$\xi(\sigma)_{C'} : H(\sigma_* C') \to \sigma_* H(C')$$

is the composite

$$FD(\sigma_* C') \xrightarrow{F(\delta(\sigma)_{C'})} F(\sigma_* D(C')) \xrightarrow{\eta(\sigma)_{D(C')}} \sigma_* FD(C'),$$

which is expanded as

$$\begin{array}{c}
(\sigma_* C'.C).X \\
\Big\downarrow {\scriptstyle (-.g(\sigma)^{-1}).-} \\
(\sigma_* C'.\sigma_* C).X \\
\Big\downarrow {\scriptstyle \psi(\sigma).-} \\
\sigma_*(C'.C).X \\
\Big\downarrow {\scriptstyle -.f(\sigma)^{-1}} \\
\sigma_*(C'.C).\sigma_* X \\
\Big\downarrow {\scriptstyle \psi(\sigma)} \\
\sigma_*((C'.C).X).
\end{array}$$

This is equal to the left vertical composite of the diagram

$$\begin{array}{ccc} (\sigma_* C'.C).X & \cong & \sigma_* C'.(C.X) \\ {\scriptstyle (-.g(\sigma)^{-1}).f(\sigma)^{-1}}\downarrow & & \downarrow{\scriptstyle -.(g(\sigma)^{-1}.f(\sigma)^{-1})} \\ (\sigma_* C'.\sigma_* C).\sigma_* X & \cong & \sigma_* C'.(\sigma_* C.\sigma_* X) \\ {\scriptstyle \psi(\sigma).-}\downarrow & & \downarrow{\scriptstyle -.\psi(\sigma)} \\ \sigma_*(C'.C).\sigma_* X & & \sigma_* C'.\sigma_*(C.X) \\ {\scriptstyle \psi(\sigma)}\downarrow & & \downarrow{\scriptstyle \psi(\sigma)} \\ \sigma_*((C'.C).X) & \cong & \sigma_*(C'.(C.X)) \end{array} \tag{19}$$

which commutes by the naturality of α and the commutativity in (6). On the other hand, relative to the action of $\mathscr{C}^G$ on $\mathscr{X}^G$ we have

$$(C, g).(X, f) = (C.X, h)$$

with

$$h(\sigma) : \sigma_*(C.X) \xrightarrow{\psi(\sigma)^{-1}} \sigma_* C.\sigma_* X \xrightarrow{g(\sigma).f(\sigma)} C.X.$$

Then $\Phi(C.X, h) = (H', \xi')$, where

$$H'(C') = C'.(C.X)$$

and

$$\xi'(\sigma)_{C'} : H'(\sigma_* C') \to \sigma_* H'(C')$$

is the composite

$$\sigma_* C'.(C.X) \xrightarrow{-.h(\sigma)^{-1}} \sigma_* C'.\sigma_*(C.X) \xrightarrow{\psi(\sigma)} \sigma_*(C'.(C.X)).$$

This is equal to the right vertical composite of (19). Also we have an isomorphism $\pi : H \to H'$ given by the associativity

$$\pi_{C'} : H(C') = (C'.C).X \cong C'.(C.X) = H'(C').$$

The diagram (19) now becomes

$$\begin{array}{ccc} H(\sigma_* C') & \xrightarrow{\pi_{\sigma_* C'}} & H'(\sigma_* C') \\ {\scriptstyle \xi(\sigma)_{C'}}\downarrow & & \downarrow{\scriptstyle \xi'(\sigma)_{C'}} \\ \sigma_* H(C') & \xrightarrow[\sigma_* \pi_{C'}]{} & \sigma_* H'(C'). \end{array}$$

Thus π gives the isomorphism

$$(H, \xi) \cong (H', \xi'),$$

that is,

$$\pi : \Phi(C, g).\Phi(X, f) \xrightarrow{\sim} \Phi((C, g).(X, f))$$

in $\mathrm{Hom}_{\mathscr{C}[G]}(\mathscr{C}, \mathscr{X})$.

Finally we have to check that π is compatible with the associativity of the actions. But this readily follows from the pentagon axiom for the action of $\mathscr{C}$ on $\mathscr{X}$. □

5. Semi-simple modules.

A k-linear category is said to be finite semi-simple if it is equivalent to the category of finite dimensional modules over a finite dimensional semi-simple algebra. We aim to show

PROPOSITION 5.1. *Suppose a $\mathscr{C}^G$-module $\mathscr{X}$ corresponds to a $\mathscr{C}[G]$-module $\mathscr{Y}$ in the* 2*-equivalence of Theorem* 4.1. *Then $\mathscr{X}$ is finite semi-simple if and only if $\mathscr{Y}$ is finite semi-simple.*

Clearly this will follow from

PROPOSITION 5.2. *Let $\mathscr{X}$ be a category with G-action. Then $\mathscr{X}$ is finite semi-simple if and only if $\mathscr{X}^G$ is finite semi-simple.*

An object M of a category $\mathscr{X}$ is called an additive generator if every object X of $\mathscr{X}$ is a direct summand of M^n for some $n > 0$. For a k-linear category $\mathscr{X}$ with direct summands, $\mathscr{X}$ is finite semi-simple if and only if $\mathscr{X}$ has an additive generator M such that $\mathrm{End}\, M$ is a finite dimensional semi-simple algebra.

Let $\mathscr{X}$ be a category with G-action. Let $U : \mathscr{X}^G \to \mathscr{X}$ be the forgetful functor $(X, f) \mapsto X$. A left and right adjoint functor T of U is defined by

$$T(X) = \left(\bigoplus_{\tau \in G} \tau_* X, g \right),$$

where $g(\sigma)$ for $\sigma \in G$ is given by the commutative diagram

$$\begin{tikzcd}
\sigma_* \left(\bigoplus_{\tau} \tau_* X \right) \arrow[r, "g(\sigma)"] \arrow[d, "\sigma_* p_\tau"'] & \bigoplus_{\tau} \tau_* X \arrow[d, "p_{\sigma\tau}"] \\
\sigma_* \tau_* X \arrow[r, "\phi(\sigma, \tau)_X^{-1}"'] & (\sigma\tau)_* X
\end{tikzcd}$$

with p_τ, $p_{\sigma\tau}$ the projections. We have trivially

$$U \circ T = \bigoplus_{\tau \in G} \tau_*.$$

For $(X, f) \in \mathscr{X}^G$, let $h : \bigoplus_{\tau \in G} \tau_* X \to X$ be the sum of $f(\tau) : \tau_* X \to X$. The diagram

$$\begin{array}{ccc} \sigma_*\left(\bigoplus_\tau \tau_* X\right) & \xrightarrow{\sigma_* h} & \sigma_* X \\ {\scriptstyle g(\sigma)}\downarrow & & \downarrow{\scriptstyle f(\sigma)} \\ \bigoplus_\tau \tau_* X & \xrightarrow[h]{} & X \end{array}$$

is commutative, because it is composed of the commutative diagrams

$$\begin{array}{ccc} \sigma_* \tau_* X & \xrightarrow{\sigma_* f(\tau)} & \sigma_* X \\ {\scriptstyle \phi(\sigma,\tau)_X^{-1}}\downarrow & & \downarrow{\scriptstyle f(\sigma)} \\ (\sigma\tau)_* X & \xrightarrow[f(\sigma\tau)]{} & X \end{array}$$

for $\tau \in G$. So h defines a morphism $TU(X, f) \to (X, f)$.

Similarly, let $l : X \to \bigoplus_{\tau \in G} \tau_* X$ be the sum of $f(\tau)^{-1}$. Then l defines a morphism $(X, f) \to TU(X, f)$ in $\mathscr{X}^G$.

The composite $\mathrm{Id} \to T \circ U \to \mathrm{Id}$ equals $|G|1$. As we are assuming $|G|$ is invertible in k, $\mathrm{Id} \to T \circ U$ is a split injection.

The following is well-known and the proof is omitted.

LEMMA 5.3. *Let A be a finite dimensional algebra with G-action. Then the skew group algebra $A[G]$ is semi-simple if and only if A is semi-simple.*

PROOF OF PROPOSITION 5.2. If $(X, f) \in \mathscr{X}^G$, then G acts on the algebra $\mathrm{End}\, X$ by

$$\sigma.u = f(\sigma) \circ \sigma_* u \circ f(\sigma)^{-1}$$

for $u \in \mathrm{End}\, X$. The isomorphisms

$$\mathrm{End}\, T(X) \cong \mathrm{Hom}(X, UT(X)) \cong \mathrm{Hom}\left(X, \bigoplus \tau_* X\right) \cong (\mathrm{End}\, X)[G]$$

give the algebra isomorphism of $\mathrm{End}\, T(X)$ to the skew group algebra $(\mathrm{End}\, X)[G]$. By Lemma 5.3, $\mathrm{End}\, T(X)$ is semi-simple if and only if $\mathrm{End}\, X$ is semi-simple. Hence for any object $Y \in \mathscr{X}^G$, $\mathrm{End}\, TU(Y)$ is semi-simple if and only if $\mathrm{End}\, U(Y)$ is semi-simple.

Suppose first that $\mathcal{X}$ is finite semi-simple. For any $Y \in \mathcal{X}^G$, $\operatorname{End} U(Y)$ is semi-simple. By the above observation, $\operatorname{End} TU(Y)$ is semi-simple. Since Y is a direct summand of $TU(Y)$, it follows that $\operatorname{End} Y$ is semi-simple. Take an additive generator $M \in \mathcal{X}$. Then $U(Y)$ is a summand of M^n for some n. Then $TU(Y)$ is a summand of $T(M)^n$, and so is Y. Thus $T(M)$ is an additive generator for $\mathcal{X}^G$. This proves that $\mathcal{X}^G$ is finite semi-simple.

Suppose next that $\mathcal{X}^G$ is finite semi-simple. For any $X \in \mathcal{X}$, $\operatorname{End} TUT(X)$ is semi-simple. Hence by the above observation $\operatorname{End} UT(X)$ is semi-simple. Since X is a summand of $UT(X)$, $\operatorname{End} X$ is semi-simple. Let N be an additive generator in $\mathcal{X}^G$. Then $T(X)$ is a summand of N^n for some n. Then $UT(X)$ is a summand of $U(N)^n$, and so is X. Thus $U(N)$ is an additive generator for $\mathcal{X}$. This proves that $\mathcal{X}$ is finite semi-simple. □

6. Modules over group tensor categories.

In this section we describe modules over a 3-cocycle deformation of $\mathcal{V}[G]$. Most of statements here are simple translations of definitions.

For $\sigma \in G$ we write the object (k, σ) of $\mathcal{V}[G]$ simply as σ. Let $w : G^3 \to k^\times$ be a 3-cocycle. We have the tensor category $\mathcal{V}[G, w]$ whose underlying k-category, tensor product and unit object are the same as those of $\mathcal{V}[G]$, and whose associativity and unit isomorphisms are given by

$$\alpha_{\sigma,\tau,\rho} = w(\sigma,\tau,\rho) 1_{\sigma\tau\rho}$$

$$\lambda_\sigma = w(1,1,\sigma)^{-1} 1_\sigma$$

$$\rho_\sigma = w(\sigma,1,1) 1_\sigma$$

for $\sigma, \tau, \rho \in G$. We call $\mathcal{V}[G, w]$ the group tensor category of the pair (G, w).

Analogously to the identification of a $\mathcal{V}[G]$-module with a category with G-action, a $\mathcal{V}[G, w]$-module is thought of as a k-category equipped with σ_*, $\phi(\sigma,\tau)$, v satisfying the commutativity of the diagrams

$$\begin{tikzcd}
(\sigma(\tau\rho))_* X \arrow[dd, "\phi(\sigma,\tau\rho)_X"'] & ((\sigma\tau)\rho)_* X \arrow[l, "w(\sigma,\tau,\rho)1"'] \arrow[d, "\phi(\sigma\tau,\rho)_X"] \\
 & (\sigma\tau)_* \rho_* X \arrow[d, "\phi(\sigma,\tau)_{\rho_* X}"] \\
\sigma_*(\tau\rho)_* X \arrow[r, "\sigma_*(\phi(\tau,\rho)_X)"'] & \sigma_* \tau_* \rho_* X
\end{tikzcd} \tag{20}$$

(instead of (1)), (2) and (3).

A k-category that is equivalent to a finite direct sum of $\mathscr{V}$ is called a 2-vector space.

All $\mathscr{V}[G,w]$-modules that are 2-vector spaces as categories can be obtained as follows. Let X be a finite G-set and $v : G \times G \times X \to k^{\times}$ a map satisfying

$$w(\sigma,\tau,\rho) = \frac{v(\sigma\tau,\rho;x)v(\sigma,\tau;\rho x)}{v(\tau,\rho;x)v(\sigma,\tau\rho;x)}$$

for $\sigma,\tau,\rho \in G$, $x \in X$. If v is viewed as a map $G \times G \to \mathrm{Map}(X,k^{\times})$, the equations read as

$$i_*(w) = \partial v^{-1}$$

in $\mathrm{Map}(G^3,\mathrm{Map}(X,k^{\times}))$, where ∂ is the coboundary operator for the group G and i_* is the map induced by the embedding $i : k^{\times} \to \mathrm{Map}(X,k^{\times})$. Let $\mathscr{V}[X]$ denote $\bigoplus_{x \in X} \mathscr{V}$, the category of X-graded vector spaces. We may regard an element $x \in X$ as a simple object of $\mathscr{V}[X]$. The action of $\mathscr{V}[G,w]$ on $\mathscr{V}[X]$ is then defined by

$$\sigma_* x = \sigma x$$

$$\phi(\sigma,\tau)_x = v(\sigma,\tau;x)1_{\sigma\tau x}$$

$$v_x = \frac{1}{v(1,1;x)}1_x$$

for $\sigma,\tau \in G$, $x \in X$. We denote by $\mathscr{V}[X,v]$ the $\mathscr{V}[G,w]$-module obtained in this way. Given two pairs (X,v), (X',v') as above, the $\mathscr{V}[G,w]$-modules $\mathscr{V}[X,v]$ and $\mathscr{V}[X',v']$ are equivalent if and only if there exists an isomorphism $f : X \to X'$ of G-sets such that $f^*(v')$ and v are cohomologue in the group $\mathrm{Map}(G^2,\mathrm{Map}(X,k^{\times}))$. Thus the equivalence class of a $\mathscr{V}[G,w]$-module which is a 2-vector space bijectively corresponds to the isomorphism class of a pair $(X,[v])$ of a finite G-set X and an element $[v]$ in the quotient set

$$\frac{\{v \in \mathrm{Map}(G^2,\mathrm{Map}(X,k^{\times})) \mid \partial v = i_*(w)^{-1}\}}{\{\partial t \mid t \in \mathrm{Map}(G,\mathrm{Map}(X,k^{\times}))\}}.$$

Here the group in the denominator acts on the set in the numerator by translation. Note that the quotient is either an empty set or a regular $H^2(G,\mathrm{Map}(X,k^{\times}))$-set.

Let $w = 1$. Then $\mathscr{V}[G,w]$-modules are just k-categories with G-action. So we know that the equivalence class of a 2-vector space $\mathscr{X}$ with G-action bijectively corresponds to the isomorphism class of a pair $(X,[v])$ of a finite G-set X and a cohomology class $[v]$ in $H^2(G,\mathrm{Map}(X,k^{\times}))$.

The category $\mathscr{V}[X,v]^G$ can be described as follows. An object of $\mathscr{V}[X,v]^G$ is a pair (V,f), where V is a family of vector spaces V_x for $x \in X$ and f is a family of linear maps $f(\sigma;x) : V_x \to V_{\sigma x}$ for $\sigma \in G$, $x \in X$ satisfying

$$f(\sigma\tau;x) = f(\sigma;\tau x) \circ f(\tau;x)v(\sigma,\tau;x)$$

for all $\sigma,\tau \in G$, $x \in X$.

Suppose X is a transitive G-set and let K be the stabilizer of an element $x_0 \in X$. The map $v_0 : K^2 \to k^\times$ defined by $v_0(\sigma,\tau) = v(\sigma,\tau;x_0)$ is a 2-cocycle on K. And we have Shapiro's isomorphism $H^2(G,\mathrm{Map}(X,k^\times)) \cong H^2(K,k^\times)$ in which $[v]$ corresponds to $[v_0]$. The pair (V,f) above is determined by the pair (V_{x_0},f_0), where $f_0 : K \to \mathrm{End}\, V_{x_0}$ is defined by $f_0(\sigma) = f(\sigma;x_0)$. Such a pair (V_{x_0},f_0) is just a module over the skew group algebra $k[K,v_0]$ relative to the 2-cocycle v_0. Thus $\mathscr{V}[X,v]^G$ is equivalent to the category of $k[K,v_0]$-modules. Also $\mathscr{V}^G$ is the category of $k[G]$-modules. The action of $\mathscr{V}^G$ on $\mathscr{V}[X,v]^G$ is given by the tensor product through the restriction to the subgroup K.

7. Group actions on group tensor categories.

In this section we apply the 2-equivalence of Theorem 4.1 to a group tensor category with G-action.

Any G-action on a group tensor category is obtained in the following way. Let A be a group with G-action denoted by $(\sigma,a) \mapsto {}^\sigma a$. Let

$$t : A \times A \times A \to k^\times$$

$$u : G \times A \times A \to k^\times$$

$$v : G \times G \times A \to k^\times$$

be maps satisfying

$$1 = \frac{t(b,c,d)t(a,bc,d)t(a,b,c)}{t(ab,c,d)t(a,b,cd)} \tag{21}$$

$$\frac{t(a,b,c)}{t({}^\sigma a,{}^\sigma b,{}^\sigma c)} = \frac{u(\sigma;b,c)u(\sigma;a,bc)}{u(\sigma;ab,c)u(\sigma;a,b)} \tag{22}$$

$$\frac{u(\sigma;{}^\tau a,{}^\tau b)u(\tau;a,b)}{u(\sigma\tau;a,b)} = \frac{v(\sigma,\tau;ab)}{v(\sigma,\tau;a)v(\sigma,\tau;b)} \tag{23}$$

$$\frac{v(\sigma\tau,\rho;a)v(\sigma,\tau;{}^\rho a)}{v(\tau,\rho;a)v(\sigma,\tau\rho;a)} = 1 \tag{24}$$

for all $\sigma, \tau, \rho \in G$, $a, b, c, d \in A$. (21) says t is a 3-cocycle of A, so we have the group tensor category $\mathscr{V}[A, t]$ of Section 6. A G-action on this tensor category is defined by

$$\sigma_*(a) = {}^{\sigma}a$$

$$\phi(\sigma, \tau)_a = v(\sigma, \tau; a)1_{{}^{\sigma\tau}a}$$

$$v_a = \frac{1}{v(1, 1; a)}1_a$$

$$\psi(\sigma)_{a,b} = u(\sigma; a, b)1_{{}^{\sigma}(ab)}$$

$$\iota(\sigma) = \frac{1}{u(\sigma; 1, 1)}1_1$$

for $\sigma, \tau \in G$, $a, b \in A$. Indeed, commutativity of (6), (8), (1) is assured by (22), (23), (24), while that of (2), (3), (7) by the definition of v, ι, respectively.

By the definition of $\mathscr{C}[G]$ in Section 2, we have $\mathscr{V}[A, t][G] = \mathscr{V}[A \rtimes G, s]$, where s is a 3-cocycle on the semi-direct product $A \rtimes G$ given by

$$s((a, \sigma), (b, \tau), (c, \rho)) = t(a, {}^{\sigma}b, {}^{\sigma\tau}c)u(\sigma; b, {}^{\tau}c)v(\sigma, \tau; c).$$

Theorem 4.1 applied to the G-tensor category $\mathscr{V}[A, t]$ says that the 2-functor

$$\mathscr{V}[A, t]^G\text{-Modk} \underset{(-)^G}{\longleftarrow} \mathscr{V}[A \rtimes G, s]\text{-Modk}$$

is a 2-equivalence. Assume k is algebraically closed. Then finite semi-simple categories are just 2-vector spaces. By Proposition 5.1 the property of being a 2-vector space is preserved under the above 2-equivalence. We saw in Section 6 that any $\mathscr{V}[A \rtimes G, s]$-module which is a 2-vector space is of the form $\mathscr{V}[X, r]$ for a finite $A \rtimes G$-set X and a map $r : (A \rtimes G)^2 \times X \to k^\times$ satisfying $i_*(s) = \partial r^{-1}$. Hence any $\mathscr{V}[A, t]^G$-module which is a 2-vector space is of the form $\mathscr{V}[X, r]^G$.

8. Pentagon identity for $\mathscr{C}[G]$.

We will show here that the associativity isomorphisms $\alpha_{(A,\sigma),(B,\tau),(C,\rho)}$ for $\mathscr{C}[G]$ defined in Section 2 satisfy the pentagon axiom of a monoidal category:

$$(A, \sigma).\alpha_{(B,\tau),(C,\rho),(D,\pi)} \circ \alpha_{(A,\sigma),(B,\tau).(C,\rho),(D,\pi)} \circ \alpha_{(A,\sigma),(B,\tau),(C,\rho)}.(D, \pi)$$

$$= \alpha_{(A,\sigma),(B,\tau),(C,\rho).(D,\pi)} \circ \alpha_{(A,\sigma).(B,\tau),(C,\rho),(D,\pi)}$$

for $\sigma, \tau, \rho, \pi \in G$, $A, B, C, D \in \mathscr{C}$.

$$
\begin{array}{ccccccc}
(((A,\sigma).(B,\tau)).(C,\rho)).(D,\pi) & = & ((A.{}^{\sigma}B,\sigma\tau).(C,\rho)).(D,\pi) & = & ((A.{}^{\sigma}B).{}^{\sigma\tau}C,\sigma\tau\rho).(D,\pi) & = & (((A.{}^{\sigma}B).{}^{\sigma\tau}C).{}^{\sigma\tau\rho}D,\sigma\tau\rho\pi) \\
{\scriptstyle \alpha_{(A,\sigma),(B,\tau),(C,\rho)}.(D,\pi)}\downarrow & & & & \downarrow & & \downarrow{\scriptstyle (\alpha(A,\sigma,B,\tau,C).{}^{\sigma\tau\rho}D,\sigma\tau\rho\pi)} \\
((A,\sigma).((B,\tau).(C,\rho))).(D,\pi) & = & ((A,\sigma).(B.{}^{\tau}C,\tau\rho)).(D,\pi) & = & (A.{}^{\sigma}(B.{}^{\tau}C),\sigma\tau\rho).(D,\pi) & = & ((A.{}^{\sigma}(B.{}^{\tau}C)).{}^{\sigma\tau\rho}D,\sigma\tau\rho\pi) \\
{\scriptstyle \alpha_{(A,\sigma),(B,\tau).(C,\rho),(D,\pi)}}\downarrow & & \downarrow & & & & \downarrow{\scriptstyle (\alpha(A,\sigma,B.{}^{\tau}C,\tau\rho,D),\sigma\tau\rho\pi)} \\
(A,\sigma).(((B,\tau).(C,\rho)).(D,\pi)) & = & (A,\sigma).((B.{}^{\tau}C,\tau\rho).(D,\pi)) & = & (A,\sigma).((B.{}^{\tau}C).{}^{\tau\rho}D,\tau\rho\pi) & = & (A.{}^{\sigma}((B.{}^{\tau}C).{}^{\tau\rho}D),\sigma\tau\rho\pi) \\
{\scriptstyle (A,\sigma).\alpha_{(B,\tau),(C,\rho),(D,\pi)}}\downarrow & & & & \downarrow & & \downarrow{\scriptstyle (A.{}^{\sigma}\alpha(B,\tau,C,\rho,D),\sigma\tau\rho\pi)} \\
(A,\sigma).((B,\tau).((C,\rho).(D,\pi))) & = & (A,\sigma).((B,\tau).(C.{}^{\rho}D,\rho\pi)) & = & (A,\sigma).(B.{}^{\tau}(C.{}^{\rho}D),\tau\rho\pi) & = & (A.{}^{\sigma}(B.{}^{\tau}(C.{}^{\rho}D)),\sigma\tau\rho\pi)
\end{array}
$$

$$
\begin{array}{ccccccc}
(((A,\sigma).(B,\tau)).(C,\rho)).(D,\pi) & = & ((A.{}^{\sigma}B,\sigma\tau).(C,\rho)).(D,\pi) & = & ((A.{}^{\sigma}B).{}^{\sigma\tau}C,\sigma\tau\rho).(D,\pi) & = & (((A.{}^{\sigma}B).{}^{\sigma\tau}C).{}^{\sigma\tau\rho}D,\sigma\tau\rho\pi) \\
{\scriptstyle \alpha_{(A,\sigma).(B,\tau),(C,\rho),(D,\pi)}}\downarrow & & \downarrow & & & & \downarrow{\scriptstyle (\alpha(A.{}^{\sigma}B,\sigma\tau,C,\rho,D),\sigma\tau\rho\pi)} \\
((A,\sigma).(B,\tau)).((C,\rho).(D,\pi)) & = & (A.{}^{\sigma}B,\sigma\tau).((C,\rho).(D,\pi)) & = & (A.{}^{\sigma}B,\sigma\tau).(C.{}^{\rho}D,\rho\pi) & = & ((A.{}^{\sigma}B).{}^{\sigma\tau}(C.{}^{\rho}D),\sigma\tau\rho\pi) \\
\| & & & & \| & & \| \\
((A,\sigma).(B,\tau)).((C,\rho).(D,\pi)) & = & ((A,\sigma).(B,\tau)).(C.{}^{\rho}D,\rho\pi) & = & (A.{}^{\sigma}B,\sigma\tau).(C.{}^{\rho}D,\rho\pi) & = & ((A.{}^{\sigma}B).{}^{\sigma\tau}(C.{}^{\rho}D),\sigma\tau\rho\pi) \\
{\scriptstyle \alpha_{(A,\sigma),(B,\tau),(C,\rho).(D,\pi)}}\downarrow & & \downarrow & & & & \downarrow{\scriptstyle (\alpha(A,\sigma,B,\tau,C.{}^{\rho}D),\sigma\tau\rho\pi)} \\
(A,\sigma).((B,\tau).((C,\rho).(D,\pi))) & = & (A,\sigma).((B,\tau).(C.{}^{\rho}D,\rho\pi)) & = & (A,\sigma).(B.{}^{\tau}(C.{}^{\rho}D),\tau\rho\pi) & = & (A.{}^{\sigma}(B.{}^{\tau}(C.{}^{\rho}D)),\sigma\tau\rho\pi)
\end{array}
$$

Figure 8.1

A pentagon diagram for $\mathscr{C}$ is

$$
\begin{tikzcd}
(A.(B.C)).D \arrow[dd, "\alpha_{A,B.C,D}"'] & ((A.B).C).D \arrow[l, "\alpha_{A,B,C}.D"'] \arrow[d, "\alpha_{A.B,C,D}"] \\
 & (A.B).(C.D) \arrow[d, "\alpha_{A,B,C.D}"] \\
A.((B.C).D) \arrow[r, "A.\alpha_{B,C,D}"'] & A.(B.(C.D)).
\end{tikzcd}
$$

We refer this diagram as $I_0(A,B,C,D)$, and also the diagram (6) as $I_1(\sigma,A,B,C)$, (8) as $I_2(\sigma,\tau,A,B)$, (1) as $I_3(\sigma,\tau,\rho,X)$. The assumption is that $I_0(A,B,C,D)$, $I_1(\sigma,A,B,C)$, $I_2(\sigma,\tau,A,B)$, $I_3(\sigma,\tau,\rho,A)$ are commutative for all $A,B,C,D \in \mathscr{C}$, $\sigma,\tau,\rho \in G$.

To save space we write here and below ${}^{\sigma}A$ instead of $\sigma_* A$.

By (11), we have the equalities in Figure 8.1 for morphisms in $\mathscr{C}[G]$. Hence what we have to prove is the equality

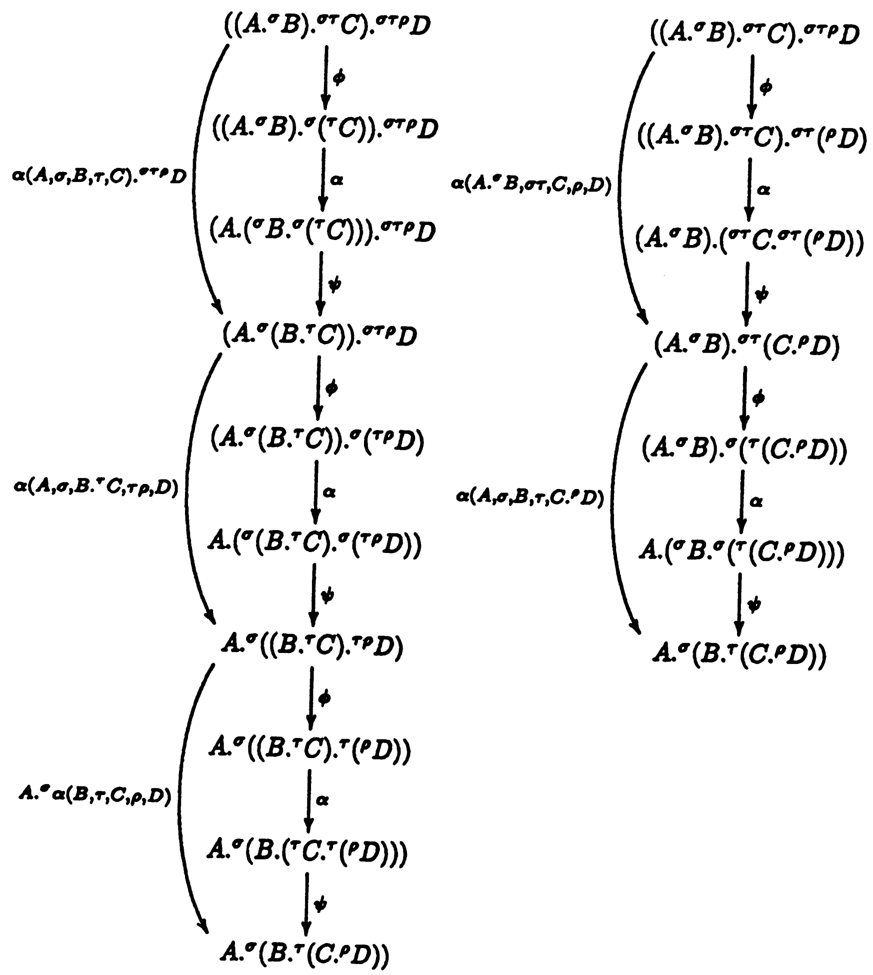

Figure 8.2

$$A.^{\sigma}\alpha(B,\tau,C,\rho,D)\circ\alpha(A,\sigma,B.^{\tau}C,\tau\rho,D)\circ\alpha(A,\sigma,B,\tau,C).^{\sigma\tau\rho}D$$
$$=\alpha(A,\sigma,B,\tau,C.^{\rho}D)\circ\alpha(A.^{\sigma}B,\sigma\tau,C,\rho,D) \tag{25}$$

in $\mathscr{C}$.

By the definition of $\alpha(-,-,-,-,-)$ in (12), the both sides of (25) are expanded as in Figure 8.2, respectively. Arrows are labeled only by their types.

Now we have the commutative diagrams in Figures 8.3 and 8.4. There are four faces induced from the diagrams of type I_0, I_1, I_2, I_3 which are commutative. Faces labeled by $[\alpha]$, $[\psi(\sigma)]$ are commutative by the naturality of α, $\psi(\sigma)$, and faces labeled by $[\otimes]$ are commutative by the functoriality of the tensor product. Hence in either diagram, the composite along the leftmost path and the composite along the rightmost path from $((A.^{\sigma}B).^{\sigma\tau}C).^{\sigma\tau\rho}D$ to $A.^{\sigma}(B.^{\tau}(C.^{\rho}D))$ are equal. Moreover the rightmost paths in the both diagrams are literally identical.

The leftmost path in Figure 8.3 and the left path in Figure 8.2 are identical, and the leftmost path in Figure 8.4 and the right path in Figure 8.2 are identical.

Hence the both sides of (21) are equal.

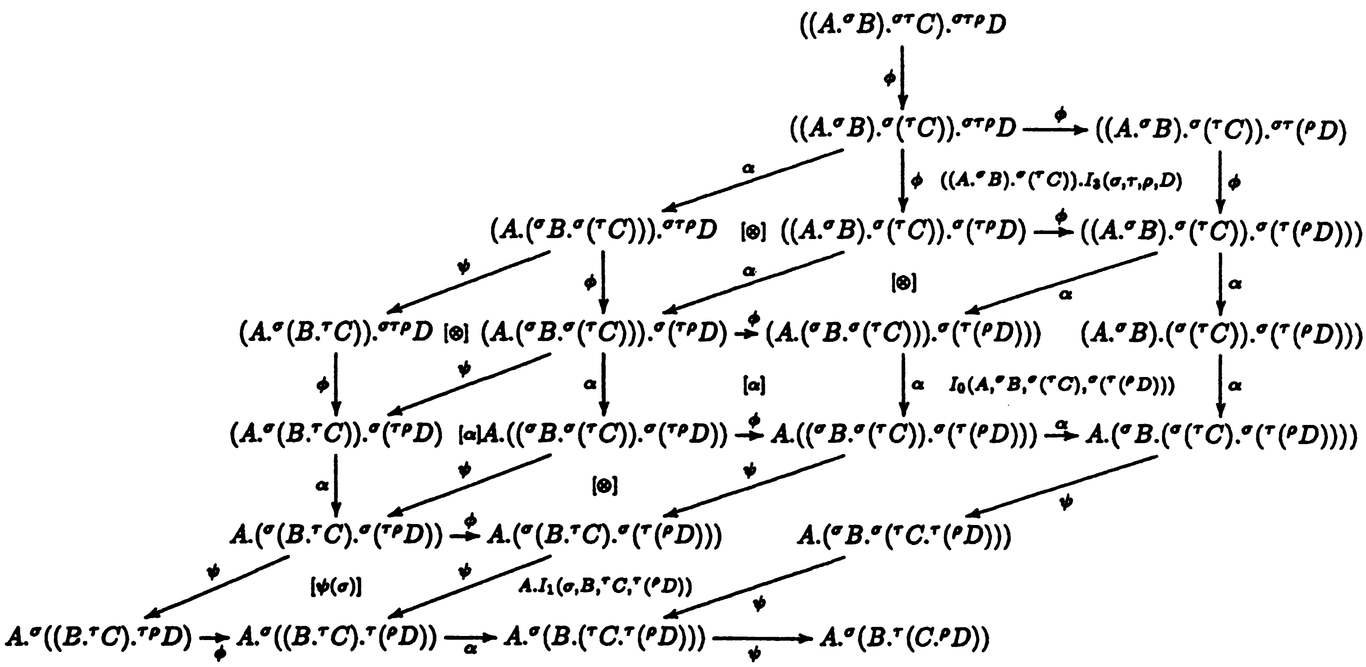

Figure 8.3

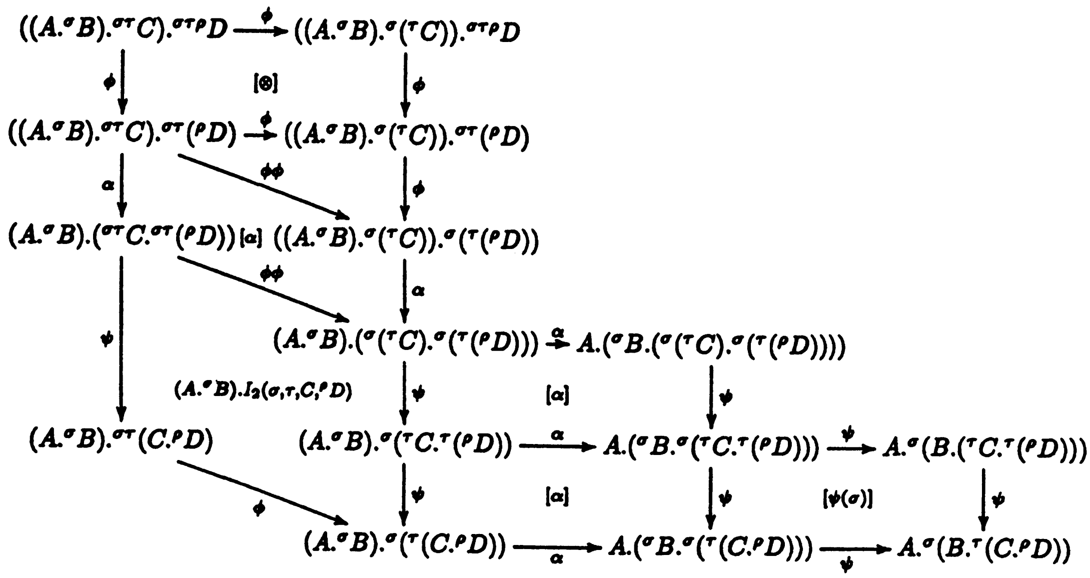

Figure 8.4

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